problem sheet 5

discussion: Tuesday, 10.11.20

5.1. What are the physical units of the following quantities: a) stress tensor $\boldsymbol{\sigma}$, b) heat flux $\mathbf{q}$, c) heat conductivity K, d) viscosities λ and μ where λ, μ are (as in the lecture) $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\varepsilon(\mathbf{v}) + \lambda\,\mathrm{tr}(\varepsilon(\mathbf{v}))\mathbf{I}$ with $\varepsilon(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\right)$

5.2. Let the velocity field of an incompressible fluid (assuming the usual NSE) is $\mathbf{v} = (-\alpha y, \alpha x, \beta)^\top$ for some α, $\beta \in \mathbb{R}$.

a) Show that $\mathbf{v}$ satisfies the continuity equation

b) Assuming that no external forces are applied, find the pressure

c) What are the path lines?

d) This is known as the “steady helical flow”. Why?

5.3. Consider a tetrahedron as in the proof of Thm. 2.7, i.e., $T \subset \mathbb{R}^d$ with faces $S_i \subset \{x_i = 0\}$ and $S_\mathbf{n}$ with normal vector $\mathbf{n}$. Let $\mathbf{n}_j > 0$, $j = 1, \ldots, d$.

a) for $d = 3$ show by elementary geometric considerations that $|S_i| = \mathbf{n}_i |S_\mathbf{n}|$.

b) Using Gauss’ theorem, show $|S_j| = \mathbf{n}_j |S_\mathbf{n}|$ for any d.

5.4. Show that the stress tensor σ is symmetric if conservation of angular momentum holds in the following form:

$$\int_{\Omega(t)} x \times \left(\partial_t(\rho v) + \mathrm{Div}(\rho v v^\top)\right)\, dx = \int_{\Omega(t)} x \times (\rho f)\, dx + \int_{\partial\Omega(t)} x \times (\sigma n)\, ds_x$$

for all (suitable) “initial volumes” $\Omega = \Omega(t_0) \subset \mathbb{R}^3$. Proceed as follows (in the following, you may use the identity $a \cdot (b \times c) = (a \times b) \cdot c$):

a) Let $a \in \mathbb{R}^3$ be a fixed vector. Show that

$$a \cdot \int_{\partial\Omega(t)} x \times (\sigma n)\, ds_x = \int_{\Omega(t)} (a \times x) \cdot \mathrm{Div}\,\sigma + \sum_{i,j=1}^{3} \partial_j\, (a \times x)_i\, \sigma_{ij}\, dx.$$

b) Using conservation of momentum and the above formulated conservation of angular momentum show that

$$\sum_{i,j=1}^{3} \partial_j\, (a \times x)_i\, \sigma_{ij} = 0.$$

(You may, of course, assume that all functions appearing are sufficiently smooth.)

c) Show the symmetry of σ by selecting a suitably.

5.5. Show that a solution of the incompressible Euler equations,

$$\nabla \cdot v = 0, \qquad \partial_t \rho + \nabla \cdot (\rho v) = 0, \qquad \partial_t(\rho v) + \mathrm{Div}(\rho v v^\top) + \nabla p = \rho f,$$

satisfies automatically the “energy equation”

$$\partial_t(\frac{1}{2}\rho|v|^2) + \nabla \cdot (\frac{1}{2}\rho|v|^2 v + pv) = \rho f \cdot v$$

What can you infer for the energy equation?

5.6. (Kelvin's Circulation Theorem) Let $C \subset \Omega \subset \mathbb{R}^d$ (e.g., $d = 3$) be a closed sufficiently smooth curve. Let $C(t)$ be the transported curve, i.e., $C(t) = \varphi(t, C)$. Let $\mathbf{v}$ be the velocity field. Denote by $D_t f(t, x) := \partial_t f + \nabla f \cdot \mathbf{v}$ the material derivative of the scalar function f; the material derivative of a vector-valued function is understood componentwise. The *circulation* is the line integral

$$\Gamma_{C(t)} = \oint_{C(t)} \mathbf{v}(t, x) \cdot d\mathbf{s}.$$

a) Show:

$$\frac{d}{dt} \oint_{C(t)} \mathbf{v} \cdot d\mathbf{s} = \int_{C(t)} D_t \mathbf{v} \cdot d\mathbf{s}.$$

b) Consider an inviscid fluid flow with $\rho = const$[1]. Assuming conservation of momentum and assuming that no external forces are present, show that the circulation is constant (in time) for every (sufficiently smooth) closed curve C.

c) Let $d = 3$. Define the *vorticity* ω of a fluid flow by

$$\omega := \nabla_x \times \mathbf{v} = \mathrm{rot}_x \mathbf{v}.$$

Consider a flow with $\omega(0, \cdot) = 0$. Show (under the assumptions of b)) that then $\omega(t, \cdot) = 0$ for all $t > 0$. That is: if the flow is irrotational at $t = 0$, then it stays irrotational. *Hint: Stokes' theorem.*
Remark: If the domain is simply connected, then such a flow field is a *potential flow*, i.e., $\mathbf{v} = \nabla_x \psi$ for a suitable scalar function ψ.

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[1] in fact, the following, slightly weaker assumption is sufficient: for the pressure field p there is a function w with $\nabla w = \frac{1}{\rho} \nabla p$