problem sheet 4

discussion: Tuesday, 3.11.20

4.1. (Buckley-Leverett equation) Consider $u_t + (f(u))_x = 0$ with

$$f(u) = \frac{u^2}{u^2 + (1-u)^2}, \qquad u_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

What is the entropy solution? You may use (without proof) that the solution is given by a rarefaction wave followed by a shock of the form $x = st$. What is the shock speed s? *Hint:* For the relevant ξ one has

$$(f')^{-1}(\xi) = \frac{1}{2}\left(1 - \sqrt{\frac{1}{\xi}\left(\sqrt{4\xi+1} - 1\right) - 1}\right)$$

4.2. (Lagrangian and Eulerian coordinates) Let the reference configuration be $\Omega = (0,1)$ and assume the motion is given by $x(t,X) = X(1+e^t)$. What is $\Omega(2)$? What is the velocity in Lagrangian coordinate $V(t,X)$ and in Eulerian coordinates $v(t,x)$? Suppose the temperature of the body is $\theta(t,x) = xt^3$. What is the rate of change in temperature when following material points?

4.3. (Lagrangian and Eulerian coordinates) Consider the 1D case. The *displacement* (in Lagrangian coordinates) is $U(t,X) = x(t,X) - X$, i.e., the deviation of the material point X at time t from its original position. Let u be the function U in Eulerian coordinates. Show: the velocity v is given by $v = D_t u$, where D_t is the material derivative. Show also $v = \frac{u_t}{1-u_x}$.

4.4. Let the following 2D velocity field be given:

$$v(x_1, x_2) = (x_2, x_1)^\top.$$

Show that the streamlines are hyperbolas (i.e., given by $a^2x_1^2 - b^2x_2^2 = c$ for suitable a, b, c). Show that the streamlines are actually path lines.

4.5. Show for matrices B, C

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\det(B + \varepsilon CB) - \det B\right) = \operatorname{tr}(C)\det B$$

Conclude for C^1-mappings $t \mapsto A(t)$ with the additional property that $A(t)$ is invertible (for each fixed t) that

$$\frac{d}{dt}\det A(t) = \operatorname{tr}(A^{-1}(t)A'(t))\det A(t).$$

4.6. Let the mapping $\Omega \ni X \mapsto x(t,X)$ from Lagrangian to Eulerian coordinates be smooth and invertible. Define $\mathbf{A}(t,X)$ by $\mathbf{A}_{ij}(t,X) = \frac{\partial_i x(t,X)}{\partial X_j}$. Set $J(t,X) = \det \mathbf{A}(t,X)$. Show:

$$\nabla_X \cdot (J\mathbf{A}^{-T}) \equiv 0. \tag{1}$$

Although this could be checked by a “direct” (somewhat messy) calculation (i.e., by product rule), proceed as follows: Let $\overline{\eta} \in C_0^\infty(\Omega)$ and define $\eta(t,x)$ by $\eta(t,x) = \overline{\eta}(X(t,x))$. Show then:

$$\int_\Omega \nabla_X \cdot (J(t,X)\mathbf{A}^{-T}(t,X))\overline{\eta}(X)\,dX = -\int_{\Omega(t)} \nabla_x \eta(t,x)\,dx.$$

Conclude (1) from this relation.