problem sheet 2

discussion: Tuesday, 20.10.20

2.1. Consider an initial value problem for the scalar function $u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ of the form

$$\partial_t u + \nabla_{\mathbf{x}} \cdot (\mathbf{f}(u)) = 0 \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \qquad u(\cdot, 0) = u_0(\cdot).$$

where $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ is sufficiently smooth and $\nabla_{\mathbf{x}}\cdot$ denotes the divergence w.r.t. the variable $\mathbf{x}$.

a) Formulate the notion of weak solution for this problem.

b) Formulate the Rankine-Hugoniot condition at points of discontinuity, if u is a piecewise smooth solution of the equation. You may assume that the discontinuity surface is parametrized by (s, t) (i.e., is of the form $(\psi_1(s,t), \psi_2(s,t), t)$ for (s,t) in a suitable subset of $\mathbb{R}^2$) and has normal vector $\mathbf{n}(s,t) = (\mathbf{n}_{x_1}(s,t), \mathbf{n}_{x_2}(s,t), \mathbf{n}_t(s,t))^\top$.

2.2. Let $u_0(x) = u_l$ for $x < 0$ and $u_0(x) = u_r$ for $x > 0$ where $u_l < u_r$. Show: For every u_m with $u_l \leq u_m \leq u_r$ and $s_m = (u_l + u_m)/2$ the function

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \leq x \leq u_m t \\ x/t & u_m t \leq x \leq u_r t \\ u_r & x > u_r t \end{cases}$$

is a weak solution of Burgers' equation $\partial_t u + \partial_x f(u)) = 0$ with $f(u) = \frac{1}{2}u^2$. Sketch the characteristics. Give a further solution with 2 jumps.

2.3. Consider the Riemann problem of Problem 2 with, however, $u_l > u_r$. Show:

a) There are no weak solutions consists of exactly 2 shocks.

b) There exist weak solution consisting of 3 shocks.

2.4. Show that a classical solution of Burgers' equation also solves

$$\partial_t(u^2) + \partial_x\left(\frac{2}{3}u^3\right) = 0. \tag{1}$$

Show that in the case $u_l > u_r$ the shock solution of the Riemann problem for Burgers' equation is *not* a weak solution of (1). Give a shock solution of (1).

2.5. Let $\Phi_{\geq 0} = \{\varphi \in C_0^1(\mathbb{R}^2) \,|\, \varphi \geq 0\}$. Let $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ and $u_0 \in L^\infty(\mathbb{R})$. Let f be sufficiently smooth. Define on $\Phi_{\geq 0}$ for every entropy-entropy flux pair (η, ψ) the functional

$$\Lambda_{(\eta,\psi)}(\varphi) := \int_{\mathbb{R}\times\mathbb{R}^+} \eta(u)\varphi_t + \psi(u)\varphi_x + \int_{\mathbb{R}} \eta(u_0)\varphi(\cdot, 0).$$

a) Let ρ_ε be the "usual" "mollifier" and $\varphi \in L^1(\mathbb{R}^2)$ with $\varphi \geq 0$. Show: $\varphi_\varepsilon := \varphi * \rho_\varepsilon$ is smooth and satisfies $\varphi_\varepsilon \geq 0$ and $\varphi_\varepsilon \to \varphi$ (in L^1).

b) Show: If for $\eta(u) = u$ (and correspondingly $\psi(u) = f(u)$) and $\eta(u) = -u$ (and correspondingly $\psi(u) = -f(u)$) the relation $\Lambda_{(\eta,\psi)}(\varphi) \geq 0$ holds for all $\varphi \in \Phi_{\geq 0}$, then u is a weak solution.
Hint: Decompose an arbitrary $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$ as $\varphi = \varphi^+ - \varphi^-$ with φ^+, $\varphi^- \geq 0$; smooth φ^+ and φ^-.