Serie 1

discussion: Tuesday, 13.10.20

1.1. Consider the scalar conservation law

$$\partial_t u + \partial_x(f(u)) = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \tag{1}$$

with initial data $u(\cdot, 0) = u_0$. Solve this equation using the method of characteristics.

1.2. Consider the "initial value problem"

$$xu_y - yu_x = u \qquad (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+, \qquad u(x,0) = h(x)$$

Determine the solution using the method of characteristics. Sketch the characteristics near the lines $\{(x,0) \,|\, x > 0\}$ and $\{(0,y) \,|\, y0\}$.

1.3. Often, the variable u in Burgers' equation (i.e., $f(u) = \frac{1}{2}u^2$ in (1)) the meaning of a velocity. Show that (at least for smooth solutions) the Burgers' equation is Galileo invariant, i.e., if $u = u(x,t)$ is a solution, then for fixed $v_0 \in \mathbb{R}$ also the function $\widetilde{u}(x,t) := v_0 + u(x - v_0 t, t)$ is a solution.

1.4. The "shallow water equations" are given by

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \partial_x \begin{pmatrix} hv \\ \frac{1}{2}v^2 + gh \end{pmatrix} = 0 \qquad \text{für } (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

Show that the system is hyperbolic in the sense of Def. 1.1. You may assume $h > 0$; g is the acceleration of gravity. *Remark:* The equation described waves in channel (filled with a fluid); h is the height of the fluid and $v(x,t)$ its velocity. The equations express conservation of mass and momentum.

1.5. For $\varepsilon > 0$ the equation

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = \varepsilon \partial_x^2 u \tag{2}$$

is called the *viscous Burgers' equation*. Solutions of (2) of the form $u(x,t) = w(x - st)$ (with $s \in \mathbb{R}$) are called "travelling wave" solutions. Show: For u_l, $u_r \in \mathbb{R}$ the profile

$$w(x) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh((u_l - u_r)x/(4\varepsilon))\right]$$

produces a "travelling wave" solution. What is s? Sketch the profile w for different values of ε.

Remark: The nonlinear equation (2) can actually be solved explicitly: The Cole-Hopf transformation ($u = -2\varepsilon\varphi_x/\varphi$) transforms (2) to a heat equation (for φ), for which a solution formula is available.