Kruzkov's theorem

We recall that the entropy condition implies: For all $\varphi \geq 0$ compactly supported in $\mathbb{R} \times \mathbb{R}^+$ there holds

$$0 \leq \int_{\mathbb{R}\times\mathbb{R}^+} |u(x,y) - k|\varphi_t(x,t) + \operatorname{sign}(u(x,t)-k)\left[f(u(x,t)) - f(k)\right]\varphi_x(x,t) \quad \forall k \in \mathbb{R}. \quad (1)$$

(Note that the support properties of φ remove the term $\int_{\mathbb{R}} |u(\cdot,0) - k|\varphi(\cdot,0)$.)

Theorem 0.1 (Kruzkov)**.** *Let $f \in C^1(\mathbb{R})$. Let $u_1,\ u_2 \in C([0,\infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}\times\mathbb{R}^+)$ be two entropy solutions. Then: for $0 \leq t_1 \leq t_2$*

$$\|u_1(\cdot,t_2) - u_2(\cdot,t_2)\|_{L^1(\mathbb{R})} \leq \|u_1(\cdot,t_1) - u_2(\cdot,t_1)\|_{L^1(\mathbb{R})}.$$

Proof. The "trick" consists in doubling the number of variables (so as to be able to use the entropy condition (1)) and then collapse integrals by a suitable choice of test functions.

We will assume throughout the proof that $0 < t_1 < t_2$. The limiting cases $0 \leq t_1 \leq t_2$ then follow.

1. step: Let $\varphi = \varphi(x,t,y,s)$ be a function of 4 arguments with compact support in $\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$ and $\varphi \geq 0$.

Fix $(y,s) \in \mathbb{R}\times\mathbb{R}^+$. Then with $u = u_1$ and $k = u_2(y,s)$ we get from (1) (note the support properties of φ)

$$0 \leq \int_{\mathbb{R}\times\mathbb{R}^+} |u_1(x,t) - u_2(y,s)|\varphi_t + \operatorname{sign}(u_1(x,t) - u_2(y,s))\left[f(u_1(x,t)) - f(u_2(y,s))\right]\varphi_x$$

Integrating in (y,s) gives

$$0 \leq \int_{\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+} |u_1(x,t) - u_2(y,s)|\varphi_t + \operatorname{sign}(u_1(x,t) - u_2(y,s))\left[f(u_1(x,t)) - f(u_2(y,s))\right]\varphi_x \quad (2)$$

Reversing the roles of u_1 and u_2, we get

$$0 \leq \int_{\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+} |u_1(x,t) - u_2(y,s)|\varphi_s + \operatorname{sign}(u_1(x,t) - u_2(y,s))\left[f(u_1(x,t)) - f(u_2(y,s))\right]\varphi_y \quad (3)$$

Adding equations (2), (3) gives

$$\begin{aligned} 0 \leq \int_{\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+} & |u_1(x,t) - u_2(y,s)|\left[\varphi_t + \varphi_s\right] \\ & + \operatorname{sign}(u_1(x,t) - u_2(y,s))\left[f(u_1(x,t)) - f(u_2(y,s))\right]\left[\varphi_x + \varphi_y\right]. \end{aligned} \quad (4)$$

By the assumptions of the support of φ we can (by extending the functions u_1, u_2 by zero for negatives times) write this as

$$0 \leq \int_{\mathbb{R}^4} |u_1(x,t) - u_2(y,s)| \left[\varphi_t + \varphi_s\right] \tag{5}$$
$$+ \operatorname{sign}(u_1(x,t) - u_2(y,s)) \left[f(u_1(x,t)) - f(u_2(y,s))\right] \left[\varphi_x + \varphi_y\right].$$

In order to collapse the integral over $\mathbb{R}^2$ into one over $\mathbb{R}^2$ we let $\psi \geq 0$ be smooth with compact support in $\mathbb{R} \times \mathbb{R}^+$ and let ρ_ε be the "usual" (non-negative) mollifier and select

$$\varphi(x,t,y,s) := \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \rho_\varepsilon(|x-y|/2)\rho_\varepsilon((t-s)/2),$$

make the change of variables

$$\widehat{x} := \frac{x+y}{2}, \quad \widehat{t} := \frac{t+s}{2}, \quad \widehat{y} := \frac{x-y}{2}, \quad \widehat{s} := \frac{t-s}{2},$$

and observe

$$\varphi_t + \varphi_s = \psi_{\widehat{t}}(\widehat{x},\widehat{t})\rho_\varepsilon(|\widehat{y}|)\rho_\varepsilon(\widehat{s}),$$
$$\varphi_x + \varphi_y = \psi_{\widehat{x}}(\widehat{x},\widehat{t})\rho_\varepsilon(|\widehat{y}|)\rho_\varepsilon(\widehat{s}).$$

Inserting the above in (5) yields

$$0 \leq \int_{(\widehat{y},\widehat{s})\in\mathbb{R}^2} \rho_\varepsilon(|\widehat{y}|)\rho_\varepsilon(\widehat{s}) \Bigg\{ \int_{(\widehat{x},\widehat{t})\in\mathbb{R}^2} \left|u_1(\widehat{x}+\widehat{y},\widehat{t}+\widehat{s}) - u_2(\widehat{x}-\widehat{y},\widehat{t}-\widehat{s})\right| \psi_{\widehat{t}}(\widehat{x},\widehat{t}) \tag{6}$$
$$+ \operatorname{sign}\left(u_1(\widehat{x}+\widehat{y},\widehat{t}+\widehat{s}) - u_2(\widehat{x}-\widehat{y},\widehat{t}-\widehat{s})\right)\left(f(u_1(\widehat{x}+\widehat{y},\widehat{t}+\widehat{s})) - f(u_2(\widehat{x}-\widehat{y},\widehat{t}-\widehat{s}))\right)\psi_{\widehat{x}}(\widehat{x},\widehat{t})\Bigg\}$$

By Lemma 0.2 the function $(\widehat{y},\widehat{s}) \mapsto \{\cdots\}$ is continuous. Hence, by letting $\varepsilon \to 0$, we obtain

$$0 \leq \int_{\mathbb{R}\times\mathbb{R}^+} |u_1(x,t) - u_2(x,t)|\psi_t + \operatorname{sign}(u_1(x,t) - u_2(x,t))(f(u_1(x,t)) - f(u_2(x,t)))\psi_x \tag{7}$$

for all $\psi \geq 0$ with compact support in $\mathbb{R} \times \mathbb{R}^+$.

In order to obtain the result stated in the theorem, we need to select the test function ψ suitably. In order to get an idea what to do, we proceed *formally* (the rigorous argument is given below by smoothing the characteristic function χ_K). Let, for $M > 0$ (which will be chosen below) and $R > 0$ the set K be the trapezoid with corners $(-R, t_1)$, (R, t_1), $(R - M(t_2 - t_1), t_2)$, $(-R + M(t_2 - t_1), t_2)$. Let χ_K be the characteristic function of K. We also introduce $\Gamma_L = \{(x,t) \,|\, t_1 < t < t_2, |x| = R - M(t - t_1)\}$ for the "lateral" sides of K. Note that the derivatives $\partial_t\chi_K$, $\partial_x\chi_K$ are Dirac distributions supported by ∂K. Finally, introduce

$$\mathbf{F} := (\operatorname{sign}(u_1 - u_2)(f(u_1) - f(u_2)), |u_1 - u_2|)^\top.$$

Then, we get from (7) with $\psi = \chi_K$

$$
\begin{aligned}
0 \le \int_{\mathbb{R}^2} \mathbf{F}\cdot\nabla_{x,t}\chi_K &\overset{\text{def.}}{=} -\int_{\mathbb{R}^2} \nabla_{x,t}\cdot\mathbf{F}\chi_K = -\int_K \nabla_{x,t}\cdot\mathbf{F} = -\int_{\partial K}\mathbf{F}\cdot\mathbf{n}_K \\
&= -\Bigg[\int_{|x|\le R-M(t_2-t_1)} |u_1(x,t_2)-u_2(x,t_2)|\,dx - \int_{|x|\le R}|u_1(x,t_1)-u_2(x,t_1)|\,dx \\
&\qquad + \int_{\Gamma_L} \operatorname{sign}(u_1-u_2)(f(u_1)-f(u_2))\frac{x}{|x|} + M|u_1-u_2|\Bigg].
\end{aligned}
$$

Since u_1, $u_2 \in L^\infty$, we choose

$$
M := \sup_{v\colon |v|\le \max\{\|u_1\|_{L^\infty},\|u_2\|_{L^\infty}\}} |f'(v)| \tag{8}
$$

Then $\operatorname{sign}(u_1-u_2)(f(u_1)-f(u_2))\frac{x}{|x|} + M|u_1-u_2| \ge 0$ so that we obtain

$$
\int_{|x|\le R-M(t_2-t_1)} |u_1(x,t_2)-u_2(x,t_2)|\,dx \le \int_{|x|\le R} |u_1(x,t_1)-u_2(x,t_1)|\,dx. \tag{9}
$$

By letting $R\to\infty$, we obtain the claimed result of the theorem.

In order to make the above formal arguments rigorous, we suitably approximate χ_K. To that end, let ρ_ε be again the "usual" non-negative molifier and set

$$
\alpha_\varepsilon(s) := \int_{-\infty}^{s} \rho_\varepsilon(z)\,dz \tag{10}
$$

(α_ε is an approximate step function.) For $\varepsilon < t_2 - t_1$ and $\delta > 0$ take

$$
\psi(x,t) := (\alpha_\varepsilon(t-t_1) - \alpha_\varepsilon(t-t_2))(1-\alpha_\delta(|x| - R + M(t-t_1)) \tag{11}
$$

(Note that ψ is smooth and compactly supported in $\mathbb{R}\times\mathbb{R}^+$.) We compute

$$
\begin{aligned}
\psi_t &= (\rho_\varepsilon(t-t_1) - \rho_\varepsilon(t-t_2))(1-\alpha_\delta(\cdots)) - M(\alpha_\varepsilon(t-t_1)-\alpha_\varepsilon(t-t_2))\rho_\delta(\cdots), \\
\psi_x &= -(\alpha_\varepsilon(t-t_1)-\alpha_\varepsilon(t-t_2))\rho_\delta(\cdots)\frac{x}{|x|}.
\end{aligned}
$$

From (7) we therefore obtain

$$
\begin{aligned}
0 \le &\int_{\mathbb{R}^2} |u_1-u_2|\,(\rho_\varepsilon(t-t_1)-\rho_\varepsilon(t-t_2))\,(1-\alpha_\delta(\cdots)) + \\
&\int_{\mathbb{R}^2} \underbrace{(\alpha_\varepsilon(t-t_1)-\alpha_\varepsilon(t-t_2))}_{\ge 0}\underbrace{\rho_\delta(\cdots)}_{\ge 0}\left(-M|u_1-u_2| - \frac{x}{|x|}\operatorname{sign}(u_1-u_2)(f(u_1)-f(u_2))\right)
\end{aligned}
$$

We note that the second integral is ≤ 0 if we select M as in (8). We obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \rho_\varepsilon(t-t_2)(1-\alpha_\delta(|x|-R+M(t-t_1)))|u_1-u_2| \\ &\quad\leq \int_{\mathbb{R}^2} \rho_\varepsilon(t-t_1)(1-\alpha_\delta(|x|-R+M(t-t_1)))|u_1-u_2|\end{aligned}$$

Letting $\delta \to 0$, we get with Lebesgue dominated convergence

$$\int_{(x,t)\colon\, |x|-R+M(t-t_1)\leq 0} \rho_\varepsilon(t-t_2)|u_1-u_2| \quad \leq \int_{(x,t)\colon\, |x|-R+M(t-t_1)\leq 0} \rho_\varepsilon(t-t_1)|u_1-u_2| \tag{12}$$

Finally, we introduce the function

$$\mu(t) := \int_{x\colon\, |x|\leq R-M(t-t_1)} |u_1(x,t)-u_2(x,t)|\, dx. \tag{13}$$

The function μ is continuous by Lemma 0.4 since $u_1,\ u_2 \in C([0,\infty); L^1) \cap L^\infty(\mathbb{R}\times\mathbb{R}^+)$. Therefore, in the limit $\varepsilon \to 0$, we arrive at

$$\int_{|x|\leq R-M(t_2-t_1)} |u_1(\cdot,t_2)-u_2(\cdot,t_2)| \leq \int_{|x|\leq R} |u_1(\cdot,t_1)-u_2(\cdot,t_1)| \tag{14}$$

□

Lemma 0.2. *For $u_1,\ u_2 \in L^\infty(\mathbb{R}^2)\cap C([0,\infty);L^1(\mathbb{R}))$ the function $\Big\{\cdots\Big\}$ in (6) is continuous.*

Proof. The proof follows from Lemma 0.3 with the following oberservations:

1. To apply Lemma 0.3, the functions u_1, u_2 should be in L^1: since one may assume that $(\widehat{y},\widehat{s})$ is in a compact set, the assumption $u_i \in C([0,\infty);L^1(\mathbb{R}))$ and the presence of the compactly supported cut-off functions ψ_t, ψ_x allows one to assume that $u_i \in L^1(\mathbb{R}^2)$.

2. Since the functions $\psi_{\widehat{x}}$ and $\psi_{\widehat{t}}$ are smooth and compactly supported so that they can take the role of χ in Lemma 0.3.

3. One applies Lemma 0.3 with

$$\begin{aligned}F_1(a,b) &= |a-b|,\\ F_2(a,b) &= \operatorname{sign}(a-b)(f(a)-f(b)) = |a-b|\frac{f(a)-f(b)}{a-b}\end{aligned}$$

   Since u_1, $u_2 \in L^\infty$, we may modify F_1, F_2 suitably to ensure *uniform* continuity.

□

Lemma 0.3. *Let $F \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be uniformly continuous. Let $\chi \in L^\infty(\mathbb{R}^d;\mathbb{R})$. Let u_1, $u_2 \in L^1(\mathbb{R}^d)$. Then the function*

$$\mathcal{F}_{u_1,u_2} : x \mapsto \int_{y\in\mathbb{R}^d} \chi(y) F(u_1(x+y), u_2(x-y))\, dy$$

is continuous.

Proof. For simplicity of notation, we only consider the case $d = 1$.

1. step: For u_1, $u_2 \in C_0^\infty(\mathbb{R})$, Lebesgue dominated convergence shows that $\mathcal{F}_{\widetilde{u}_1,\widetilde{u}_2}$ is continuous at each fixed x.

2. step: given $\widetilde{u}_1$, $\widetilde{u}_2 \in C_0^\infty(\mathbb{R})$, introduce the sets

$$\Omega_{1,\delta} := \{y \in \mathbb{R} \,|\, |u_1(y) - \widetilde{u}_1(y)| \geq \delta\}, \qquad \Omega_{2,\delta} := \{y \in \mathbb{R} \,|\, |u_2(y) - \widetilde{u}_2(y)| \geq \delta\},$$

and note that

$$\text{meas}(\Omega_{i,\delta}) = \int_{y:|u_i(y)-\widetilde{u}_i(y)|\geq\delta} 1 \leq \frac{1}{\delta} \int_{y:|u_i(y)-\widetilde{u}_i(y)|\geq\delta} |u_i - \widetilde{u}_i| \leq \frac{\|u_i - \widetilde{u}_i\|_{L^1}}{\delta}.$$

3. step: Since F is uniformly continuous, there exists, given $\varepsilon > 0$ a $\delta > 0$ such that

$$|F(a,b) - F(\widetilde{a},\widetilde{b})| \leq \varepsilon \quad \text{if} \quad |a - \widetilde{a}| + |b - \widetilde{b}| \leq \delta.$$

Hence, we write for fixed x

$$\begin{aligned}
&|\mathcal{F}_{u_1,u_2}(x) - \mathcal{F}_{\widetilde{u}_1,\widetilde{u}_2}(x)| = \left| \int_y \left(F(u_1(x+y), u_2(x-y)) - F(\widetilde{u}_1(x+y), \widetilde{u}_2(x-y))\right) \chi(y)\, dy \right| \\
&\leq \varepsilon \|\chi\|_{L^1} + 2 \int_{y:|u_1(x+y)-\widetilde{u}_1(x+y)|\geq\delta \text{ or } |u_2(x-y)-\widetilde{u}_2(x-y)|\geq\delta} \|F\|_{L^\infty} |\chi(y)|\, dy \\
&\leq \varepsilon \|\chi\|_{L^1} + 2\|F\|_{L^\infty} \|\chi\|_{L^\infty} \left(\int_{y:|u_1(x+y)-\widetilde{u}_1(x+y)|\geq\delta} 1\, dy + \int_{y:|u_2(x-y)-\widetilde{u}_2(x-y)|\geq\delta} 1\, dy \right) \\
&\leq C\left(\varepsilon + \text{meas}(\Omega_{1,\delta}) + \text{meas}(\Omega_{2,\delta})\right) \\
&\leq C\left(\varepsilon + \delta^{-1}\left(\|u_1 - \widetilde{u}_1\|_{L^1} + \|u_2 - \widetilde{u}_2\|_{L^1}\right)\right),
\end{aligned}$$

where C is independent of x.

4. step: For x, $h \in \mathbb{R}^d$ we estimate

$$\begin{aligned}
&|\mathcal{F}_{u_1,u_2}(x+h) - \mathcal{F}_{u_1,u_2}(x)| \leq \\
&|\mathcal{F}_{u_1,u_2}(x+h) - \mathcal{F}_{\widetilde{u}_1,\widetilde{u}_2}(x+h)| + |\mathcal{F}_{\widetilde{u}_1,\widetilde{u}_2}(x+h) - \mathcal{F}_{\widetilde{u}_1,\widetilde{u}_2}(x)| + |\mathcal{F}_{\widetilde{u}_1,\widetilde{u}_2}(x) - \mathcal{F}_{u_1,u_2}(x)|
\end{aligned}$$

The first and third term can be made arbitrarily small by selecting $\widetilde{u}_1$, $\widetilde{u}_2$ suitably (cf. Step 3) and then the second term can be made small by selecting h small (cf. Step 1). □

Lemma 0.4. *For fixed R, $M > 0$ let, for each t, $x \mapsto \chi(x,t)$ be the characteristic function of $\{x \colon |x| \le R - M(t-t_1)\}$. Let u_1, $u_2 \in C([0,\infty); L^1) \cap L^\infty(\mathbb{R} \times \mathbb{R}^+)$. Then the function*

$$t \mapsto \mu(t) := \int_{x\colon |x| \le R - M(t-t_1)} (u_1(x,t) - u_2(x,t))\, dx$$

is continuous.

Proof.

$$\begin{aligned}
|\mu(t+h) - \mu(t)| &= \left| \int_{\mathbb{R}} |u_1(x,t+h) - u_2(x,t+h)|\chi(x,t+h) - |u_1(x,t) - u_2(x,t)|\chi(x,t) \right| \\
&\le \int_{\mathbb{R}} \big| |u_1(x,t+h) - u_2(x,t+h)| - |u_1(x,t) - u_2(x,t)| \big| \chi(x,t) \\
&+ \int_{\mathbb{R}} |u_1(x,t) - u_2(x,t)| |\chi(x,t+h) - \chi(x,t)|
\end{aligned}$$

The first integral tends to zero as h by u_1, $u_2 \in C([0,\infty); L^1)$ (and the reverse triangle inequality); the second integral tends to zero by u_1, $u_2 \in L^\infty$. ∎