problem sheet 11

discussion: week of Monday, 6.1.2020

11.1. Newton's method converges only for initial values close to the zero x^* of f. One possibility to address this difficulty is the so-called *continuation method*: One considers a function $H(x,s)$ with $H(x,1) = f(x)$ and for which a zero x_0 of $H(x,0)$ is known. One then selects points s_i, $i = 1,\ldots,N$ and employs Newton's method to compute the zero x_{i+1} of $H(x,s_{i+1})$, taking x_i as the initial value. Perform the method for

$$f(x) = \arctan x, \qquad H(x,s) = \arctan x - (1-s)\arctan 4$$

and initial value $x_0 = 4$. Select $s_i = i/10$, $i = 0,\ldots,10$.

11.2. Show the following convergence result for the inverse iteration with shift: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be diagonalizable and $\lambda \in \mathbb{R}$. Let the eigenvalues of $\mathbf{A}$ be numbered such that $|\lambda_1 - \lambda| \geq |\lambda_2 - \lambda| \geq \cdots |\lambda_{n-1} - \lambda| > |\lambda_n - \lambda|$. Then there exists $C > 0$ such that there holds for the approximations $\widetilde{\lambda}_\ell$ of the inverse iteration:

$$|\lambda_n - \widetilde{\lambda}_\ell| \leq C \left| \frac{\lambda_n - \lambda}{\lambda_{n-1} - \lambda} \right|^\ell, \qquad \ell = 0,1,\ldots,$$

11.3. Consider the vector iteration ("power method") for the matrix $\mathbf{A}$ and the following three initial vectors $x_0^{(j)}$, $j = 0,1,2$:

$$\mathbf{A} = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}, \qquad \mathbf{x}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_0^{(3)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Discuss the behavior of the vector iteration. Do the eigenvalue approximations $\widetilde{\lambda}_\ell$ and the iterates $\mathbf{x}_\ell$ converge? If so, what do they converge to?

11.4. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a *symmetric* matrix and define the Rayleigh quotient

$$R(\mathbf{x}) := \frac{\mathbf{x}^\top \mathbf{A}\mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

a) Show that the largest eigenvalue λ_1 is the maximum of R and that smallest eigenvalue λ_n the minimum of R.

b) The minimization property of λ_n suggests to use descent methods. Formulate such a method. To that end, check that

$$\mathbf{g} := \nabla R(\mathbf{x}) = 2\frac{\mathbf{A}\mathbf{x}\|\mathbf{x}\|_2^2 - (\mathbf{x}^\top \mathbf{A}\mathbf{x})\mathbf{x}}{\|\mathbf{x}\|^4}.$$

Use the *Armijo rule* to determine the step length.
Remark: in the present case, it is possible to solve the 1-dimensional minimization problem, i.e., finding t with $R(\mathbf{x} + t\mathbf{g}) = \min_\tau R(\mathbf{x} + \tau\mathbf{g})$, explicitly.

c) Use your algorithm to find the smallest eigenvalue of the tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ given by

$$\mathbf{A} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}, \qquad h = \frac{1}{n}.$$

Plot (`semilogy`) the error versus the iteration number ℓ, where $\ell = 1,\ldots,100$. You may compute the exact value of the eigenvalue using `matlab`'s `eig` command. What do you observe in dependence on $n \in \{10,20,40,80\}$?
Remark: As in Problem 10.2 the matrix $\mathbf{A}$ is obtained as a discretization of $-\frac{d^2u}{dx^2}$.