problem sheet 10

discussion: week of Monday, 9.12.2019

10.1. **a)** Let $\mathbf{A}$ be a *symmetric* matrix. Show that $\|\mathbf{A}\|_2 = \lambda_{max}(\mathbf{A})$, where $\lambda_{max}(\mathbf{A})$ is the largest (in modulus) eigenvalue of $\mathbf{A}$. *Hint:* Write $\mathbf{A} = \mathbf{Q}^\top \mathbf{D}\mathbf{Q}$ for a diagonal matrix $\mathbf{D}$.

b) Show: For any matrix $\mathbf{A}$ one has $\|\mathbf{A}\|_2^2 = \lambda_{max}(\mathbf{A}^\top\mathbf{A})$, where $\lambda_{max}(\mathbf{A}^\top\mathbf{A})$ is the largest (in modulus) eigenvalue of the symmetric matrix $\mathbf{A}^\top\mathbf{A}$.

10.2. Consider the nonlinear system of equations $\mathbf{f}(\mathbf{x}) = 0$ given by

$$\begin{aligned} 3x_1 - \cos(x_2x_3) - 3/2 &= 0 \\ 4x_1^2 - 625x_2^2 + 2x_3 - 1 &= 0 \\ 20x_3 + e^{-x_1x_2} + 9 &= 0 \end{aligned}$$

Compute the derivative $\mathbf{f}'(\mathbf{x})$ and formulate Newton's method. Program Newton's method in `matlab`/`python`. The program should additionally estimate the error (e.g., in the $\|\cdot\|_2$-norm) and also return it. Use the initial vector $(1, 1, 1)^\top$. Plot (using `semilogy`) the estimated error versus the iteration number.

10.3. Consider the system of equations

$$\mathbf{A}x = \mathbf{b} + \varepsilon\mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

with

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} (x_1 - x_2)^2 \\ 0 \end{pmatrix}, \qquad \varepsilon = 0.01.$$

a) Formulate the Newton method and write a program to compute the iterates $\mathbf{x}_n$, $n = 1, 2, \ldots,$. Initial value: $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$.

b) Consider the following method with initial value $\mathbf{x}_0$ given by $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$: For $n = 0, 1, \ldots$ one determines $\mathbf{x}_{n+1} \in \mathbb{R}^2$ such that $\mathbf{A}\mathbf{x}_{n+1} = \mathbf{b} + \varepsilon\mathbf{f}(\mathbf{x}_n)$. Write a program to compute the iterates. Taking the last value of the Newton method as the "exact solution" you can compute the errors. Plot the error for the both methods in `loglog`-plot (error versus iteration index n). Can you relate this "simple" method to Newton's method for small ε?

c) $\|x_{n+1} - x_n\|$ is a good estimate for the error $\|x_* - x_n\|$ for the method of part b) By rewriting this in terms of x_n and x_{n-1} design an error estimator for error $\|\mathbf{x}^* - \mathbf{x}_n\|_2$ for the method of part b) that uses on the approximations $\mathbf{x}_n$ and $\mathbf{x}_{n-1}$. Plot for $n = 1, \ldots, 10$ the true error and the estimated error. What do you observe? Why is $x_{n+1} - x_n$ a good estimate for the true error $x_* - x_n$?

10.4. Consider the for $N \in \mathbb{N}$ the system of equations

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + u_i^3 = 1, \qquad i = 1, \ldots, N-1.$$

Formulate Newton's method for its solution and program it in `matlab`/`python`. Estimate the error (in the $\|\cdot\|_2$-norm) by considering the difference of two consecutive iterates. Plot the error versus the iteration number.

Remark: the above system of equations results from the numerical approximation of the "boundary value problem"

$$-u''(x) + (u(x))^3 = 1, \qquad x \in (0, 1), \qquad u(0) = u(1) = 0.$$

The values u_i are approximations to the true values $u(x_i)$ with $x_i = ih$, $i = 1, \ldots, N-1$.