problem sheet 9

discussion: week of Monday, 2.12.2019

9.1. Let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ be the SVD of $\mathbf{A} \in \mathbb{R}^{m\times n}$. Let $\sigma_1, \ldots, \sigma_{\min\{m,n\}}$ be the singular values of $\mathbf{A}$.

a) Let $r \in \mathbb{N}_0$ be such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = 0$. Show: r is the rank of $\mathbf{A}$.

b) Show: the columns of $\mathbf{U}(:, [1:r])$ are an ONB of the range of $\mathbf{A}$.

c) Show: The columns of $\mathbf{V}(:, [r+1:n])$ are an ONB of $\operatorname{Ker}\mathbf{A}$.

9.2. The Frobenius norm of a matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_F^2 = \sum_{i,j} |\mathbf{A}_{ij}|^2$.

a) Show for an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and an arbitrary matrix $\mathbf{X} \in \mathbb{R}^{n\times k}$ that $\|\mathbf{QX}\|_F^2 = \|\mathbf{X}\|_F^2$.

b) Let the SVD of $\mathbf{A} \in \mathbb{R}^{m\times n}$ be $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$, where the diagonal entries of Σ are the singular values $\sigma_1, \ldots, \sigma_{\min\{m,n\}}$. Show: $\|\mathbf{A}\|_F^2 = \sum_i \sigma_i^2$.

9.3. Let $m \geq n$ and let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ be the SVD of $\mathbf{A} \in \mathbb{R}^{m\times n}$. Decompose $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2)$ with $\mathbf{U}_1 \in \mathbb{R}^{n\times n}$, $\mathbf{U}_2 \in \mathbb{R}^{m-n,m-n}$. Define

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{V} & \mathbf{V} & 0 \\ \mathbf{U}_1 & -\mathbf{U}1 & \sqrt{2}\mathbf{U}_2 \end{pmatrix}.$$

Show: $\mathbf{Q}$ is orthogonal and

$$\mathbf{Q}^\top \begin{pmatrix} 0 & \mathbf{A}^\top \\ \mathbf{A} & 0 \end{pmatrix} \mathbf{Q} = \text{ diagonal matrix.}$$

What are the diagonal entries of this matrix?

9.4. Program Newton's method in 1D. To that end, realize a `Matlab`/`python` function `newton(x,f,df)` that realizes one step of the method. f and df are *function handles* for the function f and its derivative f'. Plot (use `semilogy`) the error versus the number of Newton steps for the following 3 functions:

$$f_1(x) = x^2, \qquad f_2(x) = e^x - 2, \qquad f_3(x) = |x|^{3/2}.$$

Use $x_0 = 0.5$ as the starting value. What do you observe? Which assumptions that underlie the proof of quadratic convergence are not satisfied? Consider Newton's method for

$$f_4(x) = \frac{1}{x} - 1$$

and initial value $x_0 = 2.1$. What do you observe? *Remark:* This reduces the problem of a division to one of multiplication