

problem sheet 8

discussion: week of Monday, 25.11.2019

8.1. Let $A \in \mathbb{R}^{n \times n}$ be a tridiagonal matrix of the form

$$A = \begin{pmatrix} d_1 & e_1 & & & & & \\ c_2 & d_2 & e_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & e_{n-1} & \\ & & & & c_n & d_n & \end{pmatrix}$$

Assume that A has an LU -factorization.

a) Show: the factors L and U have the form

$$L = \begin{pmatrix} 1 & & & & & & \\ l_2 & 1 & & & & & \\ & l_3 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & l_n & 1 & \end{pmatrix}, \quad U = \begin{pmatrix} u_1 & e_1 & & & & & \\ & u_2 & e_2 & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & e_{n-1} & \\ & & & & & u_n & \end{pmatrix}$$

b) Compute u_1 . Formulate an algorithm that computes the l_i and the u_i for $i = 2, \dots, n$

8.2. The lengths of the 3 edges (meeting at a corner) and the circumferences (orthogonal to the first and second edge) of a brick are measured. The measured values are:

edge 1: 26mm, edge 2: 38mm, edge 3: 55mm
 circumference \perp edge 1: 188mm, circumference \perp edge 2: 163mm.

Determine the edge lengths using the method of least squares.

8.3. The function $f(x) = \sin x$ is to be approximated by a polynomial of the form $\pi(x) = a_1x + a_3x^3$. To this end, the coefficients a_1, a_3 are determined using the least squares method by minimizing $\sum_{j=0}^m (\pi(x_j) - f(x_j))^2$, where x_0, \dots, x_m are given points. Set up the least squares problem for a_1 and a_3 . Write a program that computes the coefficients a_1, a_3 for the following 9 choices of knots x_j : the x_j are N randomly chosen points in the interval $[-1/N, 1/N]$ for $N = 2^n, n = 2, \dots, 10$. Do the values a_1, a_3 converge to a limit as $N \rightarrow \infty$? Which limit do you expect?

8.4. The least squares method can also be used to fit the parameters of certain nonlinear problems. How would you determine the parameters C, k to fit given data $(t_i, y_i), i = 1, \dots, N$, to the law $y(t) = Ce^{-kt}$? How do you proceed to determine C, α for the law $y(t) = Ct^\alpha$?

8.5. Let \mathbf{Q} be an orthogonal matrix. Show:

- a) $\mathbf{x}^\top \mathbf{y} = ((\mathbf{Q}\mathbf{x}))^\top (\mathbf{Q}\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- b) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m > n$ and its QR -factorization $\mathbf{A} = \mathbf{Q}\mathbf{R}$. Show: If \mathbf{A} has full rank (i.e., $\text{rank}(\mathbf{A}) = n$), then the diagonal entries of \mathbf{R} are non-zero. Show that the first n columns of \mathbf{Q} form an orthogonal basis of the range of \mathbf{A} .