problem sheet 7

discussion: week of Monday, 18.11.2019

7.1. Consider Gaussian elimination (Alg. 4.7 of the class notes). Determine the number of multiplications done by the algorithm. You may use the following facts:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

7.2. For symmetric, positive definite matrices $\mathbf{A}$, one typically makes a *Cholesky-factorization* instead of an *LU*-factorization. That is, one seeks a lower triangular matrix[1] $\mathbf{C}$ such that

$$\mathbf{C}^\top \mathbf{C} = \mathbf{A}$$

Formulate an algorithm that computes $\mathbf{C}$. *Hint:* Proceed as in Crout's method.

7.3. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For $n \times n$ matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ define a norm $\|\mathbf{A}\|$ by

$$\max_{0\neq\mathbf{x}\in\mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

a) Show that for arbitrary matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ one has $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.

b) Show for the norm $\|\mathbf{x}\|_\infty = \max_{i=1,\dots,n} |x_i|$ that $\|\mathbf{A}\|_\infty \leq \max_i \sum_{j=1}^n |a_{ij}|$. *Remark:* In fact, there holds $\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$.

c) Show for the norm $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$ that $\|\mathbf{A}\|_1 \leq \max_j \sum_{i=1}^n |a_{ij}|$. *Remark:* In fact, there holds $\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$.

7.4. ("arrowhead matrix") Let $n = 10$, $\mathbf{e} = (1, 1, \dots, 1)^\top \in \mathbb{R}^n$, $\mathbf{b} = (1, 0, \dots, 0)^\top \in \mathbb{R}^n$. Consider the matrix $\mathbf{A} = 10\mathbf{I} + \mathbf{b}\mathbf{e}^\top + \mathbf{e}\mathbf{b}^\top$ and the matrix $\widetilde{\mathbf{A}} := \mathbf{A}(n : -1 : 1, n : -1, 1)$ that is obtained from $\mathbf{A}$ by reversing the numbering of the rows and columns. Use the commands `spy` (`matplotlib.pyplot.spy`) and `lu` (`scipy.linalg.lu`) to visualize the sparsity patterns of $\mathbf{A}$, $\widetilde{\mathbf{A}}$ and the corresponding factors $\mathbf{L}$, $\mathbf{U}$ of the *LU*-factorization. What do you observe? Which variant of the numbering is to be preferred from a cost (i.e., number of floating point operations or storage requirement) point of view?

[1] not normalized, i.e., $\mathbf{C}_{ii}$ need not be 1