problem sheet 6

discussion: week of Monday, 11.11.2019

6.1. Consider quadrature rules Q^{2D} on the square $S = [0,1]^2$.

a) Show: the midpoint rule $Q(F) = F(0.5, 0.5)$ is exact for polynomials of the form $F(x,y) = a + bx + cy$.

b) Given $p \in \mathbb{N}_0$, give a quadrature formula Q^{2D} that is exact for polynomials of the form $F(x,y) = \sum_{i,j=0}^{p} a_{ij} x^i y^j$.

6.2. Develop an adaptive algorithm for the integration of functions over the rectangle $[a_x, b_x] \times [a_y, b_y]$. Base your algorithm on the midpoint rule, i.e., $Q_{[a,b]\times[c,d]}(f) = (b-a)(d-c)f((a+c)/2,(b+d)/2)$. *Hint:* adapt the ideas of the 1d-adaptive algorithm of Problem 4.4. Test your adaptive algorithm for the integration over $[0,1]^2$ of the following functions:

$$f_1(x,y) = x^2 \qquad \text{and} \qquad f_2(x,y) = \begin{cases} 0 & x < y \\ 1 & x \geq y \end{cases}$$

Use the tolerances $\tau = 2^{-i}$, $i = 0, \ldots, 15$, and make a convergence plot (quadrature error versus tolerance) in `loglog` scale.

6.3. Consider the function

$$\varphi(x) = \sqrt{x+1} - \sqrt{x}$$

a) Is the evaluation of φ well-conditioned for large x? Consider relative conditioning.

b) Formulate a stable numerical realization of φ (*Hint:* You may use that a stable realization of $\sqrt{\cdot}$ is available.)

6.4. The sequence u_k, $k = 0, 1, \ldots$, given by

$$u_1 := 2, \qquad u_{k+1} = 2^k \sqrt{2\left(1 - \sqrt{1 - (2^{-k} u_k)^2}\right)} \tag{1}$$

converges to the number $\pi = 3.1415\ldots$.

a) Compute (in `matlab`/`python`) the first 30 members of the sequence and the absolute error $|\pi - u_k|$. When is the error minimal?

b) Explain why you should expect that the error grows for $k \geq k_0$ for some k_0. *Extra Problem:* Assume that in exact arithmetic the error is $|u_k - \pi| \approx 2^{-2k}$. Use this to show that the minimal achievable error is reached for $k \approx 17$.

6.5. (Aitken Δ^2 extrapolation)

a) The Aitken Δ^2 method can be used to accelerate the convergence of a sequence $(x_n)_n$ that converges to $x_\infty = \lim_{n\to\infty} x_n$. To that end, assume that the sequence $(x_n)_n$ has the form

$$x_n = x_\infty + Cq^n \tag{2}$$

with (unknown) x_∞, C, $q \in (0,1)$. Give a formula that "extrapolates" the sought limit x_∞ from 3 successive sequence members x_n, x_{n+1}, x_{n+2} by assuming that all 3 values satisfy (2). Proceeding in this way for every n produces a new sequence $(\widetilde{x}_n)_n$ that (sometimes) converges faster to x_∞ than the original sequence. Apply this method to the sequence $(u_k)_k$ of Problem 6.4 to get a new sequence of improved approximations to π. What is the best possible error?

b) Suppose you don't know the limit π of the sequence $(u_k)_k$. How can you estimate the errors of the approximations u_k? Can you formulate a sensible stopping criterion for the iteration (1)?