problem sheet 3

discussion: week of Monday, 21.10.2019

3.1. **a)** Consider the function $f(x) = (4 - x^2)^{-1}$. Estimate $\min_{q \in \mathcal{P}_n} \|f - q\|_{\infty,[-1,1]}$ by considering the Taylor polynomial of f about a suitable point. Plot semilogarithmically (`semilogy`) the error $\|f - I_n^{Cheb}\|_{\infty,[-1,1]}$ versus n, where $I_n^{Cheb} f$ is the Chebyshev interpolant of degree n. Approximate the error $\|f - I_n^{Cheby}\|_{\infty,[-1,1]}$ by taking the maximal interpolation error in 100 uniformly distributed points in the interval $[-1, 1]$.

3.2. **a)** Show that the weights of the Newton-Cotes formulas satisfy $\sum_{i=0}^{n} w_i = 1$(= length of the interval $[0, 1]$). (*Hint:* apply the quadrature formula to a suitable function f.)

b) Show that both the closed and open Newton-Cotes quadrature formulas $\widehat{Q}_n^{cNC}$, $\widehat{Q}_n^{oNC}$ are exact for $f \in \mathcal{P}_n$.

c) Show the symmetry property $w_{n-i} = w_i$, $i = 0, \ldots, n$. (*Hint:* use the symmetry of the points, i.e., $x_j = 1 - x_{n-j}$.

d) Let $n = 2m$ be even. Consider the function $f = (x - 1/2)^{n+1}$, which is antisymmetric with respect to 1/2. Show: $\int_0^1 f(x)\,dx = 0 = \widehat{Q}_n^{cNC}(f) = \widehat{Q}_n^{oNC}(f)$. Conclude that the quadrature formulas $\widehat{Q}_n^{cNC}$ and $\widehat{Q}_n^{oNC}$ are exact for polynomials of degree $n + 1$. In particular, the midpoint rule is exact for polynomials in $\mathcal{P}_1$, and the Simpson rule is exact for polynomials in $\mathcal{P}_3$.

3.3. Write a program that realizes the composite trapezoidal rule for integration on $[a, b]$. The rule is based on a subdivision of $[a, b]$ into N subintervals of length $h = (b - a)/N$. Consider, for $[a, b] = [-1, 1]$ the three integrands

$$f_1(x) = x^2, \qquad f_2(x) = |x|, \qquad f_3(x) = \begin{cases} \frac{1}{2}e^x & x < 1/3 \\ e^x & x \geq 1/3 \end{cases}$$

Plot in `loglog`-scale the quadrature error versus h for $h = 2^{-i}$, $i = 1, 2, \ldots, 20$. What do you observe? Explain your observations.

3.4. Let two bases $\{p_0(x), \ldots, p_n(x)\}$ and $\{q_0(x), \ldots, q_n(x)\}$ of $\mathcal{P}_n$ be given and interpolation points x_i, $i = 0, \ldots, n$. Define the matrices $\mathbf{G}$ and $\mathbf{H}$ by

$$\mathbf{G}_{ij} = p_j(x_i), \qquad \mathbf{H}_{ij} = q_j(x_i).$$

Show: the matrix $\mathbf{G}^{-1}\mathbf{H}$ realizes the change of basis, i.e.,

$$\mathbf{c} = \mathbf{G}^{-1}\mathbf{H}\mathbf{d}$$

implies $\sum_i \mathbf{c}_i p_i = \sum_i \mathbf{d}_i q_i$.