problem sheet 2

discussion: week of Monday, 14.10.2019

2.1. We aim to approximate the function f on the interval $[a, b]$ by a *piecewise* polynomial of degree n. Proceed as follows: partition $[a, b]$ in N subintervals $[t_j, t_{j+1}]$, $j = 0, \ldots, N-1$, of length $h = (b-a)/N$ with $t_j = a+jh$. On each subinterval $[t_j, t_{j+1}]$ select the interpolation points $x_{i,j} := t_j + \frac{1}{n}ih$, $i = 0, \ldots, n$, and approximate f on $[t_j, t_{j+1}]$ by the polynomial that interpolates in the points $x_{i,j}$, $i = 0, \ldots, n$. In this way, one obtains a function p that is a polyomial of degree n on each subinterval. Show:

$$\|f - p\|_{\infty,[a,b]} \leq \frac{1}{(n+1)!} h^{n+1} \|f^{(n+1)}\|_{\infty,[a,b]}.$$

Sketch the function p for $n = 1$.

2.2. Let, for a function f and a point x_0,

$$D_{sym}(h) := \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

be the symmetric difference quotient. Let $h_i = 2^{-i}$, $i = 0, 1, \ldots$

a) Use your program of Problem 1.2 to generate the first 3 columns of the Neville scheme of the extrapolation of $D_{sym}(0)$ for the function $f(x) = \tan(x)$ and $x_0 = 0$. Plot in a `loglog` plot the error versus h for these 3 columns, i.e., plot for $m \in \{0, 1, 2\}$ the values `abs`$(N(:, m) - 1)$ versus $h(:)$. Include in the plot the auxiliary lines $h \mapsto h^2$, $h \mapsto h^3$, $h \mapsto h^4$. What convergence rates do you observe?

b) Modify your program of Problem 1.2 so as to exploit the fact that D_{sym} is a symmetric function with respect to $h = 0$, i.e., $D_{sym}(h) = \widetilde{D}(h^2)$ for some function $\widetilde{D}$. Again, plot in a `loglog`-plot the error versus h for these three columns. What convergence rates do you observe?

c) Repeat parts a) and b) for the function $f(x) = (\max(x, 0))^{3/2}$. What convergence rates do you observe? Explain.

2.3. ("harmonic series") The goal is the efficient evaluation/approximation of

$$S(N) := \sum_{n=1}^{N} \frac{1}{n}$$

for large N. We use the fact that $S(N)$ can be written as

$$S(N) = \ln N + a_0 + \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots \tag{1}$$

Determine the coefficients a_0, a_1, a_2 as follows: 1) Write a routine to evaluate $S(N)$. 2) Set up a linear system of equations for the coefficients a_0, a_1, a_2 that is obtained for $N = 10$, 100, 1000. (The terms $+\cdots$ in (1) are simply ignored). Solve for the coefficients (in `matlab` this is achieved with \, in `python` this can be done with `numpy.linalg.solve`).

What is the error of your approximation for $N = 10^6$ and $N = 10^8$? What is the run time of your approximation for $N = 10^6$ and $N = 10^8$? What is the run time for the evaluation of $S(10^8)$ on your computer? (Use `tic`, `toc` or `time.time()`)

2.4. The goal is to evaluate numerically the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Modify the way you proceeded in Problem 2.3 appropriately. To that end, introduce the function

$$S'(N) := \sum_{n=1}^{N} \frac{1}{n^2}$$

and approximate

$$S'(N) = a_0 + \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots$$

Use $N = 100$, 1000, 10000. What is the error $a_0 - \pi^2/6$?