Projekt 3: rational interpolation

The lecture discussed polyomial interpolation as a means to fit given data. Sometimes, it is better to fit the data to a rational function rather than a polynomial. The goal of the project is to illustrate when this is the case.

1. Define the set of rational functions by

$$\mathcal{R}(\ell, m) := \left\{ x \mapsto \frac{p(x)}{q(x)} \,|\, \partial p \leq \ell, \quad \partial q \leq m, \qquad q \neq 0 \right\}.$$

here, ∂p denotes the degree of the polynomial p. Given ℓ, $m \in \mathbb{N}_0$ we set $n := \ell + m$.

Let x_i, $i = 0, \ldots, n$, be distinct knots and $f_i \in \mathbb{R}$, $i = 0, \ldots, n$, given data. We seek (given ℓ, m) a rational function $x \mapsto R(x) = p(x)/q(x)$ that satisfies the interpolation conditions

$$R(x_i) = f_i, \qquad i = 0, \ldots, n. \tag{1}$$

This rational interpolant is called the (ℓ, m)-interpolant of the data (x_i, f_i), $i = 0, \ldots, n$, and we write, if the degrees of the numerator and the denomiator are of importance, $R^{(\ell,m)}$.

a) A necessary condition for the numerator polynomial p and the denominator polynomial q of R is

$$p(x_i) - f_i q(x_i) = 0, \qquad i = 0, \ldots, n. \tag{2}$$

Using (2) formulate a linear system of equations that allows one to determine the polynomials p and q. What can you say about solvability of the problem (2)?

b) Write a (Matlab-)code that compute the polynomials p, q for given knots x_i, values f_i, and degrees ℓ, m such that they solve (2).

c) Use your code to compute the interpolants $R^{(n,n)}$ and $R^{(2n,0)}$ (i.e., a true rational and a classical polynomial interpolant) of the function

$$r(x) := \frac{1}{1 + 25\sinh^2(2x/\pi)} \tag{3}$$

for different values n and to plot them. Use the equidistant mesh with nodes $x_i = -1 + ih$, $i = 0, \ldots, 2n$, $h = 1/n$. Plot semilogarithmically the "interpolation error" (i.e., the maximal error on a significantly finer mesh) versus n for both cases. Also plot $R^{(10,10)}$ and $R^{(20,0)}$ on $(-1, 1)$ and the pointwise error on $(-0.1, 0, 1)$ and $(0.9, 1)$.

2. A popular[1] application of polynomial or rational interpolation is extrapolation.

To approximate the derivative $f'(0)$ one can use the (one-sided) difference quotien

$$D_h f := \frac{f(0 + h) - f(0)}{h}.$$

If one has evaluated $D_h f$ for different values of h, then one can obtain by polynomial or rational extrapolation a better approximation. The general procedure is as follows: Given ν knots h_i, $i = 0, \ldots, \nu - 1$, one computes the difference quotients $y_i := D_{h_i} f$; next, one determines the interpolating polynomial $h \mapsto P(h)$ or the interpolating rational function

[1] if not the most common

$h \mapsto R(h)$; the actual approximation is then taken as $P(0)$ or $R(0)$. Note that only the value of P or R at $h = 0$ is sought so that in practice one would work with an Neville scheme (for polynomial interpolation) as discussed in class or or Neville-like scheme (not discussed in class).

Consider the functions f_1, f_2 given by

$$f_1(x) = \frac{\sin x}{0.001 + x^2}, \qquad f_2(x) = e^x$$

and the choices

$$h_i = q^i, \qquad i = 0, 1, \ldots, \nu - 1, \qquad q = 0.5$$

a) For the rational extrapolation use rational approximations of the form $R^{(\ell,\ell)}$ or $R^{(\ell,\ell+1)}$ depending on the number of knots. Plot (semilogarithmically) the error versus the number of knots ν for both polynomial and rational extrapolation. Plot, as a third curve, also the error $|f'(0) - D_{h_{\nu-1}} f|$. What do you observe? Try to qualitatively explain your observations.