Projekt 1: fast Chebyshev interpolation and applications

The Chebyshev polynomials T_n are defined by

$$T_n(x) = \cos(n \arccos x), \qquad n = 0, 1, \ldots,$$

The classical Chebyshev points $x_j^{Cheb,n} = \cos \frac{(j+1/2)\pi}{n}$, $j = 0, \ldots, n-1$ are the zeros of T_n.

1. (Properties of the T_n)

a) Show that the Chebyshev points are indeed the zeros of T_n

b) Show that the T_n satisfy a three-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

c) Check that the T_n satisfy the orthogonality

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi/2 & \text{if } n = m \neq 0 \\ \pi & \text{if } n = m = 0 \end{cases}$$

2. **a)** Check that T_i also satisfy a discrete orthogonality:

$$\sum_{k=0}^{n-1} T_i(x_k^{Cheb,n}) T_j(x_k^{Cheb,n}) = \begin{cases} 0 & \text{if } i \neq j \\ n/2 & \text{if } i = j \neq 0 \\ n & \text{if } i = j = 0 \end{cases}$$

b) Define coefficients c_j by

$$c_j = \sum_{k=0}^{n-1} f(x_k^{Cheb,n}) T_j(x_k^{Cheb,n}) = \sum_{k=0}^{n-1} f(x_k^{Cheb,n}) \cos(\frac{\pi j (k+1/2)}{n}), \qquad = j = 0, \ldots, n-1.$$

Check that $\frac{1}{N} c_0 T_0(x) + \frac{2}{N} \sum_{j=1}^{n-1} c_j T_j(x)$ is the Chebyshev interpolant of f.

3. The transformation $(x_j)_{j=0}^{n-1} \mapsto (c_j)_{j=0}^{n-1}$ is the *discrete cosine transformation*. (More precisely, the DCT-II). Up to scaling it is realized, e.g., in `matlab` or `scipy`. Show that it could be realized in terms of the FFT (if n is a power of 2).

4. Design a fast (i.e., $O(n \log n)$) quadrature algorithm to realize

$$\int_{-1}^{1} f(x)\, dx \approx Q_n(f) = \sum_{k=0}^{n-1} f(x_j^{Cheb,n}) w_j.$$

You could either realize the DCT yourself or use a `matlab` or `scipy` version (check the scaling of the implementations!). *Hint: what is* $\int_{-1}^{1} T_j(x)\, dx$*?*

5. Compare your quadrature routine with the standard Gaussian quadrature for the integrands

$$f_1(x) = e^x, \qquad f_2(x) = |x|^{0.1}.$$

6. (root finding with Chebyshev expansions) consider the polynomial

$$p(x) = \sum_{k=0}^{n} a_k T_k(x)$$

and the matrix

$$C = \begin{pmatrix} 0 & 1 & & & & \\ 1/2 & 0 & 1/2 & & & \\ & 1/2 & 0 & 1/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 1/2 \\ & & & & 1/2 & 0 \end{pmatrix} - \frac{1}{2a_n} \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}$$

Show, using the three-term recurrence relation satisfied by the T_n, that the eigenvalues of C are the zeros of p. *Hint:* consider $C\mathbf{v}$ with vector $\mathbf{v} = (T_0(x), T_1(x), \ldots, T_{n-1}(x))^\top$.

7. Write a program that computes the Gauss points, i.e., the zeros of the Legendre polynomial L_n using the technique of Problem 6. You need to be able to evaluate L_n. For that, use the fact that Legendre polynomials satisfy a three-term recurrence relation

$$L_0 = 1, \quad L_1(x) = x, \qquad (n+1)L_{n+1} = (2n+1)xL_n(x) - nL_{n-1}(x)$$

8. Write an second program to compute the Gauss points that uses Newton's method. Take as the initial guesses for the zeros the Chebyshev points. The derivatives of the Legendre polynomials satisfy

$$L'_{n+1}(x) = (n+1)L_n(x) + xL'_n(x)$$