economy size SVD

$$A = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^\top = \widetilde{U} \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix} \widetilde{V}^\top$$

economy size SVD contains the essential information about a matrix:

- $\widetilde{U} \in \mathbb{R}^{m \times r}$ is an ONB of $\mathbf{Im}(A)$
- $\widetilde{V} \in \mathbb{R}^{n \times r}$ is an ONB of $(\mathbf{Ker}(A))^\perp$

memory requirement:

A	$m \times n$
economy	$m \times r + r + n \times r = (m+n+1)r \ll m \times n \qquad \text{if } r \ll \min\{m,n\}$

idea to compreess a matrix:

- approximate $A \in \mathbb{R}^{m \times n}$ by a rank-r matrix (with an $r \ll \min\{m,n\}$)
- store this rank-r matrix in its economy SVD representation

example: data compression with SVD

$$A = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^\top = \widetilde{U}\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}\widetilde{V}^\top$$

memory requirement:

A	$m \times n$
economy	$m \times r + r + n \times r < mn \qquad \text{if } r \ll \min\{m, n\}$

data compression with SVD:

1. determine SVD of A
2. keep only the information related to the $\widehat{r} \leq r$ largest singular values $\rightarrow$ approximate A by

$$A' = \widetilde{U}(:, [1 : \widehat{r}])\widetilde{\Sigma}([1 : \widehat{r}], [1 : \widehat{r}])\widetilde{V}(:, [1 : \widehat{r}])^\top$$

3. memory requirement: $m \times \widehat{r} + \widehat{r} + n \times \widehat{r}$.

When does this work?

Exercise: For the Frobenius norm $\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2$ there holds:

$$\|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2.$$

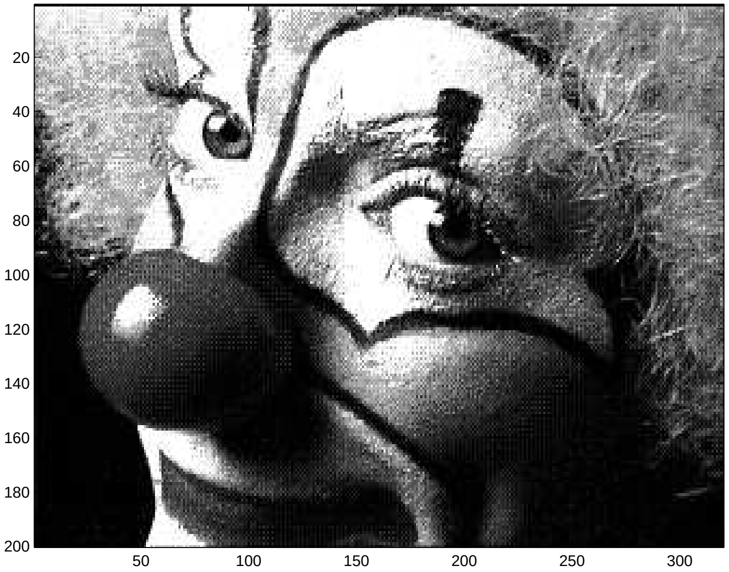
Hence for the error $A - A'$:

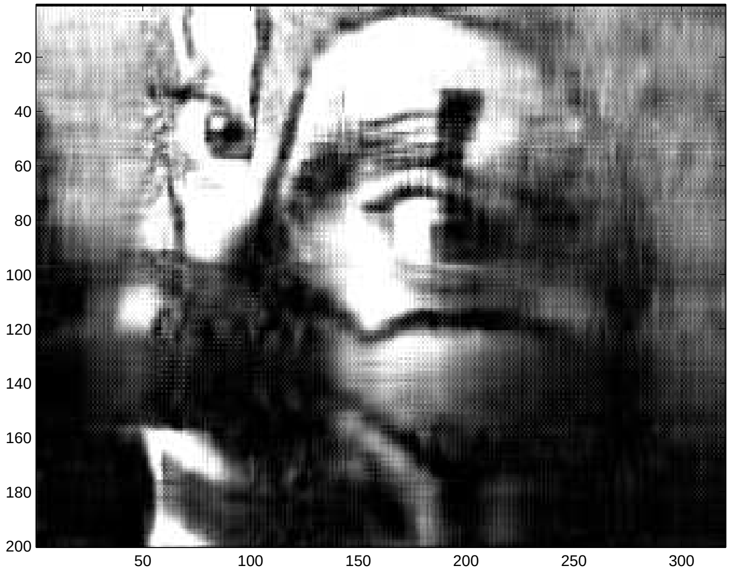
$$A - A' = \widetilde{U} \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \sigma_{\widehat{r}+1} & & \\ & & & & \ddots & \\ & & & & & \sigma_r \end{pmatrix} \widetilde{V}^\top$$

i.e., $\|A - A'\|_F^2 = \sum_{i=\widehat{r}+1}^{r} \sigma_i^2$.

data compression using the SVD works well if the singular values σ_i of the matrix A decay rapidly (as i grows).

Example: data compression using the SVD






```
load clown.mat
figure(1);    image(X);    colormap('gray');
[U,S,V] = svd(X);
k = 15
figure(2);  image(U(:,1:k)*S(1:k,1:k)*V(:,1:k)');  colormap('gray')
```

memory requirement:

X	$200 \times 320 = 64,000$
compresses	$200 \times k + 320 \times k = 7,800$ (for $k = 15$)

store merely $U(:, 1 : k)$ and $\left(S(1 : k, 1 : k) * V(:, 1 : k)'\right)$