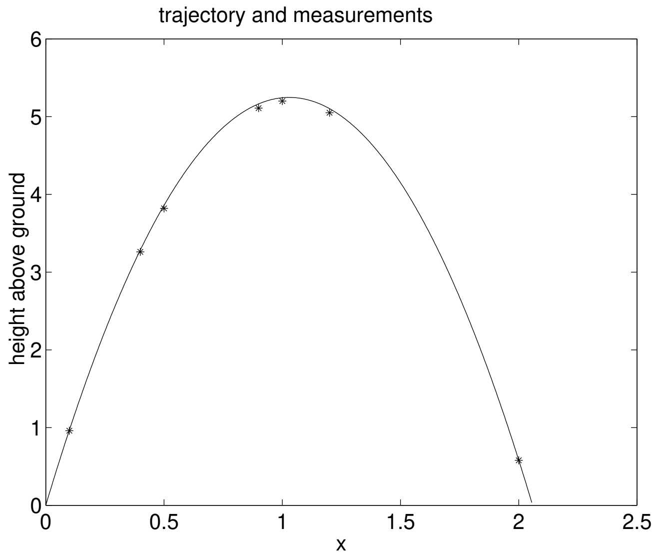
Example of Least Square Method

physical law: $y = v_y t - \frac{1}{2} g t^2$

sought: initial velocity v_y and acceleration g due to gravity.



measurements:

i	1	2	3	4	5	6	7
t_i[s]	0.1	0.4	0.5	0.9	1.0	1.2	2.0
y_i[m]	0.96	3.26	3.82	5.11	5.2	5.05	0.58

obtain an **overdetermined** system of equations:

$$t_i \, v_y - \frac{1}{2} t_i^2 \, g = y_i, \qquad i = 1, \ldots, 7$$

in matrix notation: $$\begin{pmatrix} 0.1 & -0.005 \\ 0.4 & -0.08 \\ 0.5 & -0.125 \\ 0.9 & -0.405 \\ 1.0 & -0.5 \\ 1.2 & -0.72 \\ 2.0 & -2.0 \end{pmatrix} \cdot \begin{pmatrix} v_y \\ g \end{pmatrix} = \begin{pmatrix} 0.96 \\ 3.26 \\ 3.82 \\ 5.11 \\ 5.2 \\ 5.05 \\ 0.58 \end{pmatrix}$$

solving overdetermined systems

have obtained: $$A \begin{pmatrix} v_y \\ g \end{pmatrix} = b \qquad A \in \mathbb{R}^{7\times 2}, \qquad b \in \mathbb{R}^2$$

least squares solution: find $x = (v_y, g)^\top$ s.t. the **residual** $r = b - Ax$ is minimized, i.e.,

$$\|b - Ax\|_2 \leq \|b - Ay\|_2 \qquad \forall y \in \mathbb{R}^2.$$

---

computing x using the normal equations:

The solution of the least squares problem is the solution of

$$A^\top A x = A^\top b$$

computing the least squares solution in the example:

$$A^\top A = \begin{pmatrix} 0.1 & 0.4 & 0.5 & 0.9 & 1.0 & 1.2 & 2.0 \\ -0.005 & -0.08 & -0.125 & -0.405 & -0.5 & -0.72 & -2.0 \end{pmatrix} \begin{pmatrix} 0.1 & -0.005 \\ 0.4 & -0.08 \\ 0.5 & -0.125 \\ 0.9 & -0.405 \\ 1.0 & -0.5 \\ 1.2 & -0.72 \\ 2.0 & -2.0 \end{pmatrix}$$

$$= \begin{pmatrix} 7.6700 & -5.8235 \\ -5.8235 & 4.954475 \end{pmatrix}$$

$$A^\top b = \begin{pmatrix} 0.1 & 0.4 & 0.5 & 0.9 & 1.0 & 1.2 & 2.0 \\ -0.005 & -0.08 & -0.125 & -0.405 & -0.5 & -0.72 & -2.0 \end{pmatrix} \begin{pmatrix} 0.96 \\ 3.26 \\ 3.82 \\ 5.11 \\ 5.2 \\ 5.05 \\ 0.58 \end{pmatrix} = \begin{pmatrix} 20.3290 \\ -10.20865 \end{pmatrix}$$

sought least squares solution $x = (v_y, g)^\top$ satisfies $A^\top A x = A^\top b \Longrightarrow$

continuing...

$$A^\top A \begin{pmatrix} v_y \\ g \end{pmatrix} = \begin{pmatrix} 7.6700 & -5.8235 \\ -5.8235 & 4.954475 \end{pmatrix} \begin{pmatrix} v_y \\ g \end{pmatrix} = \begin{pmatrix} 20.3290 \\ -10.20865 \end{pmatrix} = A^\top b.$$

The matrix $A^\top A$ is indeed SPD and the unique solution (v_y, g) is, to 5 digits,

$$\begin{pmatrix} v_y \\ g \end{pmatrix} = \begin{pmatrix} 10.096 \\ 9.8065 \end{pmatrix}.$$

solving using QR-factorization

$$A = \begin{pmatrix} 0.1 & -0.005 \\ 0.4 & -0.08 \\ 0.5 & -0.125 \\ 0.9 & -0.405 \\ 1.0 & -0.5 \\ 1.2 & -0.72 \\ 2.0 & -2.0 \end{pmatrix}, \qquad Q^\top A = \begin{pmatrix} \widetilde{R} \\ 0 \end{pmatrix} = \begin{pmatrix} -2.7695 & 2.1027 \\ 0 & -0.73 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$Q^\top b = Q^\top \begin{pmatrix} 0.96 \\ 3.26 \\ 3.82 \\ 5.11 \\ 5.2 \\ 5.05 \\ 0.58 \end{pmatrix} = \begin{pmatrix} -7.3404 \\ -7.1590 \\ -0.0037 \\ -0.0063 \\ 0.0058 \\ -0.0055 \\ 0.0046 \end{pmatrix}$$

continuing...

$$Q^\top A = \begin{pmatrix} \widetilde{R} \\ 0 \end{pmatrix} = \begin{pmatrix} -2.7695 & 2.1027 \\ 0 & -0.73 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q^\top b = \begin{pmatrix} -7.3404 \\ -7.1590 \\ -0.0037 \\ -0.0063 \\ 0.0058 \\ -0.0055 \\ 0.0046 \end{pmatrix} = \begin{pmatrix} r \\ z \end{pmatrix}$$

have to solve: $\widetilde{R}x = r$:

$$\begin{pmatrix} -2.7695 & 2.1027 \\ 0 & -0.73 \end{pmatrix} x = \begin{pmatrix} -7.3404 \\ -7.1590 \end{pmatrix}.$$

solving for $x = (v_y, g)^\top$ yields:

$$x = \begin{pmatrix} 10.083 \\ 9.807 \end{pmatrix}.$$

method of normal equations may fail

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ \varepsilon \\ \varepsilon \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{A}^\top \mathbf{A} = \begin{pmatrix} 1+\varepsilon^2 & 1 \\ 1 & 1+\varepsilon^2 \end{pmatrix}.$$

- $\mathbf{Ax} = \mathbf{b} \Longrightarrow \mathbf{x}$ is the exact solution of the least squares problem.
- $\kappa(\mathbf{A}^\top \mathbf{A}) = \frac{2}{\varepsilon^2} + 1$ so that $\mathbf{A}$ is ill-conditioned for small ε.

```
>> e = 1e-7;
>> A = [1 1; e 0; 0 e]; b = [2;e;e];
>> x = (A'*A)\(A'*b) %solution using normal equations
x =
   1.01123595505618O
   0.98876404494382O
>> [Q,R] = qr(A) ;
>> bb=Q'*b ;
>> xx = R(1:2,1:2)\bb(1:2) %solution using QR-factorization
xx =
   1.00000000000000
   1.00000000000000
```