(forward) stability: examples

function: $\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$

stability indicator: $\sigma = 1 + \kappa_k + \kappa_k\kappa_{k-1} + \cdots + \kappa_k\kappa_{k-1}\cdots\kappa_1$

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evaluation of $\sqrt{1+x} - \sqrt{1-x}$ for small x: in two different ways:

Alg I: $$x \overset{\varphi_1}{\mapsto} (1+x, 1-x) \overset{\varphi_2}{\mapsto} (\sqrt{1+x}, \sqrt{1-x}) \overset{\varphi_3}{\mapsto} \underbrace{(\sqrt{1+x} - \sqrt{1-x})}_{\text{cancellation}}$$

Alg II: $$x \overset{\varphi_1}{\mapsto} (2x, 1+x, 1-x) \overset{\varphi_2}{\mapsto} (2x, \sqrt{1+x}, \sqrt{1-x}) \overset{\varphi_3}{\mapsto} \frac{2x}{\sqrt{1+x} + \sqrt{1-x}}$$

κ_3 is large for Alg I. For Alg II all partial conditioning numbers κ_i are moderate.

MAPLE: with Digits := 5 yields $x = 10^{-6}$ for Alg. I 0.00000 and $0.10000 \cdot 10^{-5}$ for Alg. II.

examples of “hidden” cancellation: evaluation of $\varphi(x) = \ln(1+x)$ for small x

- evaluation of φ is well conditioned:

$$\kappa_{rel} = \frac{x}{\varphi(x)}\varphi'(x) \to 1 \qquad \text{für } x \to 0$$

- analysis: algorithmic realization using “standard functions”:

$$\varphi : x \overset{\varphi_1}{\mapsto} 1 + x \overset{\varphi_2}{\mapsto} \ln(1+x)$$

We have for the partial conditionings κ_1 and κ_2 φ_1, φ_2:

$$\begin{aligned} \kappa_1 &= 1 \qquad \text{at } x = 0 \\ \kappa_2 &= \frac{y}{|\varphi_2(y)|}|\varphi_2'(y)| = \frac{1}{\ln y} \qquad \text{large for } y \text{ close to } 1 \end{aligned}$$

- $\Longrightarrow$ expect that for small x (unavoidable) roundoff errors of the evaluation of φ_1 are dramatically amplified by the function φ_2:

```
>> x=1.234567890123456e-10;
>> w=1+x; f=log(w)
f =
      1.234568003306966e-10
```

exact value: $f = \mathbf{1.234567890047248}e - 10$

- solution: employ, e.g., Taylor expansions:
$\varphi(x) = \ln(1+x) = x - \frac{x^2}{2} + \cdots$ (suitably truncated)

example $\varphi(x) = \ln(1+x)$: simulation of floating point arithmetic

assumptions on the floating point arithmetic:

- $x \,\texttt{op}^*\, y = (x \,\texttt{op}\, y)(1+\delta)$ for a $\delta = \delta(x,y)$ with $|\delta| \le \texttt{eps}$
- $\texttt{ln}^* y = (1+\delta)\ln y$ for a $\delta = \delta(y)$ with $|\delta| \le \texttt{eps}$.

Then:

$$\begin{aligned} \texttt{ln}^*(1 +^* x) &= \texttt{ln}^*\left[(1+x)(1+\delta_1)\right] \\ &= (1+\delta_2)\ln\left[(1+x)(1+\delta_1)\right] \\ &= (1+\delta_2)\ln(1+x) + (1+\delta_2)\ln(1+\delta_1) \end{aligned}$$

and thus for the relative error

$$\left|\frac{\texttt{ln}^*(1 +^* x) - \ln(1+x)}{\ln(1+x)}\right| = \left|\delta_2 + \underbrace{\frac{(1+\delta_2)\ln(1+\delta_1)}{\ln(1+x)}}_{\text{large for } x \text{ small (and } \delta_1 \neq 0)}\right|$$

forward stability: computing the variance of a sample

$$\text{Alg I} \qquad V = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2, \qquad \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{1}$$

$$\text{Alg II} \qquad V = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right) \tag{2}$$

advantage of formula (2): 1 sweep through the data instead of 2 (think of large data sets!)

disadvange of formula (2): less stable than (1)

example: $n = 3$, $x_1 = 10000$, $x_2 = 10001$, $x_3 = 10002$. Then: $V = 1$.
A `C`-program with "single precision" (all numbers are of type float) yields:

$V_{(1)} = 1$ (correct!) and $V_{(2)} = -1.5$ (even sign is wrong).

forward stability: computing the variance of a sample (cont'd)

$$\text{Alg I} \qquad V = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2, \qquad \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

$$\text{Alg II} \qquad V = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right)$$

$$\text{Alg I} \quad (x_i) \overset{\varphi_1^I}{\mapsto} (x_i, \overline{x}) \overset{\varphi_2^I}{\mapsto} (x_i - \overline{x}) \overset{\varphi_3^I}{\mapsto} \sum (x_i - \overline{x})^2$$

$$\text{Alg II} \quad (x_i) \overset{\varphi_1^{II}}{\mapsto} \left(\sum x_i^2, \sum x_i\right) \overset{\varphi_2^{II}}{\mapsto} \sum x_i^2 - \tfrac{1}{n}\left(\sum x_i\right)^2$$

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stability indicators $\quad \sigma^I = 1 + \kappa_3^I + \kappa_3^I\kappa_2^I + \kappa_3^I\kappa_2^I\kappa_1^I, \qquad \sigma^{II} = 1 + \kappa_2^{II} + \kappa_2^{II}\kappa_1^{II}$

cond. of addition $x+y$: $\quad \kappa_{rel} = \dfrac{|x|+|y|}{|x+y|}$

forward stability: computing the variance of a sample (cont'd)

$$\text{Alg I} \quad (x_i) \overset{\varphi_1^I}{\mapsto} (x_i, \overline{x}) \overset{\varphi_2^I}{\mapsto} (x_i - \overline{x}) \overset{\varphi_3^I}{\mapsto} \sum (x_i - \overline{x})^2$$

$$\text{Alg II} \quad (x_i) \overset{\varphi_1^{II}}{\mapsto} \left(\sum x_i^2, \sum x_i\right) \overset{\varphi_2^{II}}{\mapsto} \sum x_i^2 - \tfrac{1}{n}\left(\sum x_i\right)^2$$

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conditioning of addition $x + y$: $\qquad \kappa_{rel} = \dfrac{|x| + |y|}{|x + y|}$

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numerical example:

$$x = [10000; 10001; 10002], \qquad \overline{x} = 10001, \qquad \sum_i x_i^2 \approx 3 \cdot 10^8$$

$$\kappa_{rel}(\varphi_2^I) \approx \frac{2 \cdot 10^4}{1} \to \sigma = O(10^4)$$

$$\kappa_{rel}(\varphi_2^{II}) \approx \frac{2 \cdot 3 \cdot 10^8}{2} \to \sigma = O(10^8)$$

one cannot expect in "single precision" (i.e., 8 digits) that Alg II yields a result that is correct to even a single digit