Euler-McLaurin formula

The expansion

$$T(h) = \int_a^b f(x)\,dx + c_1 h^2 + c_2 h^4 + \cdots \qquad (1)$$

is the Euler-McLaurin formula and obtained by repeated integration by part as we now show.

- For simplicity of notation, we consider

$$\int_0^N \tilde{f}(x)\,dx = \sum_{j=0}^{N-1} \int_j^{j+1} \tilde{f}(x)\,dx$$

- Integration by parts gives

$$\int_j^{j+1} \tilde{f}(x)\,dx = \Big[(x-j-1/2)\tilde{f}(x)\Big]_j^{j+1} - \int_j^{j+1} (x-j-1/2)\tilde{f}'(x)\,dx$$

- Hence

$$\frac{1}{2}\tilde{f}(j)+\frac{1}{2}\tilde{f}(j+1) = \int_j^{j+1} \tilde{f}(x)\,dx + \int_j^{j+1} \varpi_1(x)\tilde{f}'(x)\,dx, \qquad \varpi_1(x) = x-\lfloor x\rfloor-\frac{1}{2}$$

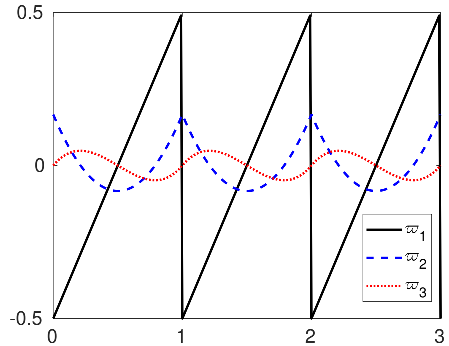
- Summation yields

$$\sum_{j=0}^{N-1} \frac{1}{2}\left(\tilde{f}(j) + \tilde{f}(j+1)\right) = \int_0^N \tilde{f}(x)\,dx + \int_0^N \varpi_1(x)\tilde{f}'(x)\,dx$$

- Let $\varpi_2(x)$ be an antiderivative of ϖ_1, i.e., $\varpi_2' = \varpi_1$. Note that ϖ_2 is 1-periodic since ϖ_1 is 1-periodic and $\int_0^1 \varpi_1(x)\,dx = 0$. This implies $\varpi_2(0) = \varpi_2(j) = \varpi_2(j+1)$ for all j. We select the integration constant of ϖ_2 such that $\int_0^1 \varpi_2(x)\,dx = 0$. (Because then, the antiderivative $\varpi_3(x)$ of $\varpi_2(x)$ will be 1-periodic).
- A two-fold integration by parts yields with an antiderivative $\varpi_3(x)$ of $\varpi_2(x)$

$$\int_j^{j+1} \varpi_1(x)\tilde{f}'(x)\,dx =$$

$$\varpi_2(0)\left(\tilde{f}'(j+1) - \tilde{f}'(j)\right) - \varpi_3(0)\left(\tilde{f}''(j+1) - \tilde{f}''(j)\right) + \int_j^{j+1} \varpi_3(x)\tilde{f}'''(x)\,dx$$



- again, one selects the integration constant of ϖ_3 such that $\int_0^1 \varpi_3(x)\,dx = 0$.
- repeating the process yields

$$\frac{1}{2}\left(\tilde{f}(j)+\tilde{f}(j+1)\right) = \int_j^{j+1} \tilde{f}(x)\,dx + \sum_{s=1}^{m-1} (-1)^{s+1} \varpi_{s+1}(0) \left(\tilde{f}^{(s)}(j+1) - \tilde{f}^{(s)}(j)\right) + (-1)^{m+1} \int_j^{j+1} \varpi_m(x) \tilde{f}^{(m)}(x)\,dx$$

- summation over j yields

$$\sum_{j=0}^{N} \frac{1}{2}(\tilde{f}(j)+\tilde{f}(j+1)) = \int_0^N \tilde{f}(x)\,dx + \sum_{s=1}^{m-1} (-1)^{s+1} \varpi_{s+1}(0) \left(\tilde{f}^{(s)}(N) - \tilde{f}^{(s)}(0)\right) + (-1)^{m+1} \int_0^N \varpi_m(x) \tilde{f}^{(m)}(x)\,dx \qquad (2)$$

- The values $\varpi_s(0)$ are the → Bernoulli numbers
- Exercise: Given a function f on $(0,1)$, define $\tilde{f}(x) := f(hx)$ on $(0,N)$ and derive the expansion (1) from (2).