Romberg extrapolation for $\int_0^1 f(x)\,dx$ with f smooth

Procedure:

1.) compute the values of the trapezoidal rule $T_{i0} := T(h_i)$ for $h_i = 2^{-i}$,

2.) evaluate the interpolating polynomial $P_{i,n} \in \mathcal{P}_n$ for the data $(h_{i+j}^2, T(h_{i+j}))$, $j = 0, \ldots, n$, at $h = 0$: $T_{in} := P_{i,n}(0)$. This is achieved with the Neville scheme:

$$
\begin{aligned}
T_{i0} &= T(h_i) \\
T_{ij} &= P_{i,j}(0) = P_{i+1,j-1}(0) - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[P_{i+1,j-1}(0) - P_{i,j-1}(0)\right] \\
&= T_{(i+1)\,(j-1)} - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[T_{(i+1)\,(j-1)} - T_{i\,(j-1)}\right], \qquad j \geq 1
\end{aligned}
$$

Romberg extrapolation for $\int_0^1 f(x)\,dx$ with f smooth, cont'd.

$$
\begin{aligned}
T_{i0} &= P_i(0) = T(h_i) \\
T_{ij} &= P_{i,j}(0) = P_{i+1,j-1}(0) - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[P_{i+1,j-1}(0) - P_{i,j-1}(0)\right] \\
&= T_{(i+1)\,(j-1)} - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[T_{(i+1)\,(j-1)} - T_{i\,(j-1)}\right], \qquad j \geq 1
\end{aligned}
$$

h	h^2	$n=0$	$n=1$	$n=2$	$n=3$
h_0	h_0^2	$T(h_0) = T_{00}$	T_{01}	T_{02}	T_{03}
h_1	h_1^2	$T(h_1) = T_{10}$	T_{11}	T_{12}	T_{13}
h_2	h_2^2	$T(h_2) = T_{20}$	T_{21}	T_{22}	T_{23}
h_3	h_3^2	$T(h_3) = T_{30}$	T_{31}	T_{32}	T_{33}
h_4	h_4^2	$T(h_4) = T_{40}$	T_{41}	T_{42}	$\vdots$
h_5	h_5^2	$T(h_5) = T_{50}$	T_{51}	$\vdots$	
h_6	h_6^2	$T(h_6) = T_{60}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$			

Romberg extrapolation for $\int_0^1 e^x\,dx$

h	h^2	$n=0$	$n=1$	$n=2$	$n=3$
1	1	1.859140914229523	1.718861151876593	1.718282687924754	1.718281828794499
2^{-1}	2^{-2}	1.753931092464825	1.718318841921747	1.718281842218437	1.718281828460412
2^{-2}	2^{-4}	1.727221904557517	1.718284154699897	1.718281828675358	1.718281828459105
2^{-3}	2^{-6}	1.720518592164302	1.718281974051892	1.718281828462428	1.718281828459017
2^{-4}	2^{-8}	1.718841128579994	1.718281837561771	1.718281828459097	1.718281828459077
2^{-5}	2^{-10}	1.718421660316327	1.718281829028016	1.718281828459049	1.718281828459047
2^{-6}	2^{-12}	1.718316786850094	1.718281828494605	1.718281828478246	
2^{-7}	2^{-14}	1.718290568083478	1.718281828461267		
2^{-8}	2^{-16}	1.718284013366820			
1	1	1.41_{-01}	5.79_{-04}	8.59_{-07}	3.35_{-10}
2^{-1}	2^{-2}	3.56_{-02}	3.70_{-05}	1.40_{-08}	1.37_{-12}
2^{-2}	2^{-4}	8.94_{-03}	2.33_{-06}	2.16_{-10}	5.95_{-14}
2^{-3}	2^{-6}	2.24_{-03}	1.46_{-07}	3.38_{-12}	-2.80_{-14}
2^{-4}	2^{-8}	5.59_{-04}	9.10_{-09}	5.20_{-14}	3.20_{-14}
2^{-5}	2^{-10}	1.40_{-04}	5.69_{-10}	3.77_{-15}	1.78_{-15}
2^{-6}	2^{-12}	3.50_{-05}	3.56_{-11}	1.92_{-11}	
2^{-7}	2^{-14}	8.74_{-06}	2.22_{-12}		
2^{-8}	2^{-16}	2.18_{-06}			
expected		$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$