6 Nonlinear equations and Newton's method

goal: determine zero $\mathbf{x}^*$ of $\mathbf{f}(\mathbf{x}^*) = 0$
Since there are typically no exact solution formulas, the zero $\mathbf{x}^*$ is approximated by iterates $\mathbf{x}_n$ with $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}^*$. The most common form is that of a *fixed point iteration*

$$\mathbf{x}_{n+1} = \Phi(\mathbf{x}_n) \tag{6.1}$$

with an initial guess $\mathbf{x}_0$ that is taken sufficiently close to $\mathbf{x}^*$. Thus, the iterative method is described by the function Φ.

Exercise 6.1 *Show: If* $\mathbf{x}_n \to \mathbf{x}^*$ *then* $\mathbf{x}^*$ *is a* fixed point *of* Φ*, i.e.,* $\mathbf{x}^* = \Phi(\mathbf{x}^*)$ *(assumption:* Φ *is continuous at* $\mathbf{x}^*$*).* ■

6.1 Newton's method in 1D

goal: Find zero x^* of $f(x^*) = 0$
Idea: *linearize* f at the current iterate x_n and find zero of the linearization.
procedure:

1. x_n = current iterate
2. $L(x) := f(x_n) + f'(x_n)(x - x_n)$ ⟦ linearization is the tangent at x_n, i.e., the Taylor expansion up to the linear term ⟧
3. x_{n+1} := zero of L, i.e.,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{6.2}$$

We recognize that the 1D-Newton method (6.2) has the form $x_{n+1} = \Phi^{Newton}(x_n)$ of a fixed point iteration with Φ^{Newton} given by

$$\Phi^{Newton}(x) = x - \frac{f(x)}{f'(x)}. \tag{6.3}$$

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Example 6.2 **slide 1**
$x^* = \sqrt{a}$ is the zero of $f(x) = x^2 - a$. With $f'(x) = 2x$, Newton's method is

$$x_{n+1} = \Phi^{Newton}(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}.$$

The rapid convergence of the method is visible in Fig. 6.1 for the choice $a = 2$ and initial value $x_0 = 2$. In fact, we observe so-called quadratic convergence *in that the error behaves like* $|x^* - x_{n+1}| \approx |x^* - x_n|^2$. ■

	Newton iterates ($x_0 = 2$)	error
x_1	1.5	8.578643762690485_{-2}
x_2	**1.41**6666666666667	2.453104293571595_{-3}
x_3	**1.414215**686274510	2.1239014147411694_{-6}
x_4	**1.41421356237**4690	1.5947243525715749_{-12}
exact:	1.414213562373095	

Figure 6.1: Newton's method for computing $\sqrt{2}$ (cf. Example 6.2)

6.2 convergence of fixed point iterations

The key property that ensures convergence of the fixed point iteration (6.1) is that Φ is a *contraction*:

Definition 6.3 *The function $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction (with respect to the norm $\|\cdot\|$) near the point $\mathbf{x}^*$ if there are $q \in (0,1)$ and $\varepsilon > 0$ such that*

$$\|\Phi(\mathbf{x}) - \Phi(\mathbf{y})\| \leq q\|\mathbf{x} - \mathbf{y}\| \qquad \forall \mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}^*). \tag{6.4}$$

Exercise 6.4 *Consider the case $d = 1$. Show: If $\Phi \in C^1$ and $|\Phi'(x^*)| < 1$ near a point x^*, then Φ is a contraction near x^*.* ■

The following result shows that the contraction property implies convergence of the fixed point iteration (6.1) if the initial value $\mathbf{x}_0$ is sufficiently close to the fixed point $\mathbf{x}^*$.

Theorem 6.5 *Let Φ be a contraction with contraction constant $q \in (0,1)$ near the fixed point $\mathbf{x}^* = \Phi(\mathbf{x}^*)$. Then there is $\varepsilon > 0$ such that for $\mathbf{x}_0 \in B_\varepsilon(\mathbf{x}^*)$ the iterates $\mathbf{x}_n$ given by (6.1) converge to $\mathbf{x}^*$. Moreover,*

$$\|\mathbf{x}^* - \mathbf{x}_{n+1}\| \leq q\|\mathbf{x}^* - \mathbf{x}_n\| \qquad \forall n \in \mathbb{N}_0. \tag{6.5}$$

Proof: Let $\varepsilon > 0$ be given by Def. 6.3 and $x_n \in B_\varepsilon(x^*)$. Then:

$$\|x^* - x_{n+1}\| = \|x^* - \Phi(x_n)\| \overset{x^* \text{ fixed pt}}{=} \|\Phi(x^*) - \Phi(x_n)\| \overset{\text{contraction property}}{\leq} q\|x^* - x_n\|.$$

Hence, if $x_0 \in B_\varepsilon(x^*)$, then by induction all iterates $x_n \in B_\varepsilon(x^*)$ and $|x^* - x_n| \to 0$. □

Exercise 6.4 gives an easy condition (in the scalar case $d = 1$) when the iteration (6.1) converges:

Exercise 6.6 *Let $d = 1$ and $\Phi \in C^1$ satisfy $|\Phi'(x^*)| < 1$ at the fixed point x^* of Φ. Then the iterates x_n given by (6.1) converge to x^* provided the initial value x_0 is sufficiently close to x^*.* Remark: *The vector-valued analog is as follows: The derivative Φ' is a $d \times d$ matrix and if there is a norm $\|\cdot\|$ such that $\|\Phi'(\mathbf{x}^*)\| < 1$ at a fixed point $\mathbf{x}^*$ of Φ, then Φ is a contraction near $\mathbf{x}^*$.* ■

Example 6.7 slide 31
We seek a solution of the nonlinear equation

$$2 - x^2 - e^x = 0. \tag{6.6}$$

n	$x_{n+1} = \Phi_1(x_n)$	$x_{n+1} = \Phi_2(x_n)$
0	0.592687716508341	0.559615787935423
1	0.437214425050104	0.522851128605001
2	0.672020792350124	0.546169619063046
3	0.204473907097276	0.531627015197373
4	0.879272743474883	0.540795632739194
5	stop: $(2 - e^{0.87} < 0)$	0.535053787215218
6		0.538664955236433
7		0.536399837485597
8		0.537823020842571
9		0.536929765486145

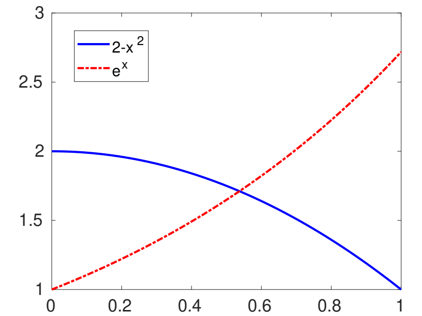

Table 6.1: Left: fixed point iteration of Example 6.7. Right: $x \mapsto e^x$ and $x \mapsto 2 - x^2$

Graphical considerations show that there is exactly one positive solution $x^* \approx 0.5$. For $x > 0$ equation (6.6) can be converted to a fixed point form in several ways:

$$x = \sqrt{2 - e^x} =: \Phi_1(x), \qquad x = \ln(2 - x^2) =: \Phi_2(x), \tag{6.7}$$

The fixed point iterations based on Φ_1 and Φ_2 behave differently when initialized with $x_0 = 0.5$ as is visible in Table 6.1: Whereas the iteration $x_{n+1} = \Phi_2(x_n)$ converges to the correct value $x^* = 0.5372744491738\ldots$ the iteration $x_{n+1} = \Phi_1(x_n)$ does not converge. The reason is that $|\Phi_1'(x^*)| \approx |-1.59| > 1$ whereas $|\Phi_2'(x^*)| \approx 0.31 < 1$. ■

Theorem 6.5 shows that if Φ is a contraction, then one has *linear* convergence, i.e., the error decreases by a factor $q \in (0,1)$ in each step. A special situation arises if $\Phi'(x^*) = 0$. Then faster convergence is possible:

Theorem 6.8 *Let $d = 1$ and $\Phi \in C^p(\mathbb{R}^d)$, $p \geq 2$. Assume $x^* = \Phi(x^*)$ and $0 = \Phi^j(x^*)$ for $j = 1, \ldots, p-1$. Then there are C, $\varepsilon > 0$ such that for $x_0 \in B_\varepsilon(x^*)$ the iterates x_n given by (6.1) converge to x^* and*

$$|x^* - x_{n+1}| \leq C|x^* - x_n|^p \qquad \forall n \in \mathbb{N}_0.$$

Proof: By Theorem 6.5 we already know that the iterates converge to x^* if ε is sufficiently small. For the estimate, we modify the proof of Theorem 6.5. By Taylor expansion around x^* we have

$$\begin{aligned}|x^* - x_{n+1}| = |\Phi(x^*) - \Phi(x_n)| = |\Phi(x^*) - \Phi(x_n)| &= \left|\frac{1}{(p-1)!}\int_{x^*}^{x_n} (x_n - t)^{p-1}\Phi^{(p)}(t)\, dt\right| \\ &\leq \frac{\|\Phi^{(p)}\|_{\infty, B_\varepsilon(x^*)}}{(p-1)!}|x^* - x_n|^p.\end{aligned}$$

□

In the setting of Theorem 6.8, we say that the iteration converges with *order p*. In particular, for $p = 2$ the method converges *quadratically*. Example 6.2 shows that the Newton method applied to the problem $f(x) = x^2 - a = 0$ convergence quadratically. This is typical of the Newton method: