High order transparent boundary conditions for the Helmholtz equation

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Abstract We consider finite element simulations of the Helmholtz equation in unbounded domains. For computational purposes, these domains are truncated to bounded domains using transparent boundary conditions at the artificial boundaries. We present here two numerical realizations of transparent boundary conditions: the complex scaling or perfectly matched layer method and the Hardy space infinite element method. Both methods are Galerkin methods, but their variational framework differs. Proofs of convergence of the methods are given in detail for one dimensional problems. In higher dimensions radial as well as Cartesian constructions are introduced with references to the known theory.

1 Introduction

We consider finite element simulations of time-harmonic, scalar waves in open systems. Since standard mesh based methods like finite element or finite difference methods are restricted to bounded domains, for these methods unbounded domains of propagation have to be truncated to a bounded computational domain. Typically, such a truncation results in artificial reflections at the truncation boundary. Due to the non-locality of the waves, the reflections may pollute the solution in the whole computational domain.

The purpose of this paper is to present some high order transparent boundary conditions such that artificial reflections are minimized. Thereby we restrict ourselves to finite element based transparent boundary conditions. For boundary element methods we refer e.g. to [34].

The simplest transparent boundary condition is the so-called first order absorbing boundary condition. It has no extra costs, but the computational domain typically has to be quite large in order to minimize artificial reflections. For a review of higher order local absorbing boundary conditions we refer to [24, 15]. For these transparent boundary conditions, as for all subsequent ones, additional unknowns are needed. Since the construction and the theoretical framework are quite complicated, we will not present them in this paper.

The so-called complex scaling or perfectly matched layer (PML) method (see e.g. [31, 19, 2, 8]) fits very well into the variational framework of finite element methods. It surrounds the computational domain with an artificial, anisotropic damping layer. It is very flexible and allows to reduce artificial reflections as much as necessary. A downside is, that it can be difficult to find optimal method parameters, since it depends on the damping profile, the thickness of the layer and on the finite element discretization in the absorbing layer.

For infinite elements no artificial truncation is needed. The unbounded domain outside of the computational domain is discretized with special basis and test functions. For classical infinite element methods (see [12, 13]) these functions fulfill the Sommerfeld radiation condition. Since the infinite elements are

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defined on an unbounded domain, integration over these basis and test functions needs to be done carefully. Moreover, the discretization matrices typically have large condition numbers.

Hardy space infinite elements (see [20, 33, 32]) also discretize the whole unbounded domain, but the basis functions are completely different to the classical ones. The basis functions are constructed using the pole condition [35, 22] as radiation condition. Roughly speaking this radiation condition characterizes outgoing waves by the poles of their Laplace transforms, which belong to a certain class of Hardy spaces. The Hardy space infinite element method is a Galerkin method applied to a variational problem in a space which is built using a Hardy space. Just as the PML method the Hardy space infinite element method allows for arbitrary small discretization errors. It is even more flexible as the PML method and can be applied to time harmonic wave equations with backward propagating modes, where standard PML methods fail (see [17, 18]).

For the Helmholtz scattering problems given in Sec. 2 we present the PML (Sec. 3) and the Hardy space infinite element method (Sec. 4). To explain the basic ideas, we start for both methods with a one dimensional model problem, even though in one dimension there exists an easy to use exact transparent boundary condition. These ideas are then generalized to higher dimensions using radial, as well as Cartesian coordinates. In Sec. 5 we compare the two methods in terms of efficiency and programming effort.

2 Helmholtz scattering problems

In this section we start with the problem setting and the most popular radiation conditions in order to control the behavior of solutions u to the Helmholtz equation for large arguments.

2.1 Problem setting

Let u be a solution to the Helmholtz equation

$$-\Delta u(x) - \omega^2(1+p(x))u(x) = 0, \qquad x \in \Omega,$$

for an unbounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, with angular frequency $\omega > 0$. p is a coefficient function with compact support $\operatorname{supp}(p) := \overline{\{x \in \mathbb{R}^d : p(x) \neq 0\}}$ in an open ball $B_R := \{x \in \mathbb{R}^d : |x| < R\}$ of radius $R > 0$. $|x| := \sqrt{\sum_{j=1}^d |x_j|^2}$ denotes the standard Euclidean norm.

Moreover, let the boundary $\partial\Omega$ be contained in B_R and let u fulfill the boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} + \alpha u = g, \qquad \text{for } x \in \partial\Omega$$

with given functions α and g and the unit normal vector $\mathbf{n}$ pointing to the exterior of Ω. We refrain from Dirichlet boundary conditions in order to simplify the presentation.

Since problems on unbounded domains Ω cannot be discretized with standard finite elements, we introduce a bounded and star shaped Lipschitz domain $D \subset \mathbb{R}^d$ such that $\partial\Omega \subset D$ and $p \equiv 1$ in $\mathbb{R}^d \setminus D$. Then Ω is the disjoint union of the bounded interior domain $\Omega_{\text{int}} := \Omega \cap D$, the unbounded exterior domain $\Omega_{\text{ext}} := \mathbb{R}^d \setminus \overline{D}$ and the interface $\Gamma := \partial\Omega_{\text{int}} \cap \partial\Omega_{\text{ext}}$. E.g. one could choose $D = B_R$.

In Ω_{ext}, we are looking for solutions u of the homogeneous problem

$$-\Delta u - \omega^2 u = 0, \qquad \text{in } \Omega_{\text{ext}}, \tag{2.1a}$$

$$u = u_0, \qquad \text{on } \Gamma, \tag{2.1b}$$

$$u \text{ is outgoing for} \qquad |x| \to \infty. \tag{2.1c}$$

The radiation condition (2.1c) ensures that (2.1) is uniquely solvable for all Dirichlet data $u_0 \in H^{1/2}(\Gamma)$ and all $\omega > 0$, and that these solutions are physically meaningful. For such a unique solution u_{u_0} we define

the so-called Dirichlet-to-Neumann operator $\mathrm{DtN} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by

$$\mathrm{DtN}\, u_0 := \frac{\partial u_{u_0}}{\partial \mathbf{n}}.$$

Here, the unit normal vector $\mathbf{n}$ on Γ points to the interior of Ω_{ext}. The interior problem for $u \in H^1(\Omega_{\mathrm{int}})$ in weak form is given by

$$\int_{\Omega_{\mathrm{int}}} \left(\nabla u \cdot \nabla v - \omega^2(1+p)uv\right) dx + \int_{\partial\Omega} \alpha uv\, ds - \int_{\Gamma} (\mathrm{DtN}\, u|_{\Gamma})\, v\, ds = \int_{\partial\Omega} gv\, ds \tag{2.2}$$

for all test functions $v \in H^1(\Omega_{\mathrm{int}})$. Representation formulas of the Dirichlet-to-Neumann operator for $d = 2,3$ will be the subject of the following subsections.

In one dimension solutions to (2.1a) are given by linear combinations of $x \mapsto \exp(\mathrm{i}\omega|x|)$ and $x \mapsto \exp(-\mathrm{i}\omega|x|)$. Using the standard convention $\exp(-\mathrm{i}\omega t)$ for the time-harmonic ansatz, $x \mapsto \exp(\mathrm{i}\omega|x|)$ is a radiating solution. Hence, the Dirichlet-to-Neumann operator in one dimension is simply given by $\mathrm{DtN}\, u_0 = \mathrm{i}\omega u_0$.

2.2 Sommerfeld radiation condition

Following [28] for time-harmonic waves of the form $\Re\left(u(x)\exp\left(-\mathrm{i}\omega t\right)\right)$, the averaged outward energy flux through the interface Γ is given by

$$J_\Gamma(u) := -\frac{1}{2\omega}\Im\left\{\int_\Gamma u\, \frac{\partial \overline{u}}{\partial \mathbf{n}} ds\right\}.$$

Using Green's first identity in a domain $B_R \cap \Omega_{\mathrm{ext}}$, it can be shown that

$$J_\Gamma(u) = \frac{1}{4\omega^2} \lim_{R\to\infty} \left(\mp \int_{\partial B_R} \left| \frac{\partial u}{\partial \mathbf{n}} \mp \mathrm{i}\omega u \right|^2 ds \pm \int_{\partial B_R} \left(\left| \frac{\partial u}{\partial \mathbf{n}} \right|^2 + \omega^2 |u|^2 \right) ds \right) \tag{2.3}$$

for all solutions $u \in H^2_{\mathrm{loc}}(\Omega_{\mathrm{ext}})$[1] to (2.1a). Using (2.3) with the minus sign in the first integral, $J_\Gamma(u)$ is non-negative for solutions to (2.1a), if u fulfills the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|^{(d-1)/2} \left(\frac{\partial u(x)}{\partial |x|} - \mathrm{i}\omega u(x) \right) = 0 \quad \text{uniformly for all directions } \frac{x}{|x|}. \tag{2.4}$$

Moreover, using (2.4) as radiation condition the problem (2.1) is uniquely solvable (see e.g. [36, Sec. 9, Theorem 1.3]). So the Sommerfeld radiation condition leads to a well defined Dirichlet-to-Neumann operator.

It can also be used to construct an approximation to the exact Dirichlet-to-Neumann operator: If the interface is a sphere of radius $R > 0$, then the so-called first order absorbing boundary condition is given by $u_0 \mapsto \mathrm{i}\omega u_0$. This Robin type boundary condition is only the exact boundary condition for $d = 1$. But since for a numerical realization no extra costs are needed, it is widely used in practice for $d = 2,3$ as well. Typically, R has to be quite large in order to guarantee, that the artificial reflections at Γ are negligible.

[1] $H^r_{\mathrm{loc}}(\Omega)$ denotes the space of functions, which belong to $H^r(\hat{\Omega})$ for each compact $\hat{\Omega} \subset \Omega$.

2.3 Dirichlet-to-Neumann operator

For $x \in \Omega_{\text{ext}} = \mathbb{R}^d \setminus \overline{B_R}$ we can use polar coordinates $x = r\hat{x}$ with $r := |x| > 0$ and $\hat{x} = x/r \in \partial B_1$ in order to construct a representation formula for solutions u to the exterior problem (2.1). In polar coordinates the Helmholtz equation (2.1a) is given by

$$-\frac{\partial^2 u(r\hat{x})}{\partial^2 r} - \frac{d-1}{r}\frac{\partial u(r\hat{x})}{\partial r} - \frac{1}{r^2}\Delta_{\hat{x}} u(r\hat{x}) - \omega^2 u(r\hat{x}) = 0, \quad r > R, \hat{x} \in \partial B_1.$$

$-\Delta_{\hat{x}}$ is the negative Laplace-Beltrami operator. As it is hermitian and positive semi-definite, all eigenvalues are non-negative. E.g. in [11] it is shown, that for $d = 3$ the eigenvalues are given by $\lambda_\nu := \nu(\nu+1)$ with multiplicities $M_\nu := 2\nu + 1$, $\nu \in \mathbb{N}_0$[2]. For $d = 2$ the eigenvalues are $\lambda_\nu := \nu^2$ with multiplicities $M_\nu := 2$ for $\nu \in \mathbb{N}$ and $M_0 := 1$ for $\nu = 0$. The corresponding eigenfunctions, the spherical harmonics $Y_\nu^{(\mu)}$, build a complete orthonormal set of $L^2(\partial B_1)$. Hence, there holds

$$u(r\hat{x}) = \sum_{\nu=0}^{\infty} \sum_{\mu=1}^{M_\nu} u_{\nu,\mu}(r) Y_\nu^{(\mu)}(\hat{x}), \qquad r > R, \quad \hat{x} \in \partial B_1 \tag{2.5}$$

with $u_{\nu,\mu}(r) := \int_{\partial B_1} u(r\hat{x})\overline{Y_\nu^{(\mu)}(\hat{x})} d\hat{x}$. The series converges for each $r > R$ in the $L^2(\partial B_1)$ sense. If u is a sufficiently smooth solution to (2.1a), we can differentiate under the integral and deduce that $u_{\nu,\mu}$ is a solution to the (spherical) Bessel equation

$$-u_\nu''(r) - \frac{d-1}{r} u_\nu'(r) + \left(\frac{\lambda_\nu}{r^2} - \omega^2\right) u_\nu(r) = 0, \qquad r > R. \tag{2.6}$$

Solutions to (2.6) with $\omega = 1$ are linear combinations of the (spherical) Hankel functions of the first and second kind. We denote the Hankel functions ($d = 2$) and the spherical Hankel functions ($d = 3$) of the first and second kind by $\mathscr{H}_\nu^{(1,2)}$. Their asymptotic behavior is given by

$$\mathscr{H}_\nu^{(1,2)}(t) = \frac{C_d}{t^{(d-1)/2}} \exp\left(\pm \mathrm{i}\left(t - \frac{\nu\pi}{2}\right)\right)\left(1 + \mathscr{O}\left(\frac{1}{t}\right)\right), \qquad t \to \infty, \tag{2.7a}$$

$$\mathscr{H}_\nu^{(1,2)'}(t) = \frac{\pm \mathrm{i} C_d}{t^{(d-1)/2}} \exp\left(\pm \mathrm{i}\left(t - \frac{\nu\pi}{2}\right)\right)\left(1 + \mathscr{O}\left(\frac{1}{t}\right)\right), \qquad t \to \infty, \tag{2.7b}$$

with $C_2 := \sqrt{2/\pi}\exp(\mp \mathrm{i}\pi/4)$ and $C_3 := \exp(\mp \mathrm{i}\pi/2)$. Hence, there holds

$$\lim_{r\to\infty} r^{(d-1)/2}\left(\mathscr{H}_\nu^{(1,2)'} \mp \mathrm{i}\mathscr{H}_\nu^{(1,2)}(r)\right) = 0.$$

In particular, the functions $u(r\hat{x}) := \mathscr{H}_\nu^{(1)}(\omega r) Y_\nu^{(\mu)}(\hat{x})$ solve the Helmholtz equation (2.1a) and satisfy the Sommerfeld radiation condition (2.4). Moreover, using (2.3) we compute

$$J_\Gamma(u) = \begin{cases} \frac{1}{\omega\pi}, & d = 2 \\ \frac{1}{2\omega^2}, & d = 3 \end{cases}, \tag{2.8}$$

i.e. the outward energy flux is positive and independent of ν and μ. So these functions radiate energy to infinity and are therefore physically meaningful.

Remark 2.1 *A second way of motivating the choice of outgoing solutions is the limiting absorption principle (see e.g. [36, Sec. 9]). Similar to the idea of shifted Laplace preconditioners, we replace the positive frequency ω in the Helmholtz equation by $\omega(1+\varepsilon\mathrm{i})$ with $\varepsilon > 0$ adding artificial absorption to the problem. Since the solutions to the perturbed problem should be bounded for $r \to \infty$, these solutions are given by*

[2] $\mathbb{N}$ denotes the set of all positive natural numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

$u_\varepsilon(r\hat{x}) := \mathscr{H}_\nu^{(1)}(\omega(1+\mathrm{i}\varepsilon)r)Y_\nu^{(\mu)}(\hat{x})$. *Passing* ε *to the limit* 0 *leads again to the Hankel functions of the first kind.*

Using the Hankel functions of the first kind in (2.5) and incorporating the boundary condition (2.1b) leads to the series representation

$$u(r\hat{x}) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{M_\nu} \frac{\int_{\partial B_1} u_0(R\hat{x})\overline{Y_\nu^{(\mu)}(\hat{x})}d\hat{x}}{\mathscr{H}_\nu^{(1)}(\omega R)}\mathscr{H}_\nu^{(1)}(\omega r)Y_\nu^{(\mu)}(\hat{x}), \quad r>R, \quad \hat{x}\in\partial B_1. \tag{2.9}$$

For $u_0 \in L^2(\partial B_R)$ it is shown in [11, Theorem 2.14], that this series as well as the series of the term by term derivatives converges absolutely and uniformly on compact subsets of $\Omega_{\text{ext}} = \mathbb{R}^3 \setminus \overline{B_R}$. The results holds true for $\Omega_{\text{ext}} = \mathbb{R}^2 \setminus \overline{B_R}$ as well. Moreover, it is indeed a solution to (2.1) with the Sommerfeld radiation condition and each solution to (2.1) satisfying the Sommerfeld radiation condition is given by (2.9). Hence, (2.9) can be used to construct the Dirichlet-to-Neumann operator on spheres of radius R by

$$\mathrm{DtN}\, u_0 := \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{M_\nu}\left(\int_{\partial B_1} u_0(R\hat{x})\overline{Y_\nu^{(\mu)}(\hat{x})}d\hat{x}\right)\frac{\omega\mathscr{H}_\nu^{(1)'}(\omega R)}{\mathscr{H}_\nu^{(1)}(\omega R)}Y_\nu^{(\mu)}(\hat{x}). \tag{2.10}$$

Note, that the roots of the (spherical) Hankel functions of the first kind have negative imaginary part and therefore the denominator never vanishes for $\omega R > 0$.

A second representation formula for solutions to (2.1) can be deduced using the fundamental solution of the Helmholtz equation

$$\Phi(x,y) := \begin{cases} \frac{\mathrm{i}}{4}\mathscr{H}_0^{(1)}(\omega|x-y|), & d=2 \\ \frac{\mathrm{i}\omega}{4\pi}\mathscr{H}_0^{(1)}(\omega|x-y|), & d=3 \end{cases}.$$

In [11, 36]) is is shown, that for smooth boundary Γ a solution u of the exterior problem (2.1) combined with the Sommerfeld radiation condition has the integral representation

$$u(x) = \int_\Gamma \left(u(y)\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)} - \frac{\partial u}{\partial\mathbf{n}}(y)\Phi(x,y)\right)ds(y), \qquad x\in\Omega_{\text{ext}}. \tag{2.11}$$

This representation can be used to construct a Dirichlet-to-Neumann operator for arbitrary smooth boundaries Γ.

Remark 2.2 *The representation formulas* (2.9) *and* (2.11) *can also be used as radiation conditions. Since the (spherical) Hankel functions are holomorphic in* $\{z\in\mathbb{C} : \Re(z) > 0\}$*, the solutions* u *to* (2.1) *using these radiation conditions are holomorphic with respect to complex frequencies* ω *with* $\Re(\omega) > 0$*. This is not the case, if the Sommerfeld radiation condition is used, since for* ω *with* $\Re(\omega) > 0$ *and* $\Im(\omega) < 0$ *the Hankel functions of the second kind fulfill the Sommerfeld radiation condition. So for resonance problems, where the frequency is the sought complex resonance, the Sommerfeld radiation condition is not useful.*

3 Complex scaling method

For test functions $v \in H^1(\Omega_{\text{ext}})$ with compact support in $\Omega_{\text{ext}} \cup \Gamma$, the variational form of (2.1) is given by

$$\int_{\Omega_{\text{ext}}} \left(\nabla u\cdot\nabla v - \omega^2 u\, v\right)dx = -\int_\Gamma \mathrm{DtN}\, u_0\, v\, ds. \tag{3.1}$$

In the complex scaling or perfectly matched layer method the left hand side of this equation is first reformulated such that the solution u and the integrand is exponentially decaying for $|x|\to\infty$. Then a truncation of the unbounded domain Ω_{ext} to a bounded layer leads to an approximation of the Dirichlet-to-Neumann

operator on the right hand side. As we will see, this approximation converges exponentially to the correct Dirichlet-to-Neumann operator with respect to the thickness of the layer.

3.1 One dimensional PML

For simplicity we start with a one dimensional problem. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$ be an outgoing solution to

$$\int_0^\infty \big(u'(x)v'(x) - \omega^2(1+p(x))u(x)v(x)\big)\,dx = -u'_0 v(0) \tag{3.2}$$

for all test functions $v \in H^1(\mathbb{R}_+)$ with compact support in $\mathbb{R}_{\geq 0}$. $u'_0 \in \mathbb{C}$ denotes a given Neumann boundary value of $u'(0)$. If $p \in L^\infty(\mathbb{R}_+)$ with $\mathrm{supp}(p) \subset [0,R)$, u is given by

$$u(x) = \begin{cases} u_{\mathrm{int}}(x), & x \in \Omega_{\mathrm{int}} := (0,R) \\ u_{\mathrm{int}}(R)\exp(\mathrm{i}\omega(x-R)), & x \in \Omega_{\mathrm{ext}} := (R,\infty) \end{cases} \tag{3.3}$$

where $u_{\mathrm{int}} \in H^1(\Omega_{\mathrm{int}})$ is a solution to the interior problem.

For the complex scaling we use a twice continuously differentiable function $\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\tau(0) = 0$ and $\tau(t) \geq Ct$, $C > 0$, for sufficiently large t. One might use simply the identity. For $\alpha \in \mathbb{R}$ and $R > 0$ the complex scaling function is defined by

$$\gamma_{\tau,\alpha,R}(r) := \begin{cases} r, & r \leq R \\ r + \alpha\mathrm{i}\tau(r-R), & r > R \end{cases}. \tag{3.4}$$

$\gamma_{\tau,\alpha,R}$ is continuous and at least twice continuously differentiable for all $r \neq R$. For the monomials $\tau(t) = t^k$, $\gamma_{\tau,\alpha,R}$ is $k-1$ times continuously differentiable at $r = R$ and arbitrary smooth elsewhere. Based on the complex scaling function we define the complex scaled variable

$$x_\gamma(x) := \begin{cases} \frac{x}{|x|}\gamma_{\tau,\alpha,R}(|x|), & x \in \mathbb{R}^d \setminus \{0\} \\ 0, & x = 0. \end{cases} \tag{3.5}$$

Since outgoing solutions u are given by (3.3) and in particular since $u|_{\Omega_{\mathrm{ext}}}$ has a holomorphic extension, the complex scaled function

$$u_\gamma := u \circ x_\gamma \tag{3.6}$$

is well defined and solves the complex scaled Helmholtz equation

$$-\frac{\partial}{\partial x}\left(\frac{u'_\gamma(x)}{\gamma'_{\tau,\alpha,R}(x)}\right) - \omega^2\gamma'_{\tau,\alpha,R}(x)u_\gamma(x) = 0, \qquad x > R. \tag{3.7}$$

Note, that u_γ inherits the regularity of $\gamma_{\tau,\alpha,R}$. Moreover, it decays exponentially for $x \to \infty$ if and only if $\alpha > 0$. Hence, partial integration on both sides yields

$$\int_{\Omega_{\mathrm{ext}}} \big(u'v' - \omega^2 uv\big)\,dx = -v(R)\,\mathrm{DtN}(u(R)) = \int_{\Omega_{\mathrm{ext}}} \left(\frac{u'_\gamma \tilde{v}'}{\gamma'_{\tau,\alpha,R}} - \omega^2\gamma'_{\tau,\alpha,R}u_\gamma\tilde{v}\right)dx$$

for all test functions $v \in H^1(\Omega_{\mathrm{ext}})$ with compact support in $[R,\infty)$ and all $\tilde{v} \in H^1(\Omega_{\mathrm{ext}})$ with $\tilde{v}(R) = v(R)$.

Theorem 3.1. *Let $\omega, \alpha, R > 0$, and $p \in L^\infty(\mathbb{R}_+)$ with $\mathrm{supp}(p) \subset [0,R)$. Moreover, let the assumptions on τ be fulfilled. Then $u \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$ is an outgoing solution to (3.2) if and only if $u_\gamma \in H^1(\mathbb{R}_+)$ defined in (3.6) is a solution to*

$$\int_0^\infty \left(\frac{u'_\gamma v'}{\gamma'_{\tau,\alpha,R}} - \omega^2(1+p)\gamma'_{\tau,\alpha,R} u_\gamma v \right) dx = -u'_0 v(0), \quad v_{\in} H^1(\mathbb{R}_+). \tag{3.8}$$

Proof. We have already shown the first direction. Vice versa, let $\tilde{u}$ be a solution to (3.8). Using test functions v with compact support in (R,∞) and elliptic regularity results, $\tilde{u}_{\text{ext}} := \tilde{u}|_{(R,\infty)} \in H^2((R,\infty))$ solves (3.7). Hence, $\tilde{u}_{\text{ext}}$ is a linear combination of $x \mapsto \exp(\pm \mathrm{i}\omega\gamma_{\tau,\alpha,R}(x))$. Since $\Re(\pm\mathrm{i}\omega\gamma_{\tau,\alpha,R}(x)) = \mp\omega\alpha\tau(x-R)$ and $\tilde{u}_{\text{ext}} \in H^2((R,\infty))$, we have $\tilde{u}_{\text{ext}}(x) = \tilde{u}_{\text{ext}}(R)\exp(\mathrm{i}\omega(\gamma_{\tau,\alpha,R}(x)-R))$. Plugging this into (3.8) and using partial integration in $[R,\infty)$ for test functions v with compact support in $[R,\infty)$ leads to

$$\int_0^R \left(\tilde{u}'v' - \omega^2(1+p))\tilde{u}v\right) dx = \mathrm{i}\omega\tilde{u}(R)v(R) - u'_0 v(0), \qquad v \in H^1(\Omega_{\text{int}}),$$

i.e. to the correct Dirichlet-to-Neumann operator at $x = R$. Thus, u defined by (3.3) with $u_{\text{int}} := \tilde{u}|_{\Omega_{\text{int}}}$ is outgoing and solves (3.2). □

Corollary 3.2 *Let u be a solution to (3.8). Then $u|_{\Omega_{\text{int}}}$ is independent of the damping function $\gamma_{\tau,\alpha,R}$.*

Of course, (3.8) is still posed on an unbounded domain $\mathbb{R}_+$ and cannot be discretized directly using standard finite element methods. But since the integrand is exponentially decaying, $\mathbb{R}_+$ is typically truncated to a bounded domain $(0,R+L)$ with $L > 0$ sufficiently large. Then, the truncated problem on $H^1((0,R+L))$ is discretized using standard finite element methods.

3.2 Convergence of a one dimensional PML

Proving convergence of a truncated and discretized PML is typically done in the following way (see e.g. [1, 23]): Similar to the last proof, the problem in the perfectly matched layer $(R,R+L)$ is solved analytically at first. This results into a perturbed Dirichlet-to-Neumann operator at the interface $x = R$. Typically the error to the correct Dirichlet-to-Neumann operator is bounded by the complex scaled function at the truncation boundary $R+L$, i.e. the truncation error decays exponentially with increasing layer thickness L. For sufficiently large $L > 0$ it is then shown, that the truncated problem is uniquely solvable if the untruncated problem is uniquely solvable.

Once this is established, compact perturbation arguments of strictly coercive operators can be used to show, that the discrete problem on the truncated domain is uniquely solvable for sufficiently fine discretization. Moreover, using the generalized Céa Lemma the discretization error can be bounded by the approximation error.

Here, we will use an approach, where truncation and discretization error are treated simultaneously. For scalar waveguides this approach was proposed in [21]. For simplicity, let us assume that τ is the identity. Then $\gamma'_{\tau,\alpha,R}(x) \equiv \sigma := 1 + \mathrm{i}\alpha$ for $x > R$ and there exists a rotation $\theta \in \{z \in \mathbb{C} : |z| = 1, \Re(z) > 0\}$ and a constant $\alpha_1 > 0$ such that

$$\Re\left(\theta \int_R^\infty \left(\frac{1}{\sigma}|u'|^2 - \omega^2\sigma|u|^2\right) dx\right) > \alpha_1 \|u\|^2_{H^1((R,\infty))}, \qquad u \in H^1((R,\infty)). \tag{3.9}$$

Since $\Re(\theta) > 0$, the Gårding inequality

$$\Re\left(\theta \int_0^\infty \left(\frac{|u'|^2}{\gamma'_{\tau,\alpha,R}} - \omega^2(1+p)\gamma'_{\tau,\alpha,R}|u|^2\right) dx + C\int_0^R |u|^2 dx\right) > \alpha\|u\|^2_{H^1(\mathbb{R}_+)} \tag{3.10}$$

holds for $u \in H^1(\mathbb{R}_+)$ with constants $\alpha := \min\{\alpha_1, \Re(\theta)\} > 0$ and $C > 0$ sufficiently large. Since $L^2((0,R))$ is compactly embedded in $H^1((0,R))$, a Fredholm operator of the form $A_{\sigma,R} + K_{\sigma,R} : H^1(\mathbb{R}_+) \to H^1(\mathbb{R}_+)$ can be associated to (3.8). $A_{\sigma,R}$ is continuous and strictly coercive and $K_{\sigma,R}$ is compact. Hence, Riesz-Fredholm theory can be used to show convergence of the truncated and discretized problem with homogeneous Dirichlet boundary condition at the truncation boundary. Note, that $L^2(\mathbb{R}_+)$ is not compactly embed-

ded in $H^1(\mathbb{R}_+)$, since $\mathbb{R}_+$ is unbounded. Hence, the compact pertubation argument for the low order term in (3.10) cannot be used for the low order term in (3.9).

Theorem 3.3. *Let $V_{h,L} \subset \{f \in H^1((0,R+L)) : f(R+L) = 0\}$ be a usual finite element discretization of the truncated domain, such that for all $v \in H^1((0,R+L))$ with $v(R+L) = 0$ the orthogonal projection converges point wise, i.e.*

$$\lim_{h\to 0} \inf_{v_h \in V_{h,L}} \|v - v_{h,L}\|_{H^1((0,R+L))} = 0. \tag{3.11}$$

If (3.8) is uniquely solvable with solution $u_\gamma \in H^1(\mathbb{R}_+)$, then there exist $h_0 > 0$ and $L_0 \in \mathbb{N}$ such that for all $h \le h_0$ and all $L \ge L_0$ a unique solution $u_{h,L} \in V_{h,L}$ to

$$\int_0^{R+L} \left(\frac{u_\gamma' v'}{\gamma'_{\tau,\alpha,R}} - \omega^2(1+p))\gamma'_{\tau,\alpha,R} u_\gamma v \right) dx = -u_0' v(0), \quad v \in V_{h,L}. \tag{3.12}$$

exists and depends continuously on u_0'. Moreover, there exists a constant $C = C(h_0, L_0) > 0$ such that

$$\|u_\gamma - u_{h,L}\|_{H^1((0,R+L))} \le C \left(\inf_{v_{h,L} \in V_{h,L}} \|u_\gamma - v_{h,L}\|_{H^1((0,R+L))} + \|u_\gamma\|_{H^1((R+L,\infty))} \right). \tag{3.13}$$

Proof. We define a finite dimensional subspace of $H^1(\mathbb{R}_+)$ by

$$\tilde{V}_{h,L} := \{f \in H^1(\mathbb{R}_+) : f|_{(0,R+L)} \in V_{h,L}, f|_{[R+L,\infty)} \equiv 0\}.$$

Since functions with compact support are dense in $H^1(\mathbb{R}_+)$, the orthogonal projection onto $\tilde{V}_{h,L}$ converges point-wise for $h \to 0$ and $L \to \infty$. So the first part of the theorem follows with [27, Theorem 13.7], since (3.12) is the projection of (3.8) to $\tilde{V}_{h,L}$. The error estimation is a consequence of Céa's Lemma (see e.g. [27, Theorem 13.6]). □

(3.13) includes truncation and discretization error. Since

$$|u_\gamma(x)| = |u_\gamma(R)| \exp(-\omega\alpha\tau(x-R)), \qquad x > R,$$

the second term of (3.13) decays exponentially with respect to L. For the first term we introduce for fixed $\varepsilon > 0$ and $k \in \mathbb{N}$ the functions

$$g_{\varepsilon,k}(x) := \begin{cases} 1, & x \le R+L-\varepsilon \\ 1 - \left(\frac{x+\varepsilon-R-L}{\varepsilon}\right)^k, & x \in (R+L-\varepsilon, R+L) \end{cases} \tag{3.14}$$

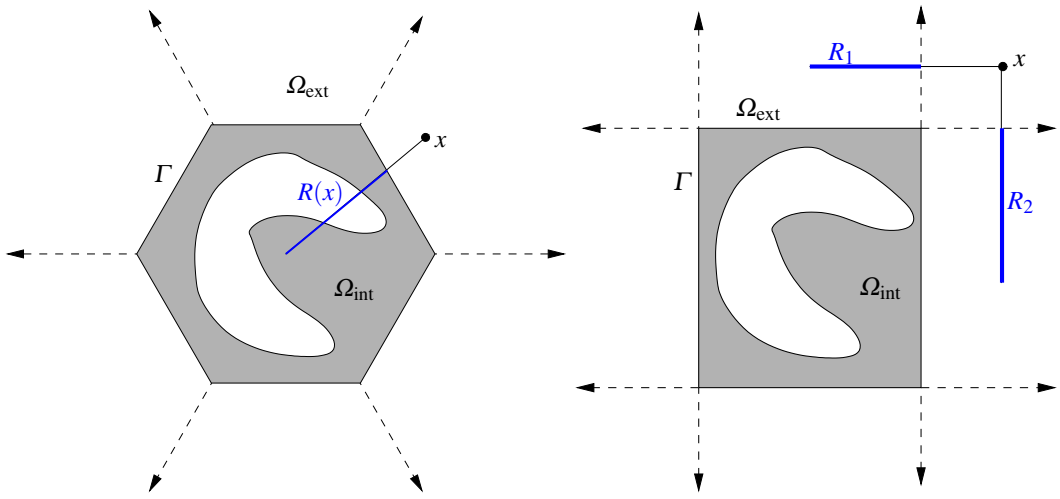
such that $x \mapsto u_\gamma(x) g_{\varepsilon,k}(x)$ belongs to $H^k((0,R+L))$ and vanishes at $R+L$. Hence, it can be approximated by functions $v_h \in V_{h,L}$ using (3.11). The remaining $H^1((0,R+L))$-error of $x \mapsto u_\gamma(x)(1 - g_{\varepsilon,k}(x))$ again decays exponentially with respect to L for fixed ε and k.

Remark 3.4 *For functions $u \in H^{k+1}(\Omega)$ the approximation error of finite element discretizations typically is bounded by*

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \le C h^k \|u\|_{H^{k+1}(\Omega)}. \tag{3.15}$$

The constant $C > 0$ is independent of the mesh size h, but depends amongst others on the order $k \in \mathbb{N}$ of the used polynomials. See e.g. [6, Sec. 4.4] or [9, Theorem 3.2.1] for sufficient conditions on finite elements such that (3.15) holds.

For those (3.11) is satisfied by density of $H^2(\Omega)$ in $H^1(\Omega)$. Moreover, (3.15) can be used to bound the approximation error of $u_\gamma g_{\varepsilon,k+1}$.



(a) radial (b) Cartesian

Fig. 1 sketch of a complex scaling. The dotted lines indicate possible discontinuities of the Jacobian $D_x x_\gamma$ in Ω_{ext}.

3.3 Radial complex scaling

For problems in higher dimensions we may use radial complex scaling. Let us assume, that the interface Γ between the interior and the exterior domain is piecewise smooth, i.e. there exists a parametrization of Γ which is piecewise k times continuously differentiable with $k \in \mathbb{N}$. Moreover, we require that for all $x \in \Gamma$ with normal vector $\mathbf{n}_\Gamma(x)$ the scalar product $x \cdot \mathbf{n}_\Gamma(x)$ does not vanish and that Γ is the boundary of a domain D, which is star shaped shaped with respect to the origin. Most often, Γ is just a sphere, but e.g. convex polyhedrons are also possible.

Using the complex scaling function of Sec. 3.1, we define as in (3.5) for all $x \in \Omega \setminus \{0\}$ the complex scaled variable

$$x_\gamma(x) := \frac{\gamma'_{\tau,\alpha,R}(|x|)}{|x|} x \quad \text{with } R(x) := \sup\{r \in \mathbb{R}_+ : r\frac{x}{|x|} \in \Omega_{\text{int}}\}. \tag{3.16}$$

If 0 is contained in Ω, we define $x_\gamma(0) = 0$. See Fig. 1(a) for a sketch of the radial complex scaling.

For a spherical complex scaling, i.e. $\Gamma = \partial B_R$, $R(x) = R$ becomes constant. It is straightforward to see, that $x_\gamma(x) = x$ for all $x \in \Omega_{\text{int}}$ and that $\Im(x_\gamma(x)) = \alpha \frac{\tau(|x|-R(x))}{|x|} x$. Since τ increases at least linearly for sufficiently large arguments, for $\alpha > 0$ the imaginary part of the Cartesian components of $x_\gamma(x)$ increase at least linearly with respect to the distance of x to the interface Γ.

Lemma 3.5. *Let $\tilde{\Gamma} \subset \Gamma$ be parametrized by a k times continuously differentiable function η and let the function τ in the definition of the complex scaling function $\gamma_{\tau,\alpha,R(x)}$ be also k times continuously differentiable. Then x_γ is k times continuously differentiable in the pyramidal frustum $\{r\hat{x} \in \mathbb{R}^d : r > 1, \hat{x} \in \tilde{\Gamma}\}$.*

On the interfaces between the pyramidal frustums and to the interior domain Ω_{int}, x_γ is at least continuous.

Proof. For $x \in \Omega_{\text{ext}} \cup \Gamma$ there exists at least one intersection point of the rays $\{rx \in \mathbb{R}^d : r > 0\}$ with Γ, since Γ is the boundary of a domain containing the origin and not containing x. This intersection point is unique, since otherwise the bounded domain would not be star shaped or there would be an $\hat{x} \in \Gamma$ with $x \cdot \mathbf{n}_\Gamma(\hat{x}) = 0$. Clearly, this intersection point depends continuously on x. Hence, $R(x)$ in the definition (3.16) of the complex scaling depends continuously on x, since it is the Euclidean norm of this intersection point. Since the complex scaling function $\gamma_{\tau,\alpha,R(x)}$ is continuous with respect to the argument and to R, $x_\gamma(x)$ is continuous in Ω.

Now, let x be in the interior of one pyramidal frustum $\{r\hat{x} \in \mathbb{R}^d : r > 1, \hat{x} \in \tilde{\Gamma}\}$ and let $\tilde{\Gamma} = \eta(S)$ with $S \subset \mathbb{R}^{d-1}$. We have to show, that for $x = r(x)\eta(\varphi(x))$, $\varphi \in S$, the function r is k times continuously

differentiable. Since $R(x)$ in the definition of the complex scaling is given by $R(x) = |x|/r(x)$, this proves the claim.

So we define $F : \Omega_{\text{ext}} \times (\mathbb{R}_+ \times S) \to \mathbb{R}^d$ by $F(x,(r,\varphi)) := x - r\eta(\varphi)$. Since the Jacobian $D_{r,\varphi}F(x,(r,\varphi)) = (-\eta(\varphi), -rD_\varphi\eta_\varphi)$ is always invertible due to the assumption $\hat{x}\cdot\mathbf{n}(\hat{x}) \neq 0$ with $\hat{x} = \eta(\varphi)$ and since F is k times continuously differentiable, the implicit function theorem guarantees the smoothness of r. □

Explicit forms of the Jacobian $J_\gamma(x) = D_x x_\gamma(x)$ are complicated, but for the most common situation $\Gamma = \partial B_R$ it is straightforward to compute

$$J_\gamma(x) = \frac{\gamma_{\tau,\alpha,R}(|x|)}{|x|}\mathrm{Id}_d + \frac{\gamma'_{\tau,\alpha,R}(|x|)|x| - \gamma_{\tau,\alpha,R}(|x|)}{|x|^3}xx^\top, \qquad x \in \Omega \setminus \{0\}. \tag{3.17}$$

$xx^\top \in \mathbb{R}^{d\times d}$ denotes the dyadic product and $\mathrm{Id}_d \in \mathbb{R}^{d\times d}$ the identity matrix. In the following we restrict ourselves to spherical interfaces in order to simplify the proof.

Theorem 3.6. *Let Γ be a sphere of radius R and let $u \in H^1_{\text{loc}}(\Omega_{\text{ext}})$ be a radiating solution to (3.1). If $\alpha > 0$ and if τ is the identity, then $u_\gamma = u \circ x_\gamma \in H^1(\Omega_{\text{ext}})$ decays exponentially and there holds*

$$\int_{\Omega_{\text{ext}}} \left(J_\gamma^{-T}\nabla u_\gamma \cdot J_\gamma^{-T}\nabla v - \omega^2 u_\gamma v\right)\det(J_\gamma)dx = -\int_\Gamma \mathrm{DtN}\, u_0\, v\, ds \tag{3.18}$$

for all $v \in H^1(\Omega_{\text{ext}})$.

Vice versa, if $\alpha > 0$ and if $\tilde{u} \in \{f \in H^1(\Omega_{\text{ext}}) : f|_\Gamma = u_0\}$ is a solution to (3.18) for all $v \in H^1_0(\Omega_{\text{ext}})$, then (3.18) holds true for all $v \in H^1(\Omega_{\text{ext}})$.

Proof. The series representation (2.9) of a solution u to (3.1) converges absolutely and uniformly on compact subsets of Ω_{ext}. The same holds true for the series of the term by term derivatives. Moreover, the spherical Hankel functions are holomorphic in $\mathbb{C}\setminus\{0\}$ and the Hankel functions are holomorphic in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$. Hence, the series representation has a holomorphic extension from $x = r\hat{x} \in \Omega_{\text{ext}}$ with $r = |x| > R$ and $\hat{x} = x/r$ to complex variables $\tilde{x} = \tilde{r}\hat{x}$ with complex radius $\tilde{r} \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

So u_γ is well defined and the last lemma guarantees, that $u_\gamma \in H^1_{\text{loc}}(\Omega_{\text{ext}})$. Based on the integral representation (2.11) for a sphere in the interior of Ω_{ext}, it can be shown that u_γ decays exponentially. So the first part of the theorem follows with the chain rule for the transformation of the gradients.

The second part can be shown using a separation into (spherical) Bessel problems. For the details see [10, Theorem 1]. □

For a different kind of complex scaling it is shown in [4], that there holds a Gårding inequality for a complex scaled bilinear form, which is similar to the one in (3.18) with Ω instead of Ω_{ext}. Hence, for this modified complex scaling the same approach as for the one dimensional problem in Sec. 3.2 can be used. The error induced by truncation and finite element discretization is again bounded by an exponentially decaying truncation error and the usual finite element approximation error.

3.4 Cartesian complex scaling

If Γ is the boundary of a rectangle ($d = 2$) or a cuboid ($d = 3$), usually a Cartesian complex scaling is used. In the radial complex scaling (3.16) basically the absolute value of $x \in \Omega_{\text{ext}}$ is scaled. Hence, all Cartesian components are scaled simultaneously with the same scaling function. In Cartesian complex scaling, each Cartesian component can be scaled individually.

W.l.o.g. we assume, that $\Omega_{\text{int}} = \Omega \cap \bigotimes_{j=1}^d (-R_j, R_j)$ with $R_j > 0$, $j = 1,\dots,d$. It general it is possible to choose $2d$ different functions $\tau_j^{(1,2)}$ in the complex scaling function and $2d$ different constants $\alpha_j^{(1,2)} > 0$, $j = 1,\dots,d$. For $x = (x_1,\dots,x_d)^\top \in \Omega$ we define the complex scaled variable $x_\gamma = ((x_\gamma)_1,\dots,(x_\gamma)_d)^\top$ by

$$\left(x_\gamma\right)_j := \begin{cases} \gamma_{\tau_j^{(1)},\alpha_j^{(1)},R_j}(x_j), & x_j > R_j \\ x_j, & x_j \in [-R_j,R_j]\,, \\ -\gamma_{\tau_j^{(2)},\alpha_j^{(2)},R_j}(-x_j), & x_j < -R_j \end{cases} \qquad j = 1,\dots,d. \tag{3.19}$$

In Fig. 1(b) a sketch of the Cartesian complex scaling is given. Since $\tau_j^{(1,2)}(0) = 0$, x_γ is continuous everywhere. The regularity of $\tau_j^{(1,2)}$ carries over to the regularity of x_γ in $\{x \in \Omega_{\text{ext}} : x_j \neq \pm R_j, j = 1,\dots,d\}$. The Jacobian $J_\gamma(x)$ for a Cartesian scaling is a diagonal matrix, where the diagonal entries are the derivatives of the scaling functions. Therefore a Cartesian complex scaling is typically much easier to implement than a radial complex scaling.

In contrast to the radial PML, the convergence theory is more involved, see e.g. [26, 5]. In [5, Theorem 5.8] it is shown, that for $\tau(t) = t$ and with some constraints on γ the truncation error decays exponentially with respect to the thickness of the layer.

3.5 Choice of complex scalings and bibliographical remarks

Cartesian complex scaling typically is easier to implement than a radial one. But if the most popular linear complex scaling $\tau(t) = t$ is used, one has to take into account the discontinuities of the Jacobian J_γ shown in Fig. 1. Since the solution u_γ in this case is only in $H^1(\Omega)$, a high order finite element method would suffer a lot (confer with Rem. 3.4). This can be avoided, if the finite element mesh is chosen such that the discontinuities of J_γ are part of the skeleton of the mesh. Hence, u_γ is smooth in the interior of each finite element, which guarantees the standard approximation error estimates of high order methods. Of course, choosing more regular damping functions also solves this issue.

The choice of the thickness of the complex scaling layer, the damping function and of the mesh in the layer is delicate. For a linear complex scaling one might use a priori error estimators for the truncation error of the form $\exp(-\omega\alpha L)$ with L being a measure for the layer thickness. Afterwards, for the truncated problem standard mesh refinement strategies can be used (see e.g. [7]).

There is a vast amount of literature using the complex scaling method. In comparison theoretical results are rare. Without claiming to be exhaustive, we mention the following references, where unique solvability and exponential convergence of the truncated complex scaling problem is shown for Helmholtz problems in free space. Note, that in most cases for the truncated problems standard finite element results can be used.

The results in [29, 30] include spherical complex scaling as in (3.16) and some additional assumptions on τ. In particular, due to an assumption $\tau''(t) > 0$, linear complex scaling is not covered from the theory there. [23] also deals with spherical complex scaling with one main difference: In this work, γ has to be at least two times continuously differentiable in a bounded transition zone $(R,\tilde{R})$. Moreover, for all $r > \tilde{R}$ the complex scaling is purely linear, i.e. $\gamma(r) = \sigma r$. So in contrast to (3.4) with $\tau(t) = t$, there is a translation by $(\sigma - 1)R$. A similar scaling is used in [4], where this translation is crucial for the existence of a Gårding inequality. [3] uses spherical complex scaling with scaling functions of the form $\gamma(r) = r + \mathrm{i}\sigma(r)$ with $\sigma(r) = \log(\hat{R} - 1) - \log(\hat{R} - r)$ for $r \in (1,\hat{R})$ with $\hat{R} > 1$. For this kind of complex scaling no truncation is needed, but the coefficients in the complex scaled variational formulation become singular. In [19] the spectral properties of untruncated radial complex scalings are investigated for $\Re(\sigma) = 1$ and two times continuously differentiable scaling functions. [25] extended this work with studies on truncated radial complex scalings.

Cartesian complex scalings were studied e.g. in a series of papers by Joseph E. Pasciak and coauthors [26, 5]. The last includes convergence results for linear complex scaling under some constraints on σ.

4 Hardy space infinite element method

Classical infinite element methods (see [12, 13]) directly discretize the exterior variational formulation (3.1) with special test and basis functions for $|x| \to \infty$. These basis functions have to satisfy the Sommerfeld radiation condition (2.4). Hardy space infinite element methods use the same idea, but they are based on the pole condition. This is another kind of radiation condition, which is to some extent equivalent to the classical ones.

4.1 One dimensional pole condition

We start with this pole condition for one dimensional problems of the form (3.2). The details can be found in [20, Sec. 2]. Arbitrary solutions u to (3.2), which do not fulfill a radiation condition, are given for $x \geq R$ by $u(x) = Cu_{\text{out}}(x-R) + Du_{\text{inc}}(x-R)$ with complex constants $C, D \in \mathbb{C}$ and

$$u_{\text{out}}(r) := \exp(\mathrm{i}\omega r), \quad u_{\text{inc}}(r) := \exp(-\mathrm{i}\omega r), \qquad r \geq 0.$$

In the following we will use the Laplace transform $(\mathscr{L}v)(s) := \int_0^\infty \exp(-sr)v(r)dr$, $\Re(s) > 0$, and for a complex constant $\kappa_0 \in \mathbb{C} \setminus \{0\}$ a Möbius transform

$$(\mathscr{M}_{\kappa_0}\hat{v})(z) := \frac{1}{z-1}\hat{v}\left(i\kappa_0\frac{z+1}{z-1}\right), \qquad z \neq 1. \tag{4.1}$$

The constant κ_0 will be the main parameter of the Hardy space infinite element method. It is somehow equivalent to the complex scaling parameter $\sigma = 1 + \alpha\mathrm{i}$ for a linear complex scaling.

Since

$$\left(\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{out}}\right)(z) = \frac{1}{\mathrm{i}(\kappa_0-\omega)z + \mathrm{i}(\kappa_0+\omega)}, \qquad z \in \mathbb{C}, \tag{4.2}$$

$$\left(\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{inc}}\right)(z) = \frac{1}{\mathrm{i}(\kappa_0+\omega)z + \mathrm{i}(\kappa_0-\omega)}, \qquad z \in \mathbb{C}, \tag{4.3}$$

$\mathscr{M}_{\kappa_0}\mathscr{L}\{u(\bullet+R)\}$ is a meromorphic function with poles at $\left(\frac{\omega+\kappa_0}{\omega-\kappa_0}\right)^{\pm 1}$. For $\omega > 0$ and $\Re(\kappa_0) > 0$, the pole of $\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{out}}$ has absolute value larger than 1. So $\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{out}}$ can be expanded into the Taylor series

$$\left(\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{out}}\right)(z) = \frac{1}{\mathrm{i}(\kappa_0+\omega)}\sum_{j=0}^{\infty}\left(\frac{\omega-\kappa_0}{\omega+\kappa_0}\right)^j z^j, \qquad z \in \mathbb{C},$$

which converges for all $|z| \leq 1$. In particular, $\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{out}}$ is holomorphic in the complex unit disk and belongs to the Hardy space [3] $H^+(S^1)$ of the complex unit sphere $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.

If $\Re(\kappa_0), \omega > 0$, $\mathscr{M}_{\kappa_0}\mathscr{L}u_{\text{inc}} \notin H^+(S^1)$, since it has a pole with absolute value smaller than 1. So we can use Hardy spaces in order to ensure, that a solution u to (3.2) only contains the outgoing solution u_{out}.

Definition 4.1 (pole condition). Let $H^+(S^1)$ denote the Hardy space of the complex unit sphere S^1 and let $\kappa_0 \in \mathbb{C}$ with positive real part be fixed. Then a function $u \in L^2_{\text{loc}}((R,\infty))$ is outgoing, if $\mathscr{M}_{\kappa_0}\mathscr{L}u(\bullet+R)$ is well defined and belongs to $H^+(S^1)$.

[3] $H^+(S^1) \subset L^2(S^1)$ consists of functions of the form $\sum_{j=0}^\infty \alpha_j z^j$, $z \in S^1$, with a square summable series (α_j). These functions are boundary values of some functions, which are holomorphic in the complex unit disk. Equipped with the $L^2(S^1)$ scalar product, $H^+(S^1)$ is a Hilbert space. For more details to Hardy spaces we refer to [14].

4.2 Hardy space variational formulation in one dimension

In order to be able to use this radiation condition, we have to reformulate (3.2). First, we define interior functions $u_{\text{int}} := u|_{(0,R)}$, $v_{\text{int}} := v|_{(0,R)}$ and shifted exterior functions $u_{\text{ext}}(r) := u(r+R)$, $v_{\text{ext}}(r) := v(r+R)$ for $r > 0$. For these functions (3.2) is split into

$$-u_0' v(0) = b_{\text{int}}(u_{\text{int}}, v_{\text{int}}) + b_{\text{ext}}(u_{\text{ext}}, v_{\text{ext}}),$$

with interior and exterior bilinear forms

$$b_{\text{int}}(u_{\text{int}}, v_{\text{int}}) := \int_0^R \left(u_{\text{int}}'(x) v_{\text{int}}'(x) - \omega^2 (1 + p(x)) u_{\text{int}}(x) v_{\text{int}}(x) \right) dx,$$
$$b_{\text{ext}}(u_{\text{ext}}, v_{\text{ext}}) := \int_0^\infty \left(u_{\text{ext}}'(r) v_{\text{ext}}'(r) - \omega^2 u_{\text{ext}}(r) v_{\text{ext}}(r) \right) dr.$$

Using test functions of the form $v_{\text{ext}}(r) = v_{\text{int}}(R) \exp(\mathrm{i}\lambda r)$ with $\Im(\lambda) > 0$ and $\Re(\lambda/\kappa_0) > 0$, we can use the identity in [20, Lemma A.1] to show

$$b_{\text{ext}}(u_{\text{ext}}, v_{\text{ext}}) = q_{\kappa_0}(\mathscr{M}_{\kappa_0} \mathscr{L} u_{ext}', \mathscr{M}_{\kappa_0} \mathscr{L} v_{ext}') - \omega^2 q_{\kappa_0}(\mathscr{M}_{\kappa_0} \mathscr{L} u_{ext}, \mathscr{M}_{\kappa_0} \mathscr{L} v_{ext}), \tag{4.4}$$

with the bilinear form $q : H^+(S^1) \times H^+(S^1) \to \mathbb{C}$ defined by

$$q_{\kappa_0}(U, V) := \frac{2\kappa_0}{2\pi\mathrm{i}} \int_0^{2\pi} U(\exp(\mathrm{i}\varphi)) V(\exp(-\mathrm{i}\varphi)) d\varphi, \qquad U, V \in H^+(S^1). \tag{4.5}$$

q_{κ_0} is almost the $L^2(S^1)$ scalar product: Let $z \mapsto \bar{z}$ denote the standard complex conjugation and let $\mathscr{C} : H^+(S^1) \to H^+(S^1)$ denote the involution defined by $(\mathscr{C} V)(z) := \overline{V(\bar{z})}$, $z \in S^1$. Then $q_{\kappa_0}(U, \mathscr{C} V) = \frac{2\kappa_0}{2\pi\mathrm{i}} (U, V)_{L^2(S^1)}$. Moreover, the monomials $z \mapsto z^j$, $j \in \mathbb{N}_0$, are orthogonal with respect to the bilinear form q_{κ_0}.

There are two main difficulties in (4.4). First we have to ensure, that our basis and test functions are continuous at the interface $x = R$. Due to (4.2), e.g. for the test functions there holds $v_{\text{int}}(R) = \frac{1}{2\mathrm{i}\kappa_0} \left(\mathscr{M}_{\kappa_0} \mathscr{L} v_{ext} \right)(1)$. The right hand side would not be well defined for an arbitrary function $V \in H^+(S^1) \subset L^2(S^1)$. The second challenge are the terms $\mathscr{M}_{\kappa_0} \mathscr{L} u_{ext}'$ and $\mathscr{M}_{\kappa_0} \mathscr{L} v_{ext}'$, which have to be computed if test functions for $\mathscr{M}_{\kappa_0} \mathscr{L} u_{ext}$ and $\mathscr{M}_{\kappa_0} \mathscr{L} v_{ext}$ are used.

Both issues can be solved with one modification. We define the operators $\mathscr{T}_\pm : \mathbb{C} \times H^+(S^1) \to H^+(S^1)$ by

$$\mathscr{T}_\pm(v_0, V)(z) := \frac{1}{2} \left(v_0 + (z \pm 1) V(z) \right), \quad z \in S^1, \qquad (v_0, V) \in \mathbb{C} \times H^+(S^1). \tag{4.6}$$

Lemma 4.2. *Let $v \in H^1_{\text{loc}}(\mathbb{R}_+) \cap C(\mathbb{R}_{\geq 0})$ be such that the Möbius and Laplace transformed function $\mathscr{M}_{\kappa_0} \mathscr{L} v$ is well defined. Moreover, we assume that $\mathscr{M}_{\kappa_0} \mathscr{L} v \in \mathscr{T}_-(\mathbb{C} \times H^+(S^1))$, i.e. there exists $(v_0, V) \in \mathbb{C} \times H^+(S^1)$ such that $\mathscr{M}_{\kappa_0} \mathscr{L} v = \frac{1}{\mathrm{i}\kappa_0} \mathscr{T}_-(v_0, V)$. Then $v_0 = v(0)$ and $\mathscr{M}_{\kappa_0} \mathscr{L} v' = \mathscr{T}_+(v_0, V)$.*

Proof. By a limit theorem of the Laplace transform, there holds

$$v(0) = \lim_{r \to 0} v(r) = \lim_{s \to \infty} s(\mathscr{L} v)(s) = \lim_{z \to 1} \mathrm{i}\kappa_0 \left((z+1) \left(\mathscr{M}_{\kappa_0} \mathscr{L} v \right)(z) \right).$$

The limit of the right hand side exists, since by assumption $\left(\mathscr{M}_{\kappa_0} \mathscr{L} v \right)(z) = 1/(2\mathrm{i}\kappa_0)(v_0 + (z-1)V(z))$ with $V \in L^2(S^1)$. Hence, $v_0 = v(0)$. The second assertion follows from direct calculations with $(\mathscr{L} v')(s) = s(\mathscr{L} v)(s) - v(0)$. □

Using this lemma, the exterior bilinear form becomes

$$b_{\text{ext},\kappa_0}((u_0, U), (v_0, V)) := q_{\kappa_0}(\mathscr{T}_+(u_0, U), \mathscr{T}_+(v_0, V)) - \omega^2 q_{\kappa_0}\left(\frac{1}{\mathrm{i}\kappa_0} \mathscr{T}_-(u_0, U), \frac{1}{\mathrm{i}\kappa_0} \mathscr{T}_-(v_0, V) \right), \tag{4.7}$$

with $(u_0,U),(v_0,V)\in\mathbb{C}\times H^+(S^1)$. u_0 and v_0 represent the Dirichlet values of $u_{\text{ext}}(0)=u_{\text{int}}(R)$ and $v_{\text{ext}}(0)=v_{\text{int}}(R)$ respectively. This allows a continuous coupling of classical finite elements for v_{int} with infinite elements for $\mathscr{M}_{\kappa_0}\mathscr{L}v_{\text{ext}}$.

Lemma 4.3. *For $\Re(\kappa_0)>0$ there exists a rotation $\theta\in\{z\in\mathbb{C}:|z|=1,\Re(z)>0\}$ and a constant $\alpha>0$ such that for all $(v_0,V)\in\mathbb{C}\times H^+(S^1)$*

$$\Re\left(\theta b_{\text{ext},\kappa_0}\left((v_0,V),(\overline{v_0},\mathscr{C}V)\right)\right)\geq\alpha\|(v_0,V)\|^2_{\mathbb{C}\times H^+(S^1)}. \tag{4.8a}$$

Moreover, b_{ext,κ_0} is continuous, i.e. there exists a constant $C>0$ such that for all $(u_0,U),(v_0,V)\in\mathbb{C}\times H^+(S^1)$

$$\left|b_{\text{ext},\kappa_0}\left((u_0,U),(v_0,V)\right)\right|\leq C\|(u_0,U)\|_{\mathbb{C}\times H^+(S^1)}\|(v_0,V)\|_{\mathbb{C}\times H^+(S^1)}. \tag{4.8b}$$

The norm on $\mathbb{C}\times H^+(S^1)$ is thereby defined as

$$\|(v_0,V)\|_{\mathbb{C}\times H^+(S^1)}:=\sqrt{|v_0|^2+\|V\|^2_{L^2(S^1)}},\qquad (v_0,V)\in\mathbb{C}\times H^+(S^1).$$

Proof. The continuity of b_{ext,κ_0} follows from the continuity of q_{κ_0} and of the operators $\mathscr{T}_\pm$. Since $2\mathscr{T}_\pm(v_0,V)(z)=v_0+zV(z)\pm V(z)$, the parallelogram identity leads to

$$\begin{aligned}\|\mathscr{T}_-(v_0,V)\|^2_{L^2(S^1)}+\|\mathscr{T}_+(v_0,V)\|^2_{L^2(S^1)}&=\frac{1}{2}\|v_0+\bullet V(\bullet)\|^2_{L^2(S^1)}+\frac{1}{2}\|V\|^2_{L^2(S^1)}\\&=\frac{1}{2}|v_0|^2+\|V\|^2_{L^2(S^1)}\geq\frac{1}{2}\|(v_0,V)\|^2_{\mathbb{C}\times L^2(S^1)}.\end{aligned}$$

The last identity yields by orthogonality of the monomials $z\mapsto z^j$, $j\in\mathbb{N}_0$, in $L^2(S^1)$. Choosing θ with $\Re(\theta)$ such that

$$\Re\left(\frac{-2\mathrm{i}\kappa_0}{2\pi}\theta\right)=\frac{1}{\pi}\Im(\kappa_0\theta)\text{ and }\Re\left(\frac{(-2\mathrm{i}\kappa_0)(-\omega^2)}{2\pi(\mathrm{i}\kappa_0)^2}\theta\right)=\frac{\omega^2}{\pi|\kappa_0|^2}\Im(\overline{\kappa_0}\theta)$$

are positive, yields the claim. □

Theorem 4.4. *Let $\omega,R>0$, $p\in L^\infty(\mathbb{R}_+)$ with $\operatorname{supp}(p)\subset[0,R)$, and $\kappa_0\in\mathbb{C}$ with positive real part. If u is an outgoing solution to (3.2) and $u_{\text{int}}:=u|_{(0,R)}$, then there exists a function $U\in H^+(S^1)$ such that $(u_{\text{int}},U)\in H^1((0,R))\times H^+(S^1)$ solves*

$$-u_0'v(0)=b_{\text{int}}(u_{\text{int}},v_{\text{int}})+b_{\text{ext},\kappa_0}\left((u_{\text{int}}(R),U),(v_{\text{int}}(R),V)\right) \tag{4.9}$$

for all test functions $(v_{\text{int}},v)\in H^1((0,R))\times H^+(S^1)$. Vice versa, if $(u_{\text{int}},U)\in H^1((0,R))\times H^+(S^1)$ is a solution to (4.9), then u_{int} is the restriction to $(0,R)$ of an outgoing solution $u\in H^1_{\text{loc}}(\mathbb{R}_+)$ to (3.2).

Proof. For a radiating solution u to (3.2), $\mathscr{M}_{\kappa_0}\mathscr{L}u(\bullet+R)=\frac{1}{\mathrm{i}\kappa_0}\mathscr{T}_-(u(R),U)$ with

$$U(z)=\frac{(\omega-\kappa_0)u(R)}{(\kappa_0-\omega)z+(\kappa_0+\omega)},\qquad z\in S^1. \tag{4.10}$$

We have already shown, that $(u_{\text{int}},U)\in H^1((0,R))\times H^+(S^1)$ solves (4.9) for a special kind of test functions. Since these test functions are dense in $H^+(S^1)$ (see [20, Lemma A.2]) and since the bilinear form in (4.9) is continuous, (4.9) holds true for all test functions in $H^1((0,R))\times H^+(S^1)$.

Conversely, let $(u_{\text{int}},U)\in H^1((0,R))\times H^+(S^1)$ be a solution to (4.9). As in the proof of Theorem 3.1, we start with test functions $(v_{\text{int}},V)\in H^1((0,R))\times H^+(S^1)$ with $v_{\text{int}}\equiv 0$. (4.9) reduces to the exterior bilinear form with $v_{\text{int}}(R)=0$, which is coercive due to the last lemma. Hence, U is unique and due to the first part of the proof given by (4.10). Plugging U into (4.7) with arbitrary test functions $(v_{\text{int}},V)\in H^1((0,R))\times H^+(S^1)$ leads to $b_{\text{ext},\kappa_0}\left((u_{\text{int}}(R),U),(v_{\text{int}}(R),V)\right)=\mathrm{i}\omega u_{\text{int}}(R)v_{\text{int}}(R)$, i.e. the correct Dirichlet-to-Neumann operator. □

Corollary 4.5 *Let $(u_{\text{int}}, U) \in H^1((0,R)) \times H^+(S^1)$ be a solution to (4.9). Then u_{int} is independent of the parameter κ_0.*

4.3 Hardy space infinite elements in one dimension

Theorem 4.6. *With the same assumptions as in Theorem 4.4 let $V_{\text{int,h}} \subset H^1((0,R))$ be a standard finite element space such that the orthogonal projection onto $V_{\text{int,h}}$ converges point wise for all $v \in H^1((0,R))$. Moreover, let $\Pi_N \subset H^+(S^1)$ denote the set of polynomials of maximal order $N \in \mathbb{N}_0$ and*

$$V_{h,N} := V_{\text{int,h}} \times \Pi_N \subset H^1((0,R)) \times H^+(S^1). \tag{4.11}$$

If (4.10) is uniquely solvable with solution $(u_{\text{int}}, U) \in H^1((0,R)) \times H^+(S^1)$, then for sufficiently small h and sufficiently large N there exists a unique solution $(u_{\text{int,h}}, U_N) \in V_{h,N}$ to (4.10) with test functions only in $V_{h,N}$. Moreover, there exist constants $C, c > 0$ independent of h and L such that

$$\|u_{\text{int,h}} - u_{\text{int}}\|_{H^1((0,R))} \leq C \left(\inf_{v_{\text{int,h}} \in V_{\text{int,h}}} \|u_{\text{int}} - v_{\text{int,h}}\|_{H^1((0,R))} + \exp(-cN) \right). \tag{4.12}$$

Proof. Similarly to the linear complex scaling in Sec. 3.2 there holds a Gårding inequality in $H^1((0,R)) \times H^+(S^1)$ and the theorem is a consequence of the projection method applied to a compact perturbation of a coercive operator [27, Theorems 13.6 and 13.7]. For $\kappa_0 = \omega$ there is no approximation error in the Hardy space. Otherwise, U has a pole at $p := \left(\frac{\omega+\kappa_0}{\omega-\kappa_0}\right)$, which has absolute value larger than one. Since $U(z) = u_{\text{int}}(R) \sum_{j=0}^{\infty} p^{-(j+1)} z^j$, $\inf_{V_N \in \Pi_N} \|V_N - U\|_{L^2(S^1)}$ converges exponentially with $p^{-(N+1)}$. □

In the one dimensional case the choice of the parameter κ_0 is obvious: If $\kappa_0 = \omega$, we have $U \equiv 0$ and the Hardy space method reduces to the correct Dirichlet-to-Neumann operator. In higher dimensions this is no longer the case, but typically $\kappa_0 \approx \omega$ remains a good choice.

In contrast to the complex scaling method, no truncation error occurs and no mesh in the exterior domain is needed. Moreover, we have exponential convergence with respect to the number of unknowns in the Hardy space. But we have to implement a new bilinear form and a new infinite element.

In the one dimensional case this is extremely easy. As basis functions for $(v_{\text{int,h}}(R), V_N) \in \mathbb{C} \times \Pi_N$ we use $\Phi_{-1}(z) := (1,0)$ and the monomials $\Phi_j(z) := (0, z^j)$, $j = 0, \ldots, N$. The operators $\mathscr{T}_{\pm,N} : \mathbb{C} \times \Pi_N \to \Pi_{N+1} = \text{span}\{z^0, \ldots, z^{N+1}\}$ in this basis are given by the bidiagonal matrices

$$T_{\pm,N} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 & & \\ & \ddots & \ddots & \\ & & 1 & \pm 1 \\ & & & 1 \end{pmatrix} \in \mathbb{R}^{(N+2)\times(N+2)}.$$

Since q_{κ_0} is orthogonal with respect to the monomials, we have

$$S_N := \left(q_{\kappa_0}(\mathscr{T}_+ \Phi_j, \mathscr{T}_+ \Phi_k) \right)_{j,k=-1}^{N} = (-2\mathrm{i}\kappa_0) T_{+,N}^{\top} T_{+,N}, \tag{4.13a}$$

$$M_N := \left(q_{\kappa_0}(1/(\mathrm{i}\kappa_0)\, \mathscr{T}_- \Phi_j, 1/(\mathrm{i}\kappa_0)\, \mathscr{T}_- \Phi_k) \right)_{j,k=-1}^{N} = \frac{2\mathrm{i}}{\kappa_0} T_{-,N}^{\top} T_{-,N} \tag{4.13b}$$

and finally

$$\left(b_{\text{ext},\kappa_0}(\Phi_j, \Phi_k) \right)_{j,k=-1}^{N} = S_N - \omega^2 M_N.$$

Only this matrix has to be implemented for Hardy space infinite elements in one dimension. The first row and the first column belong to $v_{\text{int}}(R)$ and $u_{\text{int}}(R)$ respectively. Hence, they have to be coupled with the corresponding degrees of freedoms of $V_{\text{int,h}} \subset H^1((0,R))$.

4.4 Radial Hardy space infinite elements

As for the complex scaling method there exists different ways to generalize one dimensional infinite elements to two or three dimensions. For generalized Cartesian Hardy space infinite elements in two dimensions we refer to [33, Sec. 2.3.1.]. Here, we only use radial infinite elements. Since the correct mathematical framework is rather involved, we restrict ourselves to the presentation of the numerical method. For a mathematically correct construction of the method we refer to [20] and for proof of convergence to [16].

We use the same assumptions on the interface $\Gamma = \overline{\Omega_{\text{int}}} \cap \overline{\Omega_{\text{ext}}}$ as for the radial complex scaling in Sec. 3.3. For a parametrization $\eta : S \subset \mathbb{R}^{d-1} \to \Gamma$ of the interface, we parametrize the exterior domain by $F : \mathbb{R}_+ \times S \to \Omega_{\text{ext}}$ with

$$F(r,\varphi) := (1+r)\eta(\varphi), \qquad r > 0, \varphi \in S. \tag{4.14}$$

If η is piecewise smooth, due to Lemma 3.5 F is piecewise smooth in each segment of the exterior domain (see Fig. 1(a)) and at least continuous everywhere. Hence, the exterior bilinear form in (3.1) is given by

$$\int_{\mathbb{R}_+\times S} \left(\begin{pmatrix} \partial_r u_{\text{ext}} \\ \nabla_\varphi u_{\text{ext}} \end{pmatrix}^\top J^{-1} J^{-\top} \begin{pmatrix} \partial_r v_{\text{ext}} \\ \nabla_\varphi v_{\text{ext}} \end{pmatrix} - \omega^2 u_{\text{ext}} v_{\text{ext}} \right) \det(J)\, d(r,\varphi),$$

with $u_{\text{ext}} := u \circ F$, $v_{\text{ext}} := v \circ F$ and Jacobian $J(r,\varphi) = (\eta(\varphi), (1+r)D_\varphi \eta(\varphi)) \in \mathbb{R}^{d\times d}$. Since $J(r,\varphi) = \hat{f}(\varphi) \begin{pmatrix} 1 & 0 \\ 0 & (1+r)\text{Id}_{d-1} \end{pmatrix}$ with $\hat{f}(\varphi) := (\eta(\varphi), D_\varphi \eta(\varphi))$, we define

$$\det(\hat{f}(\varphi))\hat{f}(\varphi)^{-1}\hat{f}(\varphi)^{-\top} =: \begin{pmatrix} G_{11}(\varphi) & G_{21}(\varphi)^\top \\ G_{21}(\varphi) & G_{22}(\varphi) \end{pmatrix},$$

with $G_{11}(\varphi) \in \mathbb{R}$, $G_{21}(\varphi) \in \mathbb{R}^{d-1}$ and $G_{22}(\varphi) \in \mathbb{R}^{(d-1)\times(d-1)}$ for all $\varphi \in S$. For the exterior bilinear form we have to discretize the two integrals

$$\begin{aligned} \int_{\Omega_{\text{ext}}} \nabla u \cdot \nabla v dx = \int_S \int_0^\infty \Big(& (1+r)^{d-1} \partial_r u_{\text{ext}}(r,\varphi) G_{11}(\varphi) \partial_r v_{\text{ext}}(r,\varphi) \\ & + (1+r)^{d-2} \partial_r u_{\text{ext}}(r,\varphi) G_{21}(\varphi)^\top \nabla_\varphi v_{\text{ext}}(r,\varphi) \\ & + (1+r)^{d-2} \nabla_\varphi u_{\text{ext}}(r,\varphi)^\top G_{21}(\varphi) \partial_r v_{\text{ext}}(r,\varphi) \\ & + (1+r)^{d-3} \nabla_\varphi u_{\text{ext}}(r,\varphi)^\top G_{22}(\varphi) \nabla_\varphi v_{\text{ext}}(r,\varphi) \Big) dr\, d\varphi \end{aligned} \tag{4.15a}$$

and

$$\int_{\Omega_{\text{ext}}} u v dx = \int_S \int_0^\infty u_{\text{ext}}(r,\varphi) v_{\text{ext}}(r,\varphi) (1+r)^{d-1} \det(\hat{f}(\varphi))\, dr\, d\varphi. \tag{4.15b}$$

Similar to Def. 4.1 we formulate the radiation condition in terms of the Möbius and Laplace transformed function: u_{ext} is outgoing if $\mathscr{M}_{\kappa_0}\mathscr{L} u_{\text{ext}}(\bullet,\varphi)$ exists for all $\varphi \in S$ and belongs to the Hardy space $H^+(S^1)$. In order to use this radiation condition, we transform the integrals $\int_0^\infty (\dots) dr$ in radial direction as in the one dimensional case into the bilinear form (4.5) using the identity [20, Lemma A.1]. Special attention has to be paid to the the factors $(1+r)^{\pm 1}$.

In order to treat these, we first study the Möbius and Laplace transformation of a multiplication operator. If $\mathscr{M}_{\kappa_0}\mathscr{L} v$ and $\left(\mathscr{M}_{\kappa_0}\mathscr{L} v\right)'$ belong to the Hardy space $H^+(S^1)$, then $\mathscr{M}_{\kappa_0}\mathscr{L}\{r \mapsto r v(r)\} = \frac{-1}{2i\kappa_0} \mathscr{D} \mathscr{M}_{\kappa_0}\mathscr{L} v$ with

$$(\mathscr{D} V)(z) := (z-1)^2 V'(z) + (z-1)V(z), \qquad V \in H^+(S^1).$$

Hence, we deduce

$$\mathscr{M}_{\kappa_0}\mathscr{L}\left\{r \mapsto (1+r)^{\pm 1} v(r)\right\} = \left(\mathscr{I} - \frac{1}{2i\kappa_0}\mathscr{D}\right)^{\pm 1} \mathscr{M}_{\kappa_0}\mathscr{L} v, \tag{4.16}$$

with the identity operator $\mathscr{I} : H^+(S^1) \to H^+(S^1)$. For implementation the orthogonal projection onto Π_{N+1} of $\mathscr{I} - \frac{1}{2i\kappa_0}\mathscr{D} : \Pi_{N+1} \to \Pi_{N+2}$ is useful. In the monomial basis it is given by

$$D_{\kappa_0} := \mathrm{Id}_{N+2} - \frac{1}{2i\kappa_0} \begin{pmatrix} \begin{matrix}-1 & 1\\ 1 & -3\end{matrix} & \begin{matrix}\\ 2\end{matrix} & & \\ & \begin{matrix}2\\ -5\end{matrix} & \begin{matrix}\\ 3\end{matrix} & \\ & \ddots & \ddots & \ddots \\ & & \begin{matrix}N\\ N+1\end{matrix} & \begin{matrix}-2N-1\\ -2N-3\end{matrix} \end{pmatrix} \in \mathbb{C}^{(N+2)\times(N+2)}.$$

Note, that it is symmetric, i.e. $D_{\kappa_0}^\top = D_{\kappa_0}$. For the inverse operator $(\mathscr{I} - \frac{1}{2i\kappa_0}\mathscr{D})^{-1}$ we use the inverse $D_{\kappa_0}^{-1}$ as approximation.

We are now ready for a computation of the radial Hardy space infinite elements: Let $V_h \subset H^1(\Omega_{\mathrm{int}})$ be a standard finite element discretization of $H^1(\Omega_{\mathrm{int}})$ and let $\Psi_1, \dots, \Psi_{N_\Gamma}$ with $N_\Gamma \in \mathbb{N}$ denote the non vanishing traces of the finite element basis functions in $H^1(\Omega_{\mathrm{int}})$. We need standard finite element matrices on the interface Γ

$$M_\Gamma := \left(\int_S \det(\hat{f}(\varphi))\,\Psi_j(\varphi)\Psi_k(\varphi)\,d\varphi\right)_{j,k=1}^{N_\Gamma}, \qquad S_\Gamma^{00} := \left(\int_S \Psi_j(\varphi)G_{11}(\varphi)\Psi_k(\varphi)\,d\varphi\right)_{j,k=1}^{N_\Gamma},$$
$$S_\Gamma^{10} := \left(\int_S \left(\nabla_\varphi \Psi_j(\varphi)\right)^\top G_{21}(\varphi)\,\Psi_k(\varphi)\,d\varphi\right)_{j,k=1}^{N_\Gamma}, \quad S_\Gamma^{11} := \left(\int_S \left(\nabla_\varphi \Psi_j(\varphi)\right)^\top G_{22}(\varphi)\nabla_\varphi\Psi_k(\varphi)\,d\varphi\right)_{j,k=1}^{N_\Gamma},$$

and for the basis functions $\Phi_{-1}, \dots, \Phi_{N_r+1} \in \mathbb{C} \times \Pi_{N_r}$ defined in the last subsection the non-standard Hardy space infinite elements

$$M_r := \frac{2i}{\kappa_0} T_{-,N_r}^\top D_{\kappa_0}^{d-1} T_{-,N_r}, \qquad S_r^{00} := \frac{2i}{\kappa_0} T_{-,N_r}^\top D_{\kappa_0}^{d-3} T_{-,N_r},$$
$$S_r^{10} := -2T_{+,N_r}^\top D_{\kappa_0}^{d-2} T_{-,N_r}, \qquad S_r^{11} := (-2i\kappa_0) T_{+,N_r}^\top D_{\kappa_0}^{d-1} T_{+,N_r}.$$

For the tensor product basis functions $\Phi_j \otimes \Psi_k$ the discretization of $\int_{\Omega_{\mathrm{ext}}} \nabla u \cdot \nabla v\,dx$ is due to (4.15) given by

$$S := S_r^{11} \otimes S_\Gamma^{00} + S_r^{10} \otimes S_\Gamma^{01\top} + S_r^{10\top} \otimes S_\Gamma^{10} + S_r^{00} \otimes S_\Gamma^{11}. \tag{4.17a}$$

The infinite element matrix for $\int_{\Omega_{\mathrm{ext}}} uv\,dx$ is given by

$$M := M_r \otimes M_\Gamma. \tag{4.17b}$$

As in the one dimensional case the basis functions $\Phi_{-1} \otimes \Psi_k$, $k = 1, \dots, N_\Gamma$, have to be coupled to the corresponding basis functions in $H^1(\Omega_{\mathrm{int}})$ in order to ensure continuity at the interface Γ.

Remark 4.7 *If u_{int} is a solution to (2.2), $\tilde{u}_{\mathrm{ext}}$ a solution to (2.1), $u_{\mathrm{ext}} = \tilde{u}_{\mathrm{ext}} \circ F$ and*

$$U(z,\varphi) := \frac{2i\kappa_0 \mathscr{M}_{\kappa_0} \mathscr{L}\{u_{\mathrm{ext}}(\bullet,\varphi)\}(z) - u_{\mathrm{int}}(\eta(\varphi))}{z-1},$$

then (u_{int}, U) belongs to a subspace of $H^1(\Omega_{\mathrm{int}}) \times H^+(S^1) \otimes L^2(S)$. This subspace is constructed such that the bilinear forms in (4.15) are continuous (see [20, Eq. (3.7) and Lemma A.3] for a slightly different bilinear form or [16, Eq. (3.7)]). The Hardy space infinite element method is in this space a Galerkin method with tensor product elements. In [16] a Gårding inequality is shown leading to super-algebraic convergence with respect to the number of unknowns in the Hardy space.

The numerical results in [20, 33, 32] confirm this result. The main parameters of the method are κ_0, the number of unknowns in radial direction and the choice of the interface Γ. The numerical results indicate, that $\kappa_0 \approx \omega$ is recommendable. The interface Γ has to be chosen carefully.

Of course, in order to minimize the computational effort in Ω_{int}, one would like to choose Ω_{int} as small as possible. On the other hand the numerical results show, that the number of radial unknowns has to be increased, if the distance of Γ to a source of the scattered wave becomes smaller. For a distance of

one or two wavelengths typically less than 10 radial unknowns are needed to ensure, that the error of the infinite elements is negligible. As mentioned in [16, Remark 3.3], highly anisotropic interfaces Γ should be avoided, when radial infinite elements are used. For such interfaces Cartesian infinite elements as in [33, Sec. 2.3.1.] are preferable.

5 Summary

We have presented PML and Hardy space infinite element methods for Helmholtz problems in open systems. Both methods are Galerkin methods and for both methods convergence can be shown. However, the type of convergence is different.

PML methods converge exponentially with increasing layer thickness. The convergence with respect to the finite element discretization of the perfectly matched layer depends on the used finite elements and typically is h^k for polynomials of order k. Hardy space infinite element methods converge super-algebraically with respect to the number of unknowns in radial direction and with the usual finite element convergence order for the interface unknowns. In a comparison in [33] the Hardy space infinite element method was superior to a complex scaling method for a two dimensional problem with inhomogeneous exterior domain. Of course, this might change in a different situation.

For the Hardy space infinite element method the programming effort typically is noticeable larger than for a standard PML. A non-standard infinite element with non-standard discretization matrix has to be implemented. The matrix itself is very easy and do not require a remarkable effort. On the other hand a standard PML will not converge, if the layer thickness or the damping is not increased. Realizing this in a given finite element code is not an easy task neither.

One big advantage of both methods is the flexibility. In this chapter we have only used Helmholtz problems in free space, but the methods can be used for wave-guides [1, 21] and inhomogeneous exterior domains [33, 7] as well. Moreover, they are not restricted to scalar problems in frequency domain.

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References

1. É. Bécache, A.-S. Bonnet-BenDhia, and G. Legendre, *Perfectly matched layers for the convected Helmholtz equation*, SIAM Journal on Numerical Analysis, 42 (2004), pp. 409–433.
2. J.-P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 114 (1994), pp. 185–200.
3. A. Bermúdez, L. Hervella-Nieto, A. Prieto, and R. Rodríguez, *An exact bounded perfectly matched layer for time-harmonic scattering problems*, SIAM J. Sci. Comput., 30 (2007/08), pp. 312–338.
4. J. H. Bramble and J. E. Pasciak, *Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems*, Math. Comp., 76 (2007), pp. 597–614 (electronic).
5. ———, *Analysis of a Cartesian PML approximation to acoustic scattering problems in $\mathbb{R}^2$ and $\mathbb{R}^3$*, J. Comput. Appl. Math., 247 (2013), pp. 209–230.
6. S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.
7. Z. Chen, C. Liang, and X. Xiang, *An anisotropic perfectly matched layer method for helmholtz scattering problems with discontinuous wave number*, Inverse Problems and Imaging, 7 (2013), pp. 663–678.
8. W. C. Chew and W. H. Weedon, *A 3d perfectly matched medium from modified Maxwell's equations with stretched coordinates*, Microwave Optical Tech. Letters, 7 (1994), pp. 590–604.
9. P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4.
10. F. Collino and P. Monk, *The perfectly matched layer in curvilinear coordinates*, SIAM J. Sci. Comput., 19 (1998), pp. 2061–2090 (electronic).
11. D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, vol. 93 of Applied Mathematical Sciences, Springer-Verlag, Berlin, second ed., 1998.
12. L. Demkowicz and K. Gerdes, *Convergence of the infinite element methods for the Helmholtz equation in separable domains*, Numer. Math., 79 (1998), pp. 11–42.

13. L. DEMKOWICZ AND F. IHLENBURG, *Analysis of a coupled finite-infinite element method for exterior Helmholtz problems*, Numer. Math., 88 (2001), pp. 43–73.
14. P. L. DUREN, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970.
15. D. GIVOLI, *High-order local non-reflecting boundary conditions: a review*, Wave Motion, 39 (2004), pp. 319–326.
16. M. HALLA, *Convergence of Hardy space infinite elements for Helmholtz scattering and resonance problems*, Preprint 10/2015, Institute for Analysis and Scientific Computing; TU Wien, 2013, ISBN: 978-3-902627-05-6, 2015.
17. M. HALLA, T. HOHAGE, L. NANNEN, AND J. SCHÖBERL, *Hardy space infinite elements for time harmonic wave equations with phase and group velocities of different signs*, Numerische Mathematik, (2015), pp. 1–37.
18. M. HALLA AND L. NANNEN, *Hardy space infinite elements for time-harmonic two-dimensional elastic waveguide problems*, Wave Motion, 59 (2015), pp. 94 – 110.
19. P. D. HISLOP AND I. M. SIGAL, *Introduction to spectral theory*, vol. 113 of Applied Mathematical Sciences, Springer-Verlag, New York, 1996. With applications to Schrödinger operators.
20. T. HOHAGE AND L. NANNEN, *Hardy space infinite elements for scattering and resonance problems*, SIAM J. Numer. Anal., 47 (2009), pp. 972–996.
21. ———, *Convergence of infinite element methods for scalar waveguide problems*, BIT Numerical Mathematics, 55 (2014), pp. 215–254.
22. T. HOHAGE, F. SCHMIDT, AND L. ZSCHIEDRICH, *Solving time-harmonic scattering problems based on the pole condition. I. Theory*, SIAM J. Math. Anal., 35 (2003), pp. 183–210.
23. ———, *Solving time-harmonic scattering problems based on the pole condition. II. Convergence of the PML method*, SIAM J. Math. Anal., 35 (2003), pp. 547–560.
24. F. IHLENBURG, *Finite element analysis of acoustic scattering*, vol. 132 of Applied Mathematical Sciences, Springer-Verlag, New York, 1998.
25. S. KIM AND J. E. PASCIAK, *The computation of resonances in open systems using a perfectly matched layer*, Math. Comp., 78 (2009), pp. 1375–1398.
26. ———, *Analysis of a Cartesian PML approximation to acoustic scattering problems in $\mathbb{R}^2$*, J. Math. Anal. Appl., 370 (2010), pp. 168–186.
27. R. KRESS, *Linear integral equations*, vol. 82 of Applied Mathematical Sciences, Springer-Verlag, New York, second ed., 1999.
28. R. KRESS, *Chapter 1.2.1 - specific theoretical tools*, in Scattering, R. P. Sabatier, ed., Academic Press, London, 2002, pp. 37 – 51.
29. M. LASSAS AND E. SOMERSALO, *On the existence and the convergence of the solution of the pml equations*, Computing, 60 (1998), pp. 229–241.
30. M. LASSAS AND E. SOMERSALO, *Analysis of the PML equations in general convex geometry*, Proc. Roy. Soc. Edinburgh Sect. A, 131 (2001), pp. 1183–1207.
31. N. MOISEYEV, *Quantum theory of resonances: Calculating energies, width and cross-sections by complex scaling*, Physics reports, 302 (1998), pp. 211–293.
32. L. NANNEN, T. HOHAGE, A. SCHÄDLE, AND J. SCHÖBERL, *Exact Sequences of High Order Hardy Space Infinite Elements for Exterior Maxwell Problems*, SIAM J. Sci. Comput., 35 (2013), pp. A1024–A1048.
33. L. NANNEN AND A. SCHÄDLE, *Hardy space infinite elements for Helmholtz-type problems with unbounded inhomogeneities*, Wave Motion, 48 (2010), pp. 116–129.
34. S. A. SAUTER AND C. SCHWAB, *Boundary element methods*, vol. 39 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2011.
35. F. SCHMIDT AND P. DEUFLHARD, *Discrete transparent boundary conditions for the numerical solution of Fresnel's equation*, Computers Math. Appl., 29 (1995), pp. 53–76.
36. M. E. TAYLOR, *Partial differential equations. II*, vol. 116 of Applied Mathematical Sciences, Springer-Verlag, New York, 1996. Qualitative studies of linear equations.