# Two-Scale Homogenization of the Nonlinear Eddy Current Problem with FEM

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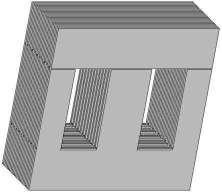
July 4, 2013

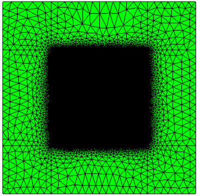
# Motivation

*Aim: Simulation of eddy current losses in large transformers*

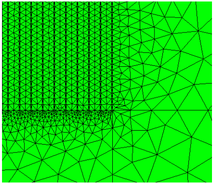
Large transformer



Laminated core



FE-Model with 100 laminates

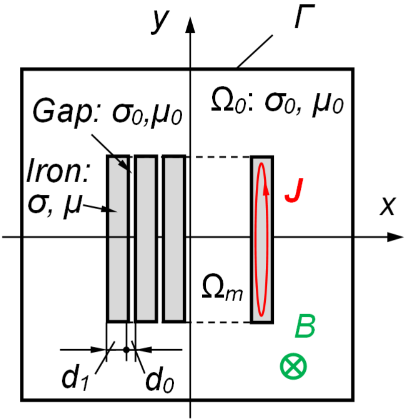


Detail, lower right corner

# Outline

# Problem Description:

*Laminated medium:*




Draft of the problem

Conductivity:

$\sigma$ ... iron
$\sigma_0$ ... air

Permeability:

$\mu$ ... iron
$\mu_0$ ... air

Dimensions:

$d = d_1 + d_0$ ... iron + air
$ff = \frac{d_1}{d}$ ... fill factor

Quantities:

**B** ... Magnetic flux density
**J** ... Current density

# Nonlinear Eddy Current Problem in the Time Domain

**Boundary value problem:**

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu(\mathbf{A})} \operatorname{curl} \mathbf{A} + \sigma \frac{\partial}{\partial t} \mathbf{A} &= \mathbf{0} \quad \text{in } \Omega = \Omega_m \cup \Omega_0, \\ \mathbf{A} \times \mathbf{n} &= \boldsymbol{\alpha} \quad \text{on } \Gamma \end{aligned}$$

**A** ... Magnetic vector potential

**Variational formulation:**

Find $\mathbf{A}_h \in V_\alpha := \{\mathbf{A}_h \in \mathcal{V}_h : \mathbf{A}_h \times \mathbf{n} = \alpha_h \text{ on } \Gamma\}$, such that

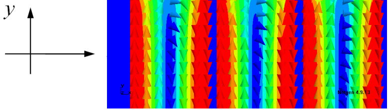
$$\int_\Omega \frac{1}{\mu(\mathbf{A}_h)} \operatorname{curl} \mathbf{A}_h \operatorname{curl} \mathbf{v}_h \, d\Omega + \frac{\partial}{\partial t} \int_\Omega \sigma \mathbf{A}_h \mathbf{v}_h \, d\Omega = 0$$

for all $\mathbf{v}_h \in V_0 := \{\mathbf{v}_h \in \mathcal{V}_h : \mathbf{v}_h \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}$, where

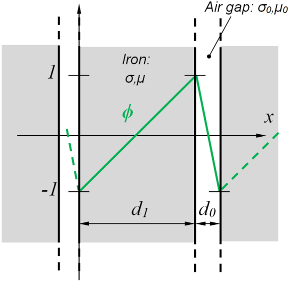
$\mathcal{V}_h \subset H(\operatorname{curl}, \Omega)$.

# Two-Scale Ansatz

**Two-Scale Ansatz:**




Eddy currents in laminates, detail




Micro-schape Function

$$\tilde{\mathbf{A}} = \mathbf{A_0} + \phi \begin{pmatrix} 0 \\ A_1 \end{pmatrix} + \nabla(\phi w)$$

Mean value $\mathbf{A_0}$
Scalar quantities $A_1$ and $w$
Micro-shape function $\phi$

# Two-Scale Ansatz

**Variational Formulation of the Homogenized Problem:**

Find $(\mathbf{A}_{0h},\, A_{1h},\, w_h) \in V_B := \{(\mathbf{A}_{0h},\, A_{1h},\, w_h) : \boldsymbol{A}_{0h} \in \mathcal{U}_h,\, A_{1h} \in \mathcal{V}_h\ ,\, w_h \in \mathcal{W}_h$ and $\boldsymbol{A}_{0h} \times \boldsymbol{n} = \alpha_h$ on $\Gamma_B\}$, such that

$$\int_\Omega \frac{1}{\mu(\tilde{\mathbf{A}}_h)} \operatorname{curl} \tilde{\mathbf{A}}_h \operatorname{curl} \tilde{\mathbf{v}}_h \, d\Omega + \frac{\partial}{\partial t} \int_\Omega \sigma \tilde{\mathbf{A}}_h \tilde{\mathbf{v}}_h \, d\Omega = 0$$

for all $(\mathbf{v}_{0h},\, v_{1h},\, q_h) \in V_0 := \{(\mathbf{v}_{0h},\, v_{1h},\, q_h) : \mathbf{v}_{0h} \in \mathcal{U}_h,\, v_{1h} \in \mathcal{V}_h,\, q_h \in \mathcal{W}_h$ and $\mathbf{v}_{0h} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_B\}$.

Finite element subspace:

$\mathbf{A}_{0h} \in \mathcal{U}_h \subset H(\operatorname{curl}, \Omega)$

$A_{1h} \in \mathcal{V}_h \subset L_2(\Omega_m)$

$w_h \in \mathcal{W}_h \subset H^1(\Omega_m)$

and

$\phi \in H_{per}(\Omega_m)$

# Numerical Methods

**Nonlinear Ordinary Differential Equation System:**

$$A(u_h(t))u_h(t) + M\frac{\partial}{\partial t}u_h(t) = f_h(t)$$

**Time Stepping Scheme:**

*Implicit Euler method* leads to

$$A(u_{h,i+1})u_{h,i+1} + M\frac{u_{h,i+1} - u_{h,i}}{\Delta t} = f_{h,i}$$

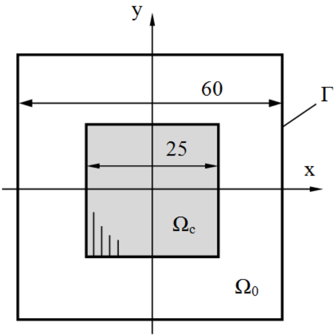
**Nonlinearity:**

*Newton's method* yields

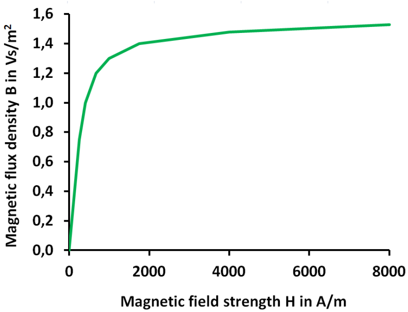
$$u_{h,i+1}^{(l+1)} = u_{h,i+1}^{(l)} - (\Delta t A'(u_{h,i+1}^{(l)}) + M)^{-1}(Mu_{h,i} - Mu_{h,i+1}^{(l)} - \Delta t A(u_{h,i+1}^{(l)}) - \Delta t f_{h,i+1})$$

# Numerical Example

**Numerical Example:**




Problem with dimensions in mm.




Magnetization curve of an electric sheet.

# Numerical Example

**Numerical Example:**

$\sigma = 2 \cdot 10^6 S/m$

$f = 50Hz$

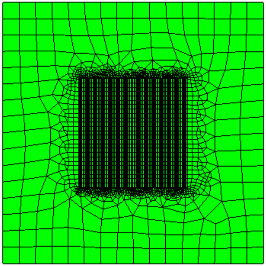
$ff = 0.9$ (Fill factor)

$\alpha = 4 \cdot 10^{-3} Vs/m$

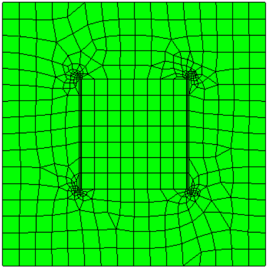
$\mu = B/H$ in $Vs/Am$

$n = 100$ No. Laminates

$d = d_1 + d_0 = 0.25mm$



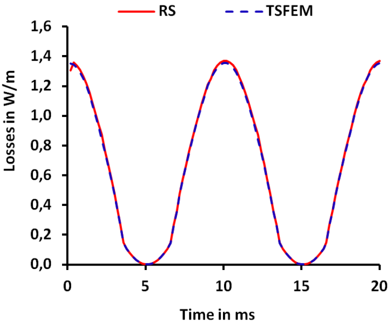
Referenz Model (RM):
Laminates are resolved



Homogenization Model (HM):
Laminates are not resolved

# Numerical Example

**Numerical Example:**



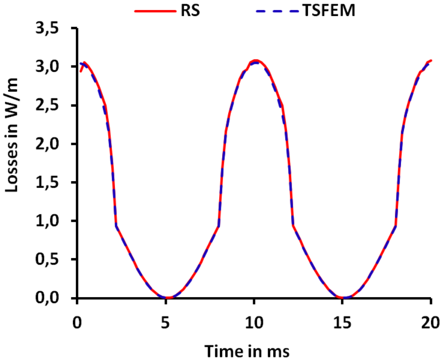

Comparison of the Eddy Current Losses: $f = 50Hz, \alpha = 0.004 Vs/m$

Numerical Data: Degrees of Freedom

| | RM | TSFEM |
|---|---|---|
| $H(\mathrm{curl}, \Omega)$ | 123564 | 3316 |
| $L_2(\Omega_m)$ | - | 692 |
| $H^1(\Omega_m)$ | - | 698 |
| Total No. DOF | 123564 | 4706 |
| Compt. Time in s | 1829 | 96 |

# Numerical Example

**Numerical Example:**

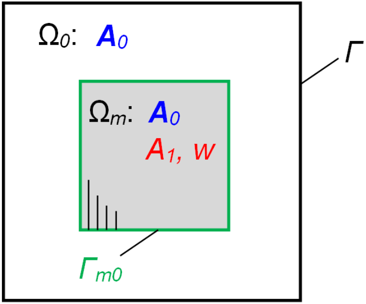



Influence of the Nonlinearity: $f = 50Hz, \alpha = 0.006Vs/m$

# Numerical Example

**Two-Scale Solution:**

$$\tilde{\mathbf{A}} = \mathbf{A_0} + \phi \begin{pmatrix} 0 \\ A_1 \end{pmatrix} + \nabla(\phi w)$$




Outer boundary Γ:

$\mathbf{A}_0 \times \mathbf{n} = \boldsymbol{\alpha}$

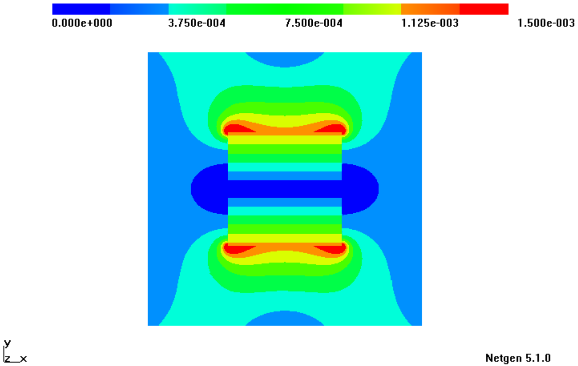
Interface $\Gamma_{m0}$:

Natural boundary conditions

# Numerical Example

**Two-Scale Solution:**

$$\tilde{\mathbf{A}} = \mathbf{A_0} + \phi \begin{pmatrix} 0 \\ A_1 \end{pmatrix} + \nabla(\phi w)$$



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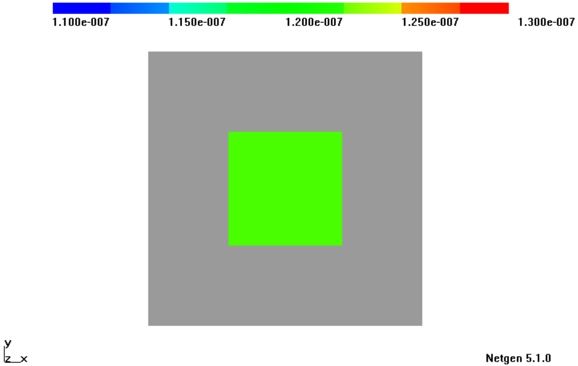
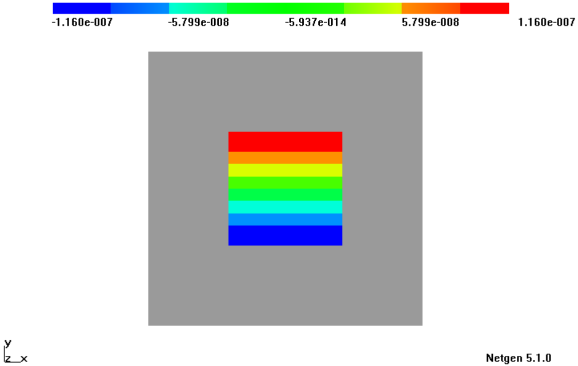


# Numerical Example

**Two-Scale Solution:**

$$\tilde{\mathbf{A}} = \mathbf{A_0} + \phi \begin{pmatrix} 0 \\ A_1 \end{pmatrix} + \nabla(\phi w)$$



# Two-Scale Ansatz

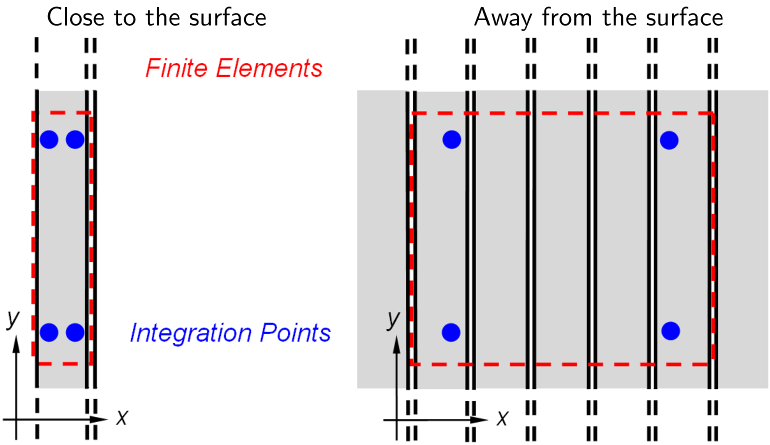
The two-scale solution

$$\tilde{\mathbf{A}} = \mathbf{A_0} + \phi \begin{pmatrix} 0 \\ A_1 \end{pmatrix} + \nabla(\phi w)$$

is rather smooth in the laminated medium.

Consequently, a very coarse finite element mesh suffices to approximate the true solution accurately!

# Finite Element Assembly with Gauss Quadrature




One Finite element comprises one or a few iron/air-layer(s) (left) and several (hundreds) iron/air-layers (right)

# Finite Element Assembly with Adapted Gauss Quadrature

For the sake of simplicity:

$$\tilde{\mathbf{A}} = \sum_{i=1}^{N^{H(curl)}} \mathbf{A}_{0,i} + \sum_{i=1}^{N^{L_2}} \begin{pmatrix} 0 \\ A_{1,i} \end{pmatrix} + \sum_{i=1}^{N^{H^1}} \nabla(\phi w_i) =: \sum_{i=1}^{N^{DOF}} \tilde{\mathbf{A}}_i$$

$$\tilde{\mathbf{v}} = \sum_{i=1}^{N^{H(curl)}} \mathbf{v}_{0,i} + \sum_{i=1}^{N^{L_2}} \begin{pmatrix} 0 \\ v_{1,i} \end{pmatrix} + \sum_{i=1}^{N^{H^1}} \nabla(\phi q_i) =: \sum_{i=1}^{N^{DOF}} \tilde{\mathbf{v}}_i$$

with

$$N^{DOF} = N^{H(curl)} + N^{L_2} + N^{H^1}$$

# Finite Element Assembly with Adapted Gauss Quadrature

**Mass matrix:**

$$\begin{aligned} M_{ij} &= \frac{1}{\Delta t}\sigma \int_{\Omega_{FE}} J^{-T}\tilde{\mathbf{A}}_i J^{-T}\tilde{\mathbf{v}}_j \det(J)\, d\Omega \\ &\approx \frac{1}{\Delta t}\sum_{k=1}^{N^{GP}} \sigma(x_k, y_k) J_k^{-T}\tilde{\mathbf{A}}_i(x_k, y_k) J_k^{-T}\tilde{\mathbf{v}}_j(x_k, y_k)\det(J_k) w_k \end{aligned}$$

$N^{GP}$ number of Gaussian points

**Interpolation:**

$$\begin{aligned} \sigma(x,y) &= \sum_{i=1}^{N^{PO}} a_i \varphi_i(x,y) \\ \phi_x(x)\sigma(x,y) &= \sum_{i=1}^{N^{PO}} a_i^{\phi_x}\varphi_i(x,y), \text{ etc.} \end{aligned}$$

$N^{PO}$ number of interpolating polynomials

# Finite Element Assembly with Adapted Gauss Quadrature

**Mass matrix:**

$$
\begin{aligned}
M_{ij} &= \frac{1}{\Delta t}\sigma \int_{\Omega_{FE}} J^{-T}\tilde{\mathbf{A}}_i J^{-T}\tilde{\mathbf{v}}_j \det(J)\, d\Omega \\
&\approx \frac{1}{\Delta t}\sum_{k=1}^{N^{GP}} \sigma(x_k, y_k) J_k^{-T}\tilde{\mathbf{A}}_i(x_k, y_k) J_k^{-T}\tilde{\mathbf{v}}_j(x_k, y_k) \det(J_k) w_k
\end{aligned}
$$

$N^{GP}$ number of Gaussian points

**Averaging:**

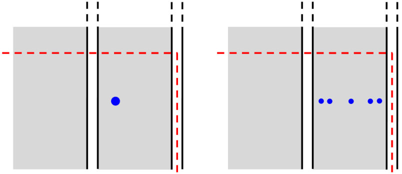
$$
\begin{aligned}
\overline{\sigma} &= \frac{1}{d_1 + d_0}\int_0^{d_1+d_0} \sigma(x)dx = \frac{\sigma_1 d_1 + \sigma_0 d_0}{d_1 + d_0} \approx \frac{\sigma_1 d_1}{d_1 + d_0} \\
\overline{\sigma\phi_x} &= \frac{1}{d_1 + d_0}\int_0^{d_1+d_0} \sigma(x)\phi_x(x)dx = 2\frac{\sigma_1 - \sigma_0}{d_1 + d_0} \approx \frac{2\sigma_1}{d_1 + d_0}, \text{ etc.}
\end{aligned}
$$

# Finite Element Assembly with Adapted Gauss Quadrature

**Stiffness matrix:**

$$
\begin{aligned}
S_{ij} &= \int_{\Omega_{FE}} \frac{1}{\mu(\tilde{\mathbf{A}})} \frac{J}{\det(J)} \operatorname{curl} \tilde{\mathbf{A}}_i \frac{J}{\det(J)} \operatorname{curl} \tilde{\mathbf{v}}_j \det(J)\, d\Omega \\
&\approx \sum_{k=1}^{N^{GP}} \frac{1}{\mu(\tilde{\mathbf{A}}_h(x_k, y_k))} \frac{J_k}{\det(J_k)} \operatorname{curl} \tilde{\mathbf{A}}_i(x_k, y_k) \\
&\quad \cdot \frac{J_k}{\det(J_k)} \operatorname{curl} \tilde{\mathbf{v}}_j(x_k, y_k) \det(J_k) w_k
\end{aligned}
$$

**Nonlinearity:** Field varies nonlinearly within one iron layer, but this variation does not change significantly from one layer to the neighbooring ones.



# Finite Element Assembly with Adapted Gauss Quadrature

**Stiffness matrix:**

$$
\begin{aligned}
S_{ij} &= \int_{\Omega_{FE}} \frac{1}{\mu(\tilde{\mathbf{A}})} \frac{J}{\det(J)} \operatorname{curl} \tilde{\mathbf{A}}_i \frac{J}{\det(J)} \operatorname{curl} \tilde{\mathbf{v}}_j \, d\Omega \\
&\approx ff \sum_{k=1}^{N^{GP}} \sum_{l=1}^{N^{1D,IRON}} \frac{1}{\mu(\tilde{\mathbf{A}}(x_{k,l}, y_{k,l}))} \frac{J_{k,l}}{\det(J_{k,l})} \operatorname{curl} \tilde{\mathbf{A}}_i(x_{k,l}, y_{k,l}) \\
&\quad \cdot \frac{J_k}{\det(J_{k,l})} \operatorname{curl} \tilde{\mathbf{v}}_j(x_{k,l}, y_{k,l}) \det(J_{k,l}) w_l w_k \\
&+ (1 - ff) \sum_{k=1}^{N^{GP}} \sum_{l=1}^{N^{1D,AIR}} \frac{1}{\mu(\tilde{\mathbf{A}}(x_{k,l}, y_{k,l}))} \frac{J_{k,l}}{\det(J_{k,l})} \operatorname{curl} \tilde{\mathbf{A}}_i(x_{k,l}, y_{k,l}) \\
&\quad \cdot \frac{J_{k,l}}{\det(J_{k,l})} \operatorname{curl} \tilde{\mathbf{v}}_j(x_{k,l}, y_{k,l}) \det(J_{k,l}) w_l w_k
\end{aligned}
$$

$n^{1D,IRON}$ number of Gaussian points, 1D quadrature in iron
$N^{1D,AIR}$ number of Gaussian points, 1D quadrature in air
$N^{GP}$ number of Gaussian points, standard quadrature

# Finite Element Assembly with Adapted Gauss Quadrature

**Numerical Example:**

Similar example as above, but 1000 laminates.

Numerical Data: Degrees of Freedom

| Adapted Quadrature | without | with |
|---|---|---|
| Simulation Time in s | 806 | 19 |

... thus, with adapted quadrature about 40 times faster!

*Thank you for your attention!*