Air Gap and Edge Effect in the 2D/1D Method with the Magnetic Vector Potential $\mathbf{A}$ Using MSFEM

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Laminated iron core of an electrical machine:

Large scale: electrical machine with overall dimensions.

Fine scale: thickness of the laminate $d$ and width of the air gap $d_0$.

- The geometric dimensions are extremely different, two scales.
- Many finite elements are required in an accurate model.
- Results in an extremely large equation system.
- The laminated core represents a quasi periodic structure with period $p = d + d_0$.  

$\textbf{Introduction}$

Laminated iron core of an electrical machine:

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$\frac{2}{27}$
Eddy current problem in the frequency domain

Boundary Value Problem:

\[
\text{curl} \mu^{-1} \text{curl} \mathbf{A} + j\omega \sigma \mathbf{A} = 0 \quad \text{in} \ \Omega = \Omega_c \cup \Omega_0, \\
\mathbf{A} \times \mathbf{n} = \alpha \quad \text{on} \ \Gamma_D \\
\mu^{-1} \text{curl} \mathbf{A} \times \mathbf{n} = 0 \quad \text{on} \ \Gamma_N 
\]

\( \mathbf{A} \) ... Magnetic vector potential, \( \Omega_c \) ... iron, \( \Omega_0 \) ... air.

Weak Form:

Find \( \mathbf{A}_h \in V_D := \{ \mathbf{A}_h \in V_h : \mathbf{A}_h \times \mathbf{n} = \alpha_h \text{ on } \Gamma_D \} \), such that

\[
\int_{\Omega} \mu^{-1} \text{curl} \mathbf{A}_h \cdot \text{curl} \mathbf{v}_h \, d\Omega + j\omega \int_{\Omega} \sigma \mathbf{A}_h \cdot \mathbf{v}_h \, d\Omega = 0
\]

for all \( \mathbf{v}_h \in V_0 := \{ \mathbf{v}_h \in V_h : \mathbf{v}_h \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_D \} \), where

\( V_h \subset H(\text{curl}, \Omega) \).
Dimensions, fluxes and currents, edge effect

Dimensions $L, W \gg d$

Fluxes and eddy currents

With edge effect

Without edge effect
Outline

1 Introduction

2 Micro-Shape Functions

3 2D/1D Multiscale Finite Element Method
   - Idea for 2D/1D MSFEM approach
   - Old 2D/1D MSFEM
   - New 2D/1D MSFEM

4 Numerical Example
   - Results

5 Higher order 2D1D-MSFEM
   - Numerical results

6 Computational Costs
Eddy currents in an infinite slab

Infinite slab with fields (left) and analytic solution (right)

Quasi-static magnetic field with the phasor convention $e^{j\omega t}$ reads as

$$\frac{\partial^2 A}{\partial z^2} = \sigma \mu \frac{\partial A}{\partial t} = j\omega \sigma \mu A,$$

with the solution

$$A(z) = A_0 \sinh(\alpha z),$$

where

$$\alpha = (1 + j)/\delta$$

with the penetration depth $\delta = \sqrt{2/(\omega \sigma \mu)}$ holds.
Gauss-Lobatto polynomials, micro-shape functions

Taylor expansion of the hyperbolic sine:

\[ \sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \]  \hspace{1cm} (3)

But, we use odd Gauss-Lobatto polynomials as micro-shape functions

\[ \phi_1(s) = s \]

\[ \phi_3(s) = \frac{1}{2} \sqrt{\frac{5}{2}} (s^2 - 1)s \]  \hspace{1cm} (4)

\[ \phi_5(s) = \frac{1}{8} \sqrt{\frac{9}{2}} (s^2 - 1)(7s^2 - 3)s \]

with the mapping \( s = 2z/d \), where \( s \in [-1, 1] \) and \( z \in [-d/2, d/2] \).
Gauss-Lobatto polynomials are the micro-shape functions, scaled to \( s \in [-1, 1] \), the laminate is grey.

The polynomials facilitate the required continuity of the unknown solution:

\[
\phi_i(-1) = 0 \quad \text{and} \quad \phi_i(1) = 0 \quad \text{with} \quad i = 3, 5
\]  

\( (5) \)
Idea, space splitting

$$\Omega = \Omega_{2D} \times \left[ -\frac{d+d_0}{2}, \frac{d+d_0}{2} \right]$$

Space splitting of the 3D-solution $\mathbf{u}$:

$$\mathbf{u}(x, y, z) \approx \sum_i L_i \phi_i(z) \mathbf{u}_i(x, y)$$

- $L_i$ is a linear differential operator.
- The unknown solution $\mathbf{u}$ depends on $(x, y)$ only.
- The solution is assumed to be odd.

Micro-shape functions $\phi_i$. 
Old 2D/1D MSFEM approach

- Use of a MVP $\mathbf{A}$. 
- Odd function in $z$. 
- No air gap is considered and the edge effect is neglected.

This motivated the 2D/1D multi-scale approach:

$$\tilde{\mathbf{A}} = \phi_1(z)\mathbf{A}_1(x, y) + \phi_3(z)\mathbf{A}_3(x, y) + \phi_5(z)\mathbf{A}_5(x, y) + \cdots$$

with $\mathbf{A}_{1h}$, $\mathbf{A}_{3h}$, $\mathbf{A}_{5h}$, $\cdots \in H(\text{curl}, \Omega_{2D})$.

Tilde marks multiscale approach.
New 2D/1D MSFEM approach

New 2D/1D multi-scale approach:

\[ \tilde{\mathbf{A}} = \phi_0^1(z) \text{grad}(u_1(x, y)) + \phi_1(z) \mathbf{A}_1(x, y) + \text{grad}(w_1(x, y)\phi_1(z)) \]  (6)

with \( \mathbf{A}_{1h} \in H(\text{curl}, \Omega_{2D}) \), \( u_{1h} \) and \( w_{1h} \in H^1(\Omega_{2D}) \).
**Excitation, boundary conditions**

A total magnetic flux $\Phi$ through the cross section $S$ can be prescribed by

$$\Phi = \int_S \mathbf{B}_m \cdot \mathbf{e}_y \, dS = \int_S \text{curl} \, \tilde{\mathbf{A}} \cdot \mathbf{e}_y \, dS = \int_S \text{curl} \left( \phi_1^0(z) \text{grad}(u_1(x, y)) \right) +$$

$$\phi_1(z) \mathbf{A}_1(x, y) \cdot \mathbf{e}_y \, dS = \int_0^w \int_{-d/2}^{d/2} \left( \phi_1^0 u_1,x + \phi_1,z (\mathbf{A}_1)_x \right) \, dz \, dx =$$

$$= \int_{-d+d_0/2}^{d+d_0/2} \phi_1^0 \, dz \int_0^w u_1,x \, dx = 2(u_1(w) - u_1(0))$$

representing a flux tube:
Weak form of the 2D/1D MSFEM

Trial function:

\[ \tilde{A} = \phi_1^0(z) \text{grad}(u_1(x, y)) + \phi_1(z)A_1(x, y) + \text{grad}(w_1(x, y)\phi_1(z)) \]

Test function:

\[ \tilde{v} = \phi_1^0(z) \text{grad}(v_1(x, y)) + \phi_1(z)v_1(x, y) + \text{grad}(q_1(x, y)\phi_1(z)) \]

Galerkin method:

Test function \( \tilde{v} \) has the same structure as the trial function.
Weak form of the 2D1D MSFEM

Replacing of $A_h$ and $v_h$ in the standard finite element method by $\tilde{A}_h$ and $\tilde{v}_h$, respectively, leads to the weak form:

Find $(u_{1h}, A_{1h}, w_{1h}) \in V_D := \{(u_{1h}, A_{1h}, w_{1h}) : u_{1h} \in U_h, A_{1h} \in V_h, w_{1h} \in W_h$ and $u_{1h} = u_D$ on $\Gamma_D\}$, such that

$$\int_\Omega \frac{1}{\mu} \text{curl}(\tilde{A}) \cdot \text{curl}(\tilde{v}) \, d\Omega + j\omega \int_\Omega \sigma(\tilde{A}) \cdot (\tilde{v}) \, d\Omega = 0$$

for all $(v_{1h}, v_{1h}, q_{1h}) \in V_0$.

Finite element subspaces for

$\tilde{A} = \phi_0^0(z) \text{grad}(u_1(x, y)) + \phi_1(z)A_1(x, y) + \text{grad}(w_1(x, y))\phi_1(z)$:

$u_{1h}, v_{1h} \in U_h \subset H^1(\Omega)$

$A_{1h}, v_{1h} \in V_h \subset H(\text{curl}, \Omega)$

$w_{1h}, q_{1h} \in W_h \subset H^1(\Omega_c)$
Numerical example

Laminate with a hole, dimensions in mm.

\[ \sigma = 2.08 \cdot 10^6 \text{ S/m} \]
\[ \mu = 1,000 \mu_0 \]
Numerical results

Re\{J_z}\: 

In the plane of symmetry \( z = 0 \), at the frequency of \( f = 1,000\text{Hz} \).
Numerical results

\[ Re\{J_x\} : \]

\[ Im\{J_x\} : \]

In the surface \( z = 1.8mm \), at the frequency of \( f = 1,000Hz \).
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**Numerical Example**

**Results**

**Numerical results**

SFEM 3D  
2D/1D MSFEM with EE

\[
\begin{align*}
Re\{J_x\} : & \\
Im\{J_x\} : & \\
\text{In the surface } z = 1.8 \text{mm}, \text{ at the frequency of } f = 1,000 \text{Hz}. & 
\end{align*}
\]
Exploiting planes of symmetry

\[ \tilde{A} = \phi_1^0(z) \nabla(u_1(x, y)) + \phi_1(z)A_1(x, y) + \nabla(w_1(x, y)\phi_1(z)) \]

Boundary conditions for \( u_1, A_1 \) and \( w_1 \).
Exploiting planes of symmetry

2D/1D MSFEM, entire

2D/1D MSFEM, quarter

$Re\{J_z\}$:

$Im\{J_z\}$:

In the plane of symmetry $z = 0$, at the frequency of $f = 1,000\text{Hz}$. 
Numerical results

The relative error of the eddy current losses versus frequency.
Higher order 2D1D-MSFEM

Higher order 2D1D MSFEM approach:

\[
\begin{align*}
\tilde{A} &= \phi_1^0(z) \text{grad}(u_1(x, y)) + \phi_1(z)A_1(x, y) + \text{grad}(w_1(x, y)\phi_1(z)) \\
&\quad + \phi_3(z)A_3(x, y) + \text{grad}(w_3(x, y)\phi_3(z)) \\
&\quad + \phi_5(z)A_5(x, y) + \text{grad}(w_5(x, y)\phi_5(z))
\end{align*}
\]

(7)

Corresponding test function:

\[
\begin{align*}
\tilde{v} &= \phi_1^0(z) \text{grad}(v_1(x, y)) + \phi_1(z)v_1(x, y) + \text{grad}(q_1(x, y)\phi_1(z)) \\
&\quad + \phi_3(z)v_3(x, y) + \text{grad}(q_3(x, y)\phi_3(z)) \\
&\quad + \phi_5(z)v_5(x, y) + \text{grad}(q_5(x, y)\phi_5(z))
\end{align*}
\]

(8)

The weak form is obtained analogue to the lowest order multi-scale approach.
Numerical results

The relative error of the eddy current losses versus frequency.
The relative error of the eddy current losses versus frequency.
Numerical results

The relative error of the eddy current losses versus frequency.
Number of unknowns

Computational costs in terms of unknowns.
Thank you for your attention!