Some Two-Dimensional Multiscale Finite Element Formulations for the Eddy Current Problem in Iron Laminates

K. Hollaus, and J. Schöberl
Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8-10, A-1040 Wien, Austria, karl.hollaus@tuwien.ac.at

Abstract—The aim of this work is to introduce and to study the performance of some multiscale finite element formulations for the eddy current problem in laminated iron in two dimensions. The case of the main magnetic flux parallel to the laminates and perpendicular to the plane of projection is considered. Multiscale approaches based on the magnetic vector potential (MVP), the single component current vector potential (SCCVP) and on a mixed formulation are presented. An approach for a multiscale formulation (MSF) with the MVP is constructed at the best by examining and representing the eddy current distribution in laminated iron of a reference solution. The associated weak form of the multiscale finite element method (MSFEM) is presented. Similarly to the MVP the SCCVP and a mixed formulation with the MVP and the current density are studied. This work does not present a mathematical analysis of MSFEM for eddy currents in laminates. The performance of MSFEM is studied by numerous numerical experiments of different examples. The paper covers the topics edge effect, averaging of coefficients and p-refinement of multiscale shape-functions and of standard finite element polynomials (SFEPs) and stability of MSFEM in a particular case. Simulations show the capability of MSFEM to efficiently and accurately approximate eddy currents in iron laminates. The simulations shall also provide a collection of numerical examples to evaluate other multiscale or homogenization methods.

Index Terms—Eddy currents in 2D, edge effect, generalized finite element method GFEM, laminated iron cores, micro-shape function, multiscale formulation MSF, multiscale finite element method MSFEM, p-refinement, linear dependence.

I. INTRODUCTION

The discretization of the micro-structure of laminated iron cores i.e. of each laminate by finite elements (FEs) would lead to prohibitively large systems of equations, whose solution is far away from being a routine task for engineers in the design of electrical devices with modern computer power or even impossible. To overcome this unpleasant fact homogenization and multiscale methods have been developed, which require basically the solution of the large scale problem.

A homogenization method has been proposed in [1] with averaging the material properties for static magnetic fields in laminates in two dimensions. Homogenization methods based on a truncated asymptotic expansion as an approach for the solution of the eddy current problem (ECP) lead to an average value and corrector terms, see for instance [2] or [3]. An asymptotic expansion with the MVP $A$ was used in [4]. To improve the local approximation first and second order corrector terms were determined in [5] solving associated cell problems for the SCCVP $T$ in two dimensions. A multiscale method has been presented in [6]. The magnetic flux density parallel to the laminations is expanded into orthogonal even polynomials, so-called orthogonal skin effect sub-basis functions, to improve the local approximation. The arising coefficients are averaged and the edge effect is neglected.

The present paper deals with multiscale finite element methods (MSFEMs) and possibly applies averaging of the coefficients. Some two-dimensional multiscale finite element formulations are introduced and analyzed. Simulations of small and simple numerical examples and of a more demanding large and nonlinear one are carried out to study the performance of the proposed MSFEMs with respect to different parameters, for example, the penetration depth. Solutions with MSFEM are compared with reference solutions (RS) [7] where each laminate is modeled by FEs.

Section II gives an introduction to MSFEM in the context of eddy currents in laminated media. The boundary value problem of the eddy current problem to be solved is described in Sec. III. Details of the small numerical examples are summarized in Sec. IV. All topics mentioned in the following are supplemented by numerical experiments. The multiscale finite element formulation with $A$ is presented in Sec. V. A physically motivated explanation of the approach for a MSF is given by the eddy current distribution in laminates. How to construct a MSF by means of a RS for $A$ is discussed in detail. Coefficients haven’t been averaged for the sake of accuracy, except in Sec. V-E and V-F where the influence of averaging is investigated and the possibility to use triangular meshes is demonstrated, respectively. The behavior of MSFEM concerning $p$-refinement of the micro-shape functions (MSFs), the power of higher order MSFEM (HMSFEM), is studied in sections V-G. Stability of MSFEM with respect to the degree of SFEPs in a particular case is addressed in Sec. V-H. A large and nonlinear problem has been studied to demonstrate the ability of the MSFEM to cope with nonlinear problems and show the benefit compared with the standard finite element method (SFEM) in Sec. VI. MSFEM for the SCCVP $T$ is shown in Sec. VII. The behavior of MSFEM concerning $p$-refinement of the SFEP bases [8] is studied in
Figure 1: Transformer core with large scale dimensions $L$ etc.

Figure 2: Small scale dimensions, $d+d_0$ represents one period, micro-shape functions $\phi_i$.

VII-E. The improvement of the accuracy increasing the degree of the SFEPs is investigated. Attention is paid to the edge effect in VII-F. Finally, a mixed MSF using the $A$ and the current density $J$ is introduced in Sec. VIII.

II. Multiscale Finite Element Method MSFEM

The problem we are confronted with exhibits two different scales. A large scale characterized by large scale dimensions, for instance the length $L$, the height $H$, the width $W$ and the dimensions of the windows, $E$ and $F$, of a transformer core shown in Fig. 1. The thickness $d$ of the laminates, the width of the air gap $d_0$ in between the laminates, see Fig. 2, and the penetration depth $\delta$ are dimensions at the small scale or at the micro-scale. The ratio of the different scales is extremely large, about $10^5$ up to $10^6$.

Mapped polynomials are used by the standard finite element method (SFEM) to approximate the unknown solution. The SFEM performs well as long as the solution or the coefficients in the equations are smooth. However, to obtain an accurate approximation also in case of equations with rough coefficients, for instance materials with a micro-structure (laminated iron core) or for problems with singularities like boundary layers, extremely fine meshes are required [9]. This is the reason for the prohibitively large equation systems which may require exorbitant amounts of computer resources to obtain an accurate solution [10]. This work addresses solely the eddy current problem in 2D with rough coefficients because of a laminated iron core.

To avoid large algebraic equation systems the generalized finite element method (GFEM) as a general framework for equations with rough coefficients or for problems with singularities seems to be a very promising option [9], [10]. Apart from the finite dimensions laminated iron cores represent a problem with a periodic micro-structure. The SFEM basis is augmented by special functions including a priori information into the ansatz space

$$u_h(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij} \varphi_i(x) \phi_j(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij} \psi_{ij}(x),$$

where $n$ is the number of the SFEPs $\varphi_i$, $m$ is the number of special functions $\phi_j$ and $u_{ij}$ are the coefficients of the approximate solution $u_h$. The special functions, which are custom tailored ansatz functions, may stem from an analytic solution or, for example, from a FE solution of a basic problem, i.e. these functions are known. The local basis of the special functions approximates well the solution locally. Micro-shape functions $\phi_j$ represent a local space and standard polynomials $\varphi_i(x)$ a global space. Multiplication of standard polynomials $\varphi_i$ with special functions $\phi_j$ yields the new basis functions $\psi_{ij}$. In the present context of micro- and macroscale we call the method MSFEM and the special functions $\phi_j$ the micro-shape functions, see Fig. 2.

III. Eddy Current Problem

The simplified eddy current problem (ECP) in two dimensions to be solved in the present work is shown in Fig. 3. The entire domain $\Omega = \Omega_m \cup \Omega_0$ consists of the laminated medium $\Omega_m$ and air $\Omega_0$. $\Omega_m$ comprises laminates $\Omega_c$, i.e. the conducting domain, and air gaps in between. Eddy currents represented by the current density $J$ occur in $\Omega_c$. The interface $\Gamma_{n0}$ separates $\Omega_m$ from $\Omega_0$. The outer boundary of $\Omega$ is denoted by $\Gamma$ and $n$ is the unit normal vector pointing out of $\Omega$. The magnetic flux density $B$ is assumed to be parallel to the laminates and perpendicular to the plane of projection. The material parameters are the magnetic permeability $\mu(B)$, which can be nonlinear, and the electric conductivity $\sigma$. Relations (2) to (6) are valid in $\Omega_m$, whereas (7) to (9) belong to air $\Omega_0$ including the air gaps in $\Omega_m$ and (10, 11) are possible

$$\text{curl } H = J \quad \text{in } \Omega_m,$$
$$\text{curl } E = -\frac{\partial B}{\partial t}$$
$$\text{div } B = 0$$
$$J = \sigma E$$
$$B = \mu H$$
$$\text{curl } H = J_0 \quad \text{in } \Omega_0$$
$$\text{div } B = 0$$
$$B = \mu H$$
$$H \times n = K \quad \text{on } \Gamma_H$$
$$B \cdot n = b \quad \text{on } \Gamma_B$$
boundary conditions on \( \Gamma \). The boundary value problem BVP (2) to (11) considers Dirichlet boundary conditions only. The BVP of the ECP in the frequency domain is straightforward and thus not shown here.

![Figure 3: Problem in 2D.](image)

**IV. Numerical Simulations**

**A. Relative errors in percentage**

To evaluate the performance of MSFEM with respect to for instance the penetration depth \( \delta \) relative errors of the potential \( u \), the flux density curl \( u \) and the eddy current losses \( P \),

\[
\frac{|u_{\text{MSFEM}}-u_{\text{RS}}|_{L_2}}{|u_{\text{RS}}|_{L_2}} \cdot 100\%, \quad \frac{|\text{curl}(u_{\text{MSFEM}})-\text{curl}(u_{\text{RS}})|_{L_2}}{|\text{curl}(u_{\text{RS}})|_{L_2}} \cdot 100\%
\]

and

\[
\frac{|P_{\text{MSFEM}}-P_{\text{RS}}|}{P_{\text{RS}}} \cdot 100\%,
\]

respectively, are computed by means of a RS and that of MSFEM. Single laminates are modeled by FEIs for the RS. Errors (12) are evaluated in \( \Omega \), i.e. iron laminates, only. The frequency was changed to get different \( \delta \).

**B. Finite element models**

A small example has been deliberately chosen to demonstrate also errors which would not be visible in large problems by means of (12). All data are given so that the simulation results can be verified. The problem is symmetric with respect to both axes, \( x \) and \( y \), assuming the origin in the center of the problem domain, compare with Fig. 3. The dimensions of the domains are \( |\Omega_0| = 20 \times 20 \text{ mm}^2 \) and \( |\Omega| = 40 \times 40 \text{ mm}^2 \). The FE models shown in Figs. 4 for the RS and for MSFEM consider 10 laminates, \( d = 1.8 \text{ mm} \), and air gaps in between, \( d_0 = 0.2 \text{ mm} \). For meaningful errors (12) the discretization in \( y \)-direction is the same for both FE models. The dimensions of the FE models in Figs. 4 are summarized in Table I. The fill factor \( f_f = \frac{d}{d_0} \cdot 100 \% \) is valid for all examples. In case of averaging \( d + d_0 = 0.5 \text{ mm} \) was selected. Most of the simulations use the models with dimensions summarized in Table I.

**V. Multiscale Finite Element Formulation with A**

In brief, the time domain is only presented here. In case of linear material properties the problem is formulated in the frequency domain with the phasor convention \( e^{j\omega t} \).

![Figure 4: FE models for a) RS (left) and b) MSFEM (right).](image)

| \( \Delta x \), RS | 9 | 1 | 0.1 | 1.8 | 0.2 | ... | b |
| \( \Delta x \), MSFEM | 9 | 1 | 2 | 8 |
| \( \Delta y \) | 9 | 1 | 0.5 | 1.5 | 8 |

\( a \) Due to the symmetry only one half of the dimensions in \( x \)- and \( y \)-direction are presented.

\( b \) After 10 laminates with air gaps in between the dimensions repeat accordingly to obtain a symmetric problem.

A. Boundary value problem with A

The magnetic vector potential (MVP) \( A \) is introduced by \( B = \text{curl } A \) with

\[
A : \quad \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (A_1(x,y), A_2(x,y))^T (13)
\]

\[
\text{curl } A : \quad \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) (x,y) (14)
\]

where \( \mathbb{R} \) stands for either real numbers \( \mathbb{R} \) or complex numbers \( \mathbb{C} \). Mapping to \( \mathbb{R} \) or \( \mathbb{C} \) belongs to the time domain or to the frequency domain, respectively. The valid domain is indicated at the given case. Considering the relevant Maxwell’s equations leads to the BVP of the ECP in the time domain

\[
\text{curl } \frac{1}{\mu(A)} \text{curl } A + \sigma \frac{\partial }{\partial t} A = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, (15)
\]

\[
A \times n = a \quad \text{on } \Gamma. (16)
\]

B. Weak form with A

The weak form of the SFEM belonging to the time domain reads:

Find \( A_h \in V_{h,\alpha} := \{ A_h \in \mathcal{V}_h : A_h \times n = \alpha_h \text{ on } \Gamma \} \), such that

\[
\int_{\Omega} \frac{1}{\mu(A_h)} \text{curl } A_h \cdot \text{curl } v_h d\Omega + \int_{\Omega} \sigma A_h \cdot v_h d\Omega = 0 \quad (17)
\]

for all \( v_h \in V_{h,0} \), where \( \mathcal{V}_h \subset H(\text{curl}, \Omega) \). Index \( h \) indicates FE discretization. The solution of (17) serves as RS to evaluate the accuracy of MSFEMs. Since \( \sigma = 0 \) in air a regularization by a penalty term [11] was applied to get a unique solution of (17).

C. An approach for the MSF with A

There is no rigorous mathematical rule to construct an approach for a MSF. Therefore, in the context of a MVP \( A \) the eddy current distribution of a RS in a laminated medium is
studied to devise an approach for the MSF. Figure 5 shows a
detail of the RS with eddy currents consisting of a laminar part,
currents are flowing parallel to the laminates, up and down in
y-direction, and a second part at the end of the laminates,
where the eddy currents turn around flowing in x-direction to
built closed loops. The later part or this fact is frequently called
the "edge effect." Due to the micro-structure. The second
due to the large scale dimensions and the second and third term
consider the solution due to the micro-structure. The second
term is a vector with the component
\( A \)
considered in (18) (for the edge e
\[ \text{ff} \]) (18) is parallel to the slab with the MVP. Without the third term in
analytic solution of eddy currents in an infinite slab where

\[ \text{for all } (\mathbf{v}_{\text{ff}}, q_{\text{ff}}, \mathbf{q}_{\text{ff}}) \in V_{\text{ff}}, \quad \mathbf{V}_{\text{ff}} = L^2(\Omega_m), \quad \mathbf{W}_h = H^1(\Omega_m) \text{ and } \phi_1 \text{ is in the}\]

space of periodic and continuous functions \( H^1_p(\Omega_m). \)
The MSFEM with \( A_1 \) in \( \Gamma_{\text{m0}} \) is analyzed in Sec. V-F.
Dirichlet boundary conditions are prescribed for \( A_{0h} \) on \( \Gamma \) and
homogenous natural boundary conditions hold for \( w_1 \) and for
\( A_1 \) in \( \Gamma_{\text{m0}}. \)
The FE approximation of the MSF (18) is consistent with the
\( H(\text{curl}, \Omega) \) space. A FE subspace of \( H(\text{curl}, \Omega) \) is selected for
the first term \( A_{0h}. \) The \( x \)-dimension of the rectangular FEs is
assumed to be an integer multiple of the period \( \mathbf{p} = \mathbf{d} + d_0 \)
and \( \phi_1 = 0 \) along the relevant FE boundaries. This means that
the second term is tangentially continuous across the laminates.
The third term is a gradient of continuous functions. Thus,
the latter two belong also to FE subspaces of \( H(\text{curl}, \Omega_m). \)
Solutions of \( A_{0h}, A_{1h} \) and \( w_{1h} \) are very smooth as can be seen in
Figs. 43 to 44, in Appendix B. Therefore, a rather coarse
FE mesh and thus much less unknowns suffice for an accurate
approximation of the solutions of MSFEM, see Figs. 4 b) or
23 b). This fact explains the big advantage of MSFEM over
SFEM and the RS.

### E. Exact integration versus averaging of the coefficients

The ECP in the frequency domain with the complex unit
\( j \) and the angular frequency \( \omega \) is considered in this section.
The bilinear forms in (19), the curl of gradients vanishes, are
written in detail in (20) in Appendix A. Two different num-
erical techniques to compute the FE matrices are compared,
averaging and exact integration, see also [6], [12], [7], [13]
and [14].

Averaging of coefficients \( \lambda \) in (19) over the period \( \mathbf{p} = \mathbf{d} + d_0 \)

\[ \overline{\lambda} = \frac{1}{\mathbf{p}} \int_0^\mathbf{p} \lambda(\mathbf{x})d\mathbf{x}, \quad (21) \]

where \( \lambda \) stands for example for \( \frac{1}{\mu}, \quad j\omega \sigma, \quad j\omega \sigma \nabla \phi_1, \quad j\omega \sigma \phi_1, \)
etc., with \( \phi_1 : = \nabla \phi_1, \) in (20) in Appendix A, leads to (22) in
Appendix A. The averaged coefficients are constant throughout
\( \Omega_m. \) Bilinear forms with averaged coefficients and in turn the
weak form are modified. Therefore, all quantities are marked
with a bar. Then, the usual Gauss integration of the modified
bilinear forms (22) is carried out. The numerical effort is
reduced significantly. Quantities of the new weak form are
marked with a bar to indicate averaging. This technique clearly
implies an error. The solution of the modified weak form with
(22) is hopefully as accurate as the original one with (19).

For the exact integration it is assumed that the \( x \)-dimension of
the rectangular FEs is an integer multiple of \( \mathbf{p} \) because \( n_1 \)
points into the $x$-direction, see Fig. 3. Exact integration means that the FEs are split into rectangular subdomains according to $d$ and $d_0$, respectively, and Gauss integration on the different subdomains is made considering the relevant standard polynomials, micro-shape function and material parameters. The contributions of the integrals over the subdomains are added up for the respective FE matrix.

The errors defined in (12) can be seen in Figs. 7 to 9. There is a significant difference between the RS $A_h$ according to (17) and the MSFEM solution $\tilde{A}_h$ obtained by (19) of the potentials and eddy current losses depending on whether the exact or the averaging integration technique is used. This is clearly visible at small errors due to the logarithmic scaling. The error in $A$ is essentially higher than that of $P$ over the investigated range. The magnetic flux density curl $A$ is hardly affect by the selected integration technique, see Fig. 8. The polynomial degrees of the SFE basis of MSFEM (18) were one for $A_0$ and $A_1$ and two for $w_1$. In general errors grow strongly once $\delta$ becomes smaller than $d + d_0$ regardless of the integration technique. The HMSFEM in Sec. V-G can solve this problem.

F. Triangular mesh

MSFEMs with averaged coefficients facilitate an easy use of triangular FE meshes. Two examples are presented in Figs. 10. The computational domain $\Omega_m \cup \Omega_0$ is uniformly subdivided into triangular FEs. The side length of the triangles is approximately $s$. In case of $s = 20\text{mm}$ $\Omega_m$ consists of only two triangular FEs. All simulations are based on the MSFEM (19) and relation (21).

Eddy current distributions are shown in Figs. 11.

The error in $A$ with respect to $s$ is large as can be seen in Fig. 12, the error in curl $A$ is almost independent from $s$ and clearly grows with the frequency (Fig. 13). The limit of the losses $P$ with decreasing $s$ is to small because the
local space is insufficient, see Fig. 14. On the other hand, the error grows with \( s \) on and on, the insufficiency of the global space becomes significant. The penetration depths are

\[
\rho_{25} = 2.25\,\text{mm}, \quad \rho_{50} = 1.59\,\text{mm} \quad \text{and} \quad \rho_{100} = 1.13\,\text{mm} \quad \text{for the frequencies} \quad f = 25\,\text{Hz}, \quad f = 50\,\text{Hz} \quad \text{and} \quad f = 100\,\text{Hz}, \quad \text{respectively.}
\]

The width of the laminates is \( d = 1.8\,\text{mm} \).

The degree of the SFEPs is one for \( A_{0h} \) and \( A_{1h} \) and two for \( w_{1h} \) in Figs. 12 to 14.

**G. Higher order MSF with \( A \), \( p \)-refinement**

To cope with problems, where \( \delta \) is essentially smaller than \( d \), approach (18) is extended. Higher order terms for the laminar part as well as for the edge effect of eddy currents are added [14]. The dimension of the local basis \( B \) is increased, from one \( B_1 = \{ \phi_1 \} \) in (18) to two or three \( B_3 = \{ \phi_1, \phi_3, \phi_5 \} \) leading to the approach of a higher order MSF

\[
\tilde{A} = A_0 + \phi_1 \left( \begin{array}{c} 0 \\ A_1 \\ 0 \\ A_3 \\ 0 \\ A_5 \end{array} \right) + \nabla (\phi_1 w_1) + \phi_3 \left( \begin{array}{c} 0 \\ A_3 \\ 0 \\ A_3 \\ 0 \end{array} \right) + \nabla (\phi_3 w_3) + \phi_5 \left( \begin{array}{c} 0 \\ A_5 \\ 0 \\ A_5 \\ 0 \end{array} \right) + \nabla (\phi_5 w_5).
\]

Since the magnetic flux density \( B \) is an even function across the laminates, only odd higher order terms for the MSF with \( A \) are considered in (23). The higher order multiscale functions \( \phi_3 \) and \( \phi_5 \), see Fig. 15, are living in \( \Omega_c \), i.e. on iron subintervals. The multiscale functions \( \phi_3 \) and \( \phi_5 \) are equal to zero in air and, thus, represent bubble functions. Therefore, they preserve the required continuity of the tangential component of \( \tilde{A} \).

The associated weak form for the higher order multiscale finite element method (HMSFEM) of (23) is a straightforward extension of (19) and, therefore, not shown here. The choice of the FE subspaces \( A_{0h} \in U_h \subset H(\text{curl}, \Omega), \quad A_{1h}, \quad A_{3h} \quad \text{and} \quad A_{5h} \in V_h \subset L_2(\Omega_m), \quad w_{1h}, \quad w_{3h} \quad \text{and} \quad w_{5h} \in W_h \subset H^1(\Omega_m) \) and \( \phi_1, \phi_3 \) and \( \phi_5 \in H_{\text{per}}(\Omega_m) \) is obvious.

**H. Stability of MSFEM**

This section deals with the stability of MSFEM with respect to the degree of SFEPs mainly with \( A \) in a particular case. The MSFEM for \( A \) in Sec. V with \( A_1 \in L_2(\Omega_m) \) offers the opportunity to apply static condensation to the degrees of

**Figures 16 to 18 show results for different dimensions of the local space. Adding of 3\textsuperscript{rd} order terms improves the accuracy enormously. The 5\textsuperscript{th} order approach performs clearly better than the 3\textsuperscript{rd} order one for small \( \delta \). The degree of the SFEPs was one for \( A_{0h} \), one for \( A_{1h}, \quad A_{3h} \quad \text{and} \quad A_{5h} \) and two for \( w_{1h}, \quad w_{3h} \quad \text{and} \quad w_{5h} \). The computational costs are compared in Table II. A fairly good improvement has been achieved increasing the dimension of the local basis for this small problem.**

**Figure 15: Odd micro-shape functions.**

**Figure 14: Eddy current losses \( P \).**

**Figure 12: Magnetic vector potential \( A \).**

**Figure 13: Magnetic flux density curl \( A \).**

\( \rho_{25} = 2.25\,\text{mm}, \quad \rho_{50} = 1.59\,\text{mm} \quad \text{and} \quad \rho_{100} = 1.13\,\text{mm} \quad \text{for the frequencies} \quad f = 25\,\text{Hz}, \quad f = 50\,\text{Hz} \quad \text{and} \quad f = 100\,\text{Hz}, \quad \text{respectively.} \)
freedom belonging to $L_2$ with which the number of unknowns can essentially be reduced. The FE-mesh described in Table I and Fig. 4 has been used for all numerical investigations. One period $p = d + d_0$ equals to 2 mm and fits exactly to one FE-layer on the left and on the right side of $\Omega_m$. Linear dependence is an important issue of multiscale methods but it is rather seldom addressed [15], [16]. Strictly speaking, the basis of functions $\psi_{ij}(x)$ of the MSFEM (19) are linear independent but not well-conditioned. Contrary to $A_1 \in L_2$ partial derivatives of $A_1$ are considered in case of $A_1 \in H_1$ introducing some degree of smoothing, compare Figs. 19. Analysis of the FE system matrices showed very similar condition numbers and eigenvalue spectra of both MSFEMs and with averaged coefficients. Surprisingly, the MSFEM with $A_1 \in H_1$ is stable with respect to the degree of SFEPs as can be seen in Figs. 20 to 22. The abscissa shows the degree of the SFEPs of $A_0$ and $A_1$, that of $w_1$ is one greater than those. The error for $A_1 \in H_1(\Omega_m)$ decreases slightly with degree of SFEPs whereas that for $A_1 \in L_2(\Omega_m)$ grow rapidly and for the losses in an uncontrolled manner. The method with averaged coefficients indicated with AC is robust against instability. The same instability can be observed for HMSFEM with $A_1$, $A_3$ and $A_5 \in L_2$. Since Dirichlet boundary conditions have to be prescribed for $T_2$ only $T_2 \in H^1$ is possible. Investigations of MSFEM with $T$ showed no instability. An effective and simple remedy for the instability is to resolve the outermost laminates by the SFEM while the MSFEM with exact integration is retained for the rest of the laminates. Two laminates ($p = 1.0$ mm) in each outermost layer is stable up to a polynomial degree of the SFE basis with four for $A_0$ and $A_1$ and five for $w_1$.

### VI. LARGE NONLINEAR PROBLEM

To demonstrate the ability of MSFEM to cope with large and nonlinear problems and to clearly point out the benefit of the MSFEM compared with the SFEM a large nonlinear problem with FE models in Figs. 23 have been studied. The nonlinear problem with the magnetization curve in Fig. 24 consists of 1,000 laminates. MSFEM of $1^{st}$ order of (19) has been studied. The implicit Euler method was used for the time discretization and Newton’s method to solve the nonlinear problem.
agreement of the eddy current losses with respect to time in Fig. 25 is very satisfactory. The reduction of computational costs by means of MSFEM compared with SFEM and the RS is impressive as can be seen in Table III. The computational requirements of MSFEM for this large problem and the linear case are almost the same as those for the small problem in Table II, Sec. V-G, for the 1st order MSFEM with only 10 laminates.

<table>
<thead>
<tr>
<th>Total No.</th>
<th>$H(\text{curl}, \Omega)$</th>
<th>$L_2(\Omega_m)$</th>
<th>$H^1(\Omega_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RS</td>
<td>164, 430</td>
<td>164, 430</td>
<td>-</td>
</tr>
<tr>
<td>MSFEM</td>
<td>1, 289 a)</td>
<td>840</td>
<td>208</td>
</tr>
</tbody>
</table>

a) For the 1st order MSF in (23)

VII. HIGHER ORDER MULTISCALE FINITE ELEMENT METHOD WITH SINGLE COMPONENT CURRENT VECTOR POTENTIAL $T$

A. Boundary value problem with $T$

The current density $\mathbf{J}$ can be represented with a current vector potential $\mathbf{T}$ by $\mathbf{J} = \text{curl} \mathbf{T}$. This section deals with the...
single component current vector potential $T$, e.g., pointing in $z$-direction $T = Te_z$ in the frequency domain:

$$ T : \quad \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{C}, \quad (x, y) \mapsto T(x, y) \quad (24) $$

$$ \text{curl } T : \quad \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{C}, \quad (x, y) \mapsto \left( \frac{\partial T_y}{\partial x} - \frac{\partial T_x}{\partial y} \right) (x, y) \quad (25) $$

A simple BVP of the ECP in the frequency domain reads, see Fig. 3:

$$ \text{curl } \frac{1}{\sigma} \text{curl } T + j\mu_0 T = 0 \quad \text{in } \Omega \subset \mathbb{R}^2 \quad (26) $$

$$ T = T_0 \quad \text{on } \Gamma \quad (27) $$

**B. Weak form with $T$**

The weak form in the frequency domain reads:

Find $T_h \in V_{h,T_0} := \{ T_h \in \mathcal{U}_h : T_h = T_0 \quad \text{on } \Gamma \}$, such that

$$ \int_{\Omega} \frac{1}{\sigma} \text{curl } T_h \cdot \text{curl } t_h \, d\Omega + j\omega \int_{\Omega} \mu T_h \partial_h d\Omega = 0 \quad (28) $$

for all $t_h \in V_{h,0}$, where $\mathcal{U}_h \subset H^1(\Omega)$. The solution of (28) serves as RS to evaluate the MSFEM with $T$.

**C. Higher order multiscale finite element method with $T$**

The MSF up to the order 4 for the single component current vector potential

$$ \tilde{T}(x, y) = T_0(x, y) + \phi_2(x)T_2(x, y) + \phi_4(x)T_4(x, y) \quad (29) $$

with even micro-shape functions $\phi_2$ and $\phi_4$ shown in Fig. 26 yields the multiscale current density

$$ \tilde{J} = \text{curl } \tilde{T} = \begin{pmatrix} T_{0y} \\ -T_{0x} \end{pmatrix} + \begin{pmatrix} \phi_2 T_{2y} \\ -\phi_2 T_{2x} \end{pmatrix} + \begin{pmatrix} -\phi_4 T_{4y} \\ \phi_4 T_{4x} \end{pmatrix}. \quad (30) $$

Simply speaking $T$ corresponds to the magnetic field strength $H$ which is an even function, therefore the multiscale functions $\phi_2$ and $\phi_4$ are used.

**D. Weak form of HMSFEM with $T$**

Find $(T_{0h}, T_{2h}, T_{4h}) \in V_{h,T_0} := \{ (T_{0h}, T_{2h}, T_{4h}) : T_{0h} \in \mathcal{U}_{h}, T_{2h} \in \mathcal{V}_{h}, T_{4h} = 0 \quad \text{on } \Gamma \}$ and $T_{2h} = 0$ and $T_{4h} = 0$ on $\Gamma_{m0,1} \subset \Gamma_{m0}$, such that

$$ \int_{\Omega} \frac{1}{\sigma} \text{curl } T_h \cdot \text{curl } t_h \, d\Omega + j\omega \int_{\Omega} \mu \tilde{T}_h \partial_h d\Omega = 0 \quad (31) $$

for all $(t_{0h}, t_{2h}, t_{4h}) \in V_{h,0}$, where $\mathcal{U}_h$ is a subspace of $H^1(\Omega)$, $\mathcal{V}_h$ of $H^1(\Omega_m)$ and $\phi_2$ and $\phi_4$ in $H^1_{per}(\Omega_m)$.

Homogeneous essential boundary conditions are prescribed for $T_2 = 0$ and $T_4 = 0$ on the interface $\Gamma_{m0,1}$ of $\Gamma_{m0} = \Gamma_{m0,1} \cup \Gamma_{m0,2}$, see Fig. 27, to ensure the edge effect that can easily be seen by the y-component in (30).

**E. p-refinement of the standard FE basis**

This time attention is paid to the accuracy with respect to the degree of the SFEPs, see Figs. 28 to 36. The degree of the SFEPs is one for $T_{0h}$, $T_{2h}$ and $T_{4h}$ in Figs. 28 to 30, one for $T_{0h}$ and two for $T_{2h}$ and $T_{4h}$ in Figs. 31 to 33 and one for $T_{0h}$ and three for $T_{2h}$ and $T_{4h}$ in Figs. 34 to 36. The errors are visibly reduced by increasing the degree of the SFEPs. The $4^{th}$ order MSFEM performs clearly better than the $2^{nd}$ order one. This holds in general for small $\frac{\delta}{d}$ (Figs. 28 to 36) and in particular for the $3^{rd}$ degree of SFEPs (Figs. 34 to 36).

![Figure 27: Support, boundary and interface conditions.](image)

The computational costs are summarized in Table IV. A fairly
good improvement has been achieved for this small problem.

It is worth mentioning that both the degree of SFEPs and the order of the MSA along with the micro-shape functions can not be increased arbitrarily because of the linear dependence due to the tensor product of MSA (18) and the use of quadrilateral FEs.

**F. Edge effect**

The difference between considering and neglecting the edge effect is shown by means of eddy currents in Figs. 37. In contrast to MSFEM with $A$ in Sec. V MSFEM with $T$ works also without prescribing homogenous essential boundary conditions for $T_2$ and $T_4$ on $\Gamma_{m0,1}$, i.e. without considering the edge effect, reasonably and offers therefore the opportunity to
study the influence of the edge effect. Dashed lines in Figs. 38 to 40 stand for edge effect neglected and solid lines for edge effect considered. Note, that the abscissa in all Figs. 38 to 40 represents the thickness $d$ of the laminates and $\delta$ was about 0.5 mm. The influence of the edge effect can easily be seen. 

The error of $\text{curl} \, T e_z$ neglecting the edge effect is relatively large throughout the range of $d$. A substantial improvement can be observed considering the edge effect for this small example. In general errors decrease due to the edge effect for smaller $d$. The 2nd order MSA shows clearly faster growing errors with large $d$ compared to the 4th order one.

The degree of the SFEPs is one for $T_{0h}$ and two for $T_{2h}$ and $T_{4h}$ in Figs. 38 to 40.
VIII. Multiscale Finite Element Method with a Mixed Formulation using \((A, J)\)

Introducing the current density

\[
J_h = -j\omega \sigma A_h
\]

also as an unknown yields the additional relation

\[
-\int_{\Omega} A_h \cdot g_h \omega d\Omega + \frac{j}{\omega} \int_{\Omega} J_h \cdot g_h \omega d\Omega = 0
\]

for the weak form of the mixed formulation:

A. Weak form of the standard mixed formulation with \((A, J)\)

Find \((A_h, J_h) \in V_{h;0} := ((A_h, J_h), A_h \in \mathcal{U}_h, J_h \in M_h \text{ and } A_h \times n = \alpha_n \text{ on } \Gamma)\), such that

\[
\begin{align*}
\int_{\Omega} \mu \text{ curl } A_h \cdot \text{ curl } \tilde{v}_h \omega d\Omega - \int_{\Omega} J_h \cdot \tilde{v}_h \omega d\Omega &= 0 \quad (34) \\
-\int_{\Omega} A_h \cdot g_h \omega d\Omega + \frac{j}{\omega} \int_{\Omega} \sigma J_h \cdot g_h \omega d\Omega &= 0 \quad (35)
\end{align*}
\]

for all \((\tilde{v}_h, g_h) \in V_{h;0}\) with \(\mathcal{U}_h \subset H(\text{curl}, \Omega)\) and \(M_h \subset H(\text{div}, \Omega)\).

B. Weak form of the mixed multiscale finite element method (MMSFEM) with \((\tilde{A}_h, \tilde{J}_h)\)

The approach of the current density

\[
\tilde{J} = J_0 + \text{curl}(\phi_2 T_2 e_3)
\]

The computational costs of the mixed formulation are certainly different from those of the corresponding standard formulation with \(A\). However, the mixed formulation performs much better in reproducing the edge effect as can easily be seen in Fig. 41. A comparison of the losses obtained by different methods are shown in Fig. 42.

References


APPENDIX A

SOME EQUATIONS

\[
\int_{\Omega} \mu \left( \nabla \phi + \phi \nabla \right) \cdot \left( \nabla v + \phi \nabla \right) d\Omega + j\omega \int_{\Omega} \sigma \left( \phi + \nabla \phi \right) \cdot \left( \nabla v + \phi \nabla \right) d\Omega = 0
\]  
(20)

\[
\int_{\Omega} \left( \nabla \phi \right) ^T \left( \frac{\nabla v}{\nabla \phi} \right) d\Omega + j\omega \int_{\Omega} \left( \nabla \phi \right) ^T \left( \frac{\nabla v}{\nabla \phi} \right) d\Omega = 0
\]  
(22)

APPENDIX B

MULTISCALE FINITE ELEMENT METHOD SOLUTIONS

Typical solutions are selected to make the advantage of MSFEM over standard FEM quite clear. In this context it is worth mentioning that the corresponding solutions of the 3rd order approach are almost the same. Results with the 5th order MSF are shown in Figs. 43 to 44.

Figure 43: MSFEM solution: Re(A_{0h}(1)) (left), Re(A_{0h}(2)) (right).

Figure 44: MSFEM solution: Re(A_{1h}) (left), Re(w_{1h}) (right).