A Mixed Multiscale Finite Element Method with A and J for Eddy Currents in Iron Laminates

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The aim of this work is to study the capability of mixed multiscale finite element methods (MMSFEMs) for eddy currents in laminated iron in two and three dimensions. The mixed multiscale approaches are constructed using the magnetic vector potential $\boldsymbol{A}$ and the electric current density $\boldsymbol{J}$. The weak form of the MMSFEMs are derived. Simulations show the capability of the MMSFEM to efficiently and accurately approximate eddy currents in iron laminates. Some numerical results are presented.

***Index Terms*—Eddy current problem, laminated iron core, mixed multiscale finite element method.**

I. Introduction

THE MAGNETIC flux density parallel to the lamination is expanded by orthogonal basis functions proposed in the multiscale finite element method [1]. Edge effects have been neglected. This work focuses on the capability to render the eddy current distribution by the proposed mixed formulations in all parts of the laminates. Contrary to [2] the magnetic vector potential $\boldsymbol{A}$ and the current density $\boldsymbol{J}$ are unknown in the present work. First, the multiscale approaches in $2D$ and $3D$ are presented and then the weak forms are derived. The MMSFEM considers also the edge effects [3] and thus reproduce the eddy current distribution very well. The different higher order finite element spaces used for the present work can be found in [4]. The capability of the MMFEMs to render the eddy current distribution is demonstrated by two numerical examples.

II. Standard Mixed Eddy Current Problem

A. Boundary Value Problem

With

$$\boldsymbol{J} = -j\omega\sigma\boldsymbol{A} \tag{1}$$

follows the mixed boundary value problem

$$\operatorname{curl}\nu\operatorname{curl}\boldsymbol{A} - \boldsymbol{J} = \boldsymbol{0}, \tag{2}$$

$$-\boldsymbol{J} + j\frac{\sigma}{\omega}\boldsymbol{A} = \boldsymbol{0} \text{ in } \Omega \subset \mathbb{R}^d \tag{3}$$

$$\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{\alpha} \text{ on } \Gamma \tag{4}$$

considered in this work, where $\boldsymbol{A}$ is the magnetic vector potential, $\boldsymbol{J}$ the current density, ν magnetic reluctivity, σ the electric conductivity, ω the angular frequency, $\Omega_c \cup \Omega_0 = \Omega \subset \mathbb{R}^d$ the conducting and non-conducting domain with the spatial dimension $d = 2$ or 3, see Fig. 1 and Γ the boundary of Ω.

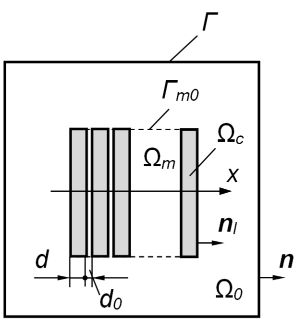

Fig. 1: Eddy current problem (sketch).

B. Weak Form

Find $(\boldsymbol{A}_h,\, \boldsymbol{J}_h) \in V_{h,\alpha} := \{(\boldsymbol{A}_h,\, \boldsymbol{J}_h) : \boldsymbol{A}_h \in \mathcal{U}_h,\, \boldsymbol{J}_h \in \mathcal{M}_h \text{ and } \boldsymbol{A}_h \times \boldsymbol{n} = \boldsymbol{\alpha}_h \text{ on } \Gamma\}$, such that

$$\int_\Omega \nu\operatorname{curl}\boldsymbol{A}_h\cdot\operatorname{curl}\boldsymbol{v}_h\,d\Omega - \int_\Omega \boldsymbol{J}_h\cdot\boldsymbol{v}_h\,d\Omega = 0 \tag{5}$$

$$-\int_\Omega \boldsymbol{A}_h\cdot\boldsymbol{g}_h\,d\Omega + \frac{j}{\omega}\int_\Omega \frac{1}{\sigma}\boldsymbol{J}_h\cdot\boldsymbol{g}_h\,d\Omega = 0 \tag{6}$$

for all $(\boldsymbol{v}_h,\, \boldsymbol{g}_h) \in V_{h,0}$ with $\mathcal{U}_h \subset H(\operatorname{curl},\Omega)$ and $\mathcal{M}_h \subset H(\operatorname{div},\Omega)$.

III. Mixed Multiscale Finite Element Method MMSFEM

The laminated domain Ω_m comprises the laminates and the air gaps in between (see Fig. 1). The micro-shape functions ϕ_1 and ϕ_2 used in the approaches below are shown in Fig. 2.

A. Mixed Multiscale Method for $2D$

The mixed multiscale approach for $2D$

$$\tilde{\boldsymbol{A}} = \boldsymbol{A}_0 + \phi_1\begin{pmatrix}0\\A_1\end{pmatrix} + \nabla(\phi_1 w_1) \tag{7}$$

$$\tilde{\boldsymbol{J}} = \boldsymbol{J}_0 + \operatorname{curl}(\phi_2 T_2 \boldsymbol{e}_z) \tag{8}$$

lead to the weak form of the mixed multiscale formulation:

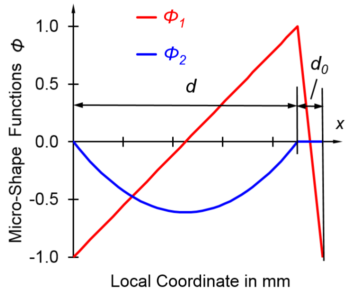

Fig. 2: Micro-shape functions.

1) Weak Form

Find $(\boldsymbol{A}_{0h}, A_{1h}, w_{1h}, \boldsymbol{J}_{0h}, T_{2h}) \in V_{h,\alpha} := \{(\boldsymbol{A}_{0h}, A_{1h}, w_{1h}, \boldsymbol{J}_{0h}, T_{2h}) : \boldsymbol{A}_{0h} \in \mathcal{U}_h, A_{1h} \in \mathcal{V}_h, w_{1h}$ and $T_{2h} \in \mathcal{W}_h, \boldsymbol{J}_{0h} \in \mathcal{M}_h$ and $\boldsymbol{A}_{0h} \times \boldsymbol{n} = \boldsymbol{\alpha}_h$ on $\Gamma\}$, such that

$$\int_\Omega \frac{1}{\mu} \operatorname{curl} \tilde{\boldsymbol{A}}_h \cdot \operatorname{curl} \tilde{\boldsymbol{v}}_h \, d\Omega - \int_\Omega \tilde{\boldsymbol{J}}_h \cdot \tilde{\boldsymbol{v}}_h \, d\Omega = 0 \quad (9)$$

$$-\int_\Omega \tilde{\boldsymbol{A}}_h \cdot \tilde{\boldsymbol{g}}_h \, d\Omega + \frac{j}{\omega} \int_\Omega \frac{1}{\sigma} \tilde{\boldsymbol{J}}_h \cdot \tilde{\boldsymbol{g}}_h \, d\Omega = 0 \quad (10)$$

$$p \int_\Omega \operatorname{div} \boldsymbol{J}_{0h} \operatorname{div} \boldsymbol{g}_{0h} \, d\Omega = 0 \quad (11)$$

for all $(\boldsymbol{v}_{0h}, v_{1h}, q_{1h}, \boldsymbol{g}_{0h}, t_{2h}) \in V_{h,0}$ with $\mathcal{U}_h \subset H(\operatorname{curl}, \Omega)$, $\mathcal{M}_h \subset H(\operatorname{div}, \Omega)$, $\mathcal{V}_h \subset L_2(\Omega_m)$, $\mathcal{W}_h \subset H^1(\Omega_m)$ and ϕ_1 and $\phi_2 \in H^1_{per}(\Omega_m)$ and for a sufficiently large $p \in \mathbb{R}$.

B. Mixed Multiscale Method for $3D$

The mixed multiscale approach for $3D$

$$\tilde{\boldsymbol{A}} = \boldsymbol{A}_0 + \phi_1 (0, A_{12}, A_{13})^T + \nabla(\phi_1 w_1) \quad (12)$$

$$\tilde{\boldsymbol{J}} = \boldsymbol{J}_0 + \operatorname{curl}(\phi_2 \boldsymbol{T}_2) \quad (13)$$

have been constructed leading to the weak form below.

1) Weak Form

Find $(\boldsymbol{A}_{0h}, A_{12h}, A_{13h}, w_{1h}, \boldsymbol{J}_{0h}, \boldsymbol{T}_{2h}) \in V_{h,\alpha} := \{(\boldsymbol{A}_{0h}, A_{12h}, A_{13h}, w_{1h}, \boldsymbol{J}_{0h}, \boldsymbol{T}_{2h}) : \boldsymbol{A}_{0h} \in \mathcal{U}_h, A_{12h}$ and $A_{13h} \in \mathcal{V}_h, w_{1h} \in \mathcal{W}_h, \boldsymbol{T}_{2h} \in \mathcal{U}_h, \boldsymbol{J}_{0h} \in \mathcal{M}_h$ and $\boldsymbol{A}_{0h} \times \boldsymbol{n} = \alpha_h$ on $\Gamma_B\}$, such that

$$\int_\Omega \frac{1}{\mu} \operatorname{curl} \tilde{\boldsymbol{A}}_h \cdot \operatorname{curl} \tilde{\boldsymbol{v}}_h \, d\Omega - \int_\Omega \tilde{\boldsymbol{J}}_h \cdot \tilde{\boldsymbol{v}}_h \, d\Omega = 0 \quad (14)$$

$$-\int_\Omega \tilde{\boldsymbol{A}}_h \cdot \tilde{\boldsymbol{g}}_h \, d\Omega + \frac{j}{\omega} \int_\Omega \frac{1}{\sigma} \tilde{\boldsymbol{J}}_h \cdot \tilde{\boldsymbol{g}}_h \, d\Omega = 0 \quad (15)$$

$$p \int_\Omega \operatorname{div} \boldsymbol{J}_{0h} \operatorname{div} \boldsymbol{g}_{0h} \, d\Omega = 0 \quad (16)$$

for all $(\boldsymbol{v}_{0h}, v_{12h}, v_{13h}, q_{1h}, \boldsymbol{g}_{0h}, \boldsymbol{t}_{2h}) \in V_{h,0}$ with $\mathcal{U}_h \subset H(\operatorname{curl}, \Omega)$, $\mathcal{V}_h \subset L_2(\Omega_m)$, $\mathcal{W}_h \subset H^1(\Omega_m)$, $\mathcal{M}_h \subset H(\operatorname{div}, \Omega)$, ϕ_1 and $\phi_2 \in H^1_{per}(\Omega_m)$ and for a sufficiently large $p \in \mathbb{R}$.

To consider the edge effect, $T_2 = 0$ in $2D$ and $\boldsymbol{T}_2 \times \boldsymbol{n} = \boldsymbol{0}$ in $3D$, respectively, have to be imposed along $\Gamma m0$.
The micro-scale currents $\operatorname{curl}(\phi_2 T_2 \boldsymbol{e}_z)$ and $\operatorname{curl}(\phi_2 \boldsymbol{T}_2)$, respectively, are divergence free per se. To get a divergence free macro-scale current density $\boldsymbol{J}_{0h}$ the penalty term (11) for $2D$ and (16) for $3D$, respectively, with the factor p are additionally required in the weak form.

IV. Numerical Examples

The mixed formulations reproduce the eddy current distribution with the edge effect accurately as can easily be seen in the Figs. 3 and 4 below. Coeffients haven't been averaged for the sake of accuracy [3].

A. Simulations in 2D

The solutions are extracted form the upper left corner of Ω_m, compare with Fig. 1. The magnetic field is perpendicular to the plane of projection. Unlike the standard mixed formulation the MMSFEM exhibits a pronounced edge effect, shown in Figs. 3.

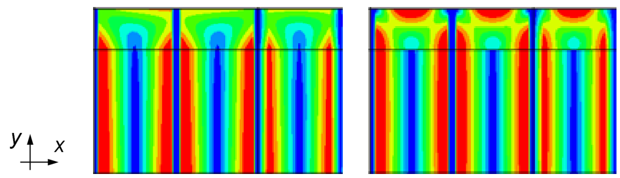

Fig. 3: Edge effect shown by $|\boldsymbol{J}|$: a) standard formulation $|\boldsymbol{J}_h|$ (left) and b) mixed formulation $|\tilde{\boldsymbol{J}}_h|$ (right).

B. Simulations in 3D

The problem consists of a laminated cube with 10 laminates immersed in a prescribed homogenous magnetic field perpendicular to a plane of symmetry. The field plots belong to the cross-section, which is also the considered plane of symmetry. The $3D$ simulations show the typical boundary layer behavior, i.e. the current density is much larger in the outer most laminate compared to the other ones.

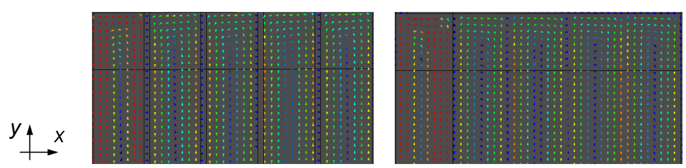

Fig. 4: Edge effect shown by $\boldsymbol{J}$: a) standard formulation $\boldsymbol{J}_h$ (left) and b) mixed formulation $\tilde{\boldsymbol{J}}_h$ (right).

Acknowledgment

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