Barrier Option

A barrier option is an option that behaves like a regular option unless the underlying asset’s price $S$ reaches a set barrier $H$ during the life of the option. The option has either a knock-in or a knock-out feature. Knock-in means that the option is worthless unless the asset price reaches the barrier. Knock-out means that the option becomes worthless as soon as the asset price reaches the barrier. Up and down indicate the direction in which the barrier has to be crossed.

The input arguments used for the valuation of barrier options are, besides $S$ and $H$, the strike price $X$, the interest rate $r$, the time to maturity $T$, the dividend rate $q$ and the volatility $\sigma$.

If $H \leq X$ the current value of a **down-and-in call** is

$$c_{di} = Se^{-qt}(H/S)^{2\lambda}N(y) - Xe^{-rT}(H/S)^{2\lambda-2}N(y - \sigma\sqrt{T})$$

where

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

and

$$y = \frac{\ln \left( \frac{H^2}{(SX)} \right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

and the value of a **down-and-out call** is given by

$$c_{do} = c - c_{di}.$$

If $H > X$, then

$$c_{do} = SN(x_1)e^{-qt} - Xe^{-rT}N(x_1 - \sigma\sqrt{T}) - Se^{-qt}(H/S)^{2\lambda}N(y_1)$$

$$+ Xe^{-rT}(H/S)^{2\lambda-2}N(y_1 - \sigma\sqrt{T})$$

and

$$c_{di} = c - c_{do}$$

with

$$x_1 = \frac{\ln(S/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

and

$$y_1 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}.$$

When $H \leq X$, the values of an **up-and-in call** and an **up-and-out call** are

$$c_{uo} = 0$$

and

$$c_{ui} = c.$$
and 
\[ c_{uo} = c - c_{ui}. \]

A put that is up-and-in and up-and-out respectively has the following value, if \( H \geq X \):
\[
p_{ui} = -Se^{-qT}(H/S)^{2\lambda}N(-y) + Xe^{-rT}(H/S)^{2\lambda-2}N(-y + \sigma\sqrt{T})
\]
and
\[
p_{uo} = p - p_{ui}.
\]

When \( H \leq X \),
\[
p_{uo} = -SN(-x_1)e^{-qT} + Xe^{-rT}N(-x_1 + \sigma\sqrt{T}) + Se^{-qT}(H/S)^{2\lambda}N(-y_1)
-Xe^{-rT}(H/S)^{2\lambda-2}N(-y_1 + \sigma\sqrt{T})
\]
and
\[
p_{ui} = p - p_{uo}.
\]

A down-and-out put has the following value, if \( H > X \):
\[
p_{do} = 0
\]
and
\[
p_{di} = p
\]
is the value of an down-and-in put. When \( H < X \),
\[
p_{di} = -SN(-x_1)e^{-qT} + Xe^{-rT}N(-x_1 + \sigma\sqrt{T}) + Se^{-qT}(H/S)^{2\lambda}(N(y) - N(y_1))
-Xe^{-rT}(H/S)^{2\lambda-2}(N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T}))
\]
and
\[
p_{do} = p - p_{di}.
\]

For more details see J.C. Hull: Options, Futures, And Other Derivatives.