GLOBAL MARTINGALE SOLUTIONS FOR A STOCHASTIC SHIGESADA–KAWASAKI–TERAMOTO POPULATION MODEL

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Abstract. The existence of global nonnegative martingale solutions to a cross-diffusion system of Shigesada–Kawasaki–Teramoto type with multiplicative noise is proven. The model describes the segregation dynamics of populations with an arbitrary number of species. The diffusion matrix is generally neither symmetric nor positive semidefinite, which excludes standard methods. Instead, the existence proof is based on the entropy structure of the model, approximated by a Wong–Zakai argument, and on suitable higher moment estimates and fractional time regularity. In the case without self-diffusion, the lack of regularity is overcome by carefully exploiting the entropy production terms.

1. Introduction

Shigesada, Kawasaki, and Teramoto (SKT) suggested in their seminal paper [29] a deterministic cross-diffusion system for two competing species, which is able to describe the segregation of the populations. A random influence of the environment or the lack of knowledge of certain biological parameters motivate the introduction of noise terms, leading to the system

$$\begin{aligned} &\mathrm{d}u_1 - \Delta\big(a_{10}u_1 + a_{11}u_1^2 + a_{12}u_1u_2\big)\mathrm{d}t = \sigma_1(u_1)\mathrm{d}W_1(t), \\ &\mathrm{d}u_2 - \Delta\big(a_{20}u_2 + a_{21}u_1u_2 + a_{22}u_2^2\big)\mathrm{d}t = \sigma_2(u_2)\mathrm{d}W_2(t) \quad \text{in } \mathcal{O},\ t>0, \end{aligned} \tag{1}$$

where $u_i = u_i(\omega, x, t)$ describes the density of the ith species ($i = 1, 2$), $\omega \in \Omega$ is the stochastic variable, $x \in \mathcal{O}$ is the spatial variable, and $t \geq 0$ is the time, $a_{ij} \geq 0$ are some parameters, (W_1, W_2) is a two-dimensional Wiener process, and $\mathcal{O} \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain. An admissible example of the stochastic diffusion term is

$$\sigma_i(u_i) = \frac{u_i}{1 + u_i^{1-\gamma}}, \quad \text{where } 0 < \gamma \leq 1,\ i = 1, 2. \tag{2}$$

Details on the stochastic framework will be given in Section 2. The equations are supplemented with initial and no-flux boundary conditions (see (4) below). The original system in [29] also contains a deterministic environmental potential, which are neglected here for simplicity.

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The key difficulty of system (1) is the fact that the diffusion matrix associated to (1) is generally neither symmetric nor positive semidefinite. In particular, standard semigroup theory is not applicable. These issues have been overcome in [8, 9] in the deterministic case by revealing a formal gradient-flow or entropy structure. The task is to extend this idea to the stochastic setting.

The aim of this work is to prove the existence of global nonnegative martingale solutions to system (1). The paper is a continuation of our previous works [14, 15]. The work [14] was concerned with a SKT-type system, in which the coefficients of the associated diffusion matrix depend quadratically on the densities (and not linearly as in (1)). This allowed us to work in a Hilbert space framework, leading to a novel result through a standard approach. The paper [15] exploited the entropy structure of a general class of cross-diffusion systems with volume filling, leading to bounded martingale solutions. The L^∞ bound follows from the entropy structure and implies higher moment estimates, which were further used to establish the tightness of laws. Unfortunately, this idea does not work for system (1), since the entropy structure is different and L^∞ bounds cannot be expected. Therefore, we need to develop new estimates to overcome this issue.

In fact, we prove the existence of martingale solutions to a SKT-type system involving an arbitrary number of species. We consider

$$\mathrm{d}u_i - \Delta\bigg(a_{i0}u_i + \sum_{j=1}^n a_{ij}u_iu_j\bigg)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u)\mathrm{d}W_j(t) \quad \text{in } \mathcal{O},\ t>0,\ i=1,\ldots,n, \tag{3}$$

with the initial and no-flux boundary conditions

$$u_i(0) = u_i^0 \quad \text{in } \mathcal{O}, \quad \nabla\bigg(a_{i0}u_i + \sum_{j=1}^n a_{ij}u_iu_j\bigg)\cdot\nu = 0 \quad \text{on } \partial\mathcal{O},\ t>0,\ i=1,\ldots,n. \tag{4}$$

Here, $a_{ij}\ge 0$ for $i=1,\ldots,n$ and $j=0,\ldots,n$, $u=(u_1,\ldots,u_n)$ is the vector of population densities, $(W_1,\ldots,W_n)$ is an n-dimensional Wiener process, ν is the exterior unit normal vector to $\partial\mathcal{O}$, and u_i^0 is a possibly random initial datum. We call a_{i0} the diffusion coefficients, a_{ii} the self-diffusion coefficients, and a_{ij} for $i\neq j$ the cross-diffusion coefficients. We say that system (3)-(4) is *with self-diffusion* if $a_{i0}\ge 0$, $a_{ii}>0$ for all $i=1,\ldots,n$, and it is *without self-diffusion* if $a_{i0}>0$, $a_{ii}=0$ for all $i=1,\ldots,n$.

The deterministic analog of (3) was formally derived from a random-walk lattice model in [32] and rigorously derived from a nonlocal population system in the triangular case in [27] and from interacting particle systems in the general case in [7].

Equations (3) can be written as

$$\mathrm{d}u_i - \operatorname{div}\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j\bigg)\mathrm{d}t = \sum_{j=1}^n \sigma_{ij}(u)\mathrm{d}W_j(t) \quad \text{in } \mathcal{O},\ t>0,\ i=1,\ldots,n,$$

with the diffusion matrix $A(u)=(A_{ij}(u))$, where

$$A_{ij}(u) = \delta_{ij}\bigg(a_{i0} + \sum_{k=1}^n a_{ik}u_k\bigg) + a_{ij}u_i, \quad i,j=1,\ldots,n. \tag{5}$$

As mentioned above, the main difficulty that this matrix is generally neither symmetric nor positive semidefinite is overcome by exploiting the entropy structure. This means that there exists a function $h : [0,\infty)^n \to [0,\infty)$, called an entropy density, such that the deterministic analog of (3) can be written in terms of the entropy variables (or chemical potentials) $w_i = \partial h/\partial u_i$ as

$$\partial_t u_i(w) - \operatorname{div}\bigg(\sum_{j=1}^n B_{ij}(w)\nabla w_j\bigg) = 0, \quad i = 1\ldots,n, \tag{6}$$

where $w = (w_1,\ldots,w_n)$, u_i depends on w, and $B(w) = A(u(w))h''(u(w))^{-1}$ with $B = (B_{ij})$ turns out to be positive semidefinite. For the deterministic analog of (3), it was shown in [10] that the entropy density is given by

$$h(u) = \sum_{i=1}^n \pi_i\big(u_i(\log u_i - 1) + 1\big), \tag{7}$$

where the numbers $\pi_i > 0$ are assumed to satisfy $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i,j = 1,\ldots,n$. This condition is the detailed-balance condition for the Markov chain associated to (a_{ij}), and $(\pi_1,\ldots,\pi_n)$ is the corresponding reversible stationary measure. Using w_i in (6) as a test function and summing over $i = 1,\ldots,n$, a formal computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} h(u)\mathrm{d}x + 2\int_{\mathcal{O}}\sum_{i=1}^n \pi_i\bigg(2a_{i0}|\nabla\sqrt{u_i}|^2 + a_{ii}|\nabla u_i|^2 + \sum_{j\neq i} a_{ij}|\nabla\sqrt{u_iu_j}|^2\bigg)\mathrm{d}x = 0. \tag{8}$$

A similar expression holds in the stochastic setting; see Lemma 5. It provides gradient estimates for $\sqrt{u_i}$ if $a_{i0} > 0$ and for u_i if $a_{ii} > 0$. Moreover, having proved the existence of a solution w to an approximate version of (3) leads to the positivity of $u_i(w) = \exp(w_i/\pi_i)$ (and nonnegativity after passing to the de-regularization limit).

In the stochastic setting, we face some technical obstacles due to Itô's lemma and the treatment of the multiplicative noise. Our idea, first used in [15], is to replace the Wiener process by a Wong–Zakai approximation and to discretize the equations by a stochastic Galerkin method. We apply a variant of the boundedness-by-entropy method [21] to the resulting system of differential equations, providing the positivity of the approximate population densities and a priori estimates uniform in the Galerkin dimension. The limit of vanishing Wong–Zakai parameter requires the existence of solutions to another Galerkin approximation, leading to strong solutions up to a stopping time. The tightness of the laws of the approximate solutions follows from the uniform estimates, and the Skorokhod–Jakubowski theorem implies the pointwise convergence of the sequence of approximate solutions.

The difference in the strategy of our proof, compared to [15], becomes apparent in the final steps. First, we need higher moment estimates, which followed in [15] from the $L^\infty(\mathcal{O})$ bound, but here we have less regularity. Second, we are proving a new fractional time regularity result, needed to show the tightness of the laws in $L^2(\mathcal{O})$. A further difference to [15] comes from the low regularity of the gradients when self-diffusion vanishes. Indeed, the regularity for $\nabla\sqrt{u_i}$ in L^2 from (8) does not allow us to define products like u_iu_j. The idea is to exploit the Laplace structure in (3) and the L^2 regularity for $\nabla\sqrt{u_iu_j}$ coming from the entropy estimate (8).

As a corollary of our main results stated in Section 2, we obtain the existence of a global martingale solution to (1) with the particular stochastic noise term (2).

Corollary 1 (Martingale solutions to the SKT model)**.** *Let $\mathcal{O}\subset\mathbb{R}^3$ be a bounded domain with smooth boundary and let either $a_{11}>0$, $a_{22}>0$, $\gamma\le 1$ or $a_{10}>0$, $a_{20}>0$, $\gamma\le 2/5$. Then there exists a global nonnegative martingale solution to* (1), (2), *and* (4).

Deterministic cross-diffusion systems of SKT type with two species have been intensively studied in the literature. First existence results were proven under restrictive conditions on the parameters, for instance in one space dimension [23], for the triangular system with $a_{21}=0$ [26], or for small cross-diffusion parameters, since in the latter situation the diffusion matrix is positive definite [13]. Amann [1] proved that a priori estimates in the $W^{1,p}(\mathcal{O})$ norm with $p>d$ are sufficient to prove the global existence of solutions to quasilinear parabolic systems, and he applied this result to the triangular SKT system. The first global existence proof without any restriction on the parameters a_{ij} (except nonnegativity) was achieved in [19] in one space dimension. This result was generalized to several space dimensions in [8, 9] and to the whole space problem in [16]. SKT-type systems with nonlinear coefficients $A_{ij}(u)$, but still for two species, were analyzed in [11, 12]. Global existence results for SKT-type models with an arbitrary number of species and under a detailed-balance condition were first proved in [10] and later generalized in [25].

There are only very few results for stochastic SKT-type systems. The first result for global martingale solutions needed quadratic diffusion coefficients, since this allows one to work in a Hilbert space framework [14]. A stronger solution concept was used in [24], leading to local-in-time pathwise mild solutions, but only for positive definite diffusion matrices $A(u)$.

The paper is organized as follows. The stochastic framework and main results are given in Section 2. In Section 3, the existence of approximate solutions is proved and uniform bounds are derived from the entropy estimate. The existence of global martingale solutions is proved in Section 4 in the case with self-diffusion and in Section 5 in the case without self-diffusion. Estimates for the deterministic SKT system without self-diffusion, which are needed for the approximate stochastic problem, are derived in Appendix A. As a by-product, we obtain an existence result for the deterministic SKT system without self-diffusion with a simpler proof and for a more general situation compared to [9].

2. Notation and main result

2.1. **Notation and stochastic framework.** Let $\mathcal{O}\in\mathbb{R}^d$ be a bounded domain. The Lebesgue and Sobolev spaces are denoted by $L^p(\mathcal{O})$ and $W^{k,p}(\mathcal{O})$, respectively, where $p\in[1,\infty]$, $k\in\mathbb{N}$, and we set $H^k(\mathcal{O})=W^{2,k}(\mathcal{O})$. We write $\|u\|_{L^2(\mathcal{O})}^2=\sum_{i=1}^n\|u_i\|_{L^2(\mathcal{O};\mathbb{R}^n)}^2$ for functions $u=(u_1,\ldots,u_n)\in L^2(\mathcal{O})$ and use this notation in related situations. We write $\langle\cdot,\cdot\rangle$ for the dual product between a Banach space and its dual. If $u\in L^p(\mathcal{O})$, $v\in L^q(\mathcal{O})$ with $1/p+1/q=1$ and $p>1$, we have $\langle u,v\rangle=\int_{\mathcal{O}}uv\mathrm{d}x$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a complete right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The space $L^2(\Omega; H)$ for a Hilbert space H consists of all H-valued random variables u such that $\mathbb{E}\|u\|_H^2 = \int_\Omega \|u(\omega)\|_H^2 \mathbb{P}(\mathrm{d}\omega) < \infty$. Let $(\eta_k)_{k=1,\ldots,n}$ be the canonical basis of $\mathbb{R}^n$. The space of Hilbert–Schmidt operators from $\mathbb{R}^n$ to $L^2(\mathcal{O})$ is defined by

$$\mathcal{L}_2(\mathbb{R}^n; L^2(\mathcal{O})) = \left\{ L : \mathbb{R}^n \to L^2(\mathcal{O}) \text{ linear continuous} : \sum_{k=1}^n \|L\eta_k\|_{L^2(\mathcal{O})}^2 < \infty \right\}$$

and endowed with the norm $\|L\|^2_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))} = \sum_{k=1}^n \|L\eta_k\|^2_{L^2(\mathcal{O})}$. The stochastic diffusion $\sigma = (\sigma_{ij}) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is assumed to be $\mathcal{B}(L^2(\mathcal{O};\mathbb{R}^n))/\mathcal{B}(\mathcal{L}_2(\mathbb{R}^n; L^2(\mathcal{O})))$-measurable and $\mathbb{F}$-adapted.

2.2. **Assumptions and main result.**

We impose the following assumptions:

(A1) Domain: $\mathcal{O} \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with $\partial\mathcal{O} \in C^\ell$ and $\ell \in \mathbb{N}$ satisfies $\ell > d/2 + 2$. Let $T > 0$ and set $Q_T = \mathcal{O} \times (0, T)$.

(A2) Initial datum: $u^0 \in L^r(\Omega; L^2(\mathcal{O}; \mathbb{R}^n))$ for $r > (2/d) \max\{8, d + 2\}$ is $\mathcal{F}_0$-measurable and $u_i^0 \geq 0$ for a.e. $x \in \mathcal{O}$ $\mathbb{P}$-a.s., $i = 1, \ldots, n$.

(A3) Diffusion matrix: $a_{ij} \geq 0$ for $i = 1, \ldots, n$, $j = 0, \ldots, n$ and the detailed-balance condition is satisfied, i.e., there exist numbers $\pi_1, \ldots, \pi_n > 0$ such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i, j = 1, \ldots, n. \tag{9}$$

(A4) Multiplicative noise: $\sigma : L^2(\mathcal{O};\mathbb{R}^n) \to \mathcal{L}_2(\mathbb{R}^n; L^2(\mathcal{O};\mathbb{R}^n))$ satisfies $\sigma_{ij}(u) = 0$ for all $u \in [0, \infty)$ with $u_i = 0$, and there is a constant $C_\sigma > 0$ such that for any u, $v \in L^2(\mathcal{O})$,

$$\begin{aligned} \|\sigma(u) - \sigma(v)\|_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))} &\leq C_\sigma \|u - v\|_{L^2(\mathcal{O})}, \\ \|\sigma(u)\|_{\mathcal{L}_2(\mathbb{R}^n;L^2(\mathcal{O}))} &\leq C_\sigma \big(1 + \|u\|^\gamma_{L^2(\mathcal{O})}\big), \end{aligned}$$

where $0 < \gamma \leq 1$ (with self-diffusion) or $0 < \gamma \leq 2/d$ (without self-diffusion).

(A5) Interaction of entropy density and noise: There exists $C_h > 0$ such that for all $u \in (0, \infty)^n$,

$$\begin{aligned} \max_{j=1,\ldots,n} \left| \sum_{i=1}^n \sigma_{ij}(u) \log u_i \right|^2 + \left| \sum_{i,j,k=1}^n \frac{\partial \sigma_{ij}}{\partial u_k}(u) \sigma_{jk}(u) \log u_i \right| & \\ + \left| \sum_{i,j=1}^n \frac{\sigma_{ij}(u)^2}{u_j} \right| \leq C_h \bigg(1 + \sum_{i=1}^n u_i(\log u_i - 1)\bigg)&. \end{aligned}$$

Furthermore, $u \mapsto \sum_{i,j,k=1}^n (\partial\sigma_{ij}/\partial u_k(u)\sigma_{jk}(u)$ is assumed to be Lipschitz continuous.

Let us discuss these assumptions. The boundary regularity in Assumption (A1) is used to define a Galerkin space embedded into $W^{2,\infty}(\mathcal{O})$, thus avoiding issues with the regularity of the diffusion coefficients. Assumption (A2) on the initial datum can be relaxed, since we only need the integrability of $u_i^0 \log u_i^0$. The detailed-balance condition in Assumption (A3) is needed to derive the entropy inequality, which provides a priori estimates (see Lemma 5). We may replace this assumption by

$$a_{ii} > \frac{1}{4}\sum_{j=1}^n \big(\sqrt{a_{ij}} - \sqrt{a_{ji}}\big)^2, \quad a_{ii} > 0 \quad \text{for } i = 1,\ldots,n,$$

which expresses that self-diffusion dominates cross-diffusion [10, Lemma 6]. The Lipschitz continuity of the stochastic diffusion term in Assumption (A4) in the case with self-diffusion is a standard condition for stochastic PDEs; see, e.g., [28]. Without self-diffusion, we have less regularity and therefore we need a sublinear condition for σ. The condition that σ_{ij} vanishes at $u_i = 0$ ensures the nonnegativity of the solution. Assumption (A5) allows us to compensate the singularity at zero for $\partial h/\partial u_i = \pi_i \log u_i$ when we derive the entropy estimate. For instance, the stochastic diffusion term

$$\sigma_{ij}(u) = \frac{u_i\delta_{ij}}{1 + u_i^{1-\gamma}} \quad \text{for all } u \in [0,\infty)^n,\ i = 1,\ldots,n,$$

satisfies Assumptions (A4)–(A5). Only finite-dimensional Wiener processes instead of infinite-dimensional ones are considered, because the specific structure of the interaction between the entropy density and stochastic diffusion in Assumption (A5) becomes clearer.

Theorem 2 (Existence, with self-diffusion)**.** *Let Assumptions (A1)–(A5) hold, $T > 0$, and let $a_{ii} > 0$ for all $i = 1,\ldots,n$. Then there exists a global martingale solution to* (3)–(4) *satisfying $\widetilde{u}_i(x,t) \ge 0$ a.e. in Q_T $\widetilde{\mathbb{P}}$-a.s., $i = 1,\ldots,n$. More precisely, there exists a triple $(\widetilde{U},\widetilde{W},\widetilde{u})$ such that $\widetilde{U} = (\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}},\widetilde{\mathbb{F}})$ is a stochastic basis with filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$, $\widetilde{W}$ is an $\mathbb{R}^n$-valued Wiener process on this filtered probability space, and $\widetilde{u}(t) = (\widetilde{u}_1(t),\ldots,\widetilde{u}_n(t))$ is a progressively measurable stochastic process for all $t \in [0,T]$ such that for $i = 1,\ldots,n$,*

$$\widetilde{u}_i \in L^2(\widetilde{\Omega}; C^0([0,T]; H^m(\mathcal{O})')) \cap L^2(\widetilde{\Omega}; L^2(0,T; H^1(\mathcal{O}))),$$

where $m > d/2+1$, the law of $\widetilde{u}_i(0)$ is the same as for u_i^0, and $\widetilde{u}$ satisfies for all $\phi_i \in H^m(\mathcal{O})$ and $i = 1,\ldots,n$,

$$\begin{aligned}\langle \widetilde{u}_i(t),\phi_i\rangle &= \langle \widetilde{u}_i(0),\phi_i\rangle + \int_0^t\int_{\mathcal{O}} \nabla\bigg(a_{i0}\widetilde{u}_i(s) + \sum_{j=1}^n a_{ij}\widetilde{u}_i(s)\widetilde{u}_j(s)\bigg)\cdot\nabla\phi_i \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=1}^n \int_{\mathcal{O}}\bigg(\int_0^t \sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s)\bigg)\phi_i\mathrm{d}x, \end{aligned} \tag{10}$$

where $\langle\cdot,\cdot\rangle$ is the dual product between $H^m(\mathcal{O})'$ and $H^m(\mathcal{O})$.

Theorem 3 (Existence, without self-diffusion)**.** *Let Assumptions (A1)–(A5) hold, $T > 0$, $d \le 3$, and let $a_{i0} > 0$, $a_{ii} = 0$ for all $i = 1,\ldots,n$. Then there exists a global martingale solution to* (3)–(4)*, satisfying $\widetilde{u}_i(x,t) \ge 0$ a.e. in Q_T $\widetilde{\mathbb{P}}$-a.s., $i = 1,\ldots,n$. More precisely, there exists a triple $(\widetilde{U},\widetilde{W},\widetilde{u})$, where $\widetilde{U}$ and $\widetilde{W}$ are as in Theorem 2, $\widetilde{u}(t) =$*

$(\widetilde{u}_1(t),\ldots,\widetilde{u}_n(t))$ *is a progressively measurable stochastic process for all* $t\in[0,T]$ *such that for* $i=1,\ldots,n$, $j\neq i$,

$$\widetilde{u}_i \in L^2(\widetilde{\Omega};C^0([0,T];H^3(\mathcal{O})')) \cap L^2(\widetilde{\Omega};L^{8/7}(0,T;W^{1,8/7}(\mathcal{O}))),$$
$$\widetilde{u}_i\widetilde{u}_j \in L^2(\widetilde{\Omega};L^{8/7}(0,T;W^{1,8/7}(\mathcal{O}))),$$

the law of $\widetilde{u}_i(0)$ *is the same as for* u_i^0, *and* $\widetilde{u}$ *satisfies the weak formulation* (10) *for all* $\phi_i\in H^3(\mathcal{O})$.

3. Approximate scheme and entropy estimates

We prove Theorems 2 and 3 by approximating system (3) by a stochastic Galerkin method, using the Wong–Zakai approximation of the Wiener process, and deriving some entropy estimates.

3.1. **Stochastic Galerkin approximation.** The existence of a strong (in the probability sense) solution to the Galerkin approximation up to a stopping time is proved by using the Banach fixed-point theorem. For this, we project (3) onto the finite-dimensional Hilbert space $H_N=\operatorname{span}\{e_1,\ldots,e_N\}$, where $N\in\mathbb{N}$ and $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{O})$ such that $H_N\subset W^{2,\infty}(\mathcal{O})$. For instance, (e_j) may consist of the eigenfunctions of $-\Delta$ on $\mathcal{O}$ with homogeneous Neumann boundary conditions. At this point, we need the regularity of $\partial\mathcal{O}$ to ensure that $e_j\in H^\ell(\mathcal{O})\hookrightarrow W^{2,\infty}(\mathcal{O})$ (which requires that $\ell>d/2+2$). Furthermore, let $\Pi_N:L^2(\mathcal{O})\to H_N$, $\Pi_N(v)=\sum_{k=1}^N\langle v,e_k\rangle e_k$ for $v\in L^2(\mathcal{O})$ be the projection onto H_N.

The approximate problem is the following system of stochastic differential equations,

$$\mathrm{d}u_i^{(N)} = \Pi_N\Delta\bigg(a_{i0}u_i^{(N)}+\sum_{j=1}^n a_{ij}u_i^{(N)}u_j^{(N)}\bigg)\mathrm{d}t + \sum_{j=1}^n \Pi_N\sigma_{ij}(u^{(N)})\mathrm{d}W_j(t) \tag{11}$$

for $i=1,\ldots,n$, with the initial conditions

$$u_i^{(N)}(0)=\Pi_N(u_i^0),\quad i=1,\ldots,n. \tag{12}$$

Given $T>0$, we introduce the space $X_T=L^2(\Omega;C^0([0,T];H_N))$ with the norm $\|u\|_{X_T}^2=\mathbb{E}(\sup_{0<t<T}\|u(t)\|_{H_N})^2$. For given $R>0$ and $u\in X_T$, we define the exit time $\tau_R:=\inf\{t\in[0,T]:\|u(t)\|_{L^2(\mathcal{O})}>R\}$. Then $\{\omega\in\Omega:\tau_R(\omega)>t\}$ belongs to $\mathcal{F}_t$ for every $t\in[0,T]$ and τ_R is an $\mathbb{F}$-stopping time. We define the fixed-point operator $S:X_T\to X_T$ by

$$\langle S_N(u)(t),\phi\rangle = \langle u^0,\phi\rangle - \int_0^t\bigg\langle\bigg(a_{i0}u_i+\sum_{j=1}^n a_{ij}u_iu_j\bigg),\Delta\phi_i\bigg\rangle\mathrm{d}s$$
$$+\int_0^t\sum_{j=1}^n\langle\sigma_{ij}(u)\mathrm{d}W_j(s),\phi_i\rangle,$$

for $u\in X_T$ and $\phi_i\in H_N$ satisfying $\nabla\phi_i\cdot\nu=0$ on $\partial\mathcal{O}$. Note that $\langle u^0,\phi\rangle=\sum_{i=1}^n\langle\Pi_N(u_i^0),\phi_i\rangle$ since $\phi\in H_N^n$.

We claim that $S:X_T\to X_T$ is a self-mapping and a contraction. The proof is similar to that one for [15, Prop. 4]. The main difference is that the definition of the stopping time is based here on the $L^2(\mathcal{O})$ norm, while the $H^1(\mathcal{O})$ norm was used in [15]. To compensate the weaker norm, we exploit the Laplace structure of (3). Indeed, for the self-mapping property, we need to verify that $\|S(u)\|_{X_{T\wedge\tau_R}} \le C(\|u\|_{X_{T\wedge\tau_R}})$. Since only the (normally) elliptic term is different, it is sufficient to estimate

$$\begin{aligned}
\mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R}\bigg|\int_0^t\bigg\langle\bigg(a_{i0}u_i(s)+\sum_{j=1}^n a_{ij}u_i(s)u_j(s)\bigg),\Delta\phi_i\bigg\rangle\mathrm{d}s\bigg|\bigg)^2 \\
\le T^2(1+C(R))\|\Delta\phi_i\|^2_{L^\infty(\mathcal{O})}\mathbb{E}\|u(t)\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} \\
\le C(T,R)\|\phi\|^2_{H_N}\mathbb{E}\|u(t)\|^2_{L^\infty(0,T;L^2(\mathcal{O}))},
\end{aligned}$$

and the other terms are essentially treated as in the proof of [15, Prop. 4]. For the contraction property, we need to estimate the difference $S(u)-S(v)$ for u, $v\in X_{T\wedge\tau_R}$. It is sufficient to consider the term

$$\begin{aligned}
\mathbb{E}\bigg(\sup_{0<t<T\wedge\tau_R}\bigg|\int_0^t\bigg\langle a_{i0}(u_i-v_i)(s)+\sum_{j=1}^n a_{ij}\big((u_i-v_i)u_j+v_i(u_j-v_j)\big)(s)\bigg),\Delta\phi_i\bigg\rangle\mathrm{d}s\bigg|\bigg)^2 \\
\le C(R)T\|\Delta\phi_i\|^2_{L^\infty(\mathcal{O})}\mathbb{E}\int_0^{T\wedge\tau_R}\|u(s)-v(s)\|^2_{L^2(\mathcal{O})}\mathrm{d}s \\
\le C(N,R)T^2\|\phi\|^2_{H_N}\|u-v\|^2_{X_{T\wedge\tau_R}}.
\end{aligned}$$

The remaining terms are estimated as in the proof of [15, Prop. 4]. This leads to

$$\|S(u)-S(v)\|_{X_{T\wedge\tau_R}} \le C(N,R)T\|u-v\|_{X_{T\wedge\tau_R}},$$

showing that $S: X_{T^*}\to X_{T^*}$ is a contraction for $0<T^*<T\wedge\tau_R$ satisfying $C(N,R)T^*<1$. This shows that (11)–(12) possesses a unique solution $u^{(N)}$ up to the stopping time τ_R.

3.2. **Wong–Zakai approximation.** We prove the existence of global-in-time solutions to another approximate system of (3) by replacing the Wiener process by the Wong–Zakai approximation, leading to a system of ordinary differential equations. This step is necessary to obtain the nonnegativity of the solutions $u_i^{(N)}$ constructed in the previous subsection.

We project (3) as in the previous subsection onto the Galerkin space H_N and introduce a uniform partition of the time interval $[0,T]$ with time step $\eta=T/M$, where $M\in\mathbb{N}$. We set $t_k=k\eta$ for $k=0,\ldots,M$. The Wiener process is approximated by the process [31]

$$W_j^{(\eta)}(t)=W_j(t_k)+\frac{t-t_k}{\eta}\big(W_j(t_{k+1})-W_j(t_k)\big),\quad t\in[t_k,t_{k+1}],\ k=0,\ldots,M.$$

The approximate system is given by

$$\frac{\mathrm{d}u^{(N,\eta)}}{\mathrm{d}t}=\Pi_N\operatorname{div}\bigg(\sum_{j=1}^n A_{ij}(u^{(N,\eta)})\nabla u_j^{(N,\eta)}\bigg)+f_i(u^{(N,\eta)},t),\quad\text{where} \tag{13}$$

$$f(u^{(N,\eta)},t)=\sum_{j=1}^n\Pi_N\big(\sigma_{ij}(u^{(N,\eta)})\big)\frac{\mathrm{d}W_j^{(\eta)}}{\mathrm{d}t}(t)-\frac12\Pi_N\sum_{j,k=1}^n\frac{\partial\sigma_{ij}}{\partial u_k}(u^{(N,\eta)})\sigma_{kj}(u^{(N,\eta)})$$

with the initial condition $u^{(N,\eta)}(0)=\Pi_N(u_i^0)$. System (13) can be written in the weak form

$$\begin{aligned}\langle u_i^{(N,\eta)}(t),\phi_i\rangle &= \langle u_i^0,\phi_i\rangle - \int_0^t\sum_{j=1}^n\big\langle A_{ij}(u^{(N,\eta)}(s))\nabla u_j^{(N,\eta)}(s),\nabla\phi_i\big\rangle\mathrm{d}s \\ &\quad + \int_0^t\sum_{j=1}^n\bigg\langle\sigma_{ij}(u^{(N,\eta)}(s))\frac{\mathrm{d}W_j^{(\eta)}}{\mathrm{d}t}(s),\phi_i\bigg\rangle\mathrm{d}s \\ &\quad - \frac12\int_0^t\bigg\langle\sum_{j,k=1}^n\frac{\partial\sigma_{ij}}{\partial u_k}(u^{(N,\eta)}(s))\sigma_{kj}(u^{(N,\eta)}(s)),\phi_i\bigg\rangle\mathrm{d}s \qquad (14)\end{aligned}$$

for any $\phi_i\in H_N$. The last term is needed, since the Wong–Zakai approximation converges to the Stratonovich noise that is related to the Itô noise by

$$\sum_{j=1}^n\sigma_{ij}(u)\circ\mathrm{d}W_j(t)=\sum_{j=1}^n\sigma_{ij}(u)\mathrm{d}W_j(t)+\frac12\sum_{j,k=1}^n\frac{\partial\sigma_{ij}}{\partial u_k}(u)\sigma_{kj}(u)\mathrm{d}t.$$

We need to distinguish the cases with and without self-diffusion. First, if $a_{ii}>0$ for $i=1,\ldots,n$, it follows from the techniques of [10] (see [15, Prop. 5] for details) that for a.e. $\omega\in\Omega$, there exists a global-in-time weak solution $u^{(N,\eta)}$ to (14) satisfying $u_i^{(N,\eta)}(\omega,\cdot,\cdot)\geq 0$ a.e. in $\Omega\times(0,T)$,

$$\begin{aligned}&u_i^{(N,\eta)}(\omega,\cdot,\cdot)\in L^2(0,T;H^1(\mathcal{O}))\cap L^\infty(0,T;L^1(\mathcal{O}))\cap L^{2+2/d}(Q_T),\\ &\partial_t u_i^{(N,\eta)}(\omega,\cdot,\cdot)\in L^{\rho_2}(0,T;W^{1,\rho_2}(\mathcal{O})'),\quad i=1,\ldots,n,\end{aligned}$$

where $\rho_2=(2d+2)/(2d+1)$, $u^{(N,\eta)}(0)=u^0$ in the sense of $W^{1,2d+2}(\mathcal{O})'$, and (14) is satisfied.

Second, if $a_{i0}>0$, $a_{ii}=0$ for $i=1,\ldots,n$, we conclude from the techniques of [9] that for a.e. $\omega\in\Omega$, there exists a global-in-time weak solution $u^{(N,\eta)}$ to (14) satisfying $u_i^{(N,\eta)}(\omega,\cdot,\cdot)\geq 0$ a.e. in $\Omega\times(0,T)$ and the weak formulation (14). However, we obtain less regularity:

$$\begin{aligned}&u_i^{(N,\eta)}(\omega,\cdot,\cdot)\in L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O}))\cap L^\infty(0,T;L^1(\mathcal{O}))\cap L^{1+2/d}(Q_T),\\ &\partial_t u_i^{(N,\eta)}(\omega,\cdot,\cdot)\in L^{\rho_2}(0,T;W^{1,\rho_2}(\mathcal{O})'),\\ &(u_i^{(N,\eta)}u_j^{(N,\eta)})(\omega,\cdot,\cdot)\in L^{\rho_2}(0,T;W^{1,\rho_2}(\mathcal{O})),\end{aligned}$$

for $i=1,\ldots,n$, $j\neq i$, where $\rho_1=(d+2)/(d+1)$; see [9] and the Appendix. In the weak formulation (14), we interpret the expression $\sum_{j=1}^n A_{ij}(u^{(N,\eta)})\nabla u^{(N,\eta)}$ here as

$$a_{i0}\nabla u_i^{(N,\eta)}+\sum_{j=1,\,j\neq i}^n a_{ij}\nabla(u_i^{(N,\eta)}u_j^{(N,\eta)}).$$

The nonnegativity of $u_i^{(N,\eta)}$ is a consequence of the entropy method (see, e.g., [10]) applied to the weak formulation (14). This formulation is important since the initial datum associated to the strong formulation (13) is projected to the Galerkin space and $\Pi_N(u_i^0)$ may have no sign. In the weak formulation, the projection is taken care of the test function and we are allowed to work with the nonnegative initial datum u_i^0.

The proof in [9, 10] provides a priori estimates for $u^{(N,\eta)}$ via the entropy inequality, but they depend on η because of the dependence of the source term f_j on η. Still, it is possible to pass to the limit $\eta \to 0$, since the solution to an ordinary differential equation involving the Wong–Zakai approximation converges in mean to the solution to the corresponding stochastic differential equation [20, Chapter 6, Theorem 7.1]. We can apply this result since the nonlinearities in the strong form associated to (14) are Lipschitz continuous (not uniform in N). We conclude that $u^{(N,\eta)} \to u^{(N)}$ in probability up to the stopping time τ_R as $\eta \to 0$, where $u^{(N)}$ is the unique solution to (11)–(12). We deduce that $u_i^{(N)}(x,t) \geq 0$ for a.e. $(x,t) \in \mathcal{O} \times (0, T \wedge \tau_R)$ $\mathbb{P}$-a.s. and $i = 1, \ldots, n$.

Remark 4. The Wong–Zakai approximation is only needed to conclude the nonnegativity of $u_i^{(N)}$. Another approach is to apply a stochastic version of the Stampacchia truncation method; see [6]. Generally, maximum principle arguments do not apply to cross-diffusion systems. For the present system, however, this is possible since the off-diagonal diffusion coefficients in (5) and the stochastic diffusion term vanish when $u_i = 0$. We leave the details to the reader. □

3.3. **Entropy estimates.** We prove some estimates uniform in the Galerkin dimension N showing that the solution is actually global in time. The starting point is a stochastic version of the entropy inequality.

Lemma 5 (Entropy inequality)**.** *The solution $u^{(N)}$ to (11)–(12) satisfies for $0 < t < T \wedge \tau_R$,*

$$\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(t))\mathrm{d}x + 2\mathbb{E}\int_0^t\int_{\mathcal{O}}\sum_{i=1}^n \pi_i\big(2a_{i0}|\nabla(u_i^{(N)})^{1/2}|^2 + a_{ii}|\nabla u_i^{(N)}|^2\big)\mathrm{d}x\mathrm{d}s$$

$$+ 2\mathbb{E}\int_0^t\int_{\mathcal{O}}\sum_{i,j=1,\,j\neq i}^n \pi_i a_{ij}|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \leq C(T) + C(T)\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(0)^+)\mathrm{d}x, \tag{15}$$

where $C(T) > 0$ depends on T but not on N or R and $(u^{(N)}(0)^+)_i = \max\{0, u_i^{(N)}(0)\}$.

Proof. Let $u^{(N)}$ be the solution to (11)–(12) up to the stopping time τ_R. Since the entropy density defined in (7) is not a C^2 function on $[0,\infty)^n$, we cannot apply the Itô lemma to this function, and we need to regularize. Let $\delta > 0$ and define

$$h_\delta(u) = \sum_{i=1}^n \pi_i\big((u_i + \delta)(\log(u_i + \delta) - 1) + 1\big) \quad \text{for } u \in [0,\infty)^n,$$

$$h_\delta^+(u) = \sum_{i=1}^n \pi_i\big((g_\delta(u_i) + \delta)(\log(g_\delta(u_i) + \delta) - 1) + 1\big) \quad \text{for } u \in \mathbb{R}^n,$$

where g_δ is a smooth regularization of $z^+ = \max\{0, z\}$ such that $g_\delta(z) \to z^+$ as $\delta \to 0$, $g_\delta(z) + \delta > 0$ for $z \in \mathbb{R}$, and $g_\delta(z) = z$ for $z \geq 0$. Then $h_\delta \in C^2([0,\infty)^n;\ [0,\infty))$ and $h_\delta^+ \in C^2(\mathbb{R}^n; [0,\infty))$. Note that these regularizations are different from that one used in

[15]. Since $u_i^{(N)}(t)\geq 0$ for $t>0$ $\mathbb{P}$-a.s., we have by definition $h_\delta^+(u^{(N)}(t))=h_\delta(u^{(N)}(t))$ $\mathbb{P}$-a.s. The second regularization h_δ^+ is needed since $u_i^{(N)}(0)=\Pi_N(u_i^0)$ may have no sign. Because of Itô's lemma and $g_\delta'(u_i^{(N)}(t))=1$, $g_\delta''(u_i^{(N)}(t))=0$ for $t>0$, we find that for $t>0$,

$$\begin{aligned}
&\int_{\mathcal{O}}h_\delta(u^{(N)}(t\wedge\tau_R))\mathrm{d}x-\int_{\mathcal{O}}h_\delta^+(u^N(0))\mathrm{d}x \\
&\quad=-\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n\pi_ia_{0i}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}+\delta}\mathrm{d}x\mathrm{d}s-\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n\pi_ia_{ij}u_j^{(N)}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}+\delta}\mathrm{d}x\mathrm{d}s \\
&\qquad-\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n\pi_ia_{ij}\frac{u_i^{(N)}}{u_i^{(N)}+\delta}\nabla u_i^{(N)}\cdot\nabla u_j^{(N)}\mathrm{d}x\mathrm{d}s \\
&\qquad+\int_0^{t\wedge\tau_R}\sum_{i,j=1}^n\bigg(\int_{\mathcal{O}}\pi_i\sigma_{ij}(u^{(N)})\log(u_i^{(N)}+\delta)\mathrm{d}x\bigg)\mathrm{d}W_j(t) \\
&\qquad+\frac12\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n\pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}+\delta}\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{16}$$

We take the expectation on both sides and observe that the expectation of the Itô integral vanishes:

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}}h_\delta(u^{(N)}(t\wedge\tau_R))\mathrm{d}x-\mathbb{E}\int_{\mathcal{O}}h_\delta^+(u^N(0))\mathrm{d}x \\
&\quad=-\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n\pi_ia_{0i}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}+\delta}\mathrm{d}x\mathrm{d}s-\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n\pi_ia_{ij}u_j^{(N)}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}+\delta}\mathrm{d}x\mathrm{d}s \\
&\qquad-\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n\pi_ia_{ij}\frac{u_i^{(N)}}{u_i^{(N)}+\delta}\nabla u_i^{(N)}\cdot\nabla u_j^{(N)}\mathrm{d}x\mathrm{d}s \\
&\qquad+\frac12\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n\pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}+\delta}\mathrm{d}x\mathrm{d}s=:I_1^\delta+\cdots+I_4^\delta.
\end{aligned}\tag{17}$$

We wish to perform the limit $\delta\to 0$ in (17). By continuity,

$$h_\delta(u^{(N)}(\omega,x,t\wedge\tau_R))\to h(u^{(N)}(\omega,x,t\wedge\tau_R))\quad\text{for a.e. }(\omega,x,t)\in\Omega\times\mathcal{O}\times(0,T).$$

Moreover, for given $(\omega,x)\in\Omega\times\mathcal{O}$, there exists $C>0$ such that for all $\delta>0$,

$$\big(u_i^{(N)}(\log u_i^{(N)}-1)+1\big)(\omega,x,t\wedge\tau_R)\leq C\big(1+u^{(N)}(\omega,x,t\wedge\tau_R)^2\big),$$

and the right-hand side is uniformly integrable in $\Omega\times\mathcal{O}$ for a fixed $t\in[0,T\wedge\tau_R]$ (because of the definition of the stopping time). We conclude from the dominated convergence theorem that

$$\mathbb{E}\int_{\mathcal{O}}h_\delta(u^{(N)}(t\wedge\tau_R))\mathrm{d}x\to\mathbb{E}\int_{\mathcal{O}}h(u^{(N)}(t\wedge\tau_R))\mathrm{d}x,$$

$$\mathbb{E}\int_{\mathcal{O}} h_\delta^+(u^N(0))\mathrm{d}x \to \mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(0)^+)\mathrm{d}x \quad \text{as } \delta\to 0,$$

recalling that $(u^{(N)}(0)^+)_i = \max\{0, u_i^{(N)}(0)\}$. The limit $\delta\to 0$ in I_1^δ, I_2^δ, and I_4^δ can be performed because of the monotone convergence theorem, while the dominated convergence theorem allows us to pass to the limit in I_3^δ. Then the limit $\delta\to 0$ in (17) leads to

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(t\wedge\tau_R))\mathrm{d}x - \mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(0)^+)\mathrm{d}x \\
&\quad = -\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n \pi_i a_{0i}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s - \mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i a_{ij}u_j^{(N)}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s \\
&\qquad - \mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i a_{ij}\nabla u_i^{(N)}\cdot\nabla u_j^{(N)}\mathrm{d}x\mathrm{d}s + \frac12\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{18}$$

Because of the detailed-balance condition (9), $(\pi_i a_{ij})$ is symmetric. Thus, the second and third integrand on the right-hand side can be formulated as

$$\begin{aligned}
&\sum_{i,j=1}^n \pi_i a_{ij}\bigg(u_j^{(N)}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}} + \nabla u_i^{(N)}\cdot\nabla u_j^{(N)}\bigg) \\
&\quad = \frac12\sum_{i,j=1}^n \pi_i a_{ij}\bigg(u_j^{(N)}\frac{|\nabla u_i^{(N)}|^2}{u_i^{(N)}} + u_i^{(N)}\frac{|\nabla u_j^{(N)}|^2}{u_j^{(N)}} + 2\nabla u_i^{(N)}\cdot\nabla u_j^{(N)}\bigg) \\
&\quad = \frac12\sum_{i,j=1}^n \pi_i a_{ij}u_i^{(N)}u_j^{(N)}\big(|\nabla\log u_i^{(N)}|^2 + |\nabla\log u_j^{(N)}|^2 + 2\nabla\log u_i^{(N)}\cdot\nabla\log u_j^{(N)}\big) \\
&\quad = \frac12\sum_{i,j=1}^n \pi_i a_{ij}u_i^{(N)}u_j^{(N)}\big|\nabla\log(u_i^{(N)}u_j^{(N)})\big|^2 = 2\sum_{i,j=1}^n \pi_i a_{ij}\big|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}\big|^2.
\end{aligned}$$

By Assumption (A5), the last integral in (18) is estimated according to

$$\frac12\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s \le C\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}(1+h(u^{(N)}))\mathrm{d}x\mathrm{d}s.$$

Inserting these expressions into (18) and applying Gronwall's inequality gives

$$\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(t\wedge\tau_R))\mathrm{d}x + 4\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i=1}^n \pi_i a_{i0}|\nabla(u_i^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \\
&\quad + 2\mathbb{E}\int_0^{t\wedge\tau_R}\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i a_{ij}|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \le C(T) + C(T)\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(0)^+)\mathrm{d}x,
\end{aligned}$$

where $C(T) > 0$ is independent of N and R. Consequently, the right-hand side does not depend on the chosen sequence of stopping times τ_R, and we can pass to the limit $R\to\infty$.

The limit $u_i^{(N)}(0) = \Pi_N(u_i^0) \to u_i^0 \geq 0$ in $L^2(\mathcal{O})$ as $N \to \infty$ yields $h(u^{(N)}(0)^+) \to h(u^0)$ in $L^1(\mathcal{O})$. Thus, the right-hand side of (15) is independent of N and R. $\square$

The entropy inequality in Lemma 5 provides a uniform bound for $\sup_{0\leq t\leq T} \mathbb{E}\|u^{(N)}(t)\|_{L^1(\mathcal{O})}$ but we need a uniform bound for $\mathbb{E}(\sup_{0<t<T} \|u^{(N)}(t)\|_{L^1(\mathcal{O})})$, which will be used later to obtain higher order moment estimates. This is shown in the following lemma.

Lemma 6. *The solution $u^{(N)}$ to* (11)–(12) *satisfies the following bounds:*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|u^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C(u^0, T), \tag{19}$$

$$\sup_{N\in\mathbb{N}} \sum_{i=1}^n \big(a_{i0}\mathbb{E}\|(u_i^{(N)})^{1/2}\|^2_{L^2(0,T;H^1(\mathcal{O}))} + a_{ii}\mathbb{E}\|u_i^{(N)}\|^2_{L^2(0,T;H^1(\mathcal{O}))}\big) \leq C(u^0, T), \tag{20}$$

$$\sup_{N\in\mathbb{N}} \sum_{j\neq i} a_{ij}\mathbb{E}\|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}\|^2_{L^2(0,T;L^2(\mathcal{O}))} \leq C(u^0, T). \tag{21}$$

In particular, the solution $u^{(N)}$ is global in time for $d \geq 1$ (with self-diffusion) and for $d \leq 3$ (without self-diffusion).

Proof. Let $u^{(N)}$ be the solution to (11)–(12) up to the stopping time τ_R and let $T < \tau_R$. The starting point of the proof is equation (16). Instead of taking first the expectation as in the proof of Lemma 5, we pass first to the limit $\delta \to 0$. This can be done as in Lemma 5 except for the stochastic integral. We claim that

$$\begin{aligned}
&\int_0^T \int_{\mathcal{O}} \sum_{i,j=1}^n \sigma_{ij}(u^{(N)}(s)) \log(u_i^{(N)}(s) + \delta)\mathrm{d}x\mathrm{d}W_j(s) \\
&\quad \to \int_0^T \int_{\mathcal{O}} \sum_{i,j=1}^n \sigma_{ij}(u^{(N)}(s)) \log(u_i^{(N)}(s))\mathrm{d}x\mathrm{d}W_j(s)
\end{aligned} \tag{22}$$

as $\delta \to 0$. To prove this limit, we use the stochastic dominated convergence theorem [17, Theorem 6.44]. For this, let

$$\begin{aligned}
F_\delta(t) &= \int_{\mathcal{O}} f_\delta(x)\mathrm{d}x = \int_{\mathcal{O}} \sum_{i=1}^n \sigma_{ij}(u^{(N)}(t)) \log(u_i^{(N)}(t) + \delta)\mathrm{d}x, \\
F(t) &= \int_{\mathcal{O}} f(x)\mathrm{d}x = \int_{\mathcal{O}} \sum_{i=1}^n \sigma_{ij}(u^{(N)}(t)) \log u_i^{(N)}(t)\mathrm{d}x.
\end{aligned}$$

It is clear that $f_\delta(x) \to f(x)$ a.e. in $\mathcal{O}$. We wish to find an integrable function g such that $|f_\delta(x)| \leq g(x)$ for $x \in \mathcal{O}$. Let $\delta \in (0, 1)$. If $z \in [0, 1-\delta)$, we have $|\log(z+\delta)| \leq |\log z|$. If $z \in [1-\delta, 1)$, it follows that $|\log(z+\delta)| \leq \log 2$. Finally, if $z > 1$,

$$\log(z+\delta) = \int_1^z \frac{\mathrm{d}r}{r} + \int_z^{z+\delta} \frac{\mathrm{d}r}{r} \leq \int_1^z \frac{\mathrm{d}r}{r} + \int_1^{1+\delta} \frac{\mathrm{d}r}{r} = \log z + \log(1+\delta).$$

Therefore, in view of Assumption (A5) and the entropy inequality in Lemma 5, the function

$$g(x) = \sum_{i=1}^n \sigma_{ij}(u^{(N)}(x,t))(\log u_i^{(N)} + \log 2)$$

is integrable in $\mathcal{O}$. We deduce from the Lebesgue dominated convergence theorem that $F_\delta(t) \to F(t)$ as $\delta \to 0$. By the definition of g and Assumption (A5), we can dominate F_δ pointwise for any $\delta > 0$ according to

$$|F_\delta(t)| \le G(x) := C_h\bigg(\int_{\mathcal{O}}(1 + h(u^{(N)}(x,t)))\mathrm{d}x\bigg)^{1/2} + C\int_{\mathcal{O}} u_i^{(N)}(x,t)\mathrm{d}x + C, \tag{23}$$

and G is square-integrable, since

$$\|G\|^2_{L^2(Q_T)} \le C(T) + C\mathbb{E}\int_0^T\int_{\mathcal{O}}\big(h(u^{(N)}(x,t)) + |u^{(N)}(x,t)|^2\big)\mathrm{d}x\mathrm{d}s < \infty.$$

By the stochastic dominated convergence theorem, we infer from the pointwise convergence $F_\delta(t) \to F(t)$ and the bound (23) that (22) holds, proving the claim.

Repeating the calculations following (18), we obtain

$$\begin{aligned}
\int_{\mathcal{O}} h(u^{(N)}(t))\mathrm{d}x &+ 4\int_0^t\int_{\mathcal{O}}\sum_{i=1}^n \pi_i a_{i0}|\nabla(u_i^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \\
&+ 2\int_0^t\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i a_{ij}|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \\
&\le \int_{\mathcal{O}} h(u^0)\mathrm{d}x + \int_0^t\int_{\mathcal{O}}\sum_{i,j=1}^n \sigma_{ij}(u^{(N)})\log u_i^{(N)}\mathrm{d}x\mathrm{d}W_j(s) \\
&+ \frac{1}{2}\int_0^t\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{24}$$

We take the supremum over $0 < t < T$ and the expectation and apply the Burkholder–Davis–Gundy inequality:

$$\begin{aligned}
\mathbb{E}\sup_{0<t<T}\int_{\mathcal{O}} h(u^{(N)}(t))\mathrm{d}x &+ 4\mathbb{E}\int_0^T\int_{\mathcal{O}}\sum_{i=1}^n \pi_i a_{i0}|\nabla(u_i^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \\
&+ 2\mathbb{E}\int_0^T\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i a_{ij}|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s \\
&\le \mathbb{E}\int_{\mathcal{O}} h(u^0)\mathrm{d}x + \mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}\sum_{i,j=1}^n \big(\sigma_{ij}(u^{(N)})\log u_i^{(N)}\big)^2\mathrm{d}x\mathrm{d}s\bigg)^{1/2} \\
&+ \frac{1}{2}\mathbb{E}\int_0^T\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$\leq \mathbb{E}\int_{\mathcal{O}} h(u^0)\mathrm{d}x + C\mathbb{E}\int_0^T \bigg(1 + \int_{\mathcal{O}} h(u^{(N)})\mathrm{d}x\bigg)\mathrm{d}s,$$

where we used Assumption (A5) in the last step. By Fubini's theorem and Gronwall's lemma, we conclude that

$$\mathbb{E}\sup_{0<t<T}\int_{\mathcal{O}} h(u^{(N)}(t))\mathrm{d}x \leq C(T) + C(T)\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(0)^+)\mathrm{d}x \quad \text{for } 0 < T < \tau_R,$$

where $C(T) > 0$ is independent of N and R. Passing to the limit $R \to \infty$ results in

$$\mathbb{E}\sup_{0<t<T}\int_{\mathcal{O}} h(u^{(N)}(t))\mathrm{d}x \leq C(T) + C(T)\mathbb{E}\int_{\mathcal{O}} h(u^{(N)}(0)^+)\mathrm{d}x \leq C(u^0, T).$$

for $T > 0$. Since the entropy density dominates the L^1 norm, this shows that

$$\sup_{N\in\mathbb{N}}\mathbb{E}\Big(\sup_{0<t<T}\|u^{(N)}(t)\|_{L^1(\mathcal{O})}\Big) \leq C(u^0, T).$$

Estimate (20) is obtained from the Poincaré–Wirtinger inequality, the previous estimate, and the gradient estimate in (15). Moreover, (21) also follows from (15).

It remains to show that $u^{(N)}$ is global in time. In case with self-diffusion, estimate (20) immediately implies that $\mathbb{E}\|u_i^{(N)}\|^2_{L^2(0,T;L^2(\mathcal{O}))} \leq C$. In case without self-diffusion, we deduce from the Gagliardo–Nirenberg inequality with $\theta = d/4$, the Hölder inequality with $p = 4/d$, $q = 4/(4-d)$ (such that $1/p + 1/q = 1$), and estimates (19) and (20) that

$$\begin{aligned}
\mathbb{E}\|u_i^{(N)}\|_{L^{4/d}(0,T;L^2(\mathcal{O}))} &= \mathbb{E}\bigg(\int_0^T \|(u_i^{(N)})^{1/2}\|^{8/d}_{L^4(\mathcal{O})}\mathrm{d}t\bigg)^{d/4} \\
&\leq C\mathbb{E}\bigg(\int_0^T \|(u_i^{(N)})^{1/2}\|^{8\theta/d}_{H^1(\mathcal{O})}\|(u_i^{(N)})^{1/2}\|^{8(1-\theta)/d}_{L^2(\mathcal{O})}\mathrm{d}t\bigg)^{d/4} \\
&\leq C\mathbb{E}\bigg\{\|u_i^{(N)}\|^{1-d/4}_{L^\infty(0,T;L^1(\mathcal{O}))}\bigg(\int_0^T \|(u_i^{(N)})^{1/2}\|^2_{H^1(\mathcal{O})}\mathrm{d}t\bigg)^{d/4}\bigg\} \\
&\leq \Big(\mathbb{E}\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}\Big)^{1-d/4}\bigg\{\mathbb{E}\int_0^T \|(u_i^{(N)})^{1/2}\|^2_{H^1(\mathcal{O})}\mathrm{d}t\bigg\}^{d/4} \leq C. \qquad (25)
\end{aligned}$$

At this point, we need the restriction $d \leq 3$. As the $L^2(\mathcal{O})$ is controlled in both cases, the stopping time τ_R equals the final time T, and the solution $u^{(N)}$ is global in time. □

3.4. **Further uniform estimates.** Next, we show some estimates for higher-order moments. This step was not necessary in [15], since the solutions in that paper are bounded.

Lemma 7 (Higher-order moments)**.** *Let $u^{(N)}$ be the solution to* (11)–(12) *and let $p \geq 2$. Then, for any $i = 1, \ldots, n$,*

$$\sup_{N\in\mathbb{N}}\mathbb{E}\|u^{(N)}\|^p_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C(p, u^0, T), \qquad (26)$$

$$\sup_{N\in\mathbb{N}}\big(a_{i0}\mathbb{E}\|(u_i^{(N)})^{1/2}\|^p_{L^2(0,T;H^1(\mathcal{O}))} + a_{ii}\mathbb{E}\|u_i^{(N)}\|^p_{L^2(0,T;H^1(\mathcal{O}))}\big) \leq C(p, u^0, T), \qquad (27)$$

where $C(p,u^0,T)>0$ *does not depend on* N*.*

Proof. We raise (24) to the power $p\geq 2$, take the expectation, apply the Burkholder–Davis–Gundy inequality to the stochastic term, and use Assumption (A5) to find that

$$
\begin{aligned}
&\mathbb{E}\bigg(\int_{\mathcal{O}} h(u^{(N)}(t))\mathrm{d}x\bigg)^p + 4\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}\sum_{i=1}^n \pi_i a_{i0}|\nabla(u_i^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s\bigg)^p \\
&\quad + 2\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i a_{ij}|\nabla(u_i^{(N)}u_j^{(N)})^{1/2}|^2\mathrm{d}x\mathrm{d}s\bigg)^p \\
&\leq C(p,u^0) + C\mathbb{E}\bigg(\int_0^T\sum_{i,j=1}^n\|\sigma_{ij}(u^{(N)})\log u_i^{(N)}\|_{L^2(\mathcal{O})}^2\mathrm{d}s\bigg)^{p/2} \\
&\quad + C\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}\sum_{i,j=1}^n \pi_i\frac{\sigma_{ij}(u^{(N)})^2}{u_i^{(N)}}\mathrm{d}x\mathrm{d}s\bigg)^p \\
&\leq C(p,u^0) + C\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}(1+h(u^{(N)}))\mathrm{d}x\mathrm{d}s\bigg)^{p/2} \\
&\quad + C\mathbb{E}\bigg(\int_0^T\int_{\mathcal{O}}(1+h(u^{(N)}))\mathrm{d}x\mathrm{d}s\bigg)^p.
\end{aligned}
$$

The lemma follows after applying Jensen's and Gronwall's inequality, using the fact that the entropy density dominates the L^1 norm, and applying the Poincaré–Wirtinger inequality. □

We derive further higher-order moment estimates from Lemma 7. For this, we distinguish the cases with and without self-diffusion.

Lemma 8 (Higher-order moments, with self-diffusion)**.** *Let* $a_{ii}>0$ *for all* $i=1,\ldots,n$*, let* $u^{(N)}$ *be the solution to* (11)–(12)*, and let* $p\geq 2$*. Then for any* $i=1,\ldots,n$*,*

$$
\sup_{N\in\mathbb{N}}\mathbb{E}\|u_i^{(N)}\|_{L^{2+2/d}(Q_T)}^p \leq C(p,u^0,T), \tag{28}
$$

$$
\sup_{N\in\mathbb{N}}\mathbb{E}\|u_i^{(N)}(t)\|_{L^{2+4/d}(0,T;L^2(\mathcal{O}))}^p \leq C(p,u^0,T). \tag{29}
$$

Proof. Applying the Gagliardo–Nirenberg inequality as in [22, p. 95] yields (28). Estimate (29) is obtained from another application of the Gagliardo–Nirenberg inequality. For this, let $\theta=d/(d+2)$. Then $\theta(2+4/d)=2$ and

$$
\begin{aligned}
\mathbb{E}\bigg(\int_0^T\|u_i^{(N)}\|_{L^2(\mathcal{O})}^{2+4/d}\mathrm{d}t\bigg)^p &\leq C\mathbb{E}\bigg(\int_0^T\|u_i^{(N)}\|_{H^1(\mathcal{O})}^{\theta(2+4/d)}\|u_i^{(N)}\|_{L^1(\mathcal{O})}^{(1-\theta)(2+4/d)}\mathrm{d}t\bigg)^p \\
&\leq C(T)\mathbb{E}\bigg(\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{4/d}\int_0^T\|u_i^{(N)}\|_{H^1(\mathcal{O})}^2\mathrm{d}t\bigg)^p
\end{aligned}
$$

$$\le C(T)\Big(\mathbb{E}\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{8p/d}\Big)^{1/2}\bigg\{\mathbb{E}\bigg(\int_0^T\|u_i^{(N)}\|_{H^1(\mathcal{O})}^2\mathrm{d}t\bigg)^{2p}\bigg\}^{1/2},$$

and the conclusion follows from estimates (26) and (27). ☐

Lemma 9 (Higher-order moments, without self-diffusion)**.** *Let $a_{i0} > 0$ for all $i = 1, \ldots, n$, let $u^{(N)}$ be the solution to* (11)*–*(12)*, and let $p \ge 2$. Then for any $i = 1, \ldots, n$,*

$$\sup_{N\in\mathbb{N}}\mathbb{E}\|u_i^{(N)}\|_{L^2(0,T;W^{1,1}(\mathcal{O}))}^p \le C(p,u^0,T), \tag{30}$$

$$\sup_{N\in\mathbb{N}}\mathbb{E}\|u_i^{(N)}\|_{L^{1+2/d}(Q_T)}^p \le C(p,u^0,T), \tag{31}$$

$$\sup_{N\in\mathbb{N}}\mathbb{E}\|u_i^{(N)}\|_{L^{4/d}(0,T;L^2(\mathcal{O}))}^p \le C(p,u^0,T) \quad \textit{for } d \le 4, \tag{32}$$

$$\sup_{N\in\mathbb{N}}\mathbb{E}\|u_i^{(N)}\|_{L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O}))}^p \le C(p,u^0,T), \tag{33}$$

where $\rho_1 = (d+2)/(d+1)$.

Proof. The identity $\nabla u_i^{(N)} = 2(u_i^{(N)})^{1/2}\nabla(u_i^{(N)})^{1/2}$ and the Hölder inequality show that

$$\begin{aligned}\mathbb{E}\|\nabla u_i^{(N)}\|_{L^2(0,T;L^1(\mathcal{O}))}^p &\le C\mathbb{E}\bigg(\int_0^T\|(u_i^{(N)})^{1/2}\|_{L^2(\mathcal{O})}^2\|\nabla(u_i^{(N)})^{1/2}\|_{L^2(\mathcal{O})}^2\mathrm{d}t\bigg)^{p/2}\\ &\le C\mathbb{E}\bigg(\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}\int_0^T\|\nabla(u_i^{(N)})^{1/2}\|_{L^2(\mathcal{O})}^2\mathrm{d}t\bigg)^{p/2}\\ &\le C\big(\mathbb{E}\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}^p\big)^{1/2}\big(\mathbb{E}\|\nabla(u_i^{(N)})^{1/2}\|_{L^2(0,T;L^2(\mathcal{O}))}^{2p}\big)^{1/2}.\end{aligned}$$

Because of (26) and (27), the right-hand side is bounded. Using (26) again, we infer that (30) holds. Estimate (31) is obtained from the Gagliardo–Nirenberg inequality with $\theta = d/(d+2)$. Indeed, taking into account estimates (26) and (27),

$$\begin{aligned}&\mathbb{E}\bigg(\int_0^T\|(u_i^{(N)})^{1/2}\|_{L^{2+4/d}(\mathcal{O})}^{2+4/d}\bigg)^p\\ &\le C\mathbb{E}\bigg(\int_0^T\|(u_i^{(N)})^{1/2}\|_{H^1(\mathcal{O})}^{\theta(2+4/d)}\|(u_i^{(N)})^{1/2}\|_{L^2(\mathcal{O})}^{(1-\theta)(2+4/d)}\mathrm{d}t\bigg)^p\\ &\le C\mathbb{E}\bigg(\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{2/d}\int_0^T\|(u_i^{(N)})^{1/2}\|_{H^1(\mathcal{O})}^2\mathrm{d}t\bigg)^p\\ &\le C\Big(\mathbb{E}\|u_i^{(N)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{4p/d}\Big)^{1/2}\bigg\{\mathbb{E}\bigg(\int_0^T\|(u_i^{(N)})^{1/2}\|_{H^1(\mathcal{O})}^2\bigg)^{2p}\bigg\}^{1/2} \le C.\end{aligned}$$

Estimate (32) can be shown as in (25). Finally, estimate (33) is a consequence of $\nabla u_i^{(N)} = 2(u_i^{(N)})^{1/2}\nabla(u_i^{(N)})^{1/2}$, the Hölder inequality, and estimates (27) and (31). ☐

Lemma 10. *Let $u^{(N)}$ be the solution to* (11)–(12) *and let $p \geq 2$. Then for any $i = 1, \dots, n$ and $j \neq i$,*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|(u_i^{(N)}u_j^{(N)})^{1/2}\|^p_{L^\infty(0,T;L^1(\mathcal{O}))} \leq C(p,u^0,T), \tag{34}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|(u_i^{(N)}u_j^{(N)})^{1/2}\|^p_{L^2(0,T;H^1(\mathcal{O}))} \leq C(p,u^0,T), \tag{35}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|(u_i^{(N)}u_j^{(N)})^{1/2}\|^p_{L^{2+2/d}(Q_T)} \leq C(p,u^0,T), \tag{36}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|u_i^{(N)}u_j^{(N)}\|^p_{L^{1+2/d}(0,T;L^1(\mathcal{O}))} \leq C(p,u^0,T), \tag{37}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|\nabla(u_i^{(N)}u_j^{(N)})\|^p_{L^{\rho_1}(0,T;L^1(\mathcal{O}))} \leq C(p,u^0,T), \tag{38}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|\nabla(u_i^{(N)}u_j^{(N)})\|^p_{L^{\rho_2}(Q_T)} \leq C(p,u^0,T), \tag{39}$$

where $\rho_1 = (d+2)/(d+1)$ and $\rho_2 = (2d+2)/(2d+1)$.

Proof. The Hölder inequality and estimate (26) yield immediately (34). By the Poincaré–Wirtinger inequality, estimates (21) and (34) lead to (35). Estimates (36) and (37) follow from the Gagliardo–Nirenberg inequality, taking into account estimates (34) and (35) (see (25) for a similar proof). Finally, estimates (34) and (35) imply that

$$\nabla(u_i^{(N)}u_j^{(N)}) = 2(u_i^{(N)}u_j^{(N)})^{1/2}\nabla(u_i^{(N)}u_j^{(N)})^{1/2}$$

is bounded in $L^{\rho_1}(0,T;L^1(\mathcal{O}))$ and in $L^{\rho_2}(Q_T)$, verifying (38) and (39). □

3.5. **Fractional time regularity.** We show that the solution $u^{(N)}$ to (11)–(12) possesses a uniform bound for a fractional time derivative. This result is used to establish the tightness of the laws of $u^{(N)}$ in some Lebesgue spaces. In our previous work [15], the tightness of the laws of the approximate solutions was proved in a different way by verifying the Aldous condition. We recall the definition of the Sobolev–Slobodeckij spaces. Let X be a vector space and let $p \geq 1$, $\alpha \in (0,1)$. Then $W^{\alpha,p}(0,T;X)$ is the set of all functions $v \in L^p(0,T;X)$ for which

$$\|v\|^p_{W^{\alpha,p}(0,T;X)} = \|v\|^p_{L^p(0,T;X)} + \int_0^T\int_0^T \frac{\|v(t)-v(s)\|_X^p}{|t-s|^{1+\alpha p}}\mathrm{d}t\mathrm{d}s$$

is finite. With this norm, $W^{\alpha,p}(0,T;X)$ becomes a Banach space. In the case without self-diffusion, we assume that $d \leq 3$.

Lemma 11 (Time regularity). *Let $u^{(N)}$ be the solution to* (11)–(12) *and let $m \in \mathbb{N}$ satisfy $m > d/2+1$. Then there exists $C(u^0,T) > 0$ such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\|u^{(N)}\|^2_{W^{\alpha,2}(0,T;H^m(\mathcal{O})')} \leq C(u^0,T), \tag{40}$$

where $\alpha < 1/2$ (with self-diffusion) and $\alpha < 1/(d+2)$ (without self-diffusion).

The time regularity of $u^{(N)}$ is restricted by the Sobolev regularity of the stochastic integral; see, e.g., [18, Lemma 2.1].

Proof. Estimate (19) and the continuous embedding $L^1(\mathcal{O}) \hookrightarrow H^m(\mathcal{O})'$ show that the sequence $(\mathbb{E}\|u_i^{(N)}\|^2_{L^2(0,T;H^m(\mathcal{O})')})$ is uniformly bounded. It remains to show that the following integral is finite:

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T \frac{\|u_i^{(N)}(t)-u_i^{(N)}(s)\|^2_{H^m(\mathcal{O})'}}{|t-s|^{1+2\alpha}}\mathrm{d}t\mathrm{d}s \\
&\le \int_0^T\int_0^T |t-s|^{-1-2\alpha}\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\operatorname{div}\sum_{j=1}^n A_{ij}(u^{(N)}(r))\nabla u_j^{(N)}(r)\mathrm{d}r\bigg\|^2_{H^m(\mathcal{O})'}\mathrm{d}t\mathrm{d}s \\
&\quad + \int_0^T\int_0^T |t-s|^{-1-2\alpha}\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n \sigma_{ij}(u^{(N)}(r))\mathrm{d}W_j(r)\bigg\|^2_{H^m(\mathcal{O})'}\mathrm{d}t\mathrm{d}s \\
&=: J_1 + J_2.
\end{aligned}$$

Before we estimate J_1 and J_2, we recall a well-known result for the sake of completeness. Let $g \in L^1(0,T)$ and $\delta < 2$, $\delta \neq 1$. We claim that

$$I := \int_0^T\int_0^T |t-s|^{-\delta}\int_{s\wedge t}^{t\vee s} g(r)\mathrm{d}r\mathrm{d}t\mathrm{d}s < \infty. \tag{41}$$

Indeed, a change of the integration domain and integration by parts gives

$$\begin{aligned}
I &= 2\int_0^T\int_s^T (t-s)^{-\delta}\bigg(\int_s^t g(r)dr\bigg)\mathrm{d}t\mathrm{d}s \\
&= -\frac{2}{1-\delta}\int_0^T\int_s^T (t-s)^{1-\delta}g(t)\mathrm{d}t\mathrm{d}s + \frac{2}{1-\delta}\int_0^T (T-s)^{1-\delta}\int_s^t g(r)\mathrm{d}r\mathrm{d}s, \tag{42}
\end{aligned}$$

observing that $\lim_{t\to s}(t-s)^{1-\delta}\int_s^t g(r)\mathrm{d}r = 0$ for $1-\delta > -1$, since the integrability of g implies that $\lim_{t\to s}(t-s)^{-1}\int_s^t g(r)\mathrm{d}r = g(s)$ for a.e. s. The claim follows as the integrals on the right-hand side of (42) are finite.

Step 1: Case with self-diffusion. Let $a_{ii} > 0$ for all $i = 1,\ldots,n$. We need some preparations before we estimate J_1. We observe that

$$\begin{aligned}
\bigg\|\sum_{j=1}^n A_{ij}(u^{(N)})\nabla u_j^{(N)}\bigg\|_{L^1(\mathcal{O})} &= \bigg\|\bigg(a_{i0} + 2\sum_{j=1}^n a_{ij}u_j^{(N)}\bigg)\nabla u_i^{(N)} + \sum_{j\neq i} a_{ij}u_i^{(N)}\nabla u_j^{(N)}\bigg\|_{L^1(\mathcal{O})} \\
&\le C\|\nabla u^{(N)}\|_{L^2(\mathcal{O})} + C\|u^{(N)}\|_{L^2(\mathcal{O})}\|\nabla u^{(N)}\|_{L^2(\mathcal{O})}.
\end{aligned}$$

Because of the embedding $L^1(\mathcal{O}) \hookrightarrow H^{m-1}(\mathcal{O})'$, it follows that

$$J_3 := \mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\operatorname{div}\sum_{j=1}^n A_{ij}(u^{(N)}(r))\nabla u_j^{(N)}(r)\mathrm{d}r\bigg\|^2_{H^m(\mathcal{O})'}$$

$$
\begin{aligned}
&\le C\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\bigg\|\sum_{j=1}^n A_{ij}(u^{(N)}(r))\nabla u_j^{(N)}(r)\bigg\|_{L^1(\mathcal{O})}\mathrm{d}r\bigg)^2 \\
&\le C\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}\mathrm{d}r\bigg)^2 + C\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\|u^{(N)}(r)\|_{L^2(\mathcal{O})}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}\mathrm{d}r\bigg)^2 \\
&=: J_{31} + J_{32}.
\end{aligned}
$$

We use the Hölder inequality to obtain

$$
\begin{aligned}
J_{31} &\le C|t-s|\mathbb{E}\int_{s\wedge t}^{t\vee s}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}^2\mathrm{d}r, \\
J_{32} &\le C\mathbb{E}\bigg\{\bigg(\int_{s\wedge t}^{t\vee s}\|u^{(N)}(r)\|_{L^2(\mathcal{O})}^{(2d+4)/d}\mathrm{d}r\bigg)^{d/(2d+4)} \\
&\qquad\times\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}^2\mathrm{d}r\bigg)^{1/2}|t-s|^{2/(2d+4)}\bigg\}^2 \\
&= C\mathbb{E}\bigg\{\bigg(\int_{s\wedge t}^{t\vee s}\|u^{(N)}(r)\|_{L^2(\mathcal{O})}^{(2d+4)/d}\mathrm{d}r\bigg)^{d/(d+2)}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}^2\mathrm{d}r\bigg)|t-s|^{2/(d+2)}\bigg\}.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
&\int_0^T\int_0^T|t-s|^{-1-2\alpha}J_{31}\mathrm{d}t\mathrm{d}s \le \int_0^T\int_0^T|t-s|^{-2\alpha}\bigg(\mathbb{E}\int_0^T\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}^2\mathrm{d}r\bigg)\mathrm{d}t\mathrm{d}s \le C, \\
&\int_0^T\int_0^T|t-s|^{-1-2\alpha}J_{32}\mathrm{d}t\mathrm{d}s \le \int_0^T\int_0^T|t-s|^{-1-2\alpha+2/(d+2)} \\
&\qquad\times\mathbb{E}\bigg\{\bigg(\int_0^T\|u^{(N)}(r)\|_{L^2(\mathcal{O})}^{(2d+4)/d}\mathrm{d}r\bigg)^{d/(d+2)}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}^2\mathrm{d}r\bigg)\bigg\}\mathrm{d}t\mathrm{d}s \\
&\quad\le\bigg\{\mathbb{E}\bigg(\int_0^T\|u^{(N)}(r)\|_{L^2(\mathcal{O})}^{(2d+4)/d}\mathrm{d}r\bigg)^{2d/(d+2)}\bigg\}^{1/2} \\
&\qquad\times\bigg\{\mathbb{E}\bigg(\int_0^T\int_0^T|t-s|^{-1-2\alpha+2/(d+2)}\int_{s\wedge t}^{t\vee s}\|\nabla u^{(N)}(r)\|_{L^2(\mathcal{O})}^2\mathrm{d}r\mathrm{d}t\mathrm{d}s\bigg)^2\bigg\}^{1/2} \le C,
\end{aligned}
$$

where we used (41) in the last step, requiring that $1+2\alpha-2/(d+2)<2$ or $\alpha<(d+4)/(2d+4)$. Consequently,

$$
J_1 \le \int_0^T\int_0^T|t-s|^{-1-2\alpha}(J_{31}+J_{32})\mathrm{d}t\mathrm{d}s \le C.
$$

To estimate J_2, we use the embedding $L^2(\mathcal{O})\hookrightarrow H^m(\mathcal{O})'$, the Itô isometry, the linear growth of σ, and the Hölder inequality:

$$
J_2 \le C\int_0^T\int_0^T|t-s|^{-1-2\alpha}\mathbb{E}\bigg\|\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n\sigma_{ij}(u^{(N)}(r))\mathrm{d}W_j(r)\bigg\|_{L^2(\mathcal{O})}^2\mathrm{d}t\mathrm{d}s
$$

$$= C\int_0^T\int_0^T|t-s|^{-1-2\alpha}\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n\|\sigma_{ij}(u^{(N)}(r))\|_{L^2(\mathcal{O})}^2\mathrm{d}r\bigg)\mathrm{d}t\mathrm{d}s$$
$$\le C\int_0^T\int_0^T|t-s|^{-1-2\alpha}\int_{s\wedge t}^{t\vee s}\mathbb{E}\sum_{j=1}^n\big(1+\|u_j^{(N)}(r)\|_{L^2(\mathcal{O})}^2\big)\mathrm{d}r\mathrm{d}t\mathrm{d}s.$$

By (41) and estimate (29), the right-hand side is finite if $-1-2\alpha>-2$ or $\alpha<1/2$.

Step 2: Case without self-diffusion. Let $a_{i0}>0$ and $a_{ii}=0$ for all $i=1,\ldots,n$. We estimate the (normally) elliptic operator in J_1 by exploiting the special structure of the diffusion term and observing that the embedding $L^1(\mathcal{O})\hookrightarrow H^{m-1}(\mathcal{O})'$ is continuous:

$$\bigg\|\operatorname{div}\sum_{j=1}^n A_{ij}(u^{(N)})\nabla u_j^{(N)}\bigg\|_{H^m(\mathcal{O})'} = \bigg\|\nabla\bigg(a_{i0}u_i^{(N)}+\sum_{j\neq i}a_{ij}u_i^{(N)}u_j^{(N)}\bigg)\bigg\|_{H^{m-1}(\mathcal{O})'}$$
$$\le a_{i0}\|\nabla u_i^{(N)}\|_{L^1(\mathcal{O})}+\sum_{j\neq i}a_{ij}\|\nabla(u_i^{(N)}u_j^{(N)})\|_{L^1(\mathcal{O})}.$$

This yields, using the definition of J_3 from the previous case and the Hölder inequality,

$$J_3\le C\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla u_i^{(N)}(r)\|_{L^1(\mathcal{O})}\mathrm{d}r\bigg)^2+C\sum_{j\neq i}\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\|\nabla(u_i^{(N)}u_j^{(N)})(r)\|_{L^1(\mathcal{O})}\mathrm{d}r\bigg)^2$$
$$\le C|t-s|\mathbb{E}\int_{s\wedge t}^{t\vee s}\|\nabla u_i^{(N)}(r)\|_{L^1(\mathcal{O})}^2\mathrm{d}r$$
$$+C|t-s|^{2/(d+2)}\bigg(\sum_{j\neq i}\mathbb{E}\int_{s\wedge t}^{t\vee s}\|\nabla(u_i^{(N)}u_j^{(N)})(r)\|_{L^1(\mathcal{O})}^{\rho_1}\mathrm{d}r\bigg)^{2(d+1)/(d+2)}.$$

It follows from (30) and (38) that

$$J_1\le C\int_0^T\int_0^T\frac{|t-s|^{2/(d+2)}}{|t-s|^{1+2\alpha}}\mathrm{d}s\mathrm{d}t,$$

and this integral is finite if $2/(d+2)-1-2\alpha>-1$ or $\alpha<1/(d+2)$.

To estimate J_2, we use, similarly as in the case with self-diffusion, the embedding $L^2(\mathcal{O})\hookrightarrow H^m(\mathcal{O})'$, the Itô isometry, the sublinear growth of σ, and the Hölder inequality:

$$J_2 = C\int_0^T\int_0^T|t-s|^{-1-2\alpha}\mathbb{E}\bigg(\int_{s\wedge t}^{t\vee s}\sum_{j=1}^n\|\sigma_{ij}(u^{(N)}(r))\|_{L^2(\mathcal{O})}^2\mathrm{d}r\bigg)\mathrm{d}t\mathrm{d}s$$
$$\le C\int_0^T\int_0^T|t-s|^{-1-2\alpha}\int_{s\wedge t}^{t\vee s}\mathbb{E}\sum_{j=1}^n\big(1+\|u_j^{(N)}(r)\|_{L^2(\mathcal{O})}^{2\gamma}\big)\mathrm{d}r\mathrm{d}t\mathrm{d}s.$$

By (41) and estimate (32), the right-hand side is finite if $2\gamma\le 4/d$ and $-1-2\alpha>-2$ or $\alpha<1/2$. This finishes the proof. □

4. Proof of Theorem 2

We prove the existence of a martingale solution in the case with self-diffusion.

4.1. **Tightness of the laws of** $(u^{(N)})$**.** We show that the laws of $u^{(N)}$ are tight in a certain sub-Polish space. For this, we introduce the following spaces, recalling that $m > d/2+1$:

- $C^0([0,T];H^m(\mathcal{O})')$ is the space of continuous functions $u:[0,T]\to H^m(\mathcal{O})'$ with the topology $\mathbb{T}_1$ induced by the norm $\|u\|_{C^0([0,T];H^m(\mathcal{O})')} = \sup_{t\in(0,T)}\|u(t)\|_{H^m(\mathcal{O})'}$;
- $L^2_w(0,T;H^1(\mathcal{O}))$ is the space $L^2(0,T;H^1(\mathcal{O}))$ with the weak topology $\mathbb{T}_2$;

We define the space

$$\widetilde{Z}_T := C^0([0,T];H^m(\mathcal{O})')\cap L^2_w(0,T;H^1(\mathcal{O})),$$

endowed with the topology $\widetilde{\mathbb{T}}$ that is the maximum of the topologies of $C^0([0,T];H^m(\mathcal{O})')$ and $L^2_w(0,T;H^1(\mathcal{O}))$. Similar to the proof of [14, Lemma 12], it can be shown that $\widetilde{Z}_T$ is a sub-Polish space.

Lemma 12. *The set of laws* $(\mathcal{L}(u^{(N)}))_{N\in\mathbb{N}}$ *is tight in* $\widetilde{Z}_T$ *and in* $L^2(0,T;L^2(\mathcal{O}))$.

Proof. The tightness in $\widetilde{Z}_T$ follows from [4, Corollary 2.6] with the spaces $U = H^m(\mathcal{O})$ and $V = H^1(\mathcal{O})$. Indeed, estimate (19) is exactly condition (a) in [4, Corollary 2.6] and estimate (20) corresponds to condition (b). Condition (c), i.e., $(u^{(N)})$ satisfies the Aldous condition in $H^m(\mathcal{O})'$, can be verified as in the proof of Lemma 11 in [14]. Thus, the set of laws of $(u^{(N)})$ is tight in $\widetilde{Z}_T$.

Next, we introduce for given $\kappa>0$ and $\alpha<1/(d+2)$ the sets $X_T = W^{\alpha,2}(0,T;H^m(\mathcal{O})')\cap L^2(0,T;H^1(\mathcal{O}))$ and $B_\kappa = \{v\in X_T : \|v\|_{X_T}\le\kappa\}$. The compact embedding $X_T\hookrightarrow L^2(0,T;L^2(\mathcal{O}))$ [30, Corollary 5] implies that B_κ is a relatively compact set in $L^2(0,T;L^2(\mathcal{O}))$. We deduce from estimates (20) and (40) and the Chebyshev inequality that

$$\begin{aligned}\mathbb{P}\{\|u^{(N)}\|_{X_T}>\kappa\} &\le \kappa^{-2}\mathbb{E}\|u^{(N)}\|^2_{X_T} \\ &\le \kappa^{-2}\big(\mathbb{E}\|u^{(N)}\|^2_{W^{\alpha,2}(0,T;H^m(\mathcal{O})')} + \mathbb{E}\|u^{(N)}\|^2_{L^2(0,T;H^1(\mathcal{O}))}\big) \le C\kappa^{-2}.\end{aligned}$$

Then the result follows directly from the definition of tightness. □

It follows from the previous lemma that the set of laws $(\mathcal{L}(u^{(N)}))$ is tight in $Z_T = \widetilde{Z}_T\cap L^2(0,T;L^2(\mathcal{O}))$ with the topology $\mathbb{T}$ that is the maximum of $\widetilde{\mathbb{T}}$ and the topology induced by the $L^2(0,T;L^2(\mathcal{O}))$ norm.

4.2. **Strong convergence of** $(u^{(N)})$**.** Since $Z_T\times C^0([0,T];\mathbb{R}^n)$ satisfies the assumptions of the Skorokhod–Jabubowski theorem [5, Theorem C1] and the set of laws $(\mathcal{L}(u^{(N)}))$ is tight in $(Z_T,\mathbb{T})$, this theorem implies the existence of a subsequence of $(u^{(N)})$, which is not relabeled, a probability space $(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and, on this space, $(Z_T\times C^0([0,T];\mathbb{R}^n))$-valued random variables $(\widetilde u,\widetilde W)$ and $(\widetilde u^{(N)},\widetilde W^{(N)})$ for $N\in\mathbb{N}$ such that $(\widetilde u^{(N)},\widetilde W^{(N)})$ has the same law as $(u^{(N)},W)$ on $\mathcal{B}(Z_T\times C^0([0,T];\mathbb{R}^n))$ and, as $N\to\infty$,

$$(\widetilde u^{(N)},\widetilde W^{(N)})\to(\widetilde u,\widetilde W)\quad\text{in } Z_T\times C^0([0,T];\mathbb{R}^n)\ \widetilde{\mathbb{P}}\text{-a.s.}$$

Because of the definition of the space Z_T, this convergence means $\widetilde{\mathbb{P}}$-a.s.

$$\begin{aligned} \widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } C^0([0,T];H^m(\mathcal{O})'), \\ \widetilde{u}^{(N)} &\rightharpoonup \widetilde{u} \quad \text{weakly in } L^2(0,T;H^1(\mathcal{O})), \\ \widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } L^2(0,T;L^2(\mathcal{O})), \\ \widetilde{W}^{(N)} &\to \widetilde{W} \quad \text{in } C^0([0,T];\mathbb{R}^n). \end{aligned} \tag{43}$$

As in [14], we derive some regularity properties for the limit $\widetilde{u}$. We infer from the facts that $C^0([0,T];\,H_N)$ is a Borel set of $C^0([0,T];H^m(\mathcal{O})')\cap L^2(0,T;L^2(\mathcal{O}))$, $u^{(N)}$ is an element of $C^0([0,T];H_N)$ $\mathbb{P}$-a.s., and $u^{(N)}$ and $\widetilde{u}^{(N)}$ have the same law on $\mathcal{B}(Z_T)$ that $\mathcal{L}(\widetilde{u}^{(N)})(C^0([0,T];H_N))=1$. Note that $\widetilde{u}$ is a Z_T-Borel random variable since $\mathcal{B}(Z_T\times C^0([0,T];\mathbb{R}^n))$ is a subset of $\mathcal{B}(Z_T)\times\mathcal{B}(C^0([0,T];\mathbb{R}^n))$. We deduce from estimates (26) and (27) and the fact that $u^{(N)}$ and $\widetilde{u}^{(N)}$ have the same laws that for any $p\ge 2$,

$$\sup_{n\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\int_0^T\|\widetilde{u}^{(N)}(t)\|_{H^1(\mathcal{O})}^2\mathrm{d}t\bigg)^p+\sup_{n\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\sup_{0<t<T}\|\widetilde{u}^{(N)}(t)\|_{H^m(\mathcal{O})'}\mathrm{d}t\bigg)^p\le C.$$

In view of the embedding $L^1(\mathcal{O})\hookrightarrow H^m(\mathcal{O})'$, we infer the existence of a subsequence of $(\widetilde{u}^{(N)})$ (not relabeled) that is weakly converging in $L^p(\widetilde{\Omega};L^2(0,T;H^1(\mathcal{O})))$ and weakly* converging in $L^p(\widetilde{\Omega};L^\infty(0,T;H^m(\mathcal{O})'))$ as $N\to\infty$. In view of the convergence of $\widetilde{u}^{(N)}\to\widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s., we infer that the limit function satisfies

$$\widetilde{\mathbb{E}}\bigg(\int_0^T\|\widetilde{u}(t)\|_{H^1(\mathcal{O})}^2\mathrm{d}t\bigg)^p+\widetilde{\mathbb{E}}\bigg(\sup_{0<t<T}\|\widetilde{u}(t)\|_{H^m(\mathcal{O})'}\bigg)^p<\infty.$$

Let $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^{(N)}$ be the filtrations generated by $(\widetilde{u},\widetilde{W})$ and $(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$, respectively. Then we conclude from [3, Lemma 7] (also see [14, Lemmas 14–15]) that $\widetilde{u}$ is progressively measurable with respect to $\widetilde{\mathcal{F}}$, $\widetilde{u}^{(N)}$ is progressively measurable with respect to $\widetilde{\mathcal{F}}^{(N)}$, and $(\widetilde{W}(t))_{t\in[0,T]}$ and $(\widetilde{W}^{(N)}(t))_{t\in[0,T]}$ are Wiener processes with respect to the corresponding filtrations.

We know that $u_i^{(N)}$ is nonnegative for $i=1,\ldots,n$. It turns out that the limit $\widetilde{u}_i$ is also nonnegative. This is proved in the following lemma.

Lemma 13 (Nonnegativity)**.** *It holds that $\widetilde{u}_i(x,t)\ge 0$ for a.e. $(x,t)\in Q_T$ $\widetilde{\mathbb{P}}$-a.s. and $i=1,\ldots,n$.*

Proof. Let $i\in\{1,\ldots,n\}$. The entropy method used in Section 3.2 implies that $u_i^{(N)}$ is nonnegative, so $\mathbb{E}\|(u_i^{(N)})^-\|_{L^2(0,T;L^2(\mathcal{O}))}=0$, where $z^-=\min\{0,z\}$. The function $u_i^{(N)}$ is Z_T-Borel measurable and so does $(u_i^{(N)})^-$. Therefore, using the equivalence of the laws of $u_i^{(N)}$ and $\widetilde{u}_i^{(N)}$ on Z_T and setting $\mu^{(N)}:=\mathrm{Law}(u_i^{(N)})=\mathrm{Law}(\widetilde{u}_i^{(N)})$, we find that

$$\begin{aligned} \widetilde{\mathbb{E}}\|(\widetilde{u}_i^{(N)})^-\|_{L^2(0,T;L^2(\mathcal{O}))} &= \int_{L^2(0,T;L^2(\mathcal{O}))}\|y^-\|_{L^2(0,T;L^2(\mathcal{O}))}\mathrm{d}\mu^{(N)}(y) \\ &= \mathbb{E}\|(u_i^{(N)})^-\|_{L^2(0,T;L^2(\mathcal{O}))}=0. \end{aligned}$$

This shows that $\widetilde{u}_i^{(N)} \geq 0$ a.e. in Q_T $\widetilde{\mathbb{P}}$-a.s. The convergence (up to a subsequence) $\widetilde{u}^{(N)} \to \widetilde{u}$ a.e. in Z_T $\widetilde{\mathbb{P}}$-a.s. then implies that $\widetilde{u}_i \geq 0$ a.e. in Q_T $\widetilde{\mathbb{P}}$-a.s. □

The following lemma is needed to verify that $(\widetilde{u}, \widetilde{W})$ is a martingale solution to (3)–(4). In view of the previous convergence results, the proof is very similar to that one of Lemma 10 in [15] and therefore, we omit it.

Lemma 14. *It holds for all s, $t \in [0,T]$ with $s \leq t$, $i = 1, \ldots, n$, and all $\phi_1 \in L^2(\mathcal{O})$ and all $\phi_2 \in H^m(\mathcal{O})$ with $m > d/2+1$ and satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \big\langle \widetilde{u}_i^{(N)}(t) - \widetilde{u}_i(t), \phi_2 \big\rangle \mathrm{d}t = 0,$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \big\langle \widetilde{u}_i^{(N)}(0) - \widetilde{u}_i(0), \phi_2 \big\rangle = 0,$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \sum_{j=1}^n \int_0^t \Big\langle A_{ij}(\widetilde{u}^{(N)}(s)) \nabla \widetilde{u}_j^{(N)}(s) - A_{ij}(\widetilde{u}(s)) \nabla \widetilde{u}_j(s), \nabla\phi_2 \Big\rangle \mathrm{d}s \bigg| \mathrm{d}t = 0,$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \bigg| \sum_{j=1}^n \int_0^t \Big\langle \sigma_{ij}(\widetilde{u}^{(N)}(s)) \mathrm{d}\widetilde{W}_j^{(N)}(s) - \sigma_{ij}(\widetilde{u}(s)) \mathrm{d}\widetilde{W}_j(s), \phi_1 \Big\rangle \bigg|^2 \mathrm{d}t = 0.$$

Next, we define for $t \in [0,T]$ and $i = 1, \ldots, n$,

$$\begin{aligned} \Lambda_i^{(N)}\big(\widetilde{u}^{(N)}, \widetilde{W}^{(N)}, \phi\big)(t) &:= \big\langle \Pi_N(\widetilde{u}_i(0)), \phi \big\rangle \\ &\quad - \int_0^t \sum_{j=1}^n \big\langle \Pi_N(A_{ij}(\widetilde{u}^{(N)}(s)) \nabla \widetilde{u}_j^{(N)}(s)), \nabla\phi \big\rangle \mathrm{d}s \\ &\quad + \sum_{j=1}^n \bigg\langle \int_0^t \Pi_N \sigma_{ij}(\widetilde{u}^{(N)}(s)) \mathrm{d}\widetilde{W}_j^{(N)}(s), \phi \bigg\rangle, \end{aligned} \tag{44}$$

$$\begin{aligned} \Lambda_i\big(\widetilde{u}, \widetilde{W}, \phi\big)(t) &:= \langle \widetilde{u}_i(0), \phi \rangle - \sum_{j=1}^n \int_0^t \big\langle A_{ij}(\widetilde{u}(s)) \nabla \widetilde{u}_j(s), \nabla\phi \big\rangle \mathrm{d}s \\ &\quad + \sum_{j=1}^n \bigg\langle \int_0^t \sigma_{ij}(\widetilde{u}(s)) \mathrm{d}\widetilde{W}_j(s), \phi \bigg\rangle. \end{aligned} \tag{45}$$

The following corollary is essentially a consequence of Lemma 14; see [14, Corollary 17] for a proof.

Corollary 15. *It holds for any $\phi_1 \in L^2(\mathcal{O})$ and any $\phi_2 \in H^m(\mathcal{O})$ with $m > d/2+1$ and satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{N\to\infty} \big\| \langle \widetilde{u}_i^{(N)}, \phi_2 \rangle - \langle \widetilde{u}_i, \phi_2 \rangle \big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0,$$

$$\lim_{N\to\infty} \big\| \Lambda_i^{(N)}\big(\widetilde{u}^{(N)}, \widetilde{W}^{(N)}, \phi_1\big) - \Lambda_i\big(\widetilde{u}, \widetilde{W}, \phi_1\big) \big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

We proceed with the proof of Theorem 2. Since $\widetilde{u}^{(N)}$ is a strong solution to (11)–(12), it satisfies

$$\langle u_i^{(N)}(t), \phi\rangle = \Lambda_i^{(N)}(u^{(N)}, W, \phi)(t) \quad \text{for a.e. } t \in [0,T] \ \mathbb{P}\text{-a.s.}, \ i = 1, \dots, n,$$

for any $\phi \in H^m(\mathcal{O})$. Hence,

$$\int_0^T \mathbb{E}\big|\langle u_i^{(N)}(t), \phi\rangle - \Lambda_i^{(N)}(u^{(N)}, W, \phi)(t)\big|\mathrm{d}t = 0, \quad i = 1, \dots, n.$$

We deduce from the equivalence of the laws of $(u^{(N)}, W)$ and $(\widetilde{u}^{(N)}, \widetilde{W}^{(N)})$ that

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle \widetilde{u}_i^{(N)}(t), \phi\rangle - \Lambda_i^{(N)}(\widetilde{u}^{(N)}, \widetilde{W}^{(N)}, \phi)(t)\big|\mathrm{d}t = 0, \quad i = 1, \dots, n.$$

By Corollary 15, we can pass to the limit $N \to \infty$ to obtain

$$\int_0^T \widetilde{\mathbb{E}}\big|\langle \widetilde{u}_i(t), \phi\rangle - \Lambda_i(\widetilde{u}, \widetilde{W}, \phi)(t)\big|\mathrm{d}t = 0, \quad i = 1, \dots, n.$$

This identity holds for all $\phi \in H^m(\mathcal{O})$ such that $\nabla\phi \cdot \nu = 0$ on $\partial\mathcal{O}$ and hence, by density, for any $\phi \in H^m(\mathcal{O})$. This shows that

$$\big|\langle \widetilde{u}_i(t), \phi\rangle - \Lambda_i(\widetilde{u}, \widetilde{W}, \phi)(t)\big| = 0 \quad \text{for a.e. } t \in [0,T] \ \widetilde{\mathbb{P}}\text{-a.s.}, \ i = 1, \dots, n.$$

We infer from the definition of Λ_i that

$$\langle \widetilde{u}_i(t), \phi\rangle = \langle \widetilde{u}_i(0), \phi\rangle - \sum_{j=1}^n \int_0^t \big\langle A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi\big\rangle \mathrm{d}s + \sum_{j=1}^n \bigg\langle \int_0^t \sigma_{ij}(\widetilde{u}(s))\mathrm{d}\widetilde{W}_j(s), \phi\bigg\rangle$$

for a.e. $t \in [0,T]$ $\widetilde{\mathbb{P}}$-a.s. and all $\phi \in H^m(\mathcal{O})$. Set $\widetilde{U} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. Then $(\widetilde{U}, \widetilde{u}, \widetilde{W})$ is a martingale solution to (3)–(4). This finishes the proof.

5. Proof of Theorem 3

Next, we prove the existence of a martingale solution in the case without self-diffusion. The proof is similar to that one of Theorem 2, but we have less regularity for $u_i^{(N)}$ than in the self-diffusion case. Therefore, we need to adapt the function spaces. Moreover, since we do not have a uniform estimate for $u_i^{(N)}$ in $L^2(Q_T)$, the convergence of $\nabla(u_i^{(N)}u_j^{(N)})$ requires some care. Recall that we assume $d \le 3$.

5.1. **Tightness of the laws.** We show that the laws of $u^{(N)}$ are tight in the sub-Polish space

$$\widetilde{Z}_T := C^0([0,T]; H^3(\mathcal{O})') \cap L_w^{8/7}(0,T; W^{1,8/7}(\mathcal{O})),$$

endowed with the topology $\widetilde{\mathbb{T}}$ with is the maximum of the topology of $C^0([0,T]; H^3(\mathcal{O})')$ and the weak topology of $L_w^{8/7}(0,T; W^{1,8/7}(\mathcal{O}))$.

Lemma 16. *The sequence of laws $(\mathcal{L}(u^{(N)}))_{N\in\mathbb{N}}$ is tight in $\widetilde{Z}_T$, $L^{5/4}(0,T; L^2(\mathcal{O}))$, and $L^2(0,T; L^{3/2}(\mathcal{O}))$.*

Proof. The tightness in $\widetilde{Z}_T$ can be shown as in Lemma 12. Furthermore, the tightness in $L^{5/4}(0,T;L^2(\mathcal{O}))$ follows similarly as in Lemma 12, observing that the embedding $W^{\alpha,2}(0,T;H^3(\mathcal{O})')\cap L^{5/4}(0,T;W^{1,5/4}(\mathcal{O}))\hookrightarrow L^{5/4}(0,T;L^2(\mathcal{O}))$ is compact. Here, we use the facts that the embedding $W^{1,5/4}(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$ is compact if $d\le 3$ and that $(u_i^{(N)})$ is bounded in $W^{\alpha,2}(0,T;H^3(\mathcal{O})')$ and $L^{5/4}(0,T;W^{1,5/4}(\mathcal{O}))$ due to estimates (40) and (33), respectively. Finally, the last statement is a consequence of the compact embedding $W^{\alpha,2}(0,T;H^3(\mathcal{O})')\cap L^2(0,T;W^{1,1}(\mathcal{O}))\hookrightarrow L^2(0,T;L^p(\mathcal{O}))$ for any $p<3/2$ as well as estimates (40) and (30). In fact, the compactness in $L^2(0,T;L^p(\mathcal{O}))$ is valid up to $p=3/2$ by taking into account the uniform bound of $u_i^{(N)}\log u_i^{(N)}$ in $L^\infty(0,T;L^1(\mathcal{O}))$; see [2, Prop. 1]. □

The previous lemma shows that $(\mathcal{L}(u^{(N)}))$ is tight in

$$Z_T=\widetilde{Z}_T\cap L^2(0,T;L^{3/2}(\mathcal{O}))\cap L^{5/4}(0,T;L^2(\mathcal{O}))$$

with the topology that is the maximum of $\widetilde{\mathbb{T}}$ and the topologies induced by the $L^2(0,T;L^{3/2}(\mathcal{O}))$ and $L^{5/4}(0,T;L^2(\mathcal{O}))$ norms.

5.2. **Strong convergence of $(u^{(N)})$.** Applying the Shorokhod–Jabubowski theorem as in Section 4.2, we obtain the existence of a subsequence of $(u^{(N)})$ (not relabeled), a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and, on this space, $(Z_T\times C^0([0,T];\mathbb{R}^n))$-valued random variables $(\widetilde{u},\widetilde{W})$ and $(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$ for $N\in\mathbb{N}$ such that $(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$ has the same laws as $(u^{(N)},W)$ on $\mathcal{B}(Z_T\times C^0([0,T];\mathbb{R}^n))$. The convergence results (43) hold with the exception that only

$$\begin{aligned}
&\widetilde{u}^{(N)}\rightharpoonup\widetilde{u}\quad\text{weakly in } L^{8/7}(0,T;W^{1,8/7}(\mathcal{O})),\\
&\widetilde{u}^{(N)}\to\widetilde{u}\quad\text{in } L^2(0,T;L^{3/2}(\mathcal{O})),\\
&\widetilde{u}^{(N)}\to\widetilde{u}\quad\text{in } L^{5/4}(0,T;L^2(\mathcal{O}))
\end{aligned}\tag{46}$$

$\widetilde{\mathbb{P}}$-a.s. as $N\to\infty$. Moreover, similarly as in Section 4.2, we deduce from estimates (33) and (26) that

$$\sup_{n\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\int_0^T\|\widetilde{u}^{(N)}(t)\|_{W^{1,8/7}(\mathcal{O})}^{8/7}\mathrm{d}t\bigg)^p+\sup_{n\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\sup_{0<t<T}\|\widetilde{u}^{(N)}(t)\|_{H^m(\mathcal{O})'}\mathrm{d}t\bigg)^p\le C.$$

Because of the measurability of the map

$$C^0([0,T];H_N)\to L^{8/7}(0,T;W^{1,8/7}(\mathcal{O}))\cap Z_T,\, w\mapsto w^2,$$

the continuity of the norm on $L^{8/7}(0,T;W^{1,8/7}(\mathcal{O}))\cap Z_T$, the identity $u_i^{(N)}u_j^{(N)}=(u_i^{(N)}+u_j^{(N)})^2-(u_i^{(N)})^2-(u_j^{(N)})^2$, the equality of the laws of $u^{(N)}$ and $\widetilde{u}^{(N)}$ on $\mathcal{B}(Z_T)$, and estimates (36) and (39) for $d=3$, we obtain

$$\sup_{N\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\int_0^T\|\widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)}\|_{W^{1,8/7}(\mathcal{O})}^{8/7}\mathrm{d}t\bigg)^p\le C.\tag{47}$$

We infer the existence of a subsequence of $(\widetilde{u}^{(N)})$ (not relabeled) that is weakly converging in $L^p(\widetilde{\Omega};L^{8/7}(0,T;W^{1,8/7}(\mathcal{O})))$ and weakly* converging in $L^p(\widetilde{\Omega};C^0([0,T];H^3(\mathcal{O})'))$ as $N\to$

∞. Thus, taking into account the convergence $\widetilde{u}^{(N)} \to \widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s., we conclude that the limit function satisfies

$$\widetilde{\mathbb{E}}\bigg(\int_0^T \|\widetilde{u}(t)\|_{W^{1,8/7}(\mathcal{O})}^{8/7}\mathrm{d}t\bigg)^p + \widetilde{\mathbb{E}}\bigg(\sup_{0<t<T}\|\widetilde{u}(t)\|_{H^3(\mathcal{O})'}\bigg)^p < \infty \quad \text{for } p < \infty.$$

We verify similarly as in Lemma 13 that $\widetilde{u}_i(x,t) \geq 0$ for a.e. $(x,t) \in Q_T$ $\widetilde{\mathbb{P}}$-a.s. and $i = 1, \ldots, n$. The only difference to the proof is that we work in the space $L^{5/4}(0,T;L^2(\mathcal{O}))$ instead of $L^2(0,T;L^2(\mathcal{O}))$.

The following lemma allows us to identify the quadratic terms.

Lemma 17. *Let $(\widetilde{u}^{(N)})$ be the sequence of Z_T-valued random variables constructed above. Then it holds for $\phi_2 \in W^{1,8}(\mathcal{O})$ and satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\lim_{N\to\infty} \widetilde{\mathbb{E}}\int_0^T \bigg|\sum_{i,j=1,\,j\neq i}^n \int_0^t \big\langle \nabla(\widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)})(s) - \nabla(\widetilde{u}_i\widetilde{u}_j)(s), \nabla\phi_2\big\rangle \mathrm{d}s\bigg|\mathrm{d}t = 0.$$

Proof. We infer from estimate (46) that, for a subsequence, $\widetilde{u}_i^{(N)}(x,t) \to \widetilde{u}_i(x,t)$ and also $(\widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)})(x,t) \to (\widetilde{u}_i\widetilde{u}_j)(x,t)$ for a.e. $(x,t) \in Q_T$ $\widetilde{\mathbb{P}}$-a.s. Then $f^{(N)} := \widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)} - \widetilde{u}_i\widetilde{u}_j \to 0$ a.e. in Q_T $\widetilde{\mathbb{P}}$-a.s. Taking into account the uniform bound for $f^{(N)}$ in $L^p(\widetilde{\Omega}; L^{8/7}(0,T; W^{1,8/7}(\mathcal{O})))$ from (47), we conclude the strong convergence $f^{(N)} \to 0$ in $L^p(\widetilde{\Omega}\times\mathcal{O}\times(0,T))$ for any $1 \leq p < 8/7$. Next, let $g^{(N)} := f^{(N)}\Delta\phi$ for $\phi \in W^{2,\infty}(\mathcal{O})$. Then $g^{(N)} \to 0$ in Q_T $\widetilde{\mathbb{P}}$-a.s. and $g^{(N)}$ is bounded in $L^{8/7}(\widetilde{\Omega}\times\mathcal{O}\times(0,T))$. Therefore, $g^{(N)} \to 0$ strongly in $L^p(\widetilde{\Omega}\times\mathcal{O}\times(0,T))$ for any $1 < p < 8/7$. This shows that

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \sum_{i,j=1,\,j\neq i}^n \int_0^t \Big|\big\langle (\widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)})(s) - (\widetilde{u}_i\widetilde{u}_j)(s), \Delta\phi\big\rangle\Big|\mathrm{d}s = 0$$

for any $\phi \in W^{2,\infty}(\mathcal{O})$ and, by integrating by parts,

$$\begin{aligned}\lim_{N\to\infty} \widetilde{\mathbb{E}}\int_0^T \bigg|\sum_{i,j=1,\,j\neq i}^n \int_0^t \big\langle \nabla(\widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)})(s) - \nabla(\widetilde{u}_i\widetilde{u}_j)(s), \nabla\phi\big\rangle \mathrm{d}s\bigg|\mathrm{d}t \\ \leq \lim_{N\to\infty} T\widetilde{\mathbb{E}} \sum_{i,j=1,\,j\neq i}^n \int_0^T \Big|\big\langle (\widetilde{u}_i^{(N)}\widetilde{u}_j^{(N)})(s) - (\widetilde{u}_i\widetilde{u}_j)(s), \Delta\phi\big\rangle\Big|\mathrm{d}s = 0\end{aligned}$$

for any $\phi \in W^{2,\infty}(\mathcal{O})$ satisfying $\nabla\phi\cdot\nu = 0$ on $\partial\mathcal{O}$. By density, the convergence also holds for $\phi \in W^{1,8}(\mathcal{O})$ satisfying $\nabla\phi\cdot\nu = 0$ on $\partial\mathcal{O}$. ☐

Lemma 14, which is needed to show that $(\widetilde{u},\widetilde{W})$ is a martingale solution to (3)–(4), also holds in the present situation. The proof is similar to that one of Lemma 10 in [15] and uses the previous convergence results, convergences (46), Lemma 17, and the sublinear growth of the multiplicative noise.

Defining $\Lambda_i^{(N)}$ and Λ_i as in (44) and (45), respectively, the same result as in Corollary 15 holds. The proof of Theorem 3 can now be finished as in Section 4.2.

Appendix A. Deterministic SKT system without self-diffusion

The proof of the existence of a global weak solution to the deterministic two-species SKT system without self-diffusion in [9] uses an $L^2\log L^2$ bound coming from the Lotka–Volterra terms. We claim that the proof can be performed without this bound. To show this claim, we recall the estimates coming from the entropy inequality proved in [9]:

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \|\nabla(u_i^{(\tau)})^{1/2}\|_{L^2(Q_T)} + \|\nabla(u_i^{(\tau)}u_j^{(\tau)})^{1/2}\|_{L^2(Q_T)} \le C,$$

where $i\neq j$ and $C>0$ does not depend on the approximation parameter τ. In particular, $(u_i^{(\tau)})$ is bounded in $L^2(0,T;H^1(\mathcal{O}))$. The function $u_i^{(\tau)}$ is the solution to an approximate problem, which we do not specify here; we refer to [9].

First, we show that $(u_i^{(\tau)})$ is bounded in $L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O}))$, where $\rho_1=(d+2)/(d+1)$. To this end, we deduce from the Gagliardo–Nirenberg inequality with $\theta=d/2-d/p$ and $p=2+4/d$ (satisfying $p\theta=2$) that

$$\begin{aligned}\|(u_i^{(\tau)})^{1/2}\|_{L^{2+4/d}(Q_T)}^{2+4/d} &\le C\int_0^T \|(u_i^{(\tau)})^{1/2}\|_{H^1(\mathcal{O})}^{\theta(2d+4)/d}\|(u_i^{(\tau)})^{1/2}\|_{L^2(\mathcal{O})}^{(1-\theta)(2d+4)/d}\mathrm{d}t \\ &\le \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\mathcal{O}))}^{(1-\theta)(d+2)/d}\int_0^T \|(u_i^{(\tau)})^{1/2}\|_{H^1(\mathcal{O})}^2\mathrm{d}t \le C.\end{aligned}$$

This bound and the $L^2(Q_T)$ bound for $\nabla(u_i^{(\tau)})^{1/2}$ show that

$$\|\nabla u_i^{(\tau)}\|_{L^{\rho_1}(Q_T)} = 2\|(u_i^{(\tau)})^{1/2}\|_{L^{2+4/d}(Q_T)}\|\nabla(u_i^{(\tau)})^{1/2}\|_{L^2(Q_T)} \le C.$$

The claim now follows from the Poincaré–Wirtinger inequality and the bound for $(u_i^{(\tau)})$ in $L^\infty(0,T;L^1(\mathcal{O}))$.

Second, we claim that $(\nabla(u_i^{(\tau)}u_j^{(\tau)})^{1/2})$ is bounded in $L^{\rho_2}(Q_T)$ for $i\neq j$, where $\rho_2=(2d+2)/(2d+1)$. Indeed, a similar argument as above, using the bounds for $(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$ in $L^\infty(0,T;L^1(\mathcal{O}))$ and $\nabla(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$ in $L^2(Q_T)$ as well as the Gagliardo–Nirenberg inequality, shows that $((u_i^{(\tau)}u_j^{(\tau)})^{1/2})$ is bounded in $L^{2+2/d}(Q_T)$. Therefore, the sequence

$$\nabla(u_i^{(\tau)}u_j^{(\tau)}) = 2(u_i^{(\tau)}u_j^{(\tau)})^{1/2}\nabla(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$$

is bounded in $L^{\rho_2}(Q_T)$, proving the claim.

For the compactness, we also need an estimate for the time derivative:

$$\|\partial_t u_i^{(\tau)}\|_{L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O})')} \le \bigg\|a_{i0}\nabla u_i^{(\tau)} + \sum_{j=1,\,j\neq i}^n a_{ij}\nabla(u_i^{(\tau)}u_j^{(\tau)})\bigg\|_{L^{\rho_2}(Q_T)} \le C.$$

(In fact, in the proof of [9], we have to replace $\partial_t u_i$ by a discrete time derivative, but this does not change the argument.) By the Aubin–Lions lemma, there exists a subsequence of $(u_i^{(\tau)})$ (not relabeled) such that $u_i^{(\tau)}\to u_i$ strongly in $L^{\rho_1}(Q_T)$ and a.e. in Q_T as $\tau\to 0$. Thus, $u_i^{(\tau)}u_j^{(\tau)}\to u_iu_j$ a.e. in Q_T. The bound for $(\nabla(u_i^{(\tau)}u_j^{(\tau)}))$ in $L^{\rho_2}(Q_T)$ for $i\neq j$ implies that $\nabla(u_i^{(\tau)}u_j^{(\tau)})\rightharpoonup\nabla(u_iu_j)$ weakly in $L^{\rho_2}(Q_T)$. Furthermore, we have the convergences

$\nabla u_i^{(\tau)} \rightharpoonup \nabla u_i$ weakly in $L^{\rho_1}(Q_T)$ and $\partial_t u_i^{(\tau)} \rightharpoonup \partial_t u_i$ weakly in $L^{\rho_2}(0,T;W^{1,\rho_2}(\mathcal{O})')$. These limits allow us to pass to the limit $\tau \to 0$ in the approximate problem. Moreover, we obtain the regularity results formulated in Section 3.2. As a corollary, we deduce the following existence result which extends [9, Theorem 1] to the n-species no-reaction case.

Theorem 18 (Existence for the deterministic system)**.** *Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$), $u^0 \in L^\infty(\mathcal{O};\mathbb{R}^n)$ with $u_i \geq 0$ a.e. in $\mathcal{O}$, let the detailed-balance condition* (9) *hold, and let $a_{i0} > 0$, $a_{ii} = 0$ for $i = 1, \ldots, n$. Then there exists a weak solution $u = (u_1, \ldots, u_n)$ to*

$$\partial_t u_i = \Delta\bigg(a_{i0}u_i + \sum_{j=1,\, j\neq i}^n a_{ij}u_iu_j\bigg) \quad \text{in } \mathcal{O},\ t > 0,\ i = 1, \ldots, n,$$
$$u_i(0) = u_i^0 \quad \text{in } \mathcal{O}, \quad \nabla u_i \cdot \nu = 0 \quad \text{on } \partial\mathcal{O},\ t > 0,$$

satisfying $u_i(t) \geq 0$ a.e. in $\mathcal{O}$, $t > 0$ and

$$u_i \in L^{\rho_1}(0,T;W^{1,\rho_1}(\mathcal{O})), \quad u_iu_j \in L^{\rho_2}(0,T;W^{1,\rho_2}(\mathcal{O})), \quad \partial_t u_i \in L^{\rho_2}(0,T;W^{1,\rho_2}(\mathcal{O})')$$

for $i = 1, \ldots, n$, where $\rho_1 = (d+2)/(d+1)$ and $\rho_2 = (2d+2)/(2d+1)$.

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