CORRIGENDUM: CROSS DIFFUSION PREVENTING BLOW UP IN THE TWO-DIMENSIONAL KELLER-SEGEL MODEL

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ABSTRACT. We correct the proof of Proposition 2.1 in the paper [1]. This proposition is used to prove the existence of global weak solutions to a Keller-Segel model with additional cross-diffusion.

In [1], the following result has been stated.

**Proposition 1** (Proposition 2.1 in [1]). Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with $\partial \Omega \in C^{0,1}$, $T > 0$, and $s \geq 0$. Furthermore, let $(u_\varepsilon)$ be a sequence of nonnegative functions satisfying

$$
\|\sqrt{u_\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|u_\varepsilon \log u_\varepsilon\|_{L^1(0,T;L^1(\Omega))} + \|\partial_t u_\varepsilon\|_{L^1(0,T;(H^s(\Omega))^*)} \leq C_0
$$

for some $C > 0$ independent of $\varepsilon$. Then, up to a subsequence, as $\varepsilon \to 0$,

$$
u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;L^{d/(d-1)}(\Omega)).$$

The proof in [1] consists in showing first that $u_\varepsilon \to u$ strongly in $L^\infty(0,T;L^1(\Omega))$ as $\varepsilon \to 0$. However, this is generally wrong as the counter-example $u_\varepsilon(x,t) = \max\{0, 1-t/\varepsilon\}$ shows. In the following, we give a corrected proof for Proposition 2.1.

**Proof.** The uniform estimate for $u_\varepsilon$ implies that $\nabla u_\varepsilon = 2\sqrt{u_\varepsilon} \nabla \sqrt{u_\varepsilon}$ is uniformly bounded in $L^2(0,T;L^1(\Omega))$. Thus, $(u_\varepsilon)$ is bounded in $L^2(0,T;W^{1,1}(\Omega))$. We observe that the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $1 < p < d/(d-1)$. Moreover, if $s \geq d/2$, the embedding $H^s(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $p^* = p/(p-1)$, and hence $L^p(\Omega) \hookrightarrow (H^s(\Omega))^*$ is continuous. Thus, we can apply the Aubin-Lions lemma with the spaces $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow (H^s(\Omega))^*$. If $0 \leq s < d/2$, we have $(H^s(\Omega))^* \hookrightarrow (H^{d/2}(\Omega))^*$ and $\|\partial_t u_\varepsilon\|_{L^1(0,T;(H^{d/2}(\Omega))^*)} \leq C\|\partial_t u_\varepsilon\|_{L^1(0,T;(H^s(\Omega))^*)} \leq C\varepsilon_0$, and the Aubin-Lions lemma can be applied with the spaces $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow (H^{d/2}(\Omega))^*$. In both cases, there exists a subsequence of $(u_\varepsilon)$, which is not relabeled, such that $u_\varepsilon \rightharpoonup u$ strongly in $L^2(0,T;L^1(\Omega))$ and also a.e. in $\Omega \times (0,T)$.

Let $L > e$ be given, and set $v^u_\varepsilon = \min\{u_\varepsilon, L\}$ and $w^L_\varepsilon = \max\{u_\varepsilon - L, 0\}$. Then $u_\varepsilon = v^u_\varepsilon + w^L_\varepsilon$. It holds that $v^u_\varepsilon \to v^u$ and $w^L_\varepsilon \to w^L$ a.e. with $v^u = \min\{u, L\}$ and $w^L = \max\{u - L, 0\}$. We thank Martin Vetter, BSc for pointing out this example to us.

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u = v^L + w^L. Then

\[\|u_\varepsilon - u\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} \leq \|v^L_\varepsilon - v^L\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} + \|w^L_\varepsilon\|_{L^2(0,T;L^{d/(d-1)}(\Omega))} + \|w^L\|_{L^2(0,T;L^{d/(d-1)}(\Omega))}\]

(1)

We first estimate the terms \(I_2\) and \(I_3\). By the Hölder’s inequality, we find that

\[
\|\nabla w^L_\varepsilon\|_{L^2(0,T;L^1(\Omega))}^2 = \int_0^T \left( \int_{\{|u_\varepsilon > L\}} |\nabla u_\varepsilon| \, dx \right)^2 \, dt \\
= 4 \int_0^T \left( \int_{\{|u_\varepsilon > L\}} \sqrt{u_\varepsilon} \sqrt{|\nabla u_\varepsilon|} \, dx \right)^2 \, dt \\
\leq 4\|\sqrt{u_\varepsilon}\|_{L^2(0,T;H^1(\Omega))}^2 \left\| \int_{\{|u_\varepsilon > L\}} u_\varepsilon \, dx \right\|_{L^\infty(0,T)} \\
\leq 4C_0^2 \left\| \int_{\{|u_\varepsilon > L\}} u_\varepsilon \frac{\log u_\varepsilon}{\log L} \, dx \right\|_{L^\infty(0,T)} \leq \frac{4C_0^3}{\log L}
\]

and

\[
\|w^L_\varepsilon\|_{L^2(0,T;L^1(\Omega))}^2 = \int_0^T \left( \int_{\{|u_\varepsilon > L\}} (u_\varepsilon - L) \, dx \right)^2 \, dt \leq \int_0^T \left( \int_{\{|u_\varepsilon > L\}} u_\varepsilon \, dx \right)^2 \, dt \\
\leq T \left\| \int_{\{|u_\varepsilon > L\}} u_\varepsilon \frac{\log u_\varepsilon}{\log L} \, dx \right\|_{L^\infty(0,T)}^2 \leq \frac{T C_0^2}{\log^2 L} \leq \frac{T C_0^2}{\log L}.
\]

Therefore,

\[
\|u_\varepsilon\|_{L^2(0,T;W^{1,1}(\Omega))} \leq \frac{2C_0^{3/2} + T^{1/2}C_0}{(\log L)^{1/2}}.
\]

A similar way, it follows that

\[
\|w^L\|_{L^2(0,T;W^{1,1}(\Omega))} \leq \frac{2C_0^{3/2} + T^{1/2}C_0}{(\log L)^{1/2}}.
\]

We conclude from the Sobolev imbedding \(W^{1,1}(\Omega) \hookrightarrow L^{d/(d-1)}(\Omega) (d \geq 2)\) with the constant \(C_d > 0\) that

\[
I_2 + I_3 \leq C_d \|w^L_\varepsilon\|_{L^2(0,T;W^{1,1}(\Omega))} + C_d \|w^L\|_{L^2(0,T;W^{1,1}(\Omega))} \leq \frac{2C_d(2C_0^{3/2} + T^{1/2}C_0)}{(\log L)^{1/2}}.
\]

For the estimate of \(I_1\), we observe that, since \(|v^L_\varepsilon(x,t) - v^L(x,t)| \leq (2L)^{d/(d-1)} \) and \(|v^L_\varepsilon(x,t) - v^L(x,t)|d/(d-1) \to 0\) a.e. in \(\Omega \times (0,T)\), the dominated convergence theorem implies that \(\int_\Omega |v^L_\varepsilon(x,t) - v^L(x,t)|d/(d-1) \, dx \to 0\) a.e. in \((0,T)\) and hence \(\|v^L_\varepsilon(\cdot,t) - v^L(\cdot,t)\|_{L^d/(d-1)(\Omega)} \to 0\) a.e. in \((0,T)\). Moreover,

\[
\left\| \|v^L_\varepsilon(\cdot,t) - v^L(\cdot,t)\|_{L^d/(d-1)(\Omega)} \right\|_{L^3(0,T)} \leq 2L|\Omega|^{(d-1)/d}T^{3/2} =: C(L).
\]
We claim that for any $L > e$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, it holds that $I_1 < 1/\log L$. Indeed, set $f^L_\varepsilon(t) := \|v^L_\varepsilon(\cdot,t) - v^L(\cdot,t)\|_{L^{d/(d-1)}(\Omega)}$. Recall that $f^L_\varepsilon(t) \to 0$ a.e. in $(0,T)$ and $\|f^L_\varepsilon\|_{L^3(0,T)} \leq C(L)$. For given $\delta > 0$, there exists $\eta > 0$ such that $C(L)^2 \eta^{1/3} \leq \delta$. We deduce for any $E \subset (0,T)$ such that $|E| \leq \eta$ and Hölder's inequality that
\[
\int_E |f^L_\varepsilon(t)|^2 dt \leq \left( \int_E |f^L_\varepsilon(t)|^3 dt \right)^{2/3} |E|^{1/3} \leq C(L)^2 \eta^{1/3} \leq \delta,
\]
which shows that $(f^L_\varepsilon)$ is uniformly integrable. As convergence a.e. in $(0,T)$ implies convergence in measure in $(0,T)$, we can apply the Vitali convergence theorem to infer that $f^L_\varepsilon \to 0$ strongly in $L^2(0,T)$ as $\varepsilon \to 0$. Thus, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, we have $I_1 = \|f^L_\varepsilon\|_{L^2(0,T)} \leq 1/\log L$. Therefore, for any $L > e$, there exists $\varepsilon_0(L) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(L)$, we infer from (1) that
\[
\|u_\varepsilon - u\|_{L^2(0;L^d/(d-1)(\Omega))} \leq \frac{1 + 2C_d(2C_0^3/2 + T^{1/2}C_0)}{(\log L)^{1/2}}.
\]
Since $L > e$ is arbitrary, this ends the proof. \hfill $\Box$

References


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