ERRATUM: ANALYSIS OF DEGENERATE CROSS-DIFFUSION POPULATION MODELS WITH VOLUME FILLING

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Abstract. This note corrects Lemma 7 in [1] on the positive (semi-) definiteness of a certain matrix product, which yields a priori estimates for the cross-diffusion system.

1. Introduction

In our paper [1], we proved the global-in-time existence of bounded weak solutions to a certain class of degenerate cross-diffusion systems for the particle densities $u(x,t) = (u_1, \ldots, u_n)$, where $x \in \Omega \subset \mathbb{R}^d$ is the spatial variable and $t \geq 0$ is the time. The proof is based on an entropy method, i.e., we introduced a scalar functional $H[u] = \int_{\Omega} h(u) dx$ (called an entropy), which turns out to be not only a Lyapunov functional along the solutions but it also provides gradient estimates. A crucial step of the proof is the observation that the product between the Hessian $H := h''(u) \in \mathbb{R}^{n \times n}$ and the diffusion matrix $A = A(u) \in \mathbb{R}^{n \times n}$ is positive definite (non-uniformly in $u$). The proof of this observation (Lemma 7 in [1]) is wrong. In this note, we will correct the proof.

We introduce the hypertriangle

$$\mathcal{D} = \left\{ u \in \mathbb{R}^n : u_i > 0 \text{ for } i = 1, \ldots, n, \sum_{j=1}^{n} u_j < 1 \right\}.$$ 

The matrix coefficients of $A(u)$ contain nonlinear functions (see (3) in [1]) for which the following structural hypotheses have been imposed: There exist functions $q : [0,1] \to \mathbb{R}$, $\chi : \mathcal{D} \to \mathbb{R}$ and a number $\gamma > 0$ such that for all $i = 1, \ldots, n$,

1. $q(s) := q_i(s) > 0$, $q'(s) \geq \gamma q(s)$ for $s \in (0,1)$, $q(0) = 0$, $q \in C^3([0,1])$,

2. $p_i(u) = \exp \left( \frac{\partial \chi(u)}{\partial u_i} \right)$ for $u \in \mathcal{D}$, $\chi \geq 0$ is convex on $\mathcal{D}$, $\chi \in C^3(\mathcal{D})$, $p_i$ is assumed to be positive on $\mathcal{D}$. We introduce the following nonnegative number:

$$\kappa = \sup_{u \in \mathcal{D}} \sup_{z \in \mathbb{R}^n, \|z\| = 1} \left( \sum_{i,j=1}^{n} \sqrt{u_i u_j} \frac{\partial^2 \chi}{\partial u_i \partial u_j} z_i z_j \right)^2.$$ 

Date: May 3, 2016.

2010 Mathematics Subject Classification. 35K51, 35K65, 35Q92, 92D25.

Key words and phrases. Cross diffusion, population dynamics.

The authors thank X. Chen (Beijing) for pointing out the mistake in the proof of Lemma 7 in [1]. They acknowledge partial support from the Austrian Science Fund (FWF), grants P24304, P27352, and W1245.
The following result replaces Lemma 7 in [1].

**Lemma 1.** Assume that (1)-(2) hold. Let \( \eta \in (0,1] \) be any number such that \( \eta \kappa < 1 \), where \( \kappa \) is defined in (3). Then it holds for all \( u \in \mathcal{D} \) and \( v \in \mathbb{R}^n \) that

\[
v^\top (HA)v \geq p_0 c_1 q(u_{n+1}) \sum_{i=1}^{n} v_i^2 \frac{1}{u_i} + p_0 c_2 \sum_{i=1}^{n} \frac{q'(u_{n+1})^2}{q(u_{n+1})} \left( \sum_{i=1}^{n} v_i \right)^2,
\]

where \( p_0 := \min_{i=1,...,n} \inf_{u \in \mathcal{D}} p_i(u) > 0 \),

\[
c_1 = 1 - \eta \kappa > 0, \quad c_2 = \min \left\{ \frac{\eta}{4q(1/2)}, \frac{2}{\sup_{1/2 \leq \sigma \leq 1} q'(\sigma)} \right\} > 0.
\]

**2. Proof of Lemma 1**

Let \( u = (u_i) \in \mathcal{D} \) and set \( \varphi = q'/q \). It is shown in the proof of Lemma 7 in [1] that

\[
\frac{1}{q}(HA)_{ij} = \delta_{ij} \frac{p_i}{u_i} + \frac{\partial p_i}{\partial u_j} + \frac{\partial p_j}{\partial u_i} + \sum_{k=1}^{n} \frac{u_k}{p_k} \frac{\partial p_k}{\partial u_i} \frac{\partial p_k}{\partial u_j}
\]

\[
+ \varphi \left( p_i + p_j + \sum_{k=1}^{n} u_k \left( \frac{\partial p_k}{\partial u_i} + \frac{\partial p_k}{\partial u_j} \right) \right) + \varphi^2 \sum_{k=1}^{n} u_k p_k.
\]

Observing that \( \partial p_i/\partial u_j = p_i \partial^2 \chi/\partial u_i \partial u_j \) and setting \( \chi_{ij} = \partial^2 \chi/\partial u_i \partial u_j \), the previous identity can be formulated as

\[
\frac{1}{q}(HA)_{ij} = \delta_{ij} \frac{p_i}{u_i} + (p_i + p_j) \chi_{ij} + \sum_{k=1}^{n} u_k p_k \chi_{ki} \chi_{kj}
\]

\[
+ \varphi \left( p_i + p_j + \sum_{k=1}^{n} u_k p_k (\chi_{ki} + \chi_{kj}) \right) + \varphi^2 \sum_{k=1}^{n} u_k p_k
\]

\[
=: I_{ij} + \varphi J_{ij} + \varphi^2 K_{ij}.
\]

Let \( v \in \mathbb{R}^n \) and define \( w_i = v_i/\sqrt{u_i} \). First, we reformulate the quadratic forms associated to \( I = (I_{ij}) \), \( J = (J_{ij}) \), and \( K = (K_{ij}) \):

\[
v^\top I v = \sum_{i=1}^{n} \frac{p_i}{u_i} v_i^2 + 2 \sum_{i,j=1}^{n} p_i \chi_{ij} v_i v_j + \sum_{k=1}^{n} u_k p_k \left( \sum_{i=1}^{n} \chi_{ki} v_i \right)^2
\]

\[
= \sum_{i=1}^{n} p_i w_i^2 + 2 \sum_{i,j=1}^{n} p_i \sqrt{u_i u_j} \chi_{ij} w_i w_j + \sum_{i=1}^{n} p_i \left( \sum_{j=1}^{n} \sqrt{u_i u_j} \chi_{ij} w_j \right)^2
\]

\[
= \sum_{i=1}^{n} p_i \left( w_i + \sum_{j=1}^{n} \sqrt{u_i u_j} \chi_{ij} w_j \right)^2,
\]

\[
v^\top J v = 2 \left( \sum_{k=1}^{n} v_k \right) \left( \sum_{i=1}^{n} p_i v_i + \sum_{i,j=1}^{n} u_i p_i \chi_{ij} v_j \right) = 2 \sum_{i,k=1}^{n} p_i v_i v_k + 2 \sum_{i,j,k=1}^{n} u_i p_i \chi_{ij} v_j v_k
\]
\[ v^\top K v = \sum_{i=1}^{n} p_{i} \left( \sum_{j=1}^{n} v_{j} \right) = \sum_{i=1}^{n} p_{i} \left( \sum_{j=1}^{n} \sqrt{u_{i} u_{j}} w_{j} \right). \]

By definition of \( p_{0} \), we deduce that
\[ \frac{1}{p_{0}} v^\top (HA) v \geq \sum_{i=1}^{n} \left( w_{i} + \sum_{j=1}^{n} \sqrt{u_{i} u_{j}} w_{j} + \varphi \sum_{j=1}^{n} \sqrt{u_{i} u_{j}} w_{j} \right)^{2}. \]

This shows that \( HA \) is positive semidefinite.

Next, we set \( M_{ij} = \sqrt{u_{i} u_{j}} \chi_{ij} \) and \( N_{ij} = \varphi \sqrt{u_{i} u_{j}} \). Then
\[ (p_{0} q)^{-1} v^\top (HA) v \geq |w|^{2} + 2 w^{\top} (M + N) w + |M w + N w|^{2}, \]

where \( w = (w_{i}) \), \( M = (M_{ij}) \), \( N = (N_{ij}) \). We employ the fact that \( M \) is symmetric positive semidefinite:
\[ (p_{0} q)^{-1} v^\top (HA) v \geq |w|^{2} + 2 w^{\top} (M + N) w + |M w + N w|^{2}, \]

where \( \eta \in (0, 1] \) is arbitrary. By definition of \( \kappa \), \( |M w|^{2} \leq \kappa |w|^{2} \), and thus, \( |M w + N w|^{2} \geq \frac{1}{2} |N w|^{2} - \kappa |w|^{2} \). We conclude that
\[ (p_{0} q)^{-1} v^\top (HA) v \geq (1 - \eta \kappa) |w|^{2} + 2 w^{\top} N w + \frac{\eta}{2} |N w|^{2}. \]

Since \( \sum_{i=1}^{n} u_{i} = 1 - u_{n+1} \), we have
\[ |w|^{2} = \sum_{i=1}^{n} \frac{v_{i}^{2}}{u_{i}}, \quad w^{\top} N w = \varphi \left( \sum_{j=1}^{n} v_{j} \right)^{2}, \quad |N w|^{2} = \varphi^{2} (1 - u_{n+1}) \left( \sum_{j=1}^{n} v_{j} \right)^{2}, \]

and consequently,
\[ (p_{0} q)^{-1} v^\top (HA) v \geq (1 - \eta \kappa) \sum_{i=1}^{n} \frac{v_{i}^{2}}{u_{i}} + \varphi \left( 2 + \frac{\eta}{2} (1 - u_{n+1}) \right) \left( \sum_{j=1}^{n} v_{j} \right)^{2}. \]

This estimate replaces (25) in [1].

Now, we proceed similarly as in the proof of Lemma 7 in [1]. The inequalities
\[ 2 + \frac{\eta}{2} (1 - s) \varphi(s) \geq \frac{\eta}{2} (1 - s) \varphi(s) \geq \frac{\eta}{4} \frac{q'(s)}{q(1/2)} \quad \text{for } 0 \leq s \leq \frac{1}{2}, \]
\[ 2 + \frac{\eta}{2} (1 - s) \varphi(s) \geq 2 \geq \frac{2 q'(s)}{\sup_{1/2 \leq \sigma \leq 1} q' \sigma} \quad \text{for } 1/2 \leq s \leq 1 \]

imply that \( 2 + \frac{\eta}{2} (1 - u_{n+1}) \varphi(u_{n+1}) \geq c_{2} q'(u_{n+1}) \), which shows the conclusion.
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