COMPACT FAMILIES OF PIECEWISE CONSTANT FUNCTIONS IN $L^p(0, T; B)$

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Abstract. A strong compactness result in the spirit of the Lions-Aubin-Simon lemma is proven for piecewise constant functions in time $(u_\tau)$ with values in a Banach space. The main feature of our result is that it is sufficient to verify one uniform estimate for the time shifts $u_\tau - u_\tau(\cdot - \tau)$ instead of all time shifts $u_\tau - u_\tau(\cdot - h)$ for $h > 0$, as required in Simon’s compactness theorem. This simplifies significantly the application of the Rothe method in the existence analysis of parabolic problems.

1. Introduction

A useful technique to prove the existence of weak solutions to nonlinear evolution equations and their systems is to semi-discretize the equations in time by the implicit Euler method (also called Rothe method [5]):

$$\frac{1}{\tau}(u_\tau(t) - u_\tau(t - \tau)) + A(u_\tau(t)) = f_\tau(t), \quad \tau \leq t < T, \quad u_\tau(0) \text{ given},$$

where $\tau > 0$ is the time step, $A$ is an abstract (nonlinear) operator defined on a certain Banach space, and $f_\tau$ is some (piecewise constant) function with values in a Banach space. In this way, nonlinear elliptic problems are obtained which are sometimes easier to solve. In order to pass to the limit of vanishing time steps, $\tau \to 0$, (relative) compactness for the sequence of piecewise constant approximate solutions $(u_\tau)$ is needed. Since the problem is nonlinear, we need strong convergence of (a subsequence of) $(u_\tau)$ to identify the limit. If the governing operator is monotone, the limit can be identified using Minty’s trick (see, e.g., [6, Lemma 2.13]). Having suitable a priori estimates at hand, strong compactness can be concluded from the Aubin (or Lions-Aubin-Simon) lemma [7] which is a consequence of a compactness criterion due to Kolmogorov. However, the results of [7] are not directly applicable. Indeed, typically one can derive the uniform estimate

$$\|u_\tau - u_\tau(\cdot - \tau)\|_{L^1(\tau, T; Y)} = \tau \|A(u_\tau) + f_\tau\|_{L^1(\tau, T; Y)} \leq C\tau,$$

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where $C > 0$ does not depend on $\tau$, and $Y$ is some Banach space. On the other hand, in order to apply the Aubin lemma, one needs \cite[Theorem 3]{Andreianov2002}

\begin{equation}
\|u_\tau - u_\tau(\cdot - h)\|_{L^1(h,T;Y)} \to 0 \quad \text{as } h \to 0, \text{ uniformly in } \tau > 0.
\end{equation}

A possible way to avoid this problem is to construct linear interpolants of $u_\tau$, say $\tilde{u}_\tau$, for which a continuous time-derivative version of the Aubin lemma can be applied, giving $\tilde{u} \to u$ in $L^1(0,T;B)$ as $\tau \to 0$ for some Banach space $B$ \cite[Corollary 4]{Andreianov2002}. Since we need strong convergence of $(u_\tau)$, one has to show that $u_\tau - \tilde{u}_\tau \to 0$ in $L^1(0,T;B)$, which might be difficult to prove (see Section 4 for a situation in which such a proof is possible).

In this note, we show that estimate (2) suffices to infer strong compactness of $(u_\tau)$. The main feature of our result is that it is sufficient to study the time shifts $u_\tau - u_\tau(\cdot - \tau)$ instead of all time shifts $u_\tau - u_\tau(\cdot - h)$ for all $h > 0$. This simplifies the proof of the limit $\tau \to 0$ in (1) significantly.

For our main results, let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, and set $t_k = k\tau$, $k = 0, \ldots, N$. Furthermore, let $(S_h u)(x,t) = u(x,t-h)$, $t \geq h > 0$, be the shift operator. We notice that quasi-uniform time steps may be considered too \cite{Berti2000}, but they are of minor interest in the existence analysis.

**Theorem 1.** Let $X$, $B$, and $Y$ be Banach spaces such that the embedding $X \hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Furthermore, let either $1 \leq p < \infty$, $r = 1$ or $p = \infty$, $r > 1$, and let $(u_\tau)$ be a sequence of functions, which are constant on each subinterval $(t_{k-1}, t_k)$, satisfying

\begin{equation}
\tau^{-1} \|u_\tau - S_\tau u_\tau\|_{L^r(\tau,T;Y)} + \|u_\tau\|_{L^p(0,T;X)} \leq C_0 \quad \text{for all } \tau > 0,
\end{equation}

where $C_0 > 0$ is a constant which is independent of $\tau$. If $p < \infty$, then $(u_\tau)$ is relatively compact in $L^p(0,T;B)$. If $p = \infty$, there exists a subsequence of $(u_\tau)$ which converges in each space $L^q(0,T;B)$, $1 \leq q < \infty$, to a limit which belongs to $C^0([0,T];B)$.

A related result in finite-dimensional spaces was recently proven by Gallouët and Latché \cite[Theorem 3.4]{Gallouet2020}. The same setting for degenerate elliptic-parabolic equations in $L^1$ was considered by Andreianov \cite{Andreianov2002}. In view of (3), one may conjecture that the condition $\|u_\tau - S_\tau u_\tau\|_{L^r(\tau,T;Y)} = O(\tau^\alpha)$ as $\tau \to 0$ with $0 < \alpha < 1$ instead of $O(\tau)$ is sufficient to obtain relative compactness. The following result shows that this is not the case (also see Theorem 5 below).

**Proposition 2.** The factor $\tau^{-1}$ in inequality (4) cannot be replaced by $\tau^{-\alpha}$ for $0 < \alpha < 1$.

This note is organized as follows. In Section 2, Theorem 1 is shown; the proof of Proposition 2 is presented in Section 3. Finally, we comment these results in Section 4.

2. **Proof of Theorem 1**

The proof of Theorem 1 is based on a characterisation of the norm of fractional Sobolev spaces. Let $1 \leq q < \infty$, $0 < \sigma < 1$, and let $Y$ be a Banach space. The fractional Sobolev
space $W^{\sigma,q}(0, T; Y)$ is the space of (equivalence classes of) functions $u \in L^q(0, T; Y)$ with finite Slobodeckii norm
\[
\|u\|_{W^{\sigma,q}(0, T; Y)} = \left( \|u\|_{L^q(0, T; Y)}^q + |u|_{W^{\sigma,q}(0, T; Y)}^q \right)^{1/q},
\]
where
\[
|u|_{W^{\sigma,q}(0, T; Y)} = \left( \int_0^T \int_0^T \frac{\|u(t) - u(s)\|_Y^q}{|t - s|^{1+\sigma q}} \, ds \, dt \right)^{1/q}
\]
is the Slobodeckii semi-norm. Fractional Sobolev spaces in time have also been proven to be a useful tool in [3].

Lemma 3. Let $1 \leq q < \infty$, $0 < \sigma < 1$ with $\sigma q < 1$ and let $u \in L^q(0, T; Y)$ be a piecewise constant function with (a finite number of) jumps of height $[u]_k \in Y$ at points $t_k$, $k = 1, \ldots, N - 1$. Then $u \in W^{\sigma,q}(0, T; Y)$ and
\[
\|u\|_{W^{\sigma,q}(0, T; Y)} \leq \|u\|_{L^q(0, T; Y)} + C_{\sigma,q,T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y,
\]
where $C_{\sigma,q,T} = 2(2^\sigma - 1)T^{1-\sigma q}/(\sigma q(1 - \sigma q))$ does not depend on $N$.

Proof. We may assume that $0 = t_0 < t_1 < \cdots < t_N-1 < t_N = T$ and that $u(t) = u_k$ for $t_{k-1} < t \leq t_k$ where $k = 1, \ldots, N$. Then $[u]_k = u_{k+1} - u_k$, $k = 1, \ldots, N - 1$, and
\[
u(t) = u_k = u_1 + \sum_{j=1}^{k-1} (u_{j+1} - u_j) = u_1 + \sum_{j=1}^{N-1} [u]_j H_{t_j}(t)
\]
for $t_{k-1} < t \leq t_k$, where $H_{t_j}$ is the shifted Heaviside function
\[
H_{t_j}(t) = \begin{cases} 
0 & \text{for } 0 < t \leq t_j, \\
1 & \text{for } t_j < t < T.
\end{cases}
\]

By definition of the $W^{\sigma,q}(0, T; Y)$ norm and the semi-norm property of $|\cdot|_{W^{\sigma,q}(0, T; Y)}$, we find that
\[
\|u\|_{W^{\sigma,q}(0, T; Y)} = \left( \|u\|_{L^q(0, T; Y)}^q + |u|_{W^{\sigma,q}(0, T; Y)}^q \right)^{1/q}
\leq \|u\|_{L^q(0, T; Y)} + |u|_{W^{\sigma,q}(0, T; Y)}
\leq \|u\|_{L^q(0, T; Y)} + |u_1|_{W^{\sigma,q}(0, T; Y)} + \sum_{j=1}^{N-1} \|[u]_j\|_Y \|H_{t_j}\|_{W^{\sigma,q}(0, T)}
\leq \|u\|_{L^q(0, T; Y)} + \sum_{j=1}^{N-1} \|[u]_j\|_Y \|H_{t_j}\|_{W^{\sigma,q}(0, T)}.
\]
It remains to compute the seminorm of $H_{t_j}$:

$$
|H_{t_j}|_{W^{\sigma,q}}^q = \frac{1}{2} \int_0^T \int_0^T \frac{|H_{t_j}(t) - H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}} \, ds \, dt = 2 \int_0^{t_j} \frac{|H_{t_j}(t) - H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}} \, ds \, dt
$$

$$
= 2 \int_0^{t_j} \int_0^T \frac{|t-s|^{-1-\sigma q}}{\sigma q (1-\sigma q)} \, ds \, dt = \frac{2}{\sigma q (1-\sigma q)} (T - t_j)^{-1-\sigma q} + t_j^{1-\sigma q} - T^{1-\sigma q}).
$$

The right-hand side can be interpreted as a function of $\vartheta = t_j \in [0, T]$ whose maximum is achieved at $\vartheta = T/2$. Therefore,

$$
|H_{t_j}|_{W^{\sigma,q}}^q \leq \frac{2}{\sigma q (1-\sigma q)} \left( (T/2)^{-1-\sigma q} - T^{-1-\sigma q} \right) = \frac{2}{\sigma q (1-\sigma q)} (2^{\sigma q} - 1) T^{1-\sigma q} = C_{\sigma,q,T}.
$$

Inserting this estimate in (5), the result follows. \hfill $\square$

For later use, we remark that the calculations in (5) and below show that

$$
|u|_{W^{\sigma,1}(0,T;Y)} \leq C_{\sigma,q,T}^{1/q} \sum_{k=1}^{N-1} \|u[k]\|_Y.
$$

**Proof of Theorem 1.** The idea of the proof is to apply Corollary 5 in [7]: If $(u_r)$ is bounded in $L^p(0,T;X) \cap W^{\sigma,q}(0,T;Y)$, where $\sigma > \max\{0, 1/q-1/p\}$, then $(u_r)$ is relatively compact in $L^p(0,T;B)$ if $p < \infty$, $q = 1$ and in $C^0([0,T];B)$ if $p = \infty$, $q > 1$.

First we consider the case $p < \infty$ and $q = 1$. Let $\sigma \in (0,1)$ satisfy $1 - 1/p < \sigma < 1$ and let $u_r(t) = u_k$ for $t_{k-1} < t < t_k$, $k = 1, \ldots, N$. Then

$$
\sum_{k=1}^{N-1} \|u_r[k]\|_Y = \sum_{k=1}^{N-1} \|u_{k+1} - u_k\|_Y = \tau^{-1} \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} \|u_{k+1} - u_k\|_Y \, dt
$$

$$
= \tau^{-1} \|u_r - S_r u_r\|_{L^1\tau(r;T;Y)} \leq C_{0}.
$$

Since $L^p(0,T;X) \hookrightarrow L^1(0,T;Y)$, Lemma 3 shows that $(u_r)$ is bounded in $W^{\sigma,q}(0,T;Y)$, and the corollary applies.

It remains to discuss the case $p = \infty$ and $q > 1$. We define the piecewise linear interpolants

$$
\tilde{u}_r(t) = \begin{cases} 
  u_1, & 0 \leq t \leq t_1, \\
  u_k - \frac{t - t_{k-1}}{\tau} (u_k - u_{k-1}), & t_{k-1} \leq t \leq t_k, \quad 2 \leq k \leq N.
\end{cases}
$$

Let $(S_r u_r)(t) = u_1$ for $0 \leq t < t_1$. We observe that

$$
\tilde{u}_r(t) = \begin{cases} 
  \frac{1}{\tau} (u_r(t) - (S_r u_r)(t)), & 0 \leq t \leq T, \quad t \neq t_k, \\
  \|\tilde{u}_r(t)\|_X \leq \|u_r(t)\|_X + \|(S_r u_r)(t)\|_X, & 0 \leq t \leq T,
\end{cases}
$$

which implies that $\|\tilde{u}_r\|_{L^p(0,T;X)} \leq 2 \|u_r\|_{L^p(0,T;X)}$. Now we apply Theorem 1 to $(u_r)$ with $p = 1$ instead of $p = \infty$, and we apply Corollary 5 in [7] to $(\tilde{u}_r)$ with $\sigma = 1$. We end up with a subsequence of $(u_r)$ (not relabeled) such that $u_r \to u^*$ in $L^1(0,T;B)$, and we may
assume that the associated subsequence \((\tilde{u}_\tau)\) of piecewise linear interpolants converges to a limit \(\tilde{u}\) in the topology of \(C^0([0, T]; B)\). Next we know, for \(k = 1, \ldots, N\) and \(t_{k-1} < t < t_k\), that

\[
\|u_\tau(t) - \tilde{u}_\tau(t)\|_Y = \frac{t_k - t}{\tau} \|u_\tau(t) - (S_\tau u_\tau)(t)\|_Y \leq \|u_\tau(t) - (S_\tau u_\tau)(t)\|_Y ,
\]

from which we infer that \(\|u_\tau - \tilde{u}_\tau\|_{L^1(0, T; Y)} \leq C_0 \tau\). Notice that the embeddings \(L^1(0, T; B) \hookrightarrow L^1(0, T; Y)\) and \(C^0([0, T]; B) \hookrightarrow L^1(0, T; Y)\) are both continuous, hence \(u^* = \tilde{u}\).

Since \((\tilde{u}_\tau)\) converges in \(C^0([0, T]; B)\) to \(\tilde{u}\), there exists a constant \(\tilde{C} > 0\) such that \(\|\tilde{u}_\tau\|_{L^\infty(0, T; B)} \leq \tilde{C}\) for all \(\tau\), and then also \(\|u^*\|_{L^\infty(0, T; B)} \leq \tilde{C}\) for all \(\tau\). The desired convergence of \((u_\tau)\) to \(u^*\) in any space \(L^q(0, T; B)\) for \(1 \leq q < \infty\) follows from interpolation between \(\|u_\tau - u^*\|_{L^1(0, T; B)} \to 0\) and \(\|u_\tau - u^*\|_{L^\infty(0, T; B)} \leq 2\tilde{C}\), which completes the proof. \(\square\)

**Remark 4.** Estimates (6) and (7) imply that, for all piecewise constant functions \(u \in L^1(0, T; Y)\) with jumps at \(t_k = k\tau\),

\[
\|u\|_{W^{\alpha, 1}(0, T; Y)} \leq C_{\alpha, q, T}^{1/q} \sum_{k=1}^{N-1} \|\|u\|_k\|_Y \leq \tau^{-1} C_{\alpha, q, T}^{1/q} \|u - S_\tau u\|_{L^1(\tau, T; Y)} .
\]

By Lemma 5 of [7], there exists an inverse inequality for all \(u \in W^{\alpha, 1}(0, T; Y)\) and all \(\sigma \in (0, 1)\):

\[
\|u - S_\tau u\|_{L^1(\tau, T; Y)} \leq C_3 \tau^\sigma \|u\|_{W^{\alpha, 1}(0, T; Y)} ,
\]

where \(C_3 > 0\) depends on \(\sigma\) and \(T\). In this sense, the chain of inequalities

\[
\tau |u|_{W^{\alpha, 1}(0, T; Y)} \leq \tau^\sigma C_{\alpha, q, T}^{1/q} C_3 |u|_{W^{\alpha, 1}(0, T; Y)}
\]

is almost sharp since we can choose \(\sigma\) as close to one as we wish. \(\square\)

### 3. Proof of Proposition 2

We construct a sequence \((u_\tau)\) satisfying the assumptions of Theorem 1 with \(\tau^{-\alpha}\) (0 < \(\alpha < 1\)) in (4) instead of \(\tau^{-1}\), but not possessing a convergent subsequence in \(L^p(0, T; B)\), where \(p < \infty\).

Take \(X = Y = B = \mathbb{C}\) and \((0, T) = (0, 1)\). For \(\beta \geq 1\), define the function

\[
f_\beta(t) := (\beta p + 1)^{1/p} t^\beta , \quad 0 \leq t \leq 1.\]

Then we have \(\|f_\beta\|_{L^p(0, T)} = 1\). For later use, we remark that

\[
\lim_{\beta \to \infty} f_\beta(t) = 0 ,
\]

for each fixed \(t \in [0, 1]\), uniformly on compact sub-intervals \([0, t_*]\) \(\subset [0, 1]\).

Since \(\alpha < 1\), we may choose a real number \(0 < \gamma \leq \min\{1, p(1 - \alpha)\}\). We set \(\beta(\tau) = \tau^{-\gamma}\) and

\[
u_\tau(t) := \begin{cases} f_{\beta(\tau)}(k\tau) & \text{for } k\tau \leq t < (k + 1)\tau, \ k \in \{0, 1, \ldots, N - 1\}, \\ f_{\beta(\tau)}((N - 1)\tau) & \text{for } t = 1. \end{cases}
\]
The function $u_\tau$ has jumps of height $[u_\tau]_k$ at the values $t = k\tau$ for $1 \leq k \leq N - 1$, and all jumps have the same sign. In particular,
\[
\sum_{k=1}^{N-1} \|[u_\tau]_k\|_{Y} = \sum_{k=1}^{N-1} [u_\tau]_k = f_{\beta(\tau)}(1 - \tau) = (\tau^{-\gamma} p + 1)^{1/p}(1 - \tau)^{\tau^{-\gamma}},
\]
\[
1 \geq (1 - \tau)^{\tau^{-\gamma}} \geq (1 - \tau)^{\tau/\gamma} \geq \frac{1}{2^e}.
\]
Therefore, it follows that
\[
\tau^{-\alpha} \|u_\tau - S_\tau u_\tau\|_{L^1(\tau, T; Y)} = \tau^{1-\alpha} \sum_{k=1}^{N-1} \|[u_\tau]_k\|_{Y} = \tau^{1-\alpha} \tau^{-\gamma} p + 1)^{1/p}(1 - \tau)^{\tau^{-\gamma}}
\]
\[
\leq \tau^{1-\alpha} (\tau^{-\gamma} p + 1)^{1/p} \leq \left(\frac{1}{2}\right)^{1-\alpha} \left(\frac{1}{2} - \gamma + 1\right)^{1/p},
\]
for all $\tau \in (0, 1/2)$, since $1 - \alpha - \gamma/p \geq 0$. Hence, (4) holds. But the sequence $(u_\tau) \subset L^p(0, T; B)$ does not possess a converging subsequence, which can be seen as follows. Fix $t \in [0, 1)$. Then $0 \leq u_\tau(t) \leq f_{\beta(\tau)}(t)$, and (10) implies the pointwise convergence $\lim_{\tau \to 0} u_\tau(t) = 0$, uniform on compact sub-intervals $[0, t_\star] \subset [0, 1)$. Thus, the pointwise limit of the subsequence must be the zero function. However, this is impossible, because of the following uniform lower bound:
\[
\int_0^1 |u_\tau(t)|^p \, dt \geq \int_0^{1-\tau} |f_{\beta(\tau)}(t)|^p \, dt = (1 - \tau)^{\tau^{-\gamma} p + 1} \geq \frac{1}{2} \left(\frac{1}{2} - \gamma p + 1\right)^p \geq \frac{1}{2}(2e)^{-p},
\]
showing the claim.

4. Comments

Let $X$, $B$, and $Y$ be Banach spaces such that the embedding $X \hookrightarrow B$ is dense and compact, the embedding $B \hookrightarrow Y$ is continuous, and there exist $0 < \theta < 1$, $C_\theta > 0$ such that for all $u \in X$, the interpolation inequality
\[
\|u\|_B \leq C_\theta \|u\|_X^{\theta} \|u\|_Y^{1-\theta}
\]
holds. The setting which we have in mind relates to (1), with given $u(0) \in B$. In this situation, a slightly weaker version of Theorem 1 can be derived directly from the Aubin lemma.\(^{1}\) Indeed, since $X$ is dense in $B$, we may approximate $u(0) \in B$ by $u_0 \in X$, and we define the piecewise linear interpolant by
\[
\tilde{u}_\tau(t) = u_k - \frac{t_k - t}{t} (u_k - u_{k-1}), \quad t_{k-1} \leq t \leq t_k, \quad 1 \leq k \leq N.
\]
We suppose that $u_0$ and $u_1$ satisfy
\[
\tau \|u_0\|^\theta_X \leq C_1, \quad \|u_0 - u_1\|_Y \leq C_1
\]
\(^{1}\)The authors are grateful to one of the referees for this observation.
for some constant $C_1 > 0$ independent of $\tau$. The first bound can always be satisfied; the second bound is a mild condition related to the construction of the sequence $(u_k)$. If this sequence is defined according to (1), the second bound can be replaced by the regularity assumption $\tau \| A(u_1) \|_Y \leq C$ for some constant $C > 0$ independent of $\tau$ since $\| u_1 - u_0 \|_Y \leq \tau \| A(u_1) \|_Y + \tau \| f(\tau) \|_Y$.

Now we make the agreement that $(S_{\tau} u_\tau)(t) = u_0$ for $0 \leq t < t_1 = \tau$. Then (8) still holds. It follows from (4) that

$$\| u_\tau \|_{L^1(\tau,T;Y)} = \| u_1 - u_0 \|_Y + \tau^{-1} \| u_\tau - S_{\tau} u_\tau \|_{L^1(\tau,T;Y)} \leq C_1 + C_0.$$  

Furthermore, using (8) and (4) again,

$$\| \tilde{u}_\tau \|_{L^p(0,T;X)} \leq \tau^{1/p} \| u_0 \|_X + 2 \| u_\tau \|_{L^p(0,T;X)} \leq C_1^{1/p} + 2C_0.$$  

Hence, by the Aubin lemma [7, Corollary 4], up to a subsequence, $\tilde{u}_\tau \rightarrow u$ in $L^p(0,T;B)$ as $\tau \rightarrow 0$. By the interpolation inequality (11) and by (9),

$$\| u_\tau - \tilde{u}_\tau \|_{L^1(0,T;B)} \leq C_0 \| u_\tau - \tilde{u}_\tau \|_{L^1(0,T;X)}^{\theta} \| u_\tau - \tilde{u}_\tau \|_{L^1(0,T;Y)}^{1-\theta} \leq C_0 (\| u_\tau \|_{L^1(0,T;X)} + \| \tilde{u}_\tau \|_{L^1(0,T;X)})^{\theta} \| u_\tau - S_{\tau} u_\tau \|_{L^1(0,T;Y)}^{1-\theta}.$$  

We remark that $\| u_\tau - S_{\tau} u_\tau \|_{L^1(0,T;Y)} \leq \tau (\| u_1 - u_0 \|_Y + C_0)$, which implies that $u_\tau - \tilde{u}_\tau \rightarrow 0$ in $L^1(0,T;B)$. Since $\tilde{u}_\tau \rightarrow u$ in $L^p(0,T;B)$, we find that $u_\tau \rightarrow u$ in $L^q(0,T;B)$ for all $q < p$. Notice, however, that Theorem 1 allows us to conclude this result up to $q = p$ without assuming (11) and (12).

Proposition 2 shows that the exponent of the factor $\tau$ in (4) cannot be raised. However, when allowing for arbitrary time shifts $S_h$, the factor can be replaced by $h^{-\alpha}$, where $0 < \alpha < 1$, under some conditions. An example, adapted to our situation, can be found in [1, Theorem 1.1]:

**Theorem 5 (Amann).** Let (11) hold. Furthermore, let $0 < s < 1$, $1 \leq p < \infty$, and $F \subset L^p(0,T;Y)$. Assume that there exists $C_2 > 0$ such that each $u \in F$ satisfies the following infinite collection of inequalities:

$$h^{-s} \| u - S_h u \|_{L^1(\tau,T;Y)} + \| u \|_{L^p(0,T;X)} \leq C_2 \quad \text{for all } h > 0.$$  

Then $F$ is relatively compact in $L^q(0,T;B)$ for all $q < p/((1-\theta)(1-s)p + \theta)$.

Notice that $q = p$ is admissible if $(1-\theta)(1-s)p + \theta < 1$ which is equivalent to $s > 1-1/p$. Thus, if we wish to allow for arbitrary large $p \geq 1$, we have to require the condition $s = 1$, which corresponds to the result of Theorem 1. On the other hand, in applications, often $p = 2$, and compactness follows even for $s < 1$, namely for any $s > 1/2$.

In the special situation when we have the triple $X \hookrightarrow B \hookrightarrow X'$, where $Y = X'$ is the dual space of $X$ and $B$ is a Hilbert space, the assumptions of Amann’s theorem hold with $\theta = 1/2$. Then $q < 2p/((1-s)p + 1)$, and we see that $2p$ is an upper bound for $q$. This corresponds to the result of Walkington [8, Theorem 3.1 (1)].
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