Small loop spaces

Žiga Virk

Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia

A R T I C L E   I N F O

Article history:
Received 14 September 2009
Received in revised form 7 October 2009
Accepted 7 October 2009

M S C:
55Q05

Keywords:
Homotopy groups general
Sets of homotopy classes

A B S T R A C T

The importance of small loops in the covering space theory was pointed out by Brodskiy, Dydak, Labuz, and Mitra in [2] and [3]. A small loop is a loop which is homotopic to a loop contained in an arbitrarily small neighborhood of its base point and a small loop space is a topological space in which every loop is small. Small loops are the strongest obstruction to semi-locally simply connectedness. We construct a small loop space using the Harmonic Archipelago. Furthermore, we define the small loop group of a space and study its impact on covering spaces, in particular its contribution to the fundamental group of the universal covering space.

© 2009 Elsevier B.V. All rights reserved.

1. Basic construction of small loop spaces

We will construct a topological space $X$ so that for every $x \in X$ and every neighborhood $U$ of $x$ the inclusion induced homomorphism $\pi_1(U, x) \rightarrow \pi_1(X, x)$ is a nontrivial epimorphism. In other words, the space $X$ admits homotopically nontrivial loops and every loop can be homotoped to an arbitrarily small neighborhood of the basepoint.

Definition 1. A loop $\alpha : (S^1, 0) \rightarrow (X, x_0)$ is small iff there exists a representative of the homotopy class $[\alpha]_{x_0} \in \pi_1(X, x_0)$ in every open neighborhood $U$ of $x_0$. A small loop is nontrivial small loop if it is not homotopically trivial. The small loop group $\pi^s_1(X, x_0)$ of $(X, x_0)$ is the subgroup of the fundamental group $\pi_1(X, x_0)$, consisting of homotopy classes of small loops.

It is easy to check that the small loop group is a group, but not necessarily a normal subgroup of $\pi_1$. Furthermore, it is a functor. The presence of small loops implies the absence of semi-locally simply connectedness. While general non semi-locally simply connected spaces may have different nontrivial loops at every neighborhood of some point, in the case of a small loop the homotopy type of one loop can be chosen for all neighborhoods of the base point.

Definition 2. A non-simply connected space $X$ is a small loop space if for every $x \in X$, every loop $\alpha : (S^1, 0) \rightarrow (X, x)$ is small.

Small loop spaces arise in the covering space theory as the opposite of the semi-locally simply connected spaces. While every point in a semi-locally simply connected space has a neighborhood that contains only homotopically trivial loops,
every neighborhood of any point of a small loop space generates the entire fundamental group. As a consequence, the naturally induced topology on the fibers of the covering space is trivial in the case of a small loop space and is discrete in the case of a semi-locally simply connected space (see [2,3,7]). Spaces without nontrivial small loops are called homotopically Hausdorff and were studied in [4,5,11].

The starting point of our construction is the Harmonic Archipelago denoted by HA (see Fig. 1). It was introduced by Bogley and Sieradsky [1] and studied by Fabel [6]. It is defined as follows. Consider the Hawaiian Earring (HE) in the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2$. Represent it as the union of oriented loops $(l_i)_{i \geq 0}$ where $l_0$ is the outermost loop and 0 is the intersection of all loops. For each pair of consecutive loops $(l_i, l_{i+1})$ in HE attach a 2-disc $B^2_i := B^2$ to the Hawaiian earring in $\mathbb{R}^3$ in the following way: identify $\partial B^2_i$ with the loop $l_i|_{l_i^{-1}}$ (where $l_i^{-1}$ is the loop $l_{i+1}$ with the opposite orientation) and stretch $B^2_i$ up from the plane so that its center is at height 1 from HE. Define the boundary loop as the loop $l_0$ (this is the outermost loop in $HE \hookrightarrow HA$) based at $0 := (0, 0, 0)$. HA is neither compact nor locally simply connected. Furthermore, all loops $(l_i, 0)$ are homotopically nontrivial (that follows from Corollary 9 by setting $l_0 = Z$). However, every loop based at 0 can be made arbitrary small (up to homotopy) by sliding over finitely many discs $B^2_i$.

There are other well-known examples of similar spaces with small loops. One of them is the union of cones of the same height in $\mathbb{R}^3$ over a sequence of circles converging to a point, with consecutive circles joined by a segment. Another example is the one-point union $C(HE) \vee C(HE)$ of two copies of the cone over the Hawaiian earring as described in [8].

The aim of this paper is to obtain the small loop property at any point of the space. We achieve it by gluing more small loops from Corollary 9 by setting $l_0 = Z$). However, every loop based at 0 can be made arbitrary small (up to homotopy) by sliding over finitely many discs $B^2_i$.

There are other well-known examples of similar spaces with small loops. One of them is the union of cones of the same height in $\mathbb{R}^3$ over a sequence of circles converging to a point, with consecutive circles joined by a segment. Another example is the one-point union $C(HE) \vee C(HE)$ of two copies of the cone over the Hawaiian earring as described in [8].

The aim of this paper is to obtain the small loop property at any point of the space. We achieve it by gluing more small loops from Corollary 9 by setting $l_0 = Z$). However, every loop based at 0 can be made arbitrary small (up to homotopy) by sliding over finitely many discs $B^2_i$.

There are other well-known examples of similar spaces with small loops. One of them is the union of cones of the same height in $\mathbb{R}^3$ over a sequence of circles converging to a point, with consecutive circles joined by a segment. Another example is the one-point union $C(HE) \vee C(HE)$ of two copies of the cone over the Hawaiian earring as described in [8].

The aim of this paper is to obtain the small loop property at any point of the space. We achieve it by gluing more small loops from Corollary 9 by setting $l_0 = Z$). However, every loop based at 0 can be made arbitrary small (up to homotopy) by sliding over finitely many discs $B^2_i$.

**Definition 3.** Let $\{(Y_i, A_i)\}_{i \geq 0}$ be a countable collection of pairs of spaces where $A_i \subseteq Y_i$ is closed for every $i$. A topological space $X$ is an **m-stratified** (map stratified) space with parameters $\{(Y_i, A_i)\}_i$ if $X$ is the direct limit of spaces $\{X_i\}_{i \geq 0}$, where the spaces $X_i$ are defined inductively as

- $X_0 := Y_0$,
- $X_i := X_{i-1} \cup f_i Y_i$, where $f_i : A_i \hookrightarrow X_{i-1}$.

The sets $Y_i$ are called **m-strata**.

**Remark.** CW complexes are m-stratified spaces with each stratum $Y_i$ being the disjoint union of $i$-cells and $f_i$ being the disjoint union of attaching maps.

**Definition 4.** Let $(Z, Y)$ be a pair where $Z$ is a topological space and $Y : (S^1, 0) \to (Z, Y(0))$ is an embedding which is a homotopically nontrivial small loop with the following property: for every map $f : K \to Z$ defined on a compact space $K$ there is a retraction $g : f(K) \cup \gamma(S^1) \to \gamma(S^1)$. The space $S(Z, Y)$ is an m-stratified space where

$$Y_0 := Z, \quad S_1 := \{\text{all maps } \alpha : (S^1, 0) \to (X_{i-1}, \alpha(0))\},$$

$$Y_i := \bigsqcup_{\alpha \in S_i} Z_{\alpha}, \quad A_i := \bigsqcup_{\alpha \in S_i} Y_{\alpha}(S^1) \quad \text{and} \quad f_i|_{Y_{\alpha}(S^1)} := \alpha \gamma_{\alpha}^{-1},$$

there is a homeomorphism $Z_{\alpha} \cong Z$ and via this homeomorphism $y_{\alpha} : S^1 \to Z_{\alpha}$ can be identified with the small loop $Y_{\alpha}$.
It is not hard to see that the construction of $S(Z,\gamma)$ is functorial. Any map $(Z,\gamma)\to (Z',\gamma')$ between pairs satisfying conditions of 4, induces a map $S(Z,\gamma)\to S(Z',\gamma')$. We will prove that $X := S(HA,\gamma)$ is a small loop space where $\gamma : S^1 \to HA$ is a boundary loop. In fact all the spaces $S(Z,\gamma)$ are small loop spaces which can be proven similarly.

Note that the space $X$ is path connected and locally path connected. The purpose of gluing $Z_0$ in the construction of 4 is to make the loop $\alpha$ small in the next step of construction. Repeating this process inductively eventually makes all loops small hence $X$ is indeed a small loop space.

The following lemma, inspired by the theory of CW complexes [9], describes the behavior of compact subsets of the small loop space $X$.

**Lemma 5.** Let $K$ be a compact space and $f : K \to X$ a continuous map. Then $f(K)$ is contained in a finite union of sets $\tilde{H}_0$ where $\tilde{H}_0 \subset X$ is the image of $H_0$ in $X$.

The lemma above follows from a more general statement.

**Lemma 6.** Suppose $Y$ is an m-stratified space so that m-strata $Y_i$ can be decomposed as $Y_i = \bigsqcup Y_i^j$ where $Y_i^j \subset Y_i$ are regular subspaces (i.e., subspaces, that are regular topological spaces). Let $K \subset Y$ be a compact space. Define $X_i^j$ to be the image of $Y_i^j$ in $Y$. Then $K$ is contained in a finite union of subsets $X_i^j \subset Y$.

**Proof.** Notice that because $Y_i$ is the disjoint union of spaces $Y_i^j$, each of the spaces $Y_i^j$ is open in $Y_i$. Furthermore, $Y$ is the disjoint union of its subsets $X_i^j \setminus X_{i-1}$.

Suppose there is a sequence $P = \{x_k\}_{k \in \mathbb{N}} \subset K$ so that $x_k \in (X_i^m - X_{i-1})$, $\forall k$ and $X_i^m \neq X_j^m$ for $k \neq m$. We will prove that $P$ is closed in $Y$. By the definition of the topology on $Y$ it is enough to show that $P \cap X_i$ is closed in $X_i$ for every $i$. We proceed by induction. Because sets $X_i^j$ are open, regular, and each of them contains at most one element of $P$, the set $P \cap X_0$ is a locally finite union of closed sets (points) hence it is closed in $X_0$.

Suppose $P \cap X_n$ is closed in $X_n$, consequently $P \cap X_n$ is closed in $X_{n+1}$. The sets $X_{n+1}^j - X_n$ are open and disjoint in $X_{n+1}$. Each of them contains at most one point of $P$. We claim that $P \cap X_{n+1}$ is a locally finite union of closed sets: points of $P \cap (X_{n+1} - X_n)$ and a closed set $P \cap X_n$. For every $j$ let $U_j$ and $V_j$ be disjoint open subsets of $X_j^j$ so that $U_j$ contains $X_{n+1}^j \cap X_n$ and $V_j$ contains $P \cap (X_{n+1}^j - X_n)$ which is at most one point set. Such choice of subsets $U_j$, $V_j$ is possible by regularity of $Y_j$. Define $U := X_n \cup \bigcup U_j$ and notice that $\{U, \{V_j\}_{j}\}$ is a collection of disjoint open subsets that guarantees that $P \cap X_{n+1}$ is a locally finite union of points of $P \cap (X_{n+1} - X_n)$ and a closed set $P \cap X_n$. It follows that $P \cap X_{n+1}$ is closed. By induction on $n$ the sequence $P$ is closed in $Y$.

Using the same argument we see that any subset of $P$ is closed in $Y$, therefore $P$ has the discrete topology. Hence $P \subset K$ is closed, infinite and discrete. There is no such set in a compact space, a contradiction. □

We have established the basic property of nice m-stratified spaces by considering the compact subsets. We now compare the homotopy groups of $X_0$ with the homotopy groups of $X$. In particular, we will prove that $S(HA,\alpha)$ is not simply connected.

**Lemma 7.** Let $f : (A, z_0) \to (Y, y_0)$ be a map where $(A, z_0) \subset (Z, z_0)$ is a closed subset. If for every compact subset $K \subset Z$ there is a retraction $r_K : (Y \cup_f K) \to Y$ then the inclusion $i : Y \to (Y \cup_f Z)$ induces an injection on the homotopy classes of pointed maps $[(M, m_0), (Y, y_0)] \to [(M, m_0), (Y \cup_f Z, y_0)]$ for each compact space $(M, m_0)$. In particular $i$ induces an injection on the homotopy groups.

**Proof.** Suppose that $f, g : (M, m_0) \to (Y, y_0)$ are pointed maps so that $f$ and $g$ are in the same class of $[(M, m_0), (Y \cup_f Z, y_0)]$. This implies the existence of a pointed homotopy $H$ between $f$ and $g$. Then $r_K H$ is a pointed homotopy between $f$ and $g$ in $Y$ where $K = H(M \times I) \subset Z$, hence $i$ is an injection. □

**Proposition 8.** Let $(l_0, 0)$ be the boundary loop in $HA$ and $K \subset HA$ a compact subspace. Then there is a retraction $r_K : (K \cup l_0) \to l_0$.

**Proof.** The subspace $K$ is compact, therefore it can contain at most finitely many peaks of the balls $B^2_0$ in $HA$ (because the peaks form a Cauchy sequence without a limit point in HA). Remove any peak point $z \notin K$ from $HA \subset Y$. We can retract $HA - \{z\}$ to $l_0$ using the projection $\pi : HA \to \mathbb{R}^2 \times \{0\}$ of $HA$ to the plane and then the radial projection from $\pi(z)$ to $l_0$. □

**Corollary 9.** Suppose $(Z, z_0)$ is a pointed topological space and $\alpha : (S^1, 0) \to (Z, z_0)$, where $(S^1, 0) \subset (HA, 0)$, is the boundary loop. Then $i$ induces an injection on the homotopy classes of pointed maps $[(M, m_0), (Z, z_0)] \to [(M, m_0), (Z \cup \alpha HA, 0)]$ for each compact space $(M, m_0)$. In particular, the inclusion induced map $\pi_1(Z, z_0) \to \pi_1(Z \cup \alpha HA, z_0)$ is a monomorphism.

**Proof.** This follows from [7] and [8]. □
Corollary 10. \( \pi_1(S(HA, \gamma)) \neq 0 \), where \( \gamma \) is the boundary loop of \( HA \).

Proof. Considering the m-stratification of \( S(HA, \gamma) \), take the boundary loop of \( X_0 \). If it is homotopically trivial in \( S(HA, \gamma) \), there must be a homotopy in \( S(HA, \gamma) \) taking that loop to the constant loop. By Lemma 6 such a homotopy takes place in a finite union of attached spaces \( HA_k \), but by Corollary 9 the boundary loop is nontrivial in the finite union of spaces \( HA_k \). Therefore the boundary loop of \( X_0 \) (and any nontrivial loop in any \( X_k \)) is nontrivial in \( X \). \( \square \)

Theorem 11. Let \( \gamma \) be the boundary loop in \( HA \). Then \( S(HA, \gamma) \) is a small loop space.

Proof. Take any loop \( \alpha : S^1 \to X \). By Lemma 6 the loop \( \alpha \) lies in \( X_k \) for some \( k \in \mathbb{N} \). By construction \((\alpha, \alpha(0))\) can be made arbitrarily small at \( \alpha(0) \) via \( HA_\alpha \subset X_{k+1} \). \( \square \)

Conjecture 12. A small loop space is neither compact nor first countable.

Conjecture 13. The fundamental group of every small loop space is uncountable.

2. The small loop group and the covering spaces

The idea of small loop spaces originates in the theory of covering spaces. We recall the definition of the universal covering space of a path connected space \((X, x)\). Let \( \Omega X \) be the space of all paths in \( X \) originating at \( x \) and let \( \hat{x} \) be the constant path in \( x \). The universal covering space \((\hat{X}, \hat{x})\) of \((X, x)\) is the quotient space of \( \Omega X \) under the equivalence of paths: \( \alpha \sim \beta \) iff \( \alpha^{-1} \beta \) is a homotopically trivial loop at \( x \). The topology is generated by the sets \( B(U, \alpha) := \{ \beta \in \Omega X : \exists \gamma : I \to U : \beta \simeq \alpha \gamma \} \) where \( \alpha \in \Omega X \) and \( U \subset X \) is an open subset. It is easy to see that the covering space is a Peano space, which means it is path connected and locally path connected.

Remark. All spaces in this section will be path connected.

The theory of the covering spaces works well for semi-locally simply connected spaces. The existence of a nontrivial small loop implies the space is not semi-locally simply connected. We study the influence of nontrivial small loops on the covering spaces.

Lemma 14. Consider the universal covering space \((\hat{X}, \hat{x})\) of a space \((X, x)\).

(i) Let \( \alpha, \beta \in \Omega X \) so that \( \alpha(1) = \beta(1) \) and \( \beta^{-1} \alpha \in \pi_1^X(\alpha(1)) \). If \( U \) is any open neighborhood of \( \alpha(1) \) then \( B(U, \alpha) = B(U, \beta) \subset (\hat{X}, \hat{x}) \).

(ii) The quotient map \( f \) defined on \((\hat{X}, \hat{x})\) by the equivalence relation \( \alpha \sim \beta \) iff \( \alpha(1) = \beta(1) \) and \( \beta^{-1} \alpha \in \pi_1^X(\alpha(1)) \) has the following property: if \( f(U) = f(V) \) for open sets \( U, V \) then \( U = V \).

Proof. (i) \( \beta^{-1} \alpha \in \pi_1^X(\alpha(1)) \) means there is \( \gamma \in \pi_1^X(\alpha(1)) \) so that \( \alpha \simeq \beta \gamma \). The loop \( \gamma \) is small and the claim follows from the definition of the topology on \((\hat{X}, \hat{x})\).

(ii) Use (i). \( \square \)

Corollary 15.

(i) If \( X \) is a small loop space then the universal covering map \( q \) is a surjection and has the following property: if \( q(U) = q(V) \) for open sets \( U, V \) then \( U = V \).

(ii) If \( X \) is a small loop space then all the well-defined lifts to the universal cover are continuous.

(iii) Suppose \( p : X \to Y \) is a continuous surjection with the property that all the well-defined lifts of the maps \( Z \to Y \) are continuous. Then \( p \) is a surjection and has the following property: if \( p(U) = p(V) \) for open sets \( U, V \), then \( U = V \).

The phenomenon in the above lemma guarantees that a path in \( \hat{X} \) remains continuous if we change its endpoint. Before we do that we introduce the following notation.

The covering space allows us to define the natural lift of a map. Suppose \( \alpha : (I, 0) \to (X, 0) \) is a path. Define its lift \( \hat{\alpha} \) in \( \hat{X} \) so that \( \hat{\alpha}(t) \) is the path \( \alpha \) restricted (and properly rescaled) to \([0, t]\). This lift is obviously continuous. In the case of covering maps for semi-locally simply connected spaces the lift of every path is unique in \( \hat{X} \).

Lemma 16. Consider the universal covering space \((\hat{X}, \hat{x})\) of \((X, x)\). Let \( \gamma \) be a nontrivial small loop based in \( \alpha(1) \) where \( \alpha \in \Omega X \). Then \( \alpha \) does not have a unique lift in \( \hat{X} \). In fact, for every loop in \( \pi_1^X(\alpha(1)) \) there is a different lift of \( \alpha \).
Proof. Define the lift \( \tilde{\alpha} \) of \( \alpha \) as \( \tilde{\alpha}(t) = \hat{\alpha}(t), \ t < 1, \ \tilde{\alpha}(1) = \alpha \gamma \). This is obviously a lift of \( \alpha \). It is different than \( \hat{\alpha} \) since \( \alpha \ncong \alpha \gamma \). It is continuous by Lemma 14. Using the same argument we obtain a different lift for a different choice of \( \gamma \). \( \square \)

The lemma above suggests that there should exist nontrivial loops in \( \hat{X} \). In order to describe some of these loops we introduce the following notation.

Definition 17. The **SG subgroup** of \( \pi_1(X, x) \) (SG stands for small generated) is denoted by \( \pi^{SG}_1(X, x) \) and is generated by the subgroups of loops \( \{ \alpha * \pi^{SG}_1(X, \alpha(1)) * \alpha^{-1} \} \alpha \in \Omega X \).

Proposition 18. Consider the covering space \( (\hat{X}, \hat{\alpha}) \) of \( (X, x) \).

(i) The subgroup \( \pi^{SG}_1(X, x) \leq \pi_1(X, x) \) is normal.

(ii) The subgroup \( \pi^{SG}_1(X, x) \) naturally embeds in \( \pi_1(\hat{X}, \hat{x}) \).

Proof. (i) This follows from the fact that the conjugation acts transitively on the set of generating subgroups.

(ii) We shall prove that for any \( \alpha \in \Omega X \) the group \( \alpha \pi^{SG}_1(X, \alpha(1)) \alpha^{-1} \) is naturally included in \( \pi_1(\hat{X}, \hat{x}) \). The construction is similar to that of Lemma 16. Pick any \( \alpha \in \Omega X \) and any nontrivial \( \gamma \in \pi^{SG}_1(X, \alpha(1)) \). Define the lift of \( \alpha \gamma \alpha^{-1} \) as the concatenation \( \tilde{\alpha} \gamma \tilde{\alpha}^{-1} \) of the natural lifts where \( \hat{\alpha} \) starts at \( \hat{x} \), \( \tilde{\alpha}^{-1} \) starts in \( \alpha(1) \) and \( \tilde{\gamma} \) is defined in the following way:

- for \( t \neq 1 \) the path \( \tilde{\gamma}(t) \) is the path \( \gamma \) restricted to \([0, t]\).
- \( \tilde{\gamma}(1) \) is the constant path in \( \alpha(1) \).

Such a lift is a continuous map even at the end of \( \tilde{\alpha} \gamma \tilde{\alpha}^{-1} \) because points \( \hat{\alpha} \gamma \) and \( \hat{\alpha} \) share the same open neighborhoods in \( \hat{X} \). The lift is a nontrivial map since its projection to \( X \) is nontrivial. \( \square \)

References