Recent characterizations of groups in which normality is a transitive relation

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A group $G$ is said to be a $T$-group if $H \trianglelefteq K \trianglelefteq G$ implies $H \trianglelefteq G$.

The study of finite groups of this type can be traced back to the 1940s, and the first explicit mention that I found of $T$-groups in the literature corresponds to a paper of Best and Taussky (Proc. Roy. Irish Acad. 1942). After, in their seminal papers G. Zacher (RicMat-1952) and W. Gaschütz (J.ReineAngMat-1957) described the structure of finite soluble $T$-groups. It turns out that

A finite group $G$ is a soluble $T$-group if and only if $G$ is a $\overline{T}$-group.

Definition: A group $G$ is said to be a $\overline{T}$ – group if every subgroup of $G$ is a $T$-group.

The structure of infinite soluble $T$-groups is more complicated and was first described by D.J.S. Robinson (Proc. Camb. Phil. Soc. (1964)). Examples of soluble $T$-groups that are not $\overline{T}$-groups can be constructed.
Let $G$ be a soluble group. Then $G$ is a $\overline{T}$-group if and only if all its ascendant subgroups are normal.
Let $\chi$ be a property referred to subgroups of a group $G$. $\chi$ is said to be a *t-property* if $X \chi G$ and $X \triangleleft \triangleleft G \Rightarrow X \triangleleft G$.

It turns out that if each subgroup of a group $G$ satisfies a *t-property* than $G$ is a $T$-group.

In literature are often considered *t-properties* $\chi$, satisfying $^*$

$$X \chi G \Rightarrow X \chi K, \quad X \leq K \leq G.$$ 

Therefore, $^*$ if each subgroup of a group $G$ satisfies a such *t-property* than $G$ is a $\overline{T}$-group.
Examples of $t$-properties

- P. Hall, 196?- Rose-ProcLMS-1967

A subgroup $X$ of a group $G$ is said to be *pronormal* if $X$ and $X^g$ are conjugate in $\langle X, X^g \rangle$, for every element $g \in G$. 
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  A subgroup $X$ of a group $G$ is said to be *pronormal* if $X$ and $X^g$ are conjugate in $\langle X, X^g \rangle$, for every element $g \in G$.

- (Muller-rsmuPadova-66)
  A subgroup $X$ of a group $G$ is said to be *weakly normal* if $X^g \leq N_G(X) \Rightarrow g \in N_G(X)$. 

- (LI-1998-jpaa)
  A subgroup $X$ of a group $G$ is called an *NE-subgroup* (Read NE as normally embedded property) if it satisfies $N_G(X) \cap X^g = X$.

- (DeGiovanni, RicMat-03) - (KurdachenkoSubbotin06-c.math) - (Mysovskikh99)
  A subgroup $X$ of a group $G$ is said to be *pseudonormal* or *transitively normal* or to satisfy the subnormaliser condition if $N_G(H) \leq N_G(X)$, for each subgroup $H$ of $G$ such that $X \leq H \leq N_G(X)$.
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  A subgroup $X$ of a group $G$ is said to be an **$H$-subgroup** or that it has the **$H$-property** in $G$ if $N_G(X) \cap X^g \leq X$ for all elements $g$ of $G$. 
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Relations between subgroup embedding $t$-properties

A group $G$ is a $\overline{T}$-group $\iff$ Every subgroup of $G$ satisfies the subnormalizer condition

Pronormality and NE-property appear as stronger properties
Relations between subgroup embedding $t$-properties

Each of the above arrows cannot be reversed.

- subnormalizer condition $\not\Rightarrow$ $\mathcal{H}$-property  Mysovskikh 99, as above
- weakly normality $\not\Rightarrow$ pronormality  Ballester-Bolinches-EstebanRomero-JAusMath-2003
- pronormality $\not\Rightarrow$ $\mathcal{H}$-property  It is enough to consider $\langle (1, 2, 3, 4) \rangle < \Sigma_4$
- $\mathcal{H}$-property $\not\Rightarrow$ NE-property  It is enough to consider a Sylow subgroups of $SL_2(3)$ EstebanRomero,--; 2017

Moreover, Pronormality and NE-property are independent  We will see ...

```
pronormal  NE-subgroup
          ↓  = = = = =
weakly normal  = = = $\mathcal{H}$-subgroup
          *  subnormalizer condition
```
The framework of the investigation

This picture and the above equivalence give the first suggestions about the difficulties related to the characterizations of $T$-groups. Infact, looking at the weakest property (subnormalizer condition) simply applying the definitions, it turns out that

- **If every** subgroup of a group $G$ satisfies one (not necessarily the same!) of the above $t$-properties, then $G$ is a $\overline{T}$-group.

  The converse is not likewise easy.
The framework of the investigation

This picture and the above equivalence give the first suggestions about the difficulties related to the characterizations of $\overline{T}$-groups. Infact, looking at the weakest property (subnormalizer condition) simply applying the definitions, it turns out that

- If **every** subgroup of a group $G$ satisfies one (not necessarily the same!) of the above $t$-properties, then $G$ is a $\overline{T}$-group.

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- Conversely, let $\chi$ be a $t$-property related to subgroups of a group $G$ (for example one of the above in the picture). The question is:

  If $G$ is a $\overline{T}$-group, can we say that every subgroup of $G$ satisfies $\chi$?
The "converse" in the universe of Finite groups

**Theorem (A)**

Let $G$ be a finite group, then are equivalent:

1. $G$ is a soluble $T$-group.
2. $G$ is a $\overline{T}$-group. (Zacher-Gaschütz-1952-1957)
6. Every subgroup of $G$ is an NE- subgroup of $G$. (Li-jgt-2006)
Examples of non $t$-properties

there are property referred to subgroups that are not $t$-property, but that are likewise connected to $T$-groups

- (Kaplan-jgt-2011) A subgroup $X$ of a group $G$ is said to be a $\varphi$-subgroup of $G$ if, for all $K, L$ maximal in $X$, it is the case that if $K, L$ are conjugate in $G$, then $K, L$ are conjugate in $X$.
- (Kaplan-jgt-2011) A subgroup $K$ of a group $G$ is said to be a $cr$-subgroup (conjugation restricted) of $G$ if there are no $A < K, g \in G$ such that $K = AA^g$.

The $CR$-property is not a $t$-property. Indeed $\langle(12)(34)\rangle$ is CR in $S_4$, and $\langle(12)(34)\rangle \triangleleft S_4$, but $\langle(12)(34)\rangle$ is not normal in $S_4$.

finally we will recall the following class of groups which also is connect to the class of $T$-groups.

- [Kaplan; 11-archiv] $G$ has the $NNM$ property (non-normal maximal)—in short: $G$ is an $NNM$-group—if each non-normal subgroup of $G$ is contained in a non-normal maximal subgroup of $G$. 
Groups whose subgroups satisfy the $CR$ property [resp. the $\varphi$-property]

**Theorem (B, Kaplan11-jgt)**

Let $G$ be a finite soluble group. Then the following statements are equivalent:

1. $G$ is a $T$-group.
2. Every subgroup of $G$ has the property $\varphi$.
3. Every subgroup of $G$ is a $cr$-subgroup.

Note that $cr$ and $\varphi$ are not $t$-properties, so that here all the implications are not trivial.

**Theorem (C, Kaplan11-archiv)**

Let $G$ be a finite group. The following conditions are equivalent:

1. $G$ is a soluble $T$-group.
2. All subgroups of $G$ are NNM-groups.
The investigation of the converse in the universe of infinite groups: If \( G \) is a \( \bar{T} \) group \( \Rightarrow \) ?

In order to avoid Tarsky groups, the first step is to restrict our study in the universe of locally graded groups.

A group \( G \) is called *locally graded* if every finitely generated non-trivial subgroup of \( G \) contains a proper subgroup of finite index.

This is a wide class of generalized soluble groups which contains every group without infinite simple sections.

Since finite \( \bar{T} \)-groups are metabelian, it follows easily that every periodic locally graded \( \bar{T} \)-group is locally finite, and so also metabelian.
The investigation of the converse in the universe of infinite groups: If $G$ is a periodic $\overline{T}$ group $\Rightarrow$ ?

We can summarize the extensions of the characterizations of finite $\overline{T}$-groups to periodic locally graded groups as follows:

**Theorem (Extension of Theorem A)**

Let $G$ be a periodic, locally graded group. The following statements are pairwise equivalent.

1. $G$ is a $\overline{T}$-group.
2. $G$ is locally soluble and all cyclic subgroups of $G$ are pronormal.
3. All subgroups of $G$ are $H$-subgroups. (—; ijgt-2017)
4. $G$ is locally soluble and all cyclic subgroups of $G$ are $H$-subgroups.
5. All subgroups of $G$ are weakly normal. (Russo; ca 2012, Romano,—; ijaa, 15)
6. $G$ is locally finite and all cyclic subgroups of $G$ are weakly normal.
7. All subgroups of $G$ are NE-subgroups. (Esteban-Romero, —; submitted)
8. $G$ is locally soluble and all cyclic subgroups of $G$ are NE-subgroups.
9. All subgroups of $G$ satisfy the subnor. condition. (de Giovanni, —; RicercheMat-03)
10. $G$ is locally soluble and all cyclic subgroups of $G$ satisfy the subnormalizer condition.

Moreover, if one of the above conditions hold, then $G$ is metabelian.
The investigation of the converse in the universe of infinite groups: If \( G \) is a **non** periodic \( \overline{T} \) group \( \Rightarrow \) ?

Non periodic soluble \( \overline{T} \)-groups are abelian (Robinson 1964)

Non periodic \( \overline{T} \)-groups without infinite simple sections are abelian (de Giovanni, — ; Q.Mat, 2001)

So that for non periodic \( \overline{T} \)-groups without infinite simple sections all the above conditions 1 — 10 are trivially satisfied. On the other hand it is unknown if Robinson’ result also holds in the universe of locally graded non periodic \( \overline{T} \)-groups. This is an open question as remarked in the Kourovka book

([Question 14.36 (F. de Giovanni]- Kourovka book-2014): Let \( G \) be a non-periodic locally graded \( \overline{T} \)-group. Can we say that \( G \) is abelian?

([Reduction of above Question]- Esteban Romero, —; submitted): Let \( G \) be a locally graded **torsion free** \( \overline{T} \)-group. Can we say that \( G \) is abelian?
(If a group $G$ has every subgroup satisfying a $t$-property ) $\Rightarrow$ $G$ is a $T$-group.

In order to attempt the solution of the question, it could be helpful to investigate the behavior of non periodic locally graded groups whose subgroups satisfy a $t$-property.
Locally graded non periodic groups whose subgroups satisfies one of the above \( t \)-property. Are they abelian?

**Theorem**

Let \( G \) be a non periodic, locally graded group. The following statements are pairwise equivalent.

1. \( G \) is abelian
2. All subgroups of \( G \) are pronormal (Robinson, Russo, \(--; \) ijac, 2007)
3. All subgroups of \( G \) are weakly normal (Russo; ca 2012; Romano, \(--; \) ijgt, 2015)
4. All subgroups of \( G \) are NE-subgroups (Esteban-Romero, \(--; \) submitted)


Let \( G \) be a non periodic, locally graded group.

1. All subgroups of \( G \) are \( \mathcal{H} \)-subgroups \( \Rightarrow \) \( G \) is abelian?
2. Let \( G \) be a group, and let \( X \) be an \( \mathcal{H} \)-subgroup of \( G \). Can we say that \( X \) is weakly normal in \( G \)? (it is true in the universe of HNN-free groups)

This conclude the main part referred to the characterizations of \( T \) groups in terms of \( t \) properties.
Characterizations of infinite $T$-groups by $cr$-subgroups, $\varphi$-subgroups and $NNM$-property: the extensions of Kaplan results to the infinite case

In this case the characterizations that we had been obtained are consequence of properties referred to the Wielandt subgroup in the universe of generalized FC-groups:

**$FC^*$**: A relevant class of generalized FC-groups

The class $FC^n$ is defined recursively as follows: $FC^0$ is the class of all finite groups $G$, and a group $G$ belongs to the class $FC^{n+1}$ if the factor group $G/C_G(\langle x \rangle^G)$ belongs to $FC^n$ for every element $x$ of $G$. Moreover, we put

$$FC^* = \bigcup_{n \geq 0} FC^n \quad (de\ Giovanni,\ Russo, \ldots;\ Ser.\ M.J.,\ 2002)$$

The class of $FC^*$ contains in particular all finite, nilpotent, and $FC$ groups, and a Sylow theory had been developed in this class ($Robinson,\ Russo, \ldots;\ J.\ Alg.\ 2011$)

Soluble Finite $\subset$ soluble $FC^* \subset$ soluble $\subset$ without infinite simple sections $\subset$ locally graded.
Characterizations of infinite $T$-groups by $cr$-subgroups and $\varphi$-subgroups: the extensions of Kaplan results to the infinite case

**Theorem**

Let $G$ be a soluble FC\(^*\)-group. Then the following are equivalent:

(i) $G$ is a $T$-group;

(ii) Every subgroup of $G$ is pronormal [(de Giovanni, — ; proc.Irish.royal.acad, 2000)];

(iii) Every subgroup of $G$ is $cr$ [(Kaplan, — ; ijac, 2014)];

(iv) Every subgroup of $G$ is $\varphi$-subgroup of $G$ [(Kaplan, — ; ijac, 2014)]

(v) All subgroups of $G$ are NNM-groups [(Esteban Romero, — ; jam, 2016)].

Note that in general [(i)] $\not\Rightarrow$ [(ii)], Infact, there are soluble $\bar{T}$-group containing non pronormal subgroups (Kov. Neu. de Vries).

Note that the equivalences [(i)] $\iff$ [(iii)] $\iff$ [(iv)] had been obtained via Wilandt’s subgroup properties, so that these equivalences perhaps could be extended to soluble groups.
<table>
<thead>
<tr>
<th>Characterizations of classes $T$-groups →</th>
<th>Finite Soluble $T$-groups</th>
<th>Soluble FC* $T$-groups</th>
<th>$\bar{T}$-Groups without infinite simple sections/ Locally graded $\bar{T}$-Groups</th>
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</thead>
<tbody>
<tr>
<td>Subnormaliser condition</td>
<td>Yes, 2002 (Ballester-Bol, Esteban Rom)</td>
<td>Yes, 2003 DeGiovanni and —</td>
<td>Yes, 2003 DeGiovanni,—; RicercheMat</td>
</tr>
<tr>
<td>Pronormality</td>
<td>Yes, 1969 Peng</td>
<td>Yes, 2000 DeGiovanni and —</td>
<td>$G$ is a $\bar{T}$-group $\iff$ every cyclic subgroup is pronormal Examples of $\bar{T}$-groups containing non-pronormal subgroups were given by Kovacs-Neumann De Vries 61, and by Kuzennyi Subbotin 89</td>
</tr>
<tr>
<td>Weakly normality</td>
<td>Yes, 2003 (Ballester-Bol, Esteban Rom)</td>
<td></td>
<td>Yes, 2012 Russo; ac., Roman, —; jaa</td>
</tr>
<tr>
<td>$H$-property</td>
<td>Yes, 2000 BianchiGilloHerzogVerardi-jgt</td>
<td></td>
<td>Yes, 2017 —; iji-gt-</td>
</tr>
<tr>
<td>NNM-group (non-normal maximal)</td>
<td>Yes, 2011 Kaplan31; archiv</td>
<td>Yes, 2016 EstebanRomero and —; Jams</td>
<td>NO, 2016 There exist examples of $T$-groups, that are hyperfinite and FC-nilpotent but that are not NNM-groups Esteban,—;ko2-genFC.</td>
</tr>
<tr>
<td>$p$-subgroup</td>
<td>Yes, 2011 Kaplan31; jgt</td>
<td>Yes, 2014 Kaplan and —; ijac</td>
<td>To investigate</td>
</tr>
<tr>
<td>cr-subgroup (conjugation restricted)</td>
<td>Yes, 2006 Li-jgt</td>
<td></td>
<td>Yes, Submitted 2018? Esteban-Romero and —</td>
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<td>NE-subgroup (normally embedded)</td>
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*FC* stands for finite conjugation.
Some References


2755065 (2012c:20038)
Thank You