Übungen zu Fana2 WS19, 1. Übung

1. Show that, given elements x, y of an allgebra with one-element,
$$\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\} .$$
Hint: Show $(\lambda e - yx)y(\lambda e - xy)^{-1}x = yx$ and conclude from show the inverse of $(yx - \lambda e)$.

2. Show that for a Banach-algebra A with one, which is denoted by e, and arbitrary $a \in A$
$$\lim_{|z|\to+\infty} z(a - ze)^{-1} = -e .$$

3. Show that the space
$$\ell_1(\mathbb{Z}) := \{(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |a_n| < +\infty\}$$
of all absolutly summable complex double series provided with the norm
$$\|(a_n)_{n\in\mathbb{Z}}\|_1 := \sum_{n\in\mathbb{Z}} |a_n|$$
and with the mapping
$$* : ((a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}}) \mapsto (c_n)_{n\in\mathbb{Z}} =: (a_n)_{n\in\mathbb{Z}} * (b_n)_{n\in\mathbb{Z}} ,$$
where
$$c_n = \sum_{j\in\mathbb{Z}} a_j b_{n-j} ,$$
is a commutative Banach-algebra with one. What is the one-element? Also show that $\sum_{j\in\mathbb{Z}} a_j b_{n-j}$ converges absolutely in $\ell_1(\mathbb{Z})$ for every n.

4. Let $A(\mathbb{T})$ be the space of all complex valued functions $f : \mathbb{T} \to \mathbb{C}$ such that f is integrable with respect to the measure μ on the Borel subsets of $\mathbb{T}$, where $2\pi \cdot \mu(A) = \lambda(T^{-1}(A))$ and $T(t) = \exp(it)$, $t \in [0, 2\pi)$, and such that is Fourier coefficients $(\alpha_n)_{n\in\mathbb{Z}}$,
$$\alpha_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta)\, \zeta^{-n}\, d\mu(\zeta) ,$$
satisfy $\sum_{n\in\mathbb{Z}} |\alpha_n| < +\infty$. Show that $A(\mathbb{T})$ provided with $\|f\| := \sum_{n\in\mathbb{Z}} |\alpha_n|$ and with pointwise multiplikation is a Banach-algebra with one. Also show prove that $\theta : \ell_1(\mathbb{Z}) \to A(\mathbb{T})$ $(\subseteq L^1(\mathbb{T}, \mathfrak{B}(\mathbb{T}), \mu))$,
$$\theta((a_n)_{n\in\mathbb{Z}})(\zeta) := \sum_{n\in\mathbb{Z}} a_n\, \zeta^n ,$$
is an isometric isomorphism, where $\ell_1(\mathbb{Z})$ is as in the previous example. In what sense does $\sum_{n\in\mathbb{Z}} a_n\, \zeta^n$ converge? Finally, show $A(\mathbb{T}) \subseteq C(\mathbb{T})$.

   Hint: You can use without proof the fact that $f \mapsto (\alpha_n)_{n\in\mathbb{Z}}$ is a one-to-one mapping from $L^1(\mu)$ into $\ell^\infty(\mathbb{Z})$. This can be shown with the help of the Riesz Representation Theorem and the fact that the trigonometric polynomials are densely contained in $C(\mathbb{T}, \mathbb{C})$.

5. For an algebra A (over $\mathbb{C}$) let $\tilde{A} := A \times \mathbb{C}$ be provided with the multiplication

$$\Big((a,\alpha),(b,\beta)\Big) := (ab + \alpha b + \beta a, \alpha\beta)\,.$$

Show that $\tilde{A}$ is an algebra with one-element. What is the one-element? Also show that $A \cong A \times \{0\}$ is a sub algebra of $\tilde{A}$ ist. What elements from A ($\subseteq \tilde{A}$)are invertible?

Show that: If A is a Banach-algebra, then $\tilde{A}$ provided with $\|(a,\lambda)\| := \|a\| + |\lambda|$ is also a Banachalgebra, which contains $A \cong A \times \{0\}$ isometrically. What is the norm of the one-element of $\tilde{A}$.

6. For an algebra A an $x \in A$ is called pseudo-invertible, if $xy+x+y = yx+x+y = 0$ for a certain $y \in A$.

Show that: If A has the one-element e, then $(e + x) \in \mathrm{Inv}(A)$ if and only if x is pseudo-invertible. If A has the one-element e, then $\sigma_A(a) \setminus \{0\} = \sigma_{\tilde{A}}(a) \setminus \{0\}$ and $r_A(a) = r_{\tilde{A}}(a)$ for all $a \in A$, where $\tilde{A}$ is as in the previous example and a is identified with $(a, 0)$.

7. Show that every multiplicative lineare functional $m : A \to \mathbb{C}$ on an algebra A can be extended to a multiplicative lineare functional $\tilde{m} : \tilde{A} \to \mathbb{C}$. Is the continuation unique?

Is the restrition of a multiplicative lineare functional on $\tilde{A}$ to A always a multiplicative lineare functional on A?

8. For a set Ω let $\mathcal{B}(\Omega,\mathbb{C})$ be the Banach-space of all bounded and complex valued functions provided with $\|.\|_\infty$. Show that $\mathcal{B}(\Omega,\mathbb{C})$ provided with pointwise addition, scalar multiplication, pointwise multiplication and pointwise conjugation forms a C^*-algebra. For $f \in \mathcal{B}(\Omega,\mathbb{C})$ calculate $\sigma(f)$ and $r(f)$.

Let $\mathcal{A}$ be a σ-algebra on Ω. Show that the subspace $\mathcal{B}_{\mathcal{A}}(\Omega,\mathbb{C})$ of all measurable $f \in \mathcal{B}(\Omega,\mathbb{C})$ is a sub-C^*-algebra of $\mathcal{B}(\Omega,\mathbb{C})$.

If $\mathcal{T}$ is a topology on ω, show that the subspace $C(\Omega,\mathbb{C})$ of all continuous $f \in \mathcal{B}(\Omega,\mathbb{C})$ is a sub-C^*-algebra of $\mathcal{B}(\Omega,\mathbb{C})$.

9. Let $(\Omega,\mathcal{A},\mu)$ be a measure space and provide $L^\infty\Omega,\mathcal{A},\mu,\mathbb{C})$ with the essential supremum. Show that $L^\infty\Omega,\mathcal{A},\mu,\mathbb{C})$ with the pointwise addition, scalar multiplication, pointwise multiplication and pointwise conjugation forms a C^*-algebra.

Find an closed ideal I in $\mathcal{B}_{\mathcal{A}}(\Omega,\mathbb{C})$ such that $L^\infty\Omega,\mathcal{A},\mu,\mathbb{C})$ is isometrically isomorphic to $\mathcal{B}_{\mathcal{A}}(\Omega,\mathbb{C})/I$. Justify your answeres! Also show that for $f \in L^\infty(\mu)$

$$\sigma_{L^\infty\Omega,\mathcal{A},\mu,\mathbb{C})}(f) = \{\lambda \in \mathbb{C} : \forall\epsilon > 0 \Rightarrow \mu(f^{-1}(U_\epsilon(\lambda))) > 0\}\,.$$