LECTURE 2

The uniform convergence and representation theorems for slowly varying functions

We saw in Lemma 1.4 that functions of the form $S[\eta_\infty, \eta, x]$ are slowly varying with locally uniform limit (1.1).

2.1. Theorem (uniform convergence theorem):

Let $r_0 > 0$ and $f : [r_0, \infty) \to (0,\infty)$ be slowly varying. Then

$$\lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = 1 \text{ locally uniformly for } \lambda \in (0,\infty).$$

Proof: Assume that f is slowly varying. We clearly have

$$\lim_{r\to\infty} \frac{f(\lambda r)}{f(r)} = 1 \text{ locally uniformly on } \lambda \iff$$

$$\forall T>0,\ \lambda_n \in [e^{-T}, e^{T}],\ r_n \to \infty : \lim_{n\to\infty} \frac{f(\lambda_n r_n)}{f(r_n)} = 1$$

and we are going to prove the latter.

Let $T>0$, $\lambda_n = e^{\omega_n}$ with $\omega_n \in [-T,T]$, $r_n \to \infty$ be given. Consider the functions

$$h_n(y) := \frac{f(e^y r_n)}{f(r_n)}, \quad k_n(y) := \frac{f(e^y \lambda_n r_n)}{f(\lambda_n r_n)}.$$

Then

$$\forall y \in [-T,T]. \lim_{n\to\infty} h_n(y) = \lim_{n\to\infty} k_n(y) = 1.$$

Egoroff's theorem provides us with sets

$$E, F \subseteq [-T,T], \quad \lambda(E), \lambda(F) \geq \tfrac{7}{8} \cdot 2T,$$

such that

$$\lim_{n\to\infty} h_n(y) = 1 \quad \text{uniformly for } y \in E,$$

$$\lim_{n\to\infty} k_n(y) = 1 \quad \text{uniformly for } y \in F.$$

We show next that

$$\forall n : E \cap (\omega_n + F) \neq \emptyset. \qquad (2.1)$$

To this end, note that

$$[-T,T] \cup \left(\omega_n + [-T,T]\right) \subseteq \begin{cases} [-T, T+\omega_n], & \omega_n \geq 0 \\ [-T+\omega_n, T], & \omega_n \leq 0 \end{cases}$$

and hence

$$\lambda\left(E \cup (\omega_n + F)\right) \leq \lambda\left([-T,T] \cup \left(\omega_n + [-T,T]\right)\right) \leq$$

$$\leq 2T + |\omega_n| \leq 3T < 2 \cdot \tfrac{7}{8} 2T \leq$$

$$\leq \lambda(E) + \lambda(F) = \lambda(E) + \lambda(\omega_n + F).$$

Thus indeed (2.1) holds.

By (2.1) we can choose $y_n \in E$ and $z_n \in F$ such that $y_n = w_n + z_n$, and write

$$\frac{f(\lambda_n r_n)}{f(r_n)} = \frac{f(\lambda_n r_n)}{f(e^{z_n}\lambda_n r_n)} \cdot \frac{f(e^{y_n} r_n)}{f(r_n)} =$$

$$= \frac{h_n(y_n)}{k_n(z_n)} \to 1 .$$

$\square$

2.2. Corollary:

Let f be slowly varying. Then there exists $R>0$ such that for all $\alpha \geq R$

$$\inf_{r\in[R,\alpha]} f(r) > 0 \quad , \quad \sup_{r\in[R,\alpha]} f(r) < \infty .$$

Proof: Choose $R>0$ such that

$$\forall\, r \geq R,\ \lambda \in [1,2] :\ \frac{1}{2} \leq \frac{f(\lambda r)}{f(r)} \leq 2 . \qquad (2.2)$$

Fix $n \in \mathbb{N}$ and consider a point $r \in [R, 2^n R]$. Choose m with $2^m R \leq r \leq 2^{m+1} R$ and write

$$f(r) = \frac{f(r)}{f(r/2)} \cdot \ldots \cdot \frac{f(r/2^{m-1})}{f(r/2^m)} \cdot \frac{f(r/2^m)}{f(R)} \cdot f(R)$$

By (2.2) each of the quotients on the right side lies in

$[\frac{1}{2}, 2]$. Note here that $\frac{r}{2^n}/R \in [1,2]$. It follows that

$$\left(\frac{1}{2}\right)^{n+1} f(R) \leq f(r) \leq 2^{n+1} f(r).$$

Since n was arbitrary, the assertion follows. $\square$

2.3. Theorem (representation theorem):

Let $r_0 > 0$ and $f : [r_0, \infty) \to (0, \infty)$ be slowly varying. Then $f = S[\eta_\infty, \eta, x]$ for some η_∞, η, x.

Proof: Choose R as in Corollary 2.2. The function $\log f(r)$ is measurable and locally bounded on $[R, \infty)$. Hence, it is also locally integrable on this interval.

We write $\log f(r)$ for $r \geq R$ by adding and subtracting suitable terms as

$$\log f(r) = \int_r^{er} \left(\log f(r) - \log f(v)\right) \frac{dv}{v} +$$

$$+ \int_R^{eR} \log f(v) \frac{dv}{v} + \left[\int_{eR}^{er} \log f(v) \frac{dv}{v} - \int_R^r \log f(v)\, dv \right] =$$

$$= \underbrace{\int_R^{eR} \log f(v) \frac{dv}{v}}_{=: \eta_\infty} + \underbrace{\int_r^{er} \log \frac{f(r)}{f(v)} \frac{dv}{v}}_{=: \eta(r)} +$$

$$+ \int_R^r \underbrace{\log \frac{f(ev)}{f(v)}}_{=: \varkappa(v)} \frac{dv}{v}$$

By Corollary 2.2 the function $\varkappa$ is locally bounded on $[R,\infty)$. Clearly, it is measurable and $\lim_{v\to\infty} \varkappa(v)=0$ (In particular, $\varkappa$ is bounded).

Write $\eta(r) = \log f(r) - \int_r^{er} \log f(v) \frac{dv}{v}$. This shows that η is measurable and locally bounded on $[R,\infty)$.

Since

$$\lim_{r\to\infty} \frac{f(r)}{f(v)} = 1 \text{ uniformly for } v\in[r,er], \quad \int_r^{er} \frac{dv}{v} = 1,$$

we have $\lim_{r\to\infty} \eta(r) = 0$.

Together we see that $f = s[\eta_\infty, \eta, \varkappa]$. $\square$