LECTURE 1

Slow- and regular variation

1.1. Definition:

Let $r_0 > 0$, let $f : [r_0, \infty) \to (0, \infty)$ be a (Borel-) measurable function, and let $\varrho \in \mathbb{R}$.
Then f is called *regularly varying with index ϱ*, if

$$\forall \lambda > 0 : \lim_{r \to \infty} \frac{f(\lambda r)}{f(r)} = \lambda^{\varrho} . \qquad (1.1)$$

The set of all regularly varying functions with index ϱ is denoted as R_{ϱ}.
If f is regularly varying with index 0, then f is called *slowly varying*.

1.2. Remark:

Let $\alpha, \varrho \in \mathbb{R}$, and let $f : [r_0, \infty) \to (0, \infty)$ be measurable. Then

$$f \in R_{\varrho} \iff r^{\alpha} f(r) \in R_{\varrho + \alpha}$$

This is clear since $(\lambda r)^{\alpha} = \lambda^{\alpha} \cdot r^{\alpha}$.

In view of this remark it is often enough to understand slowly varying functions.

Slowly varying functions can be obtained from integral representations.

1.3. Definition:

Let $r_0 > 0$, $\eta_\infty \in \mathbb{R}$, and $\eta, \varkappa : [r_0, \infty) \to (0,\infty)$ be bounded measurable functions with

$$\lim_{r\to\infty} \eta(r) = \lim_{r\to\infty} \varkappa(r) = 0.$$

Then we denote (for $r \geq r_0$)

$$S[\eta_\infty, \eta, \varkappa](r) := \exp\left(\eta_\infty + \eta(r) + \int_{r_0}^{r} \varkappa(t)\,\frac{dt}{t}\right).$$

1.4. Lemma:

Each function of the form $S[\eta_\infty, \eta, \varkappa]$ is slowly varying. In fact, the limit (1.1) is attained locally uniformly in λ.

Proof: To shorten notation write $f := S[\eta_\infty, \eta, \varkappa]$. Then, for $\lambda \geq 1$ and $r \geq r_0$,

$$\frac{f(\lambda r)}{f(r)} = \exp\left(\eta(\lambda r) - \eta(r) + \int_{r}^{\lambda r} \varkappa(t)\,\frac{dt}{t}\right).$$

We have

$$\left| \eta(\lambda r) - \eta(r) + \int_r^{\lambda r} x(t) \frac{dt}{t} \right| \le$$

$$\le 2 \cdot \sup_{t \geq r} |\eta(t)| + \log \lambda \cdot \sup_{t \geq r} |x(t)| .$$

For $r \to \infty$ the right side tends to 0 uniformly for λ in any interval of the form $[1, \lambda_0]$.

It follows that

$$\left| \frac{f(\lambda r)}{f(r)} - 1 \right| \to 0 \quad \text{uniformly for } \lambda \in [1, \lambda_0],$$

and, using $1/\lambda$ instead of λ, thus also uniformly for $\lambda \in [1/\lambda_0, 1]$.

$\square$

1.5. Example: Let $f : [r_0, \infty) \to (0, \infty)$ be continuously differentiable with

$$\lim_{r \to \infty} \frac{r f'(r)}{f(r)} = 0 . \qquad (1.2)$$

Then f is slowly varying.

To see that it is enough to note that

$$f(r) = \exp\left(\log r_0 + \int_{r_0}^{r} \frac{f'(t)}{f(t)} \cdot t \, \frac{dt}{t} \right) =$$

$$= S\left[\log r_0, 0, \frac{t f'(t)}{f(t)} \right](r) .$$

1.6. Example:

Denote by $\log^{[m]} r$ the m-times iterated logarithm, i.e.,

$$\log^{[0]} r := r, \quad \forall m \in \mathbb{N}: \log^{[m+1]} r := \log^{[m]}(\log r).$$

We refer to functions of the form (r sufficiently large)

$$f(r) := \prod_{m=0}^{N} \left(\log^{[m]} r\right)^{\alpha_m}, \qquad (1.3)$$

where $N \in \mathbb{N}$ and $\alpha_m \in \mathbb{R}$, as Lindelöf comparison functions. Each such function is regularly varying, and the order of (1.3) is α_0.

To show this we use the previous example. Clearly, every Lindelöf comparison function is even C^∞. In order to evaluate the limit in (1.2), we first note that

$$\forall n \in \mathbb{N}: \quad \frac{d}{dr} \log^{[n]} r = \left(\prod_{k=0}^{n-1} \log^{[k]} r \right)^{-1}.$$

This is checked by induction:

$$\frac{d}{dr} \log^{[0]} r = 1 = \left(\prod_{k=0}^{-1} \log^{[k]} r \right)^{-1},$$

$$\frac{d}{dr} \log^{[n+1]} r = \frac{d}{dr} \left[\log^{[n]} (\log r) \right] =$$

$$= \left(\left[\frac{d}{dr} \log^{[n]}(r) \right] \circ \log r \right) \cdot \frac{1}{r} =$$

$$= \left(\left[\prod_{k=0}^{n-1} \log^{[k]} r \right]^{-1} \circ \log r \right) \cdot \frac{1}{r} = \left(\prod_{k=0}^{n} \log^{[k]} r \right)^{-1}.$$

The first factor in $f(r)$ is r^{α_0}, and thus we must show that (1.2) holds for a function (1.3) with $\alpha_0 = 0$. However, for such a function we compute

$$r \frac{f'(r)}{f(r)} = r \cdot \frac{d}{dr} \log \prod_{m=1}^{N} \left(\log^{[m]} r \right)^{\alpha_m} =$$

$$= r \sum_{m=1}^{N} \alpha_m \frac{d}{dr} \log^{[m+1]} r = \sum_{m=1}^{N} \alpha_m \, r \left(\prod_{k=0}^{m} \log^{[k]} r \right)^{-1} =$$

$$= \sum_{m=1}^{N} \alpha_m \left(\prod_{k=1}^{m} \log^{[k]} r \right)^{-1} \to 0.$$