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Spectral Theory for Nonlinear Operators

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Chapter 1

Introduction

1.1 Spectral theory for linear operators

We first want to recapitulate some basic facts about spectral theory for a bounded linear operator T that operates on a Banach space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The set of all such operators will be denoted as $\mathcal{B}(X)$ and will be equipped with the operator norm $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$.

We start with defining the resolvent set

$$\rho(T) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ is bijective}\},$$

where I is the identity operator on X. It can easily be shown that $\rho(T)$ coincides with the set of points such that $\lambda I - T$ is invertible and its inverse is also a bounded operator. Therefore, it also coincides with the set of points where $\lambda I - T$ is a homeomorphism. The inverse operator $R(\lambda, T) := (\lambda I - T)^{-1}$ is called the resolvent operator of T at λ. Since $\lambda \in \rho(T)$ and $\mu \in \mathbb{K}$ with $|\mu - \lambda| < \|R(\lambda, T)\|^{-1}$ implies $\mu \in \rho(T)$, $\rho(T)$ is an open subset of $\mathbb{K}$.
The spectrum of the operator T is definded as

$$\sigma(T) := \mathbb{K} \backslash \rho(T),$$

which is in turn closed. In the case of $\mathbb{K} = \mathbb{C}$ the set $\rho(T)$ is also always nonempty and the spectral radius $r(T) := \sup\{|\lambda| : \lambda \in \rho(T)\}$ can be calculated by the Gelfand formula

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}.$$

The estimate $r(T) \leq \|T\|$ is true, even in the case $\mathbb{K} = \mathbb{R}$. Consequently $\sigma(T)$ is always bounded and, therefore, compact.
A very important property of the spectrum is the spectral mapping theorem. That is, for any

polynomial $p(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ we get the identity

$$\sigma(p(T)) = p(\sigma(T))$$

Here $p(T)$ denotes the operator $a_nT^n + \cdots + a_1T + a_0I$ and $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$. This theorem is one of the starting points for the spectral theorem, which allows the representation of certain linear operators as integrals over the spectrum.
Another important property of the spectrum is its upper semi-continuity. Since this fact often remains unmentioned, we will give a short proof.

Proposition 1.1 *For $T \in \mathcal{B}(X)$ let $G \subseteq \mathbb{K}$ be an open set with $\sigma(T) \subseteq G$. Then there exists a $\delta > 0$ such that $\sigma(S) \subseteq G$ for every $S \in \mathcal{B}(X)$ with $\|S - T\| < \delta$.*

Proof: We define the closed set $F = \mathbb{K}\backslash G$. Since every λ in F belongs to $\rho(T)$, $R(\lambda, T)$ lies in $\mathcal{B}(X)$. We are going to show that also $\lambda \in \rho(S)$ for all $S \in \mathcal{B}(X)$ satisfying

$$\|S - T\| < \frac{1}{\|R(\lambda, T)\|}.$$

To do so, we use the well known fact from the theory of Neumann series that $I - R$ is invertible for any R with $\|R\| < 1$. Therefore, the formula

$$\begin{aligned} \left\|I - (\lambda I - T)^{-1}(\lambda I - S)\right\| &= \|R(\lambda, T)\,((\lambda I - T) - (\lambda I - S))\| \leq \\ &\leq \|R(\lambda, T)\|\,\|(\lambda I - T) - (\lambda I - S)\| = \|R(\lambda, T)\|\,\|S - T\| < 1 \end{aligned}$$

implies that $(\lambda I - T)^{-1}(\lambda I - S)$ is invertible. Consequently, $\lambda I - S$ is invertible, i.e. $\lambda \in \rho(S)$. The fact that $\|R(\lambda, T)\| \to 0$ as $|\lambda| \to \infty$ implies that actually

$$\delta := \inf_{\lambda \in F} \frac{1}{\|R(\lambda, T)\|} > 0.$$

□

Heuristically, upper semi-continuity assures that the spectrum does not 'expand suddenly'. However it is not lower semi-continuous, which means that even under small perturbations it can 'collapse'.

1.2 Spectral theory for nonlinear operators

Since spectral theory is so fruitful in the case of linear operators, it is natural to try to extend its principles to nonlinear operators. In order to justify the label 'spectral theory for nonlinear operators', we would like the spectrum of nonlinear operators to have similar properties as in the linear case. It should even be identical to the classical spectrum when applied to a linear operator. Since this task is complicated enough, we will restrict ourselves to continuous operators.

The first question when talking about a spectral theory for nonlinear operators is then, of course, how to define the spectrum. A first straightforward attempt could simply be, as in the linear case, to define the spectrum of an operator T as the set of all points λ such that $\lambda I - T$ is not bijective. But simple examples show that in that case the spectrum would fail to have even any basic properties like being closed, bounded or nonempty. For linear operators, the linearity guarantees the linearity of its inverse and the open mapping theorem its continuity. Thus, bijectivity is equivalent to being a homeomorphism. Since this tool is not available for nonlinear operators, these two properties are not equivalent anymore. Defining the spectrum as all λ such that $\lambda I - T$ is not a homeomorphism will consequently lead to a different kind of spectrum.

Although this approach turns out to be just as dissapointing as the previous one, it points out an important difference between the spectral theory for linear and nonlinear operators. The linearity of an operator is responsible for the fact that many different properties are equivalent to bijectivity. So in the linear theory the spectrum contains information about all these properties, while at the same time one has to check only bijectivity, which is comparatively easy to handle. If we want to deal with nonlinear operators, we have to look at all these different properties seperately. So for any property A that makes sense for a nonlinear operator, we can define the A-resolvent

$$\rho_A(T) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ has property } A\}$$

and the A-spectrum

$$\sigma_A(T) = \mathbb{K} \backslash \rho_A.$$

The only restriction for this property A is that it should be equivalent to bijectivity in case of linear operators, so that the spectrum coincides with the classical one in the linear case. Therefore, there are many different spectra to be considered.

None of them has yet lead to results of the same extent as the linear spectrum. By the above considerations this is not completely unexpected. It is possible, and not very unlikely, that there is no particular nonlinear spectrum that is as all-encompassing. However, the study of these many different spectra is not in vain, and leads to interesting results.

In this thesis we will present one of these spectra, the Furi-Martelli-Vignoli-spectrum or FMV-spectrum for short. This example will show how a spectral theory for nonlinear operators can be developed and we will provide some results from this new theory.

Chapter 2

Preliminary considerations

In this chapter we discuss some results which will be used throughout this paper. We will restrict ourselves to results pertaining to the nonlinearity of operators. Well known results that strictly deal with linear operators (e.g. the open mapping theorem) will be cited without proofs throughout this thesis. We will also assume knowledge of basic theorems about topological properties of Banach spaces, like the Baire category theorem or the fact that closed balls are compact only in finite dimensional spaces.

2.1 Characteristics of nonlinear operators

Let X and Y be two Banach spaces and $T : X \to Y$ a continuous operator, which in general will be nonlinear. By $\mathcal{C}(X,Y)$ we denote the set of all continuous operators from X into Y. Of course, this set forms a linear space. $\mathcal{C}(X) := \mathcal{C}(X,X)$ is an algebra with respect to the composition. However, in the case of linear operators the space $\mathcal{B}(X,Y)$ is normed by the operator norm. As indicated in the introduction, the significance of the operator norm here is due to the linearity of the operator. In the case of nonlinear operators one has to consider multiple seminorms or other characteristics. We only present the four characteristics that we will use in the following.

Definition 2.1 *For $T \in \mathcal{C}(X,Y)$ we define*

$$[T]_Q := \limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|}$$

and

$$[T]_q := \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|},$$

as elements of $[0,\infty]$. If $[T]_Q < \infty$, we call T quasibounded. By $\mathfrak{Q}(X,Y)$ we denote the set of all quasibounded continuous maps of X into Y.

In particular, the fact that $[T]_Q = \lambda$ or $[T]_q = \lambda$ implies that there exists an unbounded sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} \|T(x_n)\|/\|x_n\| = \lambda$. Furthermore, the inequality $[T]_q \le [T]_Q$ is obviously true. Consequently, $[T]_Q = 0$ actually implies that $\lim_{n\to\infty} \|T(x_n)\|/\|x_n\| = 0$ for every sequence $(x_n)_{n\in\mathbb{N}}$ with $\|x_n\| \to \infty$.
We now gather some more properties of these characteristics.

Proposition 2.2 *Let $T, S \in \mathcal{C}(X,Y)$ and $R \in \mathcal{C}(Y,Z)$. Then the following holds:*

(i) $[T]_q > 0$ *implies that T is coercive, i.e.* $\lim\limits_{\|x\|\to\infty} \|T(x)\| = \infty$.

(ii) One of the quantities on the left being finite, $[T]_q - [S]_Q \le [T+S]_q \le [T]_q + [S]_Q$.

(iii) One of the quantities on the left being finite, $|[T]_q - [S]_q| \le [T-S]_Q$. *In particular,* $[T-S]_Q = 0$ *implies* $[T]_q = [S]_q$.

(iv) $[T^{-1}]_Q = [T]_q^{-1}$ *if T is a homeomorphism and either T is linear or X and Y are finite dimensional.*

(v) $[R \circ T]_q \ge [R]_q [T]_q$.

Proof: If $[T]_q > 0$, then for sufficiently large $\|x\|$ there exists a $k > 0$ such that $\|T(x)\| \ge k\,\|x\|$. This verifies (i).
For (ii) consider

$$[T+S]_q = \liminf_{\|x\|\to\infty} \frac{\|T(x)+S(x)\|}{\|x\|} \le \liminf_{\|x\|\to\infty} \frac{\|T(x)\|+\|S(x)\|}{\|x\|} \le \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} + \limsup_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|} = [T]_q + [S]_Q.$$

For the second inequality replace T by $T+S$ and S by $-S$.
Similarly, consider

$$[T-S]_Q = \limsup_{\|x\|\to\infty} \frac{\|T(x)-S(x)\|}{\|x\|} \ge \limsup_{\|x\|\to\infty} \left|\frac{\|T(x)\|-\|S(x)\|}{\|x\|}\right| \ge \limsup_{\|x\|\to\infty} \left(\frac{\|T(x)\|}{\|x\|} - \frac{\|S(x)\|}{\|x\|}\right)$$
$$\ge \limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} - \liminf_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|} \ge \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} - \liminf_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|} = [T]_q - [S]_q,$$

which for symmetry reasons proves (iii).
In (iv), the two assumptions that T is linear and that X and Y are finite dimensional both assure that $\|x\| \to \infty \Leftrightarrow \|T(x)\| \to \infty$. We therefore can consider the chain of equalities

$$[T^{-1}]_Q = \limsup_{\|y\|\to\infty} \frac{\|T^{-1}(y)\|}{\|y\|} = \limsup_{\|x\|\to\infty} \frac{\|x\|}{\|T(x)\|} = \left(\liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|}\right)^{-1} = \frac{1}{[T]_q}.$$

Finally, to see (v), consider

$$\begin{aligned}[R \circ T]_q &= \liminf_{\|x\|\to\infty} \frac{\|R(T(x))\|}{\|x\|} = \liminf_{\|x\|\to\infty} \frac{\|R(T(x))\|}{\|T(x)\|}\frac{\|T(x)\|}{\|x\|} \\ &\geq \liminf_{\|x\|\to\infty} \frac{\|R(T(x))\|}{\|T(x)\|} \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} \geq \liminf_{\|y\|\to\infty} \frac{\|R(y)\|}{\|y\|} \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} = [R]_q[T]_q.\end{aligned}$$

The last inequality holds true, because if $\|T(x)\| \nrightarrow \infty$, then $[T]_q = 0$ and the inequality holds, and if $\|T(x)\| \to \infty$ we can set $T(x) = y$ to see that the inequality holds. □

Next, we are going to introduce a tool that is often used in nonlinear functional analysis. In the following, $B_\epsilon(z)$ will always denote the open ball with radius ϵ and center z.

Definition 2.3 *Let X be a Banach space and $M \subseteq X$. The Hausdorff measure of noncompactness of M is defined as*

$$\alpha(M) := \inf\{\epsilon > 0 | \, \exists m \in \mathbb{N}, \{z_1, \ldots, z_m\} \subseteq X : M \subseteq B_\epsilon(z_1) \cup \cdots \cup B_\epsilon(z_m)\} \in [0, \infty].$$

A finite set $\{z_1, \ldots, z_m\} \subseteq X$ with $M \subseteq B_\epsilon(z_1) \cup \cdots \cup B_\epsilon(z_m)$ is called a finite ϵ-net for M.

So if M has a measure of noncompactness ϵ, it can be covered by finitely many open balls of any radius greater than ϵ.

Proposition 2.4 *Let X be a Banach space, $M, N \subseteq X$, $\lambda \in \mathbb{K}$, and $z \in X$. Then the measure of noncompactness has the following properties:*

(i) $\alpha(M) = \alpha(\overline{M})$.

(ii) $\alpha(M) = 0$ *if and only if M is precompact, i.e. has a compact closure.*

(iii) $\alpha(M) < \infty$ *if and only if M is bounded.*

(iv) $M \subseteq N \Rightarrow \alpha(M) \leq \alpha(N)$.

(v) $|\alpha(M) - \alpha(N)| \leq \alpha(M + N) \leq \alpha(M) + \alpha(N)$, *where for the first inequality to hold either $\alpha(M)$ or $\alpha(N)$ needs to be finite.*

(vi) $\alpha(\lambda M) = |\lambda|\alpha(M)$.

(vii) $\alpha(M + z) = \alpha(M)$.

(viii) $\alpha(M \cup N) = \max\{\alpha(M), \alpha(N)\}$.

(ix) $\alpha(\mathrm{co}(M)) = \alpha(M)$, *where $\mathrm{co}(M)$ denotes the convex hull of M.*

(x) $\alpha(B_r(z)) = 0$ *if* $\dim X < \infty$, *and* $\alpha(B_r(z)) = r$ *if* $\dim X = \infty$.

Proof: The first four assertion are straightforward to verify. To see (v), observe, that if $\{z_1, \ldots, z_m\}$ is a finite ϵ-net for M, and $\{w_1, \ldots, w_n\}$ is a finite η-net for N, then $\{z_i + w_j | i = 1, \ldots, m; j = 1, \ldots, n\}$ is a finite $(\epsilon+\eta)$-net for $M+N$, which shows $\alpha(M+N) \leq \alpha(M)+\alpha(N)$. The first inequality in (v) immediately follows from this one. Moreover, if $\{z_1, \ldots, z_m\}$ is a finite ϵ-net for M then $\{\lambda z_1, \ldots, \lambda z_m\}$ is a finite $|\lambda|\epsilon$-net for λM, so (vi) follows. Similarly, (vii) follows from the observation that $\{z_1, \ldots, z_m\}$ is a finite ϵ-net for M if and only if $\{z_1 + z, \ldots, z_m + z\}$ is a finite ϵ-net for $M + z$. For $(viii)$ we use the fact that if $\{z_1, \ldots, z_m\}$ is a finite ϵ-net for M and $\{w_1, \ldots, w_n\}$ is a finite η-net for N, then $\{z_1, \ldots, z_m\} \cup \{w_1, \ldots, w_n\}$ is a finite δ-net for $M \cup N$, where $\delta = \max\{\epsilon, \eta\}$.
To see that (ix) holds true, by (iv) it suffices to show that $\alpha(\mathrm{co}(M)) \leq \alpha(M)$. So for $\eta > \alpha(M)$ choose a finite η-net $\{z_1, \ldots z_m\}$ and define $N = \mathrm{co}(\{z_1 \ldots z_m\})$. Any $x \in \mathrm{co}(M)$ can be written as a convex combination $x = \sum a_i x_i$ with $x_i \in M$ and $\sum a_i = 1$. For every x_i there is a $z_{j(i)}$ with $|x_i - z_{j(i)}| < \eta$. Setting $z = \sum a_i z_{j(i)}$ we get

$$\|x - z\| = \left\|\sum a_i x_i - \sum a_i z_{j(i)}\right\| = \left\|\sum a_i (x_i - z_{j(i)})\right\| \leq \sum |a_i| \left\|x_i - z_{j(i)}\right\| \leq \sum |a_i| \eta = \eta.$$

Because $z \in N$, we have $\mathrm{dist}(x, N) \leq \eta$. Since N is compact, we can find a finite ϵ-net $\{w_1 \ldots w_n\}$ for N and arbitrary $\epsilon > 0$. This is then a finite $(\eta + \epsilon)$-net for $\mathrm{co}(M)$.
Since every bounded set in a finite dimensional space is precompact, the first assertion of (x) is trivial. To see the second one, by (vi) and (vii) it suffices to show that $\alpha(B_1(0)) = 1$. Since $B_1(0)$ can be covered by itself, we have $\alpha(B_1(0)) \leq 1$. Assume $\alpha(B_1(0)) < 1$. Then $B_1(0)$ can be covered by finitely many open balls with radius $\eta \in (0, 1)$. Again by (vi) and (vii) we can in turn cover each of those balls with finitely many open balls with radius η^2, which gives us a finite cover of $B_1(0)$ by such sets. Since $\eta^n \to 0$, by iterating this process we get a finite cover of $B_1(0)$ with open balls with a radius smaller than ϵ for every $\epsilon > 0$. This shows, that $B_1(0)$ is precompact, i.e. $\overline{B_1(0)}$ is compact. Thus, we get a contradiction to the well known fact that this is only the case in finite dimensional spaces. □

Unlike the characteristics we introduced before, the following characteristics can also be defined for an arbitrary subset of a Banach space.

Definition 2.5 *Let $Z \subseteq X$. For $T \in \mathcal{C}(Z, Y)$ we define*

$$[T]_A := \inf\{k : k > 0,\ \alpha(T(M)) \leq k\alpha(M) \text{ for all bounded } M \in Z\}$$

and

$$[T]_a := \sup\{k : k > 0,\ \alpha(T(M)) \geq k\alpha(M) \text{ for all bounded } M \in Z\}$$

as elements of $[0, \infty]$.
We call $[T]_A$ the measure of noncompactness of T and denote by $\mathfrak{A}(Z, Y)$ the set of all continuous maps T from Z into Y with $[T]_A < \infty$.

Note that in finite dimensional spaces we always have $[T]_A = 0$ and $[T]_a = \infty$. In infinite dimensional spaces, where this characteristic is of more use, we get the equivalent representations

$$[T]_A = \sup_{\infty > \alpha(M) > 0} \frac{\alpha(T(M))}{\alpha(M)}$$

and

$$[T]_a = \inf_{\infty > \alpha(M) > 0} \frac{\alpha(T(M))}{\alpha(M)}.$$

Sets with $\alpha(M) = 0$ can be left out here, since the continuity of T assures that also $\alpha(T(M)) = 0$. This can be seen by considering $\alpha(T(M)) \leq \alpha(T(\overline{M})) = 0$. From this representation it is also clear that an operator T with $[T]_A < \infty$ maps bounded sets into bounded sets.
These two characteristics are also closely related to two important properties of operators.

Definition 2.6 *Let $T \in \mathcal{C}(X, Y)$.*

- *The operator T is called compact, if $T(M)$ is precompact for every bounded set $M \subseteq X$.*
- *The operator T is called proper, if the preimage $T^{-1}(N)$ is compact for every compact set $N \subseteq Y$.*

Proposition 2.7 *Let X, Y, and Z be Banach spaces. For $T, S \in \mathcal{C}(X, Y)$ and $R \in \mathcal{C}(Y, Z)$ the following assertions hold true:*

(i) *T is compact if and only if $[T]_A = 0$.*

(ii) *If $[T]_a > 0$ and $[T]_q > 0$, then T is proper.*

(iii) *$[T]_a > 0$ implies that T is proper on closed bounded sets.*

(iv) *One of the quantities on the left being finite, $[T]_a - [S]_A \leq [T + S]_a \leq [T]_a + [S]_A$.*

(v) *One of the quantities on the left being finite, $|[T]_a - [S]_a| \leq [T - S]_A$. In particular, $[T - S]_A = 0$ implies $[T]_a = [S]_a$.*

(vi) *$[T^{-1}]_A = [T]_a^{-1}$ if T is a homeomorphism and either T is linear or X and Y are finite dimensional.*

(vii) *$[R]_a[T]_a \leq [R \circ T]_a \leq [R]_A[T]_a$, where the second inequality holds if $[R]_A < \infty$.*

Proof: The first assertion follows immediately from the definition of a compact operator and the definition of the measure of noncompactness of an operator.
In (ii), because $[T]_a > 0$, we may find a $k > 0$ such that $\alpha(T(M)) \geq k\alpha(M)$ for each bounded $M \in X$. As $[T]_q > 0$, Lemma 2.2,(i) shows that T is coercive. Therefore, for any compact set $N \in Y$, $T^{-1}(N)$ is bounded and

$$\alpha(T^{-1}(N)) \leq \frac{1}{k}\alpha(T(T^{-1}(N))) \leq \frac{1}{k}\alpha(N) = 0.$$

Thus $T^{-1}(N)$ is precompact. Since T is continuous, $T^{-1}(N)$ is also closed and therefore compact. The same reasoning shows that T is proper on closed bounded sets if only $[T]_a > 0$.
(iv) and (v) are proven similarly as in Lemma 2.2.
(vi) is trivial if X and Y are finite dimensional. If one is infinite dimensional, the assumption that T is linear assures that T maps bounded sets into bounded sets. Also, due to T being a homeomorphism, $\alpha(T(M)) = 0$ if and only if $\alpha(M) = 0$. We therefore get the chain of equalities

$$[T^{-1}]_A = \sup_{\infty>\alpha(N)>0} \frac{\alpha(T^{-1}(N))}{\alpha(N)} = \sup_{\infty>\alpha(M)>0} \frac{\alpha(M)}{\alpha(T(M))} = \left(\inf_{\infty>\alpha(M)>0} \frac{\alpha(T(M))}{\alpha(M)} \right)^{-1} = \frac{1}{[T]_a}.$$

Finally, let $k_T > 0$ such that $\alpha(T(M)) \geq k_T \alpha(M)$ for all bounded $M \in X$. Further, let $k_R > 0$ such that $\alpha(R(M)) \geq k_R \alpha(M)$ for all bounded $M \in Y$. Then $\alpha(R \circ T(M)) \geq k_R \alpha(T(M)) \geq k_R k_T \alpha(M)$ for all bounded $M \subseteq X$. This shows the first inequality in (vii). For the second inequality, consider

$$\begin{aligned} [R \circ T]_a &= \inf_{\infty>\alpha(M)>0} \frac{\alpha(R \circ T(M))}{\alpha(M)} = \inf_{\substack{\infty>\alpha(M)>0 \\ \infty>\alpha(T(M))\neq 0}} \frac{\alpha(R \circ T(M))}{\alpha(T(M))} \frac{\alpha(T(M))}{\alpha(M)} \\ &\leq \sup_{\infty>\alpha(N)>0} \frac{\alpha(R(N))}{\alpha(N)} \inf_{\infty>\alpha(M)>0} \frac{\alpha(T(M))}{\alpha(M)} = [R]_A [T]_a. \end{aligned}$$

□

For linear operators the characteristic $[T]_q$ and $[T]_a$ are linked to the injectivity and bijectivity of the operator T. $[T]_A$ and $[T]_Q$ on the other hand can be linked to the operator norm.

Lemma 2.8 *Let $T : X \to Y$ be linear and bounded. Then the following holds*

(i) $[T]_Q = \|T\|$.

(ii) $[T]_A \leq \|T\|$.

(iii) If $[T]_q > 0$, then T is injective.

(iv) If T is bijective, then $[T]_q > 0$ and $[T]_a > 0$.

Proof: The first assertion follows immediately from the linearity of T. Also, if $\{z_1, \ldots, z_m\}$ is a finite ϵ-net for a bounded set M, then $\{Tz_1, \ldots, Tz_m\}$ is obviously a finite $\|T\|\,\epsilon$-net for $T(M)$, i.e. $\alpha(T(M)) \leq \|T\|\,\alpha(M)$. Hence, $[T]_A \leq \|T\|$.
For (iii), assume T is not injective. Then there exist $x \neq y \in X$ with $Tx = Ty$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences with $x_n \to x$ and $y_n \to y$. Then $\|n(x_n - y_n)\| \to \infty$ for $n \to \infty$. Using the linearity of T we get the contradiction

$$[T]_q = \liminf_{\|z\|\to\infty} \frac{\|Tz\|}{\|z\|} \leq \lim_{n\to\infty} \frac{\|T(n(x_n - y_n))\|}{\|n(x_n - y_n)\|} = \lim_{n\to\infty} \frac{n\,\|T(x_n - y_n)\|}{n\,\|x_n - y_n\|} = \frac{\|Tx - Ty\|}{\|x - y\|} = 0.$$

To see (iv), note that because T is linear, bijective, and bounded, it is invertible and its inverse is also a linear and bounded operator. Using Proposition 2.2,(iv) we get

$$[T]_q = [T^{-1}]_Q^{-1} = \left\|T^{-1}\right\|^{-1} = \frac{1}{\|T^{-1}\|} > 0,$$

and using Proposition 2.7,(vi) we get

$$[T]_a = \frac{1}{[T^{-1}]_A} \geq \frac{1}{\|T^{-1}\|} > 0.$$

□

2.2 The Dugundji Extension Theorem

Dugundjis extension theorem assures that for any continuous map $T : A \to Y$ that is defined on a closed set $A \subseteq X$, there exists a continuous extension $\tilde{T}$ of T to X, i.e. $\tilde{T} : X \to Y$ and $\tilde{T}(x) = T(x)$ for $x \in A$, with the additional restriction that the range of $\tilde{T}$ is in some sense constrained by the original range of T.
In its original version X is allowed to be an arbitrary metric space and Y a locally convex linear space. We will give a simpler version of this theorem for mappings between two Banach spaces X and Y. First we need some definitions.

Definition 2.9 *(i) Let $\mathcal{A} = (A_i)_{i\in I}$ and $\mathcal{B} = (B_j)_{j\in J}$ be systems of subsets of a set X. $\mathcal{B}$ is called a refinement of $\mathcal{A}$, if for every $j \in J$ there is an $i \in I$ with $B_j \subseteq A_i$.*

(ii) Let X be a Hausdorff space and $\mathcal{U} = (U_i)_{i\in I}$ be a cover of X. $\mathcal{U}$ is called locally finite, if every point $x \in X$ has a neighborhood $O(x)$ such that $O(x)$ intersects at most finitely many elements of $\mathcal{U}$.

(iii) A Hausdorff space X is called paracompact if for every open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X, such that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$.

The following definition will be needed in the next section, but is closely related to the definitions above.

Definition 2.10 *Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. A system of real valued continuous functions $(f_i)_{i\in I}$ is called a partition of unity subordinate to $\mathcal{U}$ if*

(a) $f_i(x) \geq 0$ for all $x \in X$ and all $i \in I$,

(b) $(\operatorname{supp}(f_i))_{i\in I}$ is a locally finite system, where $\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$,

(c) $\operatorname{supp}(f_i) \subseteq U_i$ for all $i \in I$,

(d) $\sum_{i\in I} f_i(x) = 1$ *for all* $x \in X$.

The following theorem contains two well known topological results.

Theorem 2.11 *(i) Every metric space is paracompact.*

(ii) To every open cover $\mathcal{U}$ *of a paracompact space there exists a a partition of unity subordinate to* $\mathcal{U}$.

Proof: For example [Q]. □

We are now able to formulate our version of Dugundjis theorem.

Theorem 2.12 (Dugundji Extension Theorem) *Let* X *and* Y *be Banach spaces and let* $T : C \to K$ *be a continuous map, where* $C \subseteq X$ *is closed and* $K \subseteq Y$ *is convex.*
Then there exists a continuous mapping $\tilde{T} : X \to K$ *such that* $\tilde{T}(x) = T(x)$ *for* $x \in C$.

Proof: For each x in $X\backslash C$ set $r_x = \frac{1}{3}\text{dist}(x, C)$. Then $\text{diam}B_{r_x}(x) \leq \text{dist}(B_{r_x}, C)$.
The collection $(B_{r_x}(x))_{x\in X\backslash C}$ is an open cover of $X\backslash C$. By Theorem 2.11,(i) it has a locally finite refinement $(O_i)_{i\in I}$ which is an open cover of $X\backslash C$.
Define $q : X\backslash C \to (0, \infty)$ by

$$q(x) = \sum_{i\in I} \text{dist}(x, X\backslash O_i).$$

Since $(O_i)_{i\in I}$ is locally finite, the sum contains only finitely many not vanishing terms and q is a continuous function. Further, since the $(O_i)_{i\in I}$ form an open cover of $X\backslash C$, $q(x) > 0$.
Next we define for $i \in I$ and $x \in X\backslash C$

$$\rho_i(x) = \frac{\text{dist}(x, X\backslash O_i)}{q(x)}.$$

We then have $0 \leq \rho_i(x) \leq 1$ and $\sum_{i\in I} \rho_i(x) = 1$.
Note that since $(O_i)_{i\in I}$ is a refinement of $(B_{r_x}(x))_{x\in X\backslash C}$, for any O_i there exists $x_0 \in X\backslash C$ with

$$\text{dist}(C, O_i) \geq \text{dist}(C, B_{r_{x_0}}(x_0)) \geq \text{diam}B_{r_{x_0}}(x_0) > 0.$$

For each $i \in I$, we can therefore choose an $x_i \in C$ such that $\text{dist}(x_i, O_i) \leq 2\text{dist}(C, O_i)$, and define

$$\tilde{T}(x) = \begin{cases} T(x) & for \quad x \in C, \\ \sum_{i\in I} \rho_i(x)T(x_i) & for \quad x \notin C. \end{cases}$$

Obviously $\tilde{T} : X \to K$ and $\tilde{T}$ is an extension of T. Further, $\tilde{T}$ is continuous on the interior of C as well as on $X\backslash C$. In order to show that $\tilde{T}$ is continuous, it suffices to prove that $\tilde{T}$ is continuous on ∂C.

Let $x \in \partial C$. Since T is continuous, for given $\epsilon > 0$ we find a $\delta > 0$ such that $\|T(x) - T(y)\| \leq \epsilon$ for $y \in C$ with $\|x - y\| \leq \delta$. For $y \in X \backslash C$ we have

$$\left\|\tilde{T}(x) - \tilde{T}(y)\right\| = \left\|T(x) - \sum_{i \in I} \rho_i(y) T(x_i)\right\| \leq \sum_{i \in I} \rho_i(y) \left\|T(x) - T(x_i)\right\|.$$

If $\rho_i(y) \neq 0$ then $\mathrm{dist}(y, X \backslash O_i) > 0$, i.e. $y \in O_i$. Taking the infimum over all $w \in O_i$ in $\|y - x_i\| \leq \|y - w\| + \|w - x_i\|$ we obtain

$$\|y - x_i\| \leq \mathrm{diam} O_i + \mathrm{dist}(x_i, O_i).$$

Now, $O_i \subseteq B_{r_{x_0}}(x_0)$ for some $x_0 \in X \backslash C$. Since

$$\mathrm{diam} O_i \leq \mathrm{diam} B_{r_{x_0}}(x_0) \leq \mathrm{dist}(B_{r_{x_0}}, C) \leq \mathrm{dist}(C, O_i),$$

we get

$$\|y - x_i\| \leq 3 \mathrm{dist}(C, O_i) \leq 3 \|y - x\|.$$

Thus, for i such that $\rho_i(y) \neq 0$ we get $\|x - x_i\| \leq \|y - x\| + \|y - x_i\| \leq 4 \|x - y\|$. Hence, $\|x - y\| \leq \delta/4$ implies $\|x - x_i\| \leq \delta$, and, therefore, $\|T(x) - T(x_i)\| \leq \epsilon$. Finally,

$$\left\|\tilde{T}(x) - \tilde{T}(y)\right\| \leq \sum_{i \in I} \rho_i(y) \left\|T(x) - T(x_i)\right\| \leq \epsilon \sum_{i \in I} \rho_i(y) = \epsilon.$$

□

This theorem has two very useful corollaries.

Corollary 2.13 *Let X and Y be Banach spaces and let $T : C \to Y$ be continuous, where $C \subseteq X$ is closed.*
Then there exists a continuous extension $\tilde{T}$ of T to X with $\tilde{T}(X) \subseteq \mathrm{co}(T(C))$.

Corollary 2.14 *Let X and Y be Banach spaces and let $T : C \to Y$ be compact, where $C \subseteq X$ is closed and bounded.*
Then there exists a compact extension $\tilde{T}$ of T to X with $\tilde{T}(X) \subseteq \mathrm{co}(T(C))$.

Proof: Since T is compact and C bounded, $T(C)$ is precompact. By Lemma 2.4(ix), so is $\mathrm{co}(T(C))$. Therefore, an extension $\tilde{T}$ of T as in Theorem 2.12 is also compact. □

2.3 Set-valued maps

For a set X we will denote the power set of X by $\mathscr{P}(X)$. By a multivalued (or set-valued) map between two sets X and Y we mean a map $T : X \to \mathscr{P}(Y)$. It assigns to each point of X a subset of Y. Every map $T : X \to Y$ can be identified with a set-valued map $T' : X \to \mathscr{P}(Y)$ by setting $T'(x) = \{T(x)\}$. Such maps are then referred to as single-valued maps.
For a set-valued map we define the image of a set M as

$$T(M) := \bigcup_{x \in M} T(x)$$

and the preimage of a set A as

$$T^{-1}(A) := \{x \in X : T(x) \cap A \neq \emptyset\}.$$

Note that unlike for single-valued maps the inclusion $T(T^{-1}(A)) \subseteq A$ need not hold. However, unless $T^{-1}(A) = \emptyset$, we have $T(T^{-1}(A)) \cap A \neq \emptyset$. For single-valued maps this definition coincides with the classical preimage of a set. Furthermore, we call the set

$$G(T) := \{(x, y) : x \in X, y \in T(x)\}$$

the graph of T.
For set-valued maps we have the following notions of continuity:

Definition 2.15 *Let X and Y be topological spaces and $T : X \to \mathscr{P}(Y)$ a set-valued map.*

(i) T is called upper semi-continuous if $T^{-1}(A)$ is closed for all closed sets $A \subseteq Y$.

(ii) T is called lower semi-continuous if $T^{-1}(A)$ is open for all open sets $A \subseteq Y$.

For single-valued maps both upper and lower semi-continuity are obviously equivalent to continuity. In the following we will also need a different characterization of semi-continuity.

Proposition 2.16 *Let X and Y be topological spaces and $T : X \to \mathscr{P}(Y)$ a set-valued map.*

(i) T is upper semi-continuous if and only if for every $x \in X$ and every open set V in Y with $T(x) \subseteq V$ there exists a neighbourhood $U(x)$ such that $T(U(x)) \subseteq V$.

(ii) T is lower semi-continuous if and only if for every $x \in X$, $y \in T(x)$ and every neighbourhood $V(y)$ of y there exists a neighbourhood $U(x)$of x such that

$$T(u) \cap V(y) \neq \emptyset, \quad \text{for all } u \in U(x).$$

Proof:

(i) Suppose T fulfills the assumptions and $A \subseteq Y$ is closed. Choose $x \in (T^{-1}(A))^C$, then $T(x) \subseteq A^C$. Since A^C is open, there exists a neighbourhood $U(x)$ of x, such that

$T(U(x)) \subseteq A^C$. Therefore $U(x) \subseteq (T^{-1}(A))^C$ and it follows that $T^{-1}(A)$ is closed.
Conversely, suppose $T^{-1}(A)$ is closed for all closed $A \subseteq Y$. Let $x \in X$ and $V \subseteq Y$ be open with $T(x) \subseteq V$. Then V^C is closed and by assumption so is $T^{-1}(V^C)$. Moreover, $x \notin T^{-1}(V^C)$. Hence, there exists a neighbourhood of x with $U(x) \subseteq (T^{-1}(V^C))^C$. This neighbourhood apparently satisfies $T(U(x)) \subseteq V$.

(ii) Suppose T fulfills the assumptions and $A \subseteq Y$ is open. Assume that $x \in T^{-1}(A)$ and choose $y \in T(x) \cap A$. Since A is open, there exists a neighbourhood of y with $V(y) \subseteq A$. Because of our assumption, there exists a neighbourhood $U(x)$ of x with $T(u) \cap V(y) \neq \emptyset$ for all $u \in U(x)$. This means $U(x) \subseteq T^{-1}(V(y)) \subseteq T^{-1}(A)$. It follows that $T^{-1}(A)$ is open.
Conversely, suppose $T^{-1}(A)$ is open for every open set $A \subseteq Y$. Assume that $x \in X$, $y \in T(x)$, and $V(y)$ is an open neighbourhood of y. Then $T^{-1}(V(y))$ is open and $x \in T^{-1}(V(y))$. Therefore, $U(x) = T^{-1}(V(y))$ is a neighbourhood of x. It follows that $T(u) \cap V(y) \neq \emptyset$ for all $u \in U(x)$.

□

With this characterization of upper semi-continuity it is easy to see that Proposition 1.1 indeed shows the upper semi-continuity of the classical spectrum. It also clarifies what we meant with the heuristical explanation that the values of a lower semi-continuous map cannot 'collapse' and the values of an upper semi-continuous map cannot 'suddenly expand'.
We now give a condition for a set-valued map to be upper semi-continuous, which will be needed later on. For this, let X be linear space and p be a seminorm on X. For $x \in X$ we denote by $U_\delta(x)$ the p-neighbourhood $U_\delta(x) := \{y \in X : p(x-y) < \delta\}$.

Definition 2.17 *A set-valued map $T : X \to \mathscr{P}(\mathbb{K})$ is called closed if the graph of T is closed in $X \times \mathbb{K}$, i.e. $y_n \in T(x_n)$, $y_n \to y$ and $p(x_n - x) \to 0$ imply that $y \in T(x)$.*

Lemma 2.18 *Let $T : X \to \mathscr{P}(\mathbb{K})$ be closed. Then for every $x \in X$ and $y \in \mathbb{K}\backslash T(x)$ there exists $\delta > 0$ and an open set $V_y \subseteq \mathbb{K}$ such that $y \in V_y$ and $T(U_\delta(x)) \cap V_y = \emptyset$.*

Proof: Assume the assertion does not hold true for $x \in X$ and $y \in \mathbb{K}\backslash T(x)$. Let $(\delta_n)_{n\in\mathbb{N}}$ be a null sequence. Since $T(U_{\delta_n}(x)) \cap B_{\delta_n}(y) \neq \emptyset$, we can find $x_n \in U_{\delta_n}(x)$ and $y_n \in B_{\delta_n}(y)$ such that $y_n \in T(x_n)$. Further, we get that $y_n \to y$ and $p(x_n - x) \to 0$. Since T is closed, this shows that $y \in T(x)$, which contradicts our choice of y. □

Lemma 2.19 *Let $T : X \to \mathscr{P}(\mathbb{K})$ be a closed map. If*

$$\sup_{y \in T(x)} |y| \leq p(x) \quad \textit{for all } x \in X,$$

then T is upper semi-continuous.

Proof: Let $x \in X$ and let $V \subseteq \mathbb{K}$ be open with $T(x) \subseteq V$. By Proposition 2.16,(i), we have to show that there exists a $\delta > 0$ with $T(U_\delta(x)) \subseteq V$.
Choose $\eta > 0$ with $T(U_\eta(x))\backslash V \neq \emptyset$ (if this is not possible, there is nothing to prove). For $z \in U_\eta(x)$ we have

$$\sup_{\lambda \in T(z)} |\lambda| \leq p(z) \leq p(x) + \eta.$$

Consequently, $T(U_\eta(x))$ is bounded, and so the set $C := \overline{T(U_\eta(x))}\backslash V$ is compact.
Let $y \in C$ be arbitrary. Since $y \notin T(x)$ and T is closed, by Lemma 2.18 we find $\delta(y) > 0$ and an open $V_y \subseteq \mathbb{K}$ with $y \in V_\lambda$ and $T(U_{\delta(y)}(x)) \cap V_y = \emptyset$. Obviously, $\{V_y : y \in C\}$ is an open cover of C. Since C is compact, we get $C \subseteq V_{y_1} \cup \cdots \cup V_{y_m}$ for suitable $y_1, \ldots, y_m \in C$. Putting $\delta = \min\{\eta, \delta(y_1), \ldots, \delta(y_m)\}$ we see that $T(U_\delta(x)) \cap C = \emptyset$. Since $T(U_\delta(x)) \subseteq T(U_\eta(x))$, we get $T(U_\delta(x)) \subseteq V$. □

For the rest of this section we will only deal with lower semi-continuity.

Lemma 2.20 *Let $T : X \to \mathscr{P}(Y)$ be lower semi-continuous, where X and Y are topological spaces. If $S : X \to \mathscr{P}(Y)$ is such that $\overline{T(x)} = \overline{S(x)}$ for all $x \in X$, then S is lower semi-continuous.*

Proof: Assume S is not lower semi-continuous. Then there exists an $x \in X$, $y \in S(x)$, and an open set $V(y) \subseteq Y$ with $y \in V(y)$ such that for all neighbourhoods $U(x)$ of x there exists an $u \in U(x)$ with $S(u) \cap V(y) = \emptyset$. Since $V(y)$ is open, this also means $\overline{T(u)} \cap V(y) = \overline{S(u)} \cap V(y) = \emptyset$, which consequently implies $T(u) \cap V(y) = \emptyset$. If $y \in T(x)$, this contradicts the lower semicontinuity of T.
If $y \notin T(x)$, we have $y \in \overline{T(x)}$. Hence, $V(y) \cap T(x) \neq \emptyset$. Further, $V(y)$ is an open neighbourhood $V(z)$ of every $z \in V(y) \cap T(x)$. As above we get $T(u) \cap V(z) = T(u) \cap V(y) = \emptyset$. □

Lemma 2.21 *Let $T : X \to \mathscr{P}(Y)$, where Y is a Banach space. Let T be lower semi-continuous, $O \subseteq Y$ open, $f : X \to Y$ a continuous map, and suppose that $T(x) \cap (f(x) + O) \neq \emptyset$ for all $x \in X$. Then $S : X \to \mathscr{P}(Y)$, defined by $S(x) = T(x) \cap (f(x) + O)$, is lower semi-continuous.*

Proof: Let $x \in X$, $y \in S(x)$, and $V(y)$ be an open set with $y \in V(y)$. Then $y \in T(x)$ and $(f(x) + O) \cap V(y)$ is an open neighbourhood of y. Therefore, there exists an $o \in O$ with $y = f(x) + o$ and an $\epsilon > 0$ such that $f(x) + B_\epsilon(o) \subseteq (f(x) + O) \cap V(y)$. Since T is lower semi-continuous, there exists a neighbourhood $\tilde{U}(x)$ of x such that for all $u \in \tilde{U}(x)$ we have $T(u) \cap (f(x) + B_{\epsilon/2}(o)) \neq \emptyset$.
Since f is continuous, there exists a neighbourhood $\hat{U}(x)$ of x, such that $\|f(x) - f(u)\| < \epsilon/2$ for all $u \in \hat{U}(x)$. Let $a \in f(x) + B_{\epsilon/2}(o)$. Then for all $u \in \hat{U}(x)$ we get $\|a - (f(u) + o)\| \leq \|a - f(x) - o\| + \|f(x) + o - f(u) - o\| < \epsilon$, i.e. $a \in f(u) + B_\epsilon(o) \subseteq (f(u) + O)$.
Set $U(x) = \tilde{U}(x) \cap \hat{U}(x)$. Then for any $u \in U(x)$ we have $T(u) \cap (f(x) + B_{\epsilon/2}(o)) \neq \emptyset$, since

$u \in \tilde{U}(x)$. But for any $a \in T(u) \cap (f(x) + B_{\epsilon/2}(o))$ we have $a \in (f(x) + B_{\epsilon/2}(o))$, hence $a \in V(y)$. Further, $u \in \hat{U}(x)$ yields $a \in (f(u) + O)$. Hence, $a \in T(u) \cap (f(u) + O) \cap V(y) = S(u) \cap V(y)$. Therefore, $S(u) \cap V(y) \neq \emptyset$ for all $u \in U(x)$. □

Lemma 2.22 *Let $T = X \to \mathscr{P}(Y)$ be lower semi-comtinuous and Y be a Banach space. Suppose $f : X \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in X$. Define $S : X \to \mathscr{P}(Y)$ by*

$$S(x) = \begin{cases} T(x) \cap B_{f(x)}(0) & if \quad f(x) > 0, \\ \{0\} & if \quad f(x) = 0. \end{cases}$$

If $S(x) \neq \emptyset$ for all $x \in X$, then S is lower semi-continuous.

Proof: Let $x \in X$, $y \in S(x)$, and $V(y)$ be open with $y \in V(y)$. First, suppose $f(x) > 0$. Then $f(x) > \|y\|$. Hence, there exists an $\epsilon > 0$ with $f(x) > \|y\| + \epsilon$. Since f is continuous, there exists an open neighbourhood $\hat{U}(x)$ of x such that $f(u) > \|y\| + \epsilon$ for all $u \in \hat{U}(x)$. Moreover, there exists an open neighbourhood $\tilde{U}(x)$ of x such that $T(u) \cap B_{\|y\|+\epsilon}(0) \cap V(y) \neq \emptyset$ for all $u \in \tilde{U}(x)$, because T is lower semi-continuous and $B_{\|y\|+\epsilon}(0) \cap V(y)$ is an open neighbourhood of y. So for u in the open set $U(x) = \hat{U}(x) \cap \tilde{U}(x)$ we get

$$S(u) \cap V(y) = T(u) \cap B_{f(u)} \cap V(y) \supseteq T(u) \cap B_{\|y\|+\epsilon} \cap V(y) \neq \emptyset.$$

Now suppose $f(x) = 0$. Then $y = 0$ and $V(y)$ is a neighbourhood of 0, i.e. there is an $\epsilon > 0$ such that $B_\epsilon(0) \subseteq V(y)$. Since f is continuous, there is a open neighbourhood $U(x)$ of x such that $f(u) < \epsilon$ for $u \in U(x)$. For $f(u) > 0$ we have $S(u) = T(u) \cap B_{f(u)}(0) \subseteq B_{f(u)}(0) \subseteq B_\epsilon(0) \subseteq V(y)$. For $f(u) = 0$ we have $S(u) = \{0\} \subseteq V(y)$. In any case, $S(u) \cap V(y) \neq \emptyset$. □

Finally, we define the objects that will be studied for the rest of this section.

Definition 2.23 *Let $T : X \to \mathscr{P}(Y)$ be a set-valued map. A single-valued map $t : X \to Y$ is called a selection of T, if*

$$t(x) \in T(x) \quad \textit{for all } x \in X.$$

The existence of a selection is obviously equivalent to the fact, that $T(x) \neq \emptyset$ for all $x \in X$. Selections are an important tool when dealing with set-valued maps. Therefore, it is of interest to show the existence of selections with additional properties. The following important theorem by Michael provides conditions for the existence of a continuous selection.

Theorem 2.24 (Michael's Selection Theorem) *Let X be a paracompact space, Y a Banach spaces, and $T : X \to \mathscr{P}(Y)$ a lower semi-continuous set-valued map. If $T(x)$ is nonempty, closed, and convex for all $x \in X$, then there exists a continuous selection $t : X \to Y$ of T.*

Proof: For $y \in Y$ and $A \subseteq Y$ set $d(y, A) = \inf_{a \in A} \|y - a\|$. As a first step, we show that for each $\epsilon > 0$ there exists a continuous map $f : X \to Y$ such that

$$d(f(x), T(x)) < \epsilon \quad \text{for all } x \in X. \tag{2.1}$$

Fix $\epsilon > 0$ and choose any selection $m : X \to Y$. Since T is lower semi-continuous, for each $x \in X$ there exists an open neighbourhood $U(x)$ of x such that

$$T(u) \cap B_\epsilon(m(x)) \neq \emptyset \quad \text{for all}\, u \in U(x). \tag{2.2}$$

Let $(f_i)_{i \in I}$ be a partition of unity subordinate to the open cover $(U(x))_{x \in X}$ of X. For every $i \in I$ choose an $x_i \in X$ such that $\operatorname{supp}(f_i) \subseteq U(x_i)$ if $\operatorname{supp}(f_i) \neq \emptyset$ and set

$$f(x) = \sum_{i \in I} f_i(x) m(x_i).$$

Then f is a continuous function of X into Y. If $f_i(x) > 0$ for some i, then $x \in U(x_i)$ and by (2.2)

$$m(x_i) \in T(x) + B_\epsilon(0).$$

Since $T(x) + B_\epsilon(0)$ is convex, we get $f(x) \in T(x) + B_\epsilon(0)$. Hence, $d(f(x), T(x)) < \epsilon$, i.e. f satisfies (2.1).
In the second step we construct the requested selection. Set $\epsilon_n = 2^{-n}$. We will inductively define a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous mappings $f_n : X \to Y$ with

$$d(f_n(x), T(x)) < \epsilon_n, \quad x \in X,\, n = 1, 2, \ldots \tag{2.3}$$

$$d(f_n(x), f_{n-1}(x)) < \epsilon_{n-1}, \quad x \in X,\, n = 2, 3, \ldots \tag{2.4}$$

As we showed in the first step, there exists f_1 with $d(f_1(x), T(x)) < 1/2$, $x \in X$.
Assume that $n \geq 2$ and we already have constructed $f_1, \ldots, f_{n-1}$. For each $x \in X$ we define

$$G(x) = (f_{n-1}(x) + B_{\epsilon_{n-1}}(0)) \cap T(x)$$

By the induction hypothesis, $G(x)$ is not empty. Since $T(x)$ convex, so is $G(x)$. Furthermore, $G : X \to \mathscr{P}(Y)$ is lower semi-continuous by Lemma 2.21. So we can apply the first part of our proof also to G, since the only additional property of T is that $T(x)$ is closed, which was not used in the first part. Therefore, there exists a continuous map $f_n : X \to Y$ such that (2.3) holds. By construction, f_n also satisfies (2.4).
Since $\sum_{n=1}^{\infty} \epsilon_n$ converges, $(f_n)_{n \in \mathbb{N}}$ is a uniform Cauchy sequence and, therefore, converges to a continuous map $t : X \to Y$. Since $T(x)$ is closed, $d(t(x), T(x)) = 0$ implies that t is a selection of T. □

Corollary 2.25 *Let X be paracompact, Y a Banach space, and $T : X \to \mathscr{P}(Y)$ a lower semi-continuous map such that $T(x)$ is nonempty, closed, and convex for every $x \in X$. Set $m(x) = \inf\{\|y\| : y \in T(x)\}$ and suppose $f : X \to \mathbb{R}$ is continuous weith $f(x) \geq 0$ for all x, and $f(x) > m(x)$ whenever $m(x) > 0$. Then there exists a continuous selection t of T such that $\|t(x)\| \leq f(x)$ for all $x \in X$.*

Proof: The map $S : X \to \mathscr{P}(Y)$ defined by

$$S(x) = \begin{cases} T(x) \cap B_{f(x)}(0) & if \quad f(x) > 0, \\ \{0\} & if \quad f(x) = 0, \end{cases}$$

is lower semi-continuous by Lemma 2.22. Hence, $R : X \to \mathscr{P}(Y)$, defined by $R(x) = \overline{S(x)}$, is lower semi-continuous by Lemma 2.20. Furthermore, $R(x)$ is nonempty, closed, and convex for all $x \in X$. By Theorem 2.24, there exists a selection t of R. By construction t is also a selection of T and fulfils $\|t(x)\| \leq f(x)$. □

Theorem 2.26 *Let $T : X \to Y$ be a continuous linear surjection from a Banach space X onto a Banach space Y. Then there exists a continous function $s : Y \to X$ and a constant $M > 0$ such that for every $y \in Y$*

- $s(y) \in T^{-1}(y)$,
- $\|s(y)\| \leq M \|y\|$.

Proof: First, we show that there exists an $M > 0$, such that $m(y) = \inf\{\|x\| : x \in T^{-1}(y)\} \leq M \|y\|$ and $m(y) < M \|y\|$ whenever $m(y) > 0$. Since T is surjective, by the open mapping theorem the image of $B_1(0) \subseteq X$ under T is an open neighbourhood of $0 \in Y$. In particular, there exists a $\delta > 0$ such that $S_\delta = \{y \in Y : \|y\| = \delta\} \subseteq T(B_1(0))$. For an arbitrary $0 \neq y \in Y$ we have $y_0 = \delta y \|y\|^{-1} \in S_\delta$. Hence, there exists a $x_0 \in B_1(0)$ with $T(x_0) = y_0$ and, further, $T(x_0 \|y\| \delta^{-1}) = y$. Therefore, $m(y) \leq \|x_0 \|y\| / \delta\| \leq \delta^{-1} \|y\|$. Now set $M = \lambda\delta^{-1}$ with a $\lambda > 1$. Define $S : S_1 \to \mathscr{P}(X)$ by $S(x) = T^{-1}(\{x\})$. Then for any open $U \subseteq X$ the set $S^{-1}(U) = \{y \in S_1 : T^{-1}(\{y\}) \cap U \neq \emptyset\} = T(U) \cap S_1$ is open in S_1 by the open mapping theorem. Hence, S is lower semi-continuous. Moreover, $S(y)$ is nonempty, closed, and convex for all $y \in S_1$. So we can apply Corollary 2.25 to S with $f(x) = M \|x\|$ and get a continuous function $\tilde{s} : S_1 \to X$ with $\tilde{s}(y) \in T^{-1}(y)$ and $\|\tilde{s}(y)\| \leq M \|y\|$. Setting

$$s(y) = \begin{cases} \|y\| \, \tilde{s}(\frac{y}{\|y\|}) & if \quad y \neq 0, \\ 0 & if \quad y = 0, \end{cases}$$

we get a map from the whole space Y into X with the mentioned properties. □

2.4 The Antipodal Theorem

To formulate and prove the Antipodal Theorem we first need to establish the fixed point index and other preliminary results.

Definition 2.27

- *Let G be a nonempty and open bounded set in a Banach space X. Then $V(G,X)$ denotes the set of all compact maps $T : \overline{G} \to X$ such that T has no fixed points on the boundary ∂G of G, i.e. $\nexists x \in \partial G : T(x) = x$.*
- *Two maps $T, S \in V(G,X)$ are called compactly homotopic on ∂G if there exists a continuous map H with the following properties*

  (i) $H : \partial G \times [0,1] \to X$ *is compact,*

  (ii) $H(x,0) = T(x)$ *and* $H(x,1) = S(x)$ *for* $x \in \partial G$*,*

  (iii) $H(x,t) \neq x$ *for all* $(x,t) \in \partial G \times [0,1]$.

  We write $T \cong S$. The map H is called a compact homotopy.

Proposition 2.28 *Let G be a nonempty open bounded subset of a Banach space X and let $T, S \in V(G,X)$. Then the following holds true:*

(i) *The relation $\cong$ is an equivalence relation.*

(ii) $\inf_{x\in\partial G} \|x - T(x)\| > 0$.

(iii) *If* $\sup\limits_{x\in\partial G} \|T(x) - S(x)\| < \inf\limits_{x\in\partial G} \|x - T(x)\|$, *then* $T \cong S$.

Proof: The relation $\cong$ is reflexive, since $T \cong T$ by $H(x,t) = T(x)$. If $T \cong S$ by H, then $S \cong T$ by $H_1 = H(x, 1-t)$, so $\cong$ is symmetric. To see that $\cong$ is also transitive, let $T \cong S$ by H_1 and $S \cong R$ by H_2, then $T \cong R$ by

$$H(x,t) = \begin{cases} H_1(x,2t) & for \quad 0 \le t \le \frac{1}{2}, \\ H_2(x,2t-1) & for \quad \frac{1}{2} < t \le 1. \end{cases}$$

To prove (ii), we show that $(I-T)(\partial G)$ is closed. Becaus of $0 \notin (I-T)(\partial G)$, this gives the assertion. So let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $(I-T)(\partial G)$ that converges to a $y \in X$. By Corollary 2.14 we can extend T to a compact map $\hat{T}$ on X. Since ∂G is closed and bounded, and $[I-\hat{T}]_a \geq [I]_a - [\hat{T}]_A = 1 > 0$ by Lemma 2.7,(iii), point (ii) of the same lemma tells us that $(I-\hat{T})$ is proper on ∂G, so $I-T$ is also proper there. Because $(y_n)_{n\in\mathbb{N}} \cup \{y\}$ is compact, also $(I-T)^{-1}((y_n)_{n\in\mathbb{N}} \cup \{y\})$ is compact. If we choose $x_n \in (I-T)^{-1}(y_n) \cap \partial G$, then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x \in \partial G$. Because $I-T$ is continuous, we get $(I-T)(x) = y$. Thus, $y \in (I-T)(\partial G)$.

For (iii) let $H(x,t) = (1-t)T(x) + tS(x)$. Then H is continuous and compact on $\partial G \times [0,1]$, $H(x,0) = T(x)$ and $H(x,1) = S(x)$ for $x \in \partial G$, and for all $(x,t) \in \partial G \times [0,1]$ we have

$$\|H(x,t) - x\| = \|T(x) - x - t(T(x) - S(x))\| \geq \|T(x) - x\| + t\,\|T(x) - S(x)\| > 0.$$

□

The next lemma is especially interesting in connection with point (iii) of the previous proposition.

Lemma 2.29 *Let X and Y be Banach spaces and $M \subseteq X$ be nonempty and bounded. Let $T : M \to Y$ be a compact operator. Then to each $\epsilon > 0$, there exists a compact operator $P : M \to Y$ such that $\sup_{x\in M} \|T(x) - P(x)\| \leq \epsilon$ and $P(M)$ is contained in finite dimensional subspace of Y.*

Proof: Fix $\epsilon > 0$. Since M is bounded, $T(M)$ is precompact. Hence, there exist $y_i \in T(M)$, $i = 1, \ldots, N$, such that $\min_{i=1,\ldots,N} \|T(x) - y_i\| < \epsilon$ for all $x \in M$. Define continuous functions $p_i : M \to \mathbb{R}$ by $p_i(x) = \max(\epsilon - \|T(x) - y_i\|, 0)$. Then for each $x \in M$ there exists at least one i such that $p_i(x) \neq 0$. Therefore, we can define $P : M \to Y$ by

$$P(x) = \frac{\sum_{i=1}^N p_i(x) y_i}{\sum_{i=1}^N p_i(x)}.$$

For all $x \in M$ we then have

$$\|P(x) - T(x)\| = \left\| \frac{\sum_{i=1}^N p_i(x)(y_i - T(x))}{\sum_{i=1}^N p_i(x)} \right\| \leq \frac{\sum_{i=1}^N p_i(x)\epsilon}{\sum_{i=1}^N p_i(x)} \leq \epsilon.$$

By construction, $P(M)$ lies in the finite dimensional subspace spanned by $\{y_i : i = 1, \ldots, N\}$. Finally, since $T(M)$ is bounded, so is $P(M)$. Since a bounded set in a finite dimensional space is precompact, P is compact. □

Definition 2.30 *Let X be a Banach space. An integer valued function $i(T,G)$, where G is a nonempty open bounded subset of X and $T \in V(G,X)$, is called the fixed point index of T on G if the following axioms are satisfied.*

(A1) If $T(x) = x_0$ for all $x \in \overline{G}$ and some fixed $x_0 \in G$, then $i(T,G) = 1$.

(A2) If $i(T,G) \neq 0$, then there exists an $x \in G$ such that $T(x) = x$.

(A3) If there are open $G_i \subseteq X$, $i = 1, \ldots, n$, such that $\overline{G} = \bigcup_{i=1}^n \overline{G_i}$ and $G_i \cap G_j = \emptyset$ for $i \neq j$, then

$$i(T,G) = \sum_{i=1}^n i(T,G_i)$$

whenever $T \in V(G,X)$ and $T|_{\overline{G_i}} \in V(G_i,X)$ for all i.

(A4) If $T \cong S$, then $i(T,G) = i(S,G)$.

Despite their importance, we will not give a proof of the following three results, since they are beyond the scope of the present thesis. The proof of the first of these theorems includes a lengthy construction of the fixed point index, while the other two proofs necessitate a detailed knowledge of this construction. For a proof of these theorems see for example Zeidler [Z].

Theorem 2.31 *For every Banach space X there exists a unique fixed point index.*

Theorem 2.32 *Let G be a nonempty open bounded region and $T \in V(G,X)$. Let $\Omega \subseteq X$ be a bounded region with $\overline{G} \subseteq \Omega$ and $h : \Omega \to X$ be a linear map such that $I - h$ is compact. Suppose $h : \Omega \to h(\Omega)$ is a homeomorphism, $T(\overline{G}) \subseteq \Omega$, and $0 \notin h \circ (I - T)(\partial G)$. Then*

$$i(h \circ T \circ h^{-1}, h(G)) = i(T,G).$$

Theorem 2.33 *If $T \in V(G,X)$ and if $T(\overline{G})$ lies entirely in a closed linear subspace Y of X, then $i(T,G) = i(T, G \cap Y)$, where the right hand side is the fixed point index in Y.*

The last two lemmas that we need before we are able to proof the Antipodal Theorem deal with fixed point free extensions of fixed point free maps.

Lemma 2.34 *Let A and B be closed and bounded subsets of a Banach space X with $A \subseteq B$. Let $H : A \times [0,1] \to X$ be compact and $H(x,t) \neq x$ for all $(x,t) \in A \times [0,1]$.
If $H(.,1)$ has a fixed point free compact extension $H_1 : B \to X$, then H has a compact extension $\tilde{H} : B \times [0,1] \to X$ with $\tilde{H}(x,t) \neq x$ for all $(x,t) \in B \times [0,1]$*

Proof: Set

$$\tilde{H}_0(x,t) = \begin{cases} H_1(x) & for \quad x \in B,\ t = 0 \\ H(x,t) & for \quad x \in A,\ t \in [0,1], \end{cases}$$

and extend $\tilde{H}_0$ by Corollary 2.14 to a compact map $H_0 : B \times [0,1] \to X$. Let $B_0 = \{x \in B : \exists t \in [0,1] \text{ with } H_0(x,t) = x\}$. Then B_0 is closed and $A \cap B_0 = \emptyset$. Therefore, by Corollary 2.13, we can extend the map

$$\tilde{a}(x) = \begin{cases} 0 & for \quad x \in B_0 \\ 1 & for \quad x \in A \end{cases}$$

to a continuous map $a : B \to [0,1]$. Finally, we set $\tilde{H}(x,t) = H_0(x, a(x)t)$ for $(x,t) \in B \times [0,1]$. Then $\tilde{H}$ is the desired extension of H. In fact, if $\tilde{H}(x,t) = x$, then $H_0(x,\tau) = x$ for some τ, i.e. $x \in B_0$. Therefore, $a(x) = 0$ and further $H_0(x,0) = x$, i.e. $H_1(x) = x$, which is impossible by hypothesis. □

Lemma 2.35 *Let A and B be compact subsets of a proper linear subspace of $\mathbb{R}^N$ with $N \geq 2$ and $A \subseteq B$. Then every continuous fixed point free map $f : A \to \mathbb{R}^N$ has a continuous fixed point free extension $\tilde{f}$ to B.*

Proof: We may assume without loss of generality that A and B lie in the subspace with coordinates $(\zeta_1, \ldots, \zeta_M, 0, \ldots, 0)$. We also set $F(x) = f(x) - x$.
Since F has no zeroes on A, we get $\alpha = \inf_{x\in A} |F(x)| > 0$. We extend F continuously to $\overline{F} : B \to \mathbb{R}^N$ by Corollary 2.13. By the Weierstrass approximation theorem, there exists a map $G : B \to \mathbb{R}^N$ with $\sup_{x\in B} |\overline{F} - G| < \alpha/3$, and such that the components of G are polynomials. Since $\partial G_i(x)/\partial \zeta_j = 0$ for $j = M + 1, \ldots, N$, we have $\det(G'(x)) = 0$ on B. By Sard's Theorem, $G(B)$ has no interior points. Hence, there exists an x_0 such that $x_0 \notin G(B)$ and $|x_0| < \alpha/3$.
For $(x, t) \in (A \times [0, 1]) \cup (B \times 1)$ define $h(x, t) = (1 - t)\overline{F}(x) + t(G(x) - x_0)$. Then $h(x, t) \neq 0$ since $|h(x, t)| \geq |\overline{F}(x)| - |\overline{F}(x) - G(x)| - |x_0| > 0$. This also implies $h(x, 1) \neq 0$ for $x \in B$. Thus we can apply Lemma 2.34 to $H(x, t) = x + h(x, t)$. In particular, we get that the extention $\tilde{H}(x, 0) : B \to \mathbb{R}^N$ is fixed point free and $\tilde{H}(x, 0) = f(x)$ for $x \in A$. □

Theorem 2.36 (Antipodal Theorem) *Let X be a Banach space and let $T : \overline{B_R(0)} \to X$ be compact. If T has no fixed points on $\partial B_R(0)$ and if T satisfies the antipodal condition $T(-x) = -T(x)$ for $x \in \partial B_R(0)$, then the fixed point index $i(T, B_R(0))$ is odd.*

Proof: First, we proof this theorem for $\dim(X) < \infty$. For this we can assume that $X = \mathbb{R}^N$ for some $N \geq 1$.
Q is called an orthant of $\mathbb{R}^N$ if

$$Q = \{x \in \mathbb{R}^N : 0 \leq e_j x_j \text{ for } j = 1, \ldots, N\},$$

where $(e_1, \ldots e_N) \in \{-1, 1\}^N$ is fixed. Per definition, there are 2^N orthants in $\mathbb{R}^N$. Let $Q_1, \ldots, Q_{2^{N-1}}$ be the 2^{N-1} orthants of $\mathbb{R}^N$ with $x_N \geq 0$ for all $x \in Q_j$. Then $\mathbb{R}^N = Q_1 \cup \cdots \cup Q_{2^{N-1}} \cup -Q_1 \cup \cdots \cup -Q_{2^{N-1}}$.[1]
Now fix $r \in (0, R)$ and define B_j as the interior of $Q_j \cap B_R(0) \backslash B_r(0)$. Then

$$\overline{B_R(0)} = \overline{B_r(0)} \cup \overline{B_1} \cup \cdots \cup \overline{B_{2^{N-1}}} \cup \overline{-B_1} \cup \cdots \cup \overline{-B_{2^{N-1}}}.$$

We show that there exists a continuous function $S : \overline{B_R(0)} \to \mathbb{R}^N$ with

$$S(x) = T(x) \quad \text{for all } x \in \partial B_R(0), \tag{2.5}$$

$$S(x) = 0 \quad \text{for all } x \in \overline{B_r(0)}, \tag{2.6}$$

[1] This is the multidimensional equivalent of the following easy construction in $\mathbb{R}^2$: Divide the plane into its four quadrants. Choose Q_1 and Q_2 as the first and second quadrant, respectively. Then $-Q_1$ is the third quadrant and $-Q_2$ the fourth. So $\mathbb{R}^2 = Q_1 \cup Q_2 \cup -Q_1 \cup -Q_2$.

$$S(x) \neq x \quad \text{for all } x \in \partial(\pm B_j),\ j = 1, \ldots, 2^{N-1}, \tag{2.7}$$

$$S(-x) = -S(x) \quad \text{for all } x \in B_R(0). \tag{2.8}$$

Because of (2.6) we only have to define S on $\overline{\pm B_j}$. There S is already defined on $\partial B_j \cap \partial B_R(0)$ and $\partial B_j \cap \partial B_r(0)$ by (2.5) and (2.6), and S is fixed point free there. The remaining border of B_j can be divided into N parts, each of which lies in a $N-1$ dimensional hyperplane of $\mathbb{R}^N$. We start by applying Lemma 2.35 to each of these parts in order to extend S continuously to ∂B_1 such that (2.7) is fulfilled. We now can extend S continuously from ∂B_1 to $\overline{B_1}$ using Theorem 2.12. Further, we define S on $\overline{-B_1}$ by $S(x) = -S(-x)$ for $x \in \overline{-B_1}$, such that (2.8) is fulfilled. We repeat this procedure consecutively with B_2 to $B_{2^{N-1}}$. Note that when considering B_j, S is already defined on $\overline{B_j} \cap \overline{B_i}$ for $i < j$, so we can skip defining S on that part of the border of B_j.

Now (2.6) implies that $i(S, B_r(0)) = 1$ by axiom (A1) of the fixed point index. Using the homeomorphism $h : x \mapsto -x$ we can use Theorem 2.32 (with $\Omega = \overline{B_R(0)}$) to see that $i(S(x), B_j) = i(-S(-x), -B_j) = i(S(x), -B_j)$. Axiom (A3) of the fixed point index then gives

$$i(S, B_R(0)) = i(S, B_r(0)) + \sum_{j=1}^{2^{N-1}} (i(S, B_j) + i(S, -B_j)) = i(S, B_r(0)) + 2 \sum_{j=1}^{2^{N-1}} i(S, B_j),$$

i.e. $i(S, B_R(0))$ is odd. Since $S(x) = T(x)$ for $x \in \partial B_R(0)$, Proposition 2.28 together with axiom (A4) of the fixed point index show that $i(T, B_R(0)) = i(S, B_R(0))$.

Now let $\dim(X) = \infty$. First, we observe that $(T(x) - T(-x))/2$ is compact and coincides with $T(x)$ for $x \in \partial B_R(0)$. By Proposition 2.28 and (A4), this yields $i(T(x), B_R(0)) = i((T(x) - T(-x))/2, B_R(0))$. By Lemma 2.29, there is a compact operator $S : B_R(0) \to Y$, where Y is a finite dimensional subspace of X, such that $\sup_{x \in B_R(0)} \|T(x) - S(x)\| < \inf_{x \in \partial B_R(0)} \|T(x) - x\|$. As above, for $P(x) = (S(x) - S(-x))/2$ we have $i(P, B_R(0)) = i((T(x) - T(-x))/2, B_R(0))$. But since P maps $B_R(0)$ into a finite dimensional subspace Y, Theorem 2.33 shows that $i(P, B_R(0)) = i(P, B_R(0) \cap Y)$, the latter of which is odd by the first part of the proof, since $P(-x) = -P(x)$. □

Chapter 3

The FMV-spectrum

3.1 FMV-regular operators

FMV-regularity, named after its inventors M. Furi, M. Martelli, and A. Vignoli, will play the essential role in the definition of the FMV-spectrum as explained in the introduction.

3.1.1 Stable solvability

Stable solvability is a property of operators that is important in the definition of FMV-regularity and of the FMV-spectrum. It assures solvability of certain types of equations.
In the following, X and Y will always denote Banach-spaces.

Definition 3.1 *A continuous operator $T : X \to Y$ is called stably solvable, if, given any compact operator $S : X \to Y$ with $[S]_Q = 0$, the equation $T(x) = S(x)$ has a solution $x \in X$.*

Lemma 3.2 *Let $T \in \mathcal{C}(X, Y)$ be stably solvable. Then T is surjective.*

Proof: For $y \in Y$ the operator $S(x) \equiv y$ fulfils $[S]_A = [S]_Q = 0$. By the stable solvability of T, there is an $x \in X$ with $T(x) = S(x) = y$. □

In general, the converse of this lemma does not hold true. We will however show later that in the case of linear operators stable solvability reduces to surjectivity.
The notion of stable solvability can be extended further. This more general definition will be usefull in the study of FMV-regular operators.

Definition 3.3 *For $k \geq 0$, we call an operator $T \in \mathcal{C}(X, Y)$ k-stably solvable, if, given any operator $S \in \mathcal{C}(X, Y)$ with $[S]_A \leq k$ and $[S]_Q \leq k$, the equation $T(x) = S(x)$ has a solution $x \in X$.*

Obviously, 0-stably solvable operaters are exactly the stably solvable operators. Moreover, every k-stably solvable operator is certainly k'-stably solvable for $0 \leq k' < k$. This motivates the following definition.

Definition 3.4 *For $T \in \mathcal{C}(X,Y)$ we call*

$$\mu(T) := \inf\{k : k \geq 0,\, T \text{ is not } k\text{-stably solvable}\}$$

the measure of stable solvability of T.
We call an operator $T \in \mathcal{C}(X,Y)$ strictly stably solvable if $\mu(T) > 0$, i.e. T is k-stably solvably for some $k > 0$.

For the rest of this section we return to stably solvable operators and show some of their properties.

Lemma 3.5 *Let $T \in \mathcal{C}(X,Y)$ with $[T]_q > 0$. Then T is stably solvable if and only if the equation $T(x) = S(x)$ has a solution $x \in X$ for every compact operator $S : X \to Y$ for which the set $\{x \in X : S(x) \neq 0\}$ is bounded.*

Proof: Since every operator S with $S(x) = 0$ outside a bounded set fulfils $[S]_Q = 0$, this direction of the proof is trivial.
So let S be a compact operator from X into Y with $[S]_Q = 0$. For $n \in \mathbb{N}$ define the operator $S_n(x) = d_n(\|x\|)S(x)$, where

$$d_n(t) = \begin{cases} 1 & if \quad 0 \leq t \leq n, \\ 2 - \frac{1}{n}t & if \quad n \leq t \leq 2n, \\ 0 & if \quad t \geq 2n. \end{cases}$$

Then $\{x \in X : S_n(x) \neq 0\}$ is bounded and S_n compact. Hence, by assumption, there exists an $x_n \in X$ with $S_n(x_n) = T(x_n)$. If $\|x_n\| \leq n$ for some $n \in \mathbb{N}$, then $T(x_n) = S_n(x_n) = S(x_n)$ and we are finished.
So assume $\|x_n\| > n$ for all $n \in \mathbb{N}$. Then $\|x_n\| \to \infty$ as $n \to \infty$, and we get the contradiction

$$\begin{aligned} [T]_q = \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|} \leq \lim_{n\to\infty} \frac{\|T(x_n)\|}{\|x_n\|} &= \lim_{n\to\infty} d_n(\|x_n\|)\frac{\|S(x_n)\|}{\|x_n\|} \\ &\leq \lim_{n\to\infty} \frac{\|S(x_n)\|}{\|x_n\|} \leq \limsup_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|} = 0. \end{aligned}$$

□

Corollary 3.6 *The identity operator is stably solvable.*

Proof: Since $[I]_q = 1$ we can restrict ourselves to compact maps S with $\{x \in X : S(x) \neq 0\}$ is bounded to prove stable solvability. For such S, there obviously exists an $r > 0$ such that $S(\overline{B_r(0)}) \subseteq \overline{B_r(0)}$. But then Schauders fixed point theorem tells us that S has a fixed point, i.e. $S(x) = I(x)$. □

Theorem 3.7 *Let $T \in \mathcal{C}(X)$ be stably solvable, $B \subseteq X$ a nonempty closed subset, and let $H : B \to X$ be a continuous operator. Assume that $H(B)$ is bounded and*

$$T^{-1}(\overline{\text{co}}(H(B))) \subseteq B, \ \textit{i.e.}\ T(y) \in \overline{\text{co}}(H(B)) \Rightarrow y \in B.$$

Moreover, suppose that the equality

$$\alpha(T(M)) = \alpha(H(M))$$

implies the precompactness of M for any $M \subseteq B$.
Then the equation $T(x) = H(x)$ has a solution $\hat{x} \in X$.

Proof: We construct a sequence $(x_n)_{n\in\mathbb{N}}$ as follows. Let $x_0 \in B$ be arbitrary. Due to Lemma 3.2 we can choose inductively $x_n \in T^{-1}(H(x_{n-1}))$ for $n \geq 1$, so $T(x_n) = H(x_{n-1})$. By construction, the set $A = \{x_0, x_1, \dots\}$ satisfies $T(A) = \{T(x_o)\} \cup H(A)$. Therefore, we get $\alpha(T(A)) = \alpha(\{T(x_o)\} \cup H(A)) = \alpha(H(A))$. Hence, A is precompact by assumption. Moreover, by construction $A \subseteq B$ and consequently $H(A) \subseteq H(B)$ is bounded.
Next, we show that $A' \subseteq T^{-1}(H(A'))$, where A' denotes the set of all accumulation points of A. Note that $A' \neq \emptyset$ since A is precompact. So for $x \in A'$ we find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$, hence, also $H(x_{n_k-1}) = T(x_{n_k}) \to T(x)$ as $k \to \infty$. Therefore, if we choose $y \in A'$ to be a limit point of $(x_{n_k-1})_{k\in\mathbb{N}}$, we get $H(y) = T(x)$.
Next, we consider the family

$$\mathfrak{M} = \{M : M \supseteq A',\, \overline{M} = M,\, T^{-1}(\overline{\text{co}}(H(M))) \subseteq M\}.$$

Since $B \in \mathfrak{M}$, this family is nonempty. Denote by M_0 the intersection of all $M \in \mathfrak{M}$. Note that M_0 is nonempty and closed, since $M \supseteq A' \neq \emptyset$ for each $M \in \mathfrak{M}$ and each M is closed.
Set $M_1 = T^{-1}(\overline{\text{co}}(H(M_0)))$. Then for all $M \in \mathfrak{M}$ we get

$$M_1 = T^{-1}(\overline{\text{co}}(H(M_0))) \subseteq T^{-1}(\overline{\text{co}}(H(M))) \subseteq M,$$

hence $M_1 \subseteq M_0$.
Further, M_1 is closed and satisfies both

$$M_1 = T^{-1}(\overline{\text{co}}(H(M_0))) \supseteq T^{-1}(\overline{\text{co}}(H(A'))) \supseteq A'$$

and

$$T^{-1}(\overline{\text{co}}(H(M_1))) \subseteq T^{-1}(\overline{\text{co}}(H(M_0))) = M_1.$$

Therefore, $M_1 \in \mathfrak{M}$ and $M_0 \subseteq M_1$. Consequently, $T^{-1}(\overline{\text{co}}(H(M_0))) = M_1 = M_0$. Since T is stably solvable and, therefore, surjective by Lemma 3.2, this in turn implies $T(M_0) = \overline{\text{co}}(H(M_0))$

and, hence,

$$\alpha(T(M_0)) = \alpha(\overline{\text{co}}(H(M_0))) = \alpha(H(M_0)).$$

Thus, by assumption, M_0 is compact, and by the continuity of H also $H(M_0)$ is compact.
By Corollary 2.13, we find a continuous operator $G : X \to X$ with $G(x) = H(x)$ for $x \in M_0$ and $G(X) \subseteq \overline{\text{co}}(H(M_0))$. This shows that $G(X)$ is precompact, and in turn that G is a compact operator. Furthermore, $[G]_Q = 0$ since $G(X)$ is bounded. Since T is stably solvable, there exists an $\hat{x} \in X$ with $T(\hat{x}) = G(\hat{x})$. But from $T(\hat{x}) \in \overline{\text{co}}(H(M_0))$ it follows that $\hat{x} \in T^{-1}(\overline{\text{co}}(H(M_0))) = M_0$. Thus, $T(\hat{x}) = G(\hat{x}) = H(\hat{x})$. □

Corollary 3.8 (Darbo fixed point theorem) *Let $B \subseteq X$ be nonempty, bounded, closed, and convex. Let $S \in \mathcal{C}(B, B)$ and suppose $[S]_A < 1$. Then S has a fixed point.*

Proof: Choose $H = S$ and $T = I$ in Theorem 3.7, the latter of which is possible because of Corollary 3.6 □

Corollary 3.9 *Let $T \in \mathcal{C}(X)$ with $[T]_A < 1$ and $[T]_Q < 1$. Then T has a fixed point.*

Proof: Let $b \in ([T]_Q, 1)$. Then $\|T(x)\| \leq b\|x\|$ for all $x \in X$ with $\|x\| \geq r$ for sufficiently large $r > 0$. Because of $[T]_A < \infty$, $T(B_r(0))$ is bounded. Hence, there exists a $d > 0$ with $\|T(x)\| < d$ for $x \in B_r(0)$. Altogether we have $\|T(x)\| \leq b\|x\| + d$ for all $x \in X$.
Setting $R = d/(1 - b)$, for $x \in \overline{B_R(0)}$ we get

$$\|T(x)\| \leq b\|x\| + d \leq bR + d = \frac{bd + d(1-b)}{1-b} = \frac{d}{1-b} = R.$$

This shows, that T maps $\overline{B_R(0)}$ into itself and the assertion follows from Corollary 3.8. □

Theorem 3.10 *Let $T \in \mathcal{B}(X, Y)$. Then T is stably solvable if and only if T is surjective.*

Proof: By Lemma 3.2, we only have to prove that a surjective operator is stably solvable. So let $T \in \mathcal{B}(X, Y)$ be surjective. By Theorem 2.26 we may find a continuous function $s : Y \to X$ such that $s(y) \in T^{-1}(y)$ for all $y \in Y$ and $\|s(y)\| \leq M\|y\|$ for some $M > 0$.
Now let $G : X \to Y$ be compact with $[G]_Q = 0$. Since s maps bounded sets onto bounded sets, the map $G \circ s : Y \to Y$ is compact.
Since $[G]_A = [G]_Q = 0$, we can use the same technique as at the beginning of the proof of Corollary 3.9 to show that for every $\epsilon > 0$ we can find a $d > 0$ such that $\|G(x)\| \leq \epsilon\|x\| + d$ for

all $x \in X$. For any sequence $(y_n)_{n\in\mathbb{N}}$ with $\|y_n\| \to \infty$ we then get

$$\lim_{n\to\infty} \frac{\|G \circ s(y_n)\|}{\|y_n\|} \leq \lim_{n\to\infty} \frac{\epsilon\|s(y_n)\| + d}{\|y_n\|} \leq \lim_{n\to\infty} \frac{\epsilon M\|y_n\| + d}{\|y_n\|} = \lim_{n\to\infty} \left(\epsilon M + \frac{d}{\|y_n\|}\right) = \epsilon M.$$

Since $\epsilon > 0$ was arbitrary, we get $\lim_{n\to\infty} \frac{\|G\circ s(y_n)\|}{\|y_n\|} = 0$. Hence, $[G \circ s]_Q = 0$.
Thus, we can use Corollary 3.9 to see that $G\circ s$ has a fixed point $y \in Y$. Consequently, $x = s(y)$ satisfies

$$Tx = Ts(y) = y = G(s(y)) = G(x),$$

i.e. T is stably solvable. □

3.1.2 FMV-regularity

Definition 3.11 *An operator $T \in \mathcal{C}(X, Y)$ is called FMV-regular, if T is stably solvable and fulfils $[T]_q > 0$ and $[T]_a > 0$.*

By what we have shown in chapter 2, amongst the properties of FMV-regular operators are, that they are proper and coercive. Both of these properties are due to the fact, that $[T]_q > 0$, and $[T]_a > 0$. But the positivity of these two characteristics also has an important influence on the stable solvability of the operator.

Lemma 3.12 *Every FMV-regular operator is strictly stably solvable. More precisely, the estimate*

$$\mu(T) \geq \min\{[T]_q, [T]_a\}$$

holds true.

Proof: Fix k with $k < [T]_q$ and $k < [T]_a$. We have to show that the equation $T(x) = S(x)$ has a solution $\hat{x} \in X$ for any $S \in \mathcal{C}(X, Y)$ with $[S]_Q \leq k$ and $[S]_A \leq k$.
Choose two numbers b and c such that $[S]_Q < b < c < [T]_q$. Then $\|S(x)\| \leq b\|x\|$ and $\|T(x)\| \geq c\|x\|$ for $\|x\| \geq r$ with sufficiently large $r > 0$. Since $[S]_A < \infty$, $S(B_r(0))$ is bounded by some constant $R > 0$. Hence, $\|S(x)\| \leq R + b\|x\|$ for all $x \in X$. Put

$$\rho = \frac{R}{c - b}.$$

By choosing R sufficiently large, we may assume without loss of generality that $\rho \geq r$. We now want to apply Theorem 3.7 with $B = \overline{B_\rho(0)}$. It is clear that $S(\overline{B_\rho(0)})$ is bounded, because $S(\overline{B_\rho(0)}) \subseteq \overline{B_{R+b\rho}(0)}$.
Fix $x \in X$ with $T(x) \in \overline{B_{R+b\rho}(0)}$. If $\|x\| > \rho$ were true (which would also imply $\|x\| > r$), we

would get

$$R + b\rho \geq \|T(x)\| \geq c\|x\| > c\rho,$$

i.e.

$$\frac{R}{c-b} > \rho,$$

contraditing our choice of ρ. Therefore,

$$T^{-1}(\overline{\mathrm{co}}(S(\overline{B_\rho(0)}))) \subseteq T^{-1}(\overline{\mathrm{co}}(\overline{B_{R+b\rho}(0)})) = T^{-1}(\overline{B_{R+b\rho}(0)}) \subseteq \overline{B_\rho(0)}.$$

Furthermore, for all $M \subseteq \overline{B_\rho(0)}$ with $\alpha(M) > 0$, by assumption we have

$$\alpha(T(M)) \geq [T]_a \alpha(M) > [S]_A \alpha(M) \geq \alpha(S(M)).$$

Thus, all hypotheses of Theorem 3.7 are satisfied. Hence the equation $T(x) = S(x)$ has a solution $\hat{x} \in X$. □

Lemma 3.13 *Let $T, S \in \mathcal{C}(X,Y)$, let T be k-stably solvable. Further, assume that $k \geq [S]_A$ and $k \geq [S]_Q$. Then $T + S$ is k'-stably solvable for $0 \leq k' \leq k - \max\{[S]_A, [S]_Q\}$. In particular,*

$$\mu(T+S) \geq \mu(T) - \max\{[S]_A, [S]_Q\}.$$

Proof: Let $H \in \mathcal{C}(X,Y)$ with $[H]_A \leq k - \max\{[S]_A, [S]_Q\}$ and $[H]_Q \leq k - \max\{[S]_A, [S]_Q\}$. We have to show that the equation $T(x) + S(x) = H(x)$ has a solution $\hat{x} \in X$. But $[H - S]_A \leq [H]_A + [S]_A \leq k$ and $[H - S]_Q \leq [H]_Q + [S]_Q \leq k$. Since T is k-stably solvable, the equation $T(x) = (H - S)(x)$ has a solution $\hat{x} \in X$. □

Theorem 3.14 *Let $T, S \in \mathcal{C}(X,Y)$. If T is FMV-regular with $\min\{[T]_q, [T]_a\} > [S]_A$ and $\min\{[T]_q, [T]_a\} > [S]_Q$, then $T + S$ is also FMV-regular.*

Proof: By Lemma 3.12, T is k-stably solvable for $k < \min\{[T]_q, [T]_a\}$. By Lemma 3.13, $T+S$ is stably solvable. Moreover, $[T+S]_q \geq [T]_q - [S]_Q > 0$ and $[T+S]_a \geq [T]_a - [S]_A > 0$, by Lemma 2.2,(*ii*) and Lemma 2.7,(*iv*) respectively. □

Heuristically, this theorem shows that FMV-regularity is stable under perturbations that are sufficiently compact and quasibounded.
FMV-regularity is also invariant under linear isomorphisms.

Theorem 3.15 *Let $T \in \mathcal{C}(X,Y)$ and let $S \in \mathcal{B}(Y,Z)$ be bijective. Then the operator $S \circ T \in \mathcal{C}(X,Z)$ is FMV-regular if and only if T is FMV-regular.*

Proof: First, let T be FMV-regular. By definition $[T]_a > 0$ and $[T]_q > 0$. Moreover $[S]_a > 0$ and $[S]_q > 0$ by Lemma 2.8,(iv). So Proposition 2.2,(v) and Proposition 2.7,(vii) show that

$$[S \circ T]_a \geq [S]_a[T]_a > 0$$

and

$$[S \circ T]_q \geq [S]_q[T]_q > 0,$$

respectively. It remains to show that $S \circ T$ is stably solvable. So let $G \in \mathcal{C}(X,Z)$ satisfy $[G]_A = [G]_Q = 0$. Then obviously $S^{-1} \circ G$ is still compact and $[S^{-1} \circ G]_Q \leq \|S^{-1}\|[G]_Q = 0$. Since T is stably solvable, there exists an $x \in X$ with $T(x) = S^{-1} \circ G(x)$. Hence, $S \circ T(x) = G(x)$ and $S \circ T$ is stably solvable.
Conversely, suppose $S \circ T$ is FMV-regular. Since S^{-1} is a linear bijection, we can use the first part of the proof to see that $T = S^{-1} \circ (S \circ T)$ is FMV-regular. □

Finally, we want to show, that for linear operators FMV-regularity reduces to a very simple property.

Theorem 3.16 *Let $T \in \mathcal{B}(X,Y)$. Then T is FMV-regular if and only if T is bijective.*

Proof: If T is FMV-regular, then $[T]_q > 0$. By Lemma 2.8,(iii), T is injective. Since T is also stably solvable, T is surjective by Theorem 3.10.
Conversely, let T be a bijection. Then $[T]_a > 0$ and $[T]_q > 0$ by Lemma 2.8,(iv). Again by Theorem 3.10, T is also stably solvable. □

3.2 Topological properties

Definition 3.17 *For $T \in \mathcal{C}(X)$ we call the set*

$$\rho_{FMV}(T) := \{\lambda \in \mathbb{K} : \lambda I - T \textit{ is FMV-regular}\}$$

the FMV-resolvent set and its complement

$$\sigma_{FMV}(T) := \mathbb{K} \backslash \rho_{FMV}(T)$$

the FMV-spectrum of T.

Thanks to our extensive preparations, the following proofs concerning topological properties of $\sigma_{FMV}(T)$ are very simple. One of the most important properties that we wanted a nonlinear spectrum to have, has already been proven for the FMV-spectrum.

Proposition 3.18 *$\sigma(T) = \sigma_{FMV}(T)$ for $T \in \mathcal{B}(X)$.*

Proof: This is a simple consequence of Theorem 3.16. □

Proposition 3.19 *The FMV-spectrum $\sigma_{FMV}(T)$ is closed for all $T \in \mathcal{C}(X)$.*

Proof: Fix $\lambda \in \rho_{FMV}(T)$ and let $0 < \delta < \min\{[\lambda I - T]_a, [\lambda I - T]_q\}$. Now let $\mu \in \mathbb{K}$ with $|\mu - \lambda| < \delta$. Because of

$$[(\mu - \lambda)I]_A = |\mu - \lambda| < [\lambda I - T]_a$$

and

$$[(\mu - \lambda)I]_Q = |\mu - \lambda| < [\lambda I - T]_q,$$

it follows from Theorem 3.14 that $\mu I - T = (\lambda I - T) + (\mu - \lambda)I$ is FMV-regular. This shows that λ is an interior point of $\rho_{FMV}(T)$. Thus, $\rho_{FMV}(T)$ is open in $\mathbb{K}$. □

Proposition 3.20 *Let $T \in \mathcal{C}(X)$ and suppose $[T]_A < \infty$ and $[T]_Q < \infty$. Then $\sigma_{FMV}(T)$ is bounded. In fact, it is contained in $\overline{B_{\max\{[T]_A,[T]_Q\}}(0)}$.*

Proof: Let $\lambda \in \mathbb{K}$ with $|\lambda| > [T]_Q$ and $|\lambda| > [T]_A$. We will show, that $\lambda \in \rho_{FMV}(T)$.
First of all, we have $[\lambda I - T]_q \geq |\lambda| - [T]_Q > 0$ and $[\lambda I - T]_a \geq |\lambda| - [T]_a > 0$ by Lemma 2.2,(*ii*) and Lemma 2.7,(*iv*), respectively. It remains to show that $\lambda I - T$ is stably solvable. So let $S \in \mathcal{C}(X)$ be compact with $[S]_Q = 0$. Then the operator $H := (T + S)/\lambda$ satisfies both

$$[H]_A = \left[\frac{T+S}{\lambda}\right]_A = \frac{1}{|\lambda|}[T+S]_A \leq \frac{1}{|\lambda|}([T]_A + [S]_A) = \frac{[T]_A}{|\lambda|} < 1$$

and

$$[H]_Q = \left[\frac{T+S}{\lambda}\right]_Q = \frac{1}{|\lambda|}[T+S]_Q \leq \frac{1}{|\lambda|}([T]_Q + [S]_Q) = \frac{[T]_Q}{|\lambda|} < 1.$$

By Corollary 3.9, H has a fixed point $x \in X$, from which the assertion follows. □

The previous proposition motivates the following definitions.

Definition 3.21 *We define the set*

$$\mathfrak{AQ}(X) := \mathfrak{A}(X,X) \cap \mathfrak{Q}(X,X) = \{T \in \mathcal{C}(X) : [T]_A < \infty \text{ and } [T]_Q < \infty\}.$$

For $T \in \mathfrak{AQ}(X)$ *define*

$$p_{AQ}(T) := \max\{[T]_A, [T]_Q\}.$$

Further, for $T \in \mathcal{C}(X)$ *we call*

$$r_{FMV}(T) := \sup\{|\lambda| : \lambda \in \sigma_{FMV}(T)\}$$

the FMV spectral radius of T*. If* $\sigma_{FMV}(T) = \emptyset$*, we set* $r_{FMV}(T) = 0$*.*

The following assertion has been proven in Proposition 3.20 and 3.19

Corollary 3.22 *Let* $T \in \mathfrak{AQ}(X)$*. Then* $\sigma_{FMV}(T)$ *is compact. Furthermore,*

$$r_{FMV}(T) \leq p_{AQ}(T).$$

It is easy to see, that p_{AQ} is a seminorm on $\mathfrak{AQ}$. Also, note that $\mathcal{B}(X) \subseteq \mathfrak{AQ}(X)$ and $p_{AQ}(T) = \|T\|$ for $T \in \mathcal{B}(X)$. This follows from Lemma 2.8,(i) and (ii). Hence, Corollary 3.22 includes the well known result that for a linear operator the spectral radius of the classical spectrum is bounded by its operator norm.
Further, $\mathfrak{AQ}(X)$ can be equipped with a topology defined by the seminorm p_{AQ}. The discussion above then also shows that the following proposition includes the result of Proposition 1.1.

Proposition 3.23 *The multivalued map* $\sigma_{FMV} : \mathfrak{AQ}(X) \to \mathscr{P}(\mathbb{K})$*, which assignes to each* $T \in \mathfrak{AQ}(X)$ *its FMV-spectrum, is upper semi-continuous.*

Proof: First, we want to show, that the map σ_{FMV} is closed. To do so, choose a sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathfrak{AQ}(X)$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\subseteq \mathbb{K}$ such that

$$\lambda_n \in \sigma_{FMV}(T_n), \qquad \lambda_n \to \lambda, \qquad p_{AQ}(T - T_n) \to 0.$$

Since

$$p_{AQ}((\lambda_n I - T_n) - (\lambda I - T)) \leq |\lambda_n - \lambda| + p_{AQ}(T - T_n) \to 0,$$

the sequence $(\lambda_n I - T_n)_{n\in\mathbb{N}}$ converges to $\lambda I - T$ in $\mathfrak{AQ}(X)$. If $\lambda I - T$ were FMV-regular, then for sufficiently large n, $\lambda_n I - T_n = (\lambda I - T) + ((\lambda_n I - T_n) - (\lambda I - T))$ would also be FMV-regular by Theorem 3.14, contradicting our choice of λ_n. Therefore, $\lambda \in \sigma_{FMV}(T)$ and σ_{FMV} is closed. The assertion now follows from Lemma 2.19, since by Corollary 3.22 for all $T \in \mathfrak{AQ}(X)$ we have

$$\sup_{\lambda \in \sigma_{FMV}(T)} |\lambda| = r_{FMV}(T) \leq p_{AQ}(T).$$

□

3.3 Subdivision of the FMV-spectrum

Definition 3.24 *For $T \in \mathcal{C}(X)$ we set*

$$\sigma_\delta(T) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not stably solvable}\},$$
$$\sigma_a(T) := \{\lambda \in \mathbb{K} : [\lambda I - T]_a = 0\},$$
$$\sigma_q(T) := \{\lambda \in \mathbb{K} : [\lambda I - T]_q = 0\},$$

and

$$\sigma_\pi(T) := \sigma_a(T) \cup \sigma_q(T).$$

By definition, we then have

$$\sigma_{FMV} = \sigma_\delta \cup \sigma_\pi = \sigma_\delta \cup \sigma_a \cup \sigma_q.$$

However, this decomposition need not be disjoint.
The seminorm p_{AQ} can be usefull to compare the spectrum und subspectra of two operators.

Lemma 3.25 *Let $T, S \in \mathcal{C}(X)$ such that $p_{AQ}(T-S) = 0$. Then $\sigma_q(T) = \sigma_q(S)$, $\sigma_a(T) = \sigma_a(S)$, and $\sigma_\delta(T) = \sigma_\delta(S)$, and hence $\sigma_{FMV}(T) = \sigma_{FMV}(S)$.*

Proof: By Proposition 2.2,(iii), $[T - S]_Q = 0$ implies $\sigma_q(T) = \sigma_q(S)$, while $[T - S]_A$ implies $\sigma_a(T) = \sigma_a(S)$ by Proposition 2.7,(v).
Assume $\lambda I - S$ is stably solvable and let $H \in \mathcal{C}(x)$ with $p_{AQ}(H) = 0$. Since $p_{AQ}(H+(T-S)) = 0$, there exists an $x \in X$ with $\lambda x - S(x) = H(x) + T(x) - S(x)$, i.e. $\lambda x - T(x) = H(x)$. So $\lambda I - T$ is also stably solvable. Hence, $\sigma_\delta(T) = \sigma_\delta(S)$. □

We now want to show that these subdivisons have their counterparts in the linear theory.

Definition 3.26 *For $T \in \mathcal{B}(X)$ we define the approximate point spectrum*

$$\sigma_{app}(T) := \{\lambda \in \mathbb{K} : \exists\, x_n,\, n \in \mathbb{N},\, \|x_n\| = 1,\, \|(\lambda I - T)(x_n)\| \to 0\}$$

and the defect spectrum

$$\sigma_d(T) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not surjective}\}.$$

Just as the FMV-spectrum is the union of $\sigma_\delta(T)$ and $\sigma_\pi(T)$, for linear operators we have $\sigma(T) = \sigma_{app}(T) \cup \sigma_d(T)$. We show that $\sigma_{app}(T)$ and $\sigma_d(T)$ correspond with $\sigma_\delta(T)$ and $\sigma_\pi(T)$, respectively. In particular, they are the same in the case of linear operators.

Lemma 3.27 *If $T \in \mathcal{B}(X)$, then $\sigma_a(T) \subseteq \sigma_q(T)$.*

Proof: If $\dim(X) < \infty$, then $\sigma_a(T) = \emptyset$. So we can restrict ourselves to infinite dimensional spaces. Let $\lambda \in \sigma_a(T)$. Then for every $n \in \mathbb{N}$ there exists a bounded set M_n with $\alpha(M_n) > 0$

and $\alpha((\lambda I - T)(M_n)) < \alpha(M_n)/(2n)$.
If $(\lambda I - T)$ is not injective on M_n, choose $x_1, \neq x_2 \in M_n$ with $(\lambda I - T)(x_1) = (\lambda I - T)(x_2)$ and set $y_n = x_1 - x_2$.
If $(\lambda I - T)$ is injective on M_n, define

$$K = \inf_{\substack{x \neq y \\ x,y \in M_n}} \frac{\|(\lambda I - T)(x) - (\lambda I - T)(y)\|}{\|x - y\|}.$$

Assume $K > 0$ and fix $k \in (0, K)$. Then for $x, y \in M_n$ we have

$$\|(\lambda I - T)(x) - (\lambda I - T)(y)\| \geq k\|x - y\|.$$

If $\{z_1, \ldots, z_n\}$ is a finite ϵ-net for $(\lambda I - T)(M_n)$, then $\{(\lambda I - T)^{-1}(z_1), \ldots, (\lambda I - T)^{-1}(z_n)\}$ is a finite $\epsilon/k-$net for M_n, i.e. $\alpha((\lambda I - T)(M_n)) \geq k\alpha(M_n)$. This in turn means $1/(2n) > k$. Hence, $1/n > K$.
We therefore can choose $x_1, x_2 \in M_n$ with $\|(\lambda I - T)(x_1) - (\lambda I - T)(x_2)\| < \|x_1 - x_2\|/n$. Set $y_n = x_1 - x_2$. This is of course also possible in the case of $K = 0$.
We need to show that

$$\liminf_{\|y\| \to \infty} \frac{\|(\lambda I - T)(y)\|}{\|y\|} = 0.$$

For now we have

$$\lim_{n \to \infty} \frac{\|(\lambda I - T)(y_n)\|}{\|y_n\|} = 0.$$

If $(y_n)_{n \in \mathbb{N}}$ has a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ with $\|y_{n_k}\| \to \infty$, then $\lambda \in \sigma_q(T)$ and we are done. If it has subsequence $(y_{n_k})_{n \in \mathbb{N}}$ that is both bounded and bounded away from zero, set $x_k = k y_{n_k}$ to see that $\lambda \in \sigma_q(T)$. If $\|y_{n_k}\| \to 0$, set $x_k = y_{n_k}/\|y_{n_k}\|^2$. We then get $\|x_k\| \to \infty$ and

$$\lim_{k \to \infty} \frac{\|(\lambda I - T)(x_k)\|}{\|x_k\|} = \lim_{k \to \infty} \frac{\|y_{n_k}\|^{-2}\|(\lambda I - T)(y_{n_k})\|}{\|y_{n_k}\|^{-1}} = \lim_{k \to \infty} \frac{\|(\lambda I - T)(y_{n_k})\|}{\|y_{n_k}\|} = 0,$$

hence, $\lambda \in \sigma_q(T)$. □

Theorem 3.28 *If $T \in \mathcal{B}(X)$, then $\sigma_\delta(T) = \sigma_d(T)$ and $\sigma_\pi(T) = \sigma_{app}(T)$.*

Proof: That $\sigma_\delta(T) = \sigma_d(T)$ is a simple consequence of Theorem 3.10. To see that $\sigma_q(T) = \sigma_{app}(T)$ consider the equation

$$\liminf_{\|x\| \to \infty} \frac{\|\lambda x - T(x)\|}{\|x\|} = \liminf_{\|x\| \to \infty} \left\| \lambda \frac{x}{\|x\|} - T\left(\frac{x}{\|x\|}\right) \right\| = \liminf_{\|y\| = 1} \|\lambda y - T(y)\|.$$

Since $\sigma_q(T) = \sigma_\pi(T)$ by Lemma 3.27, we are done. $\square$

One of the important properties of the approximate point spectrum is that it is closed and contains the boundary of the classical spectrum. The same is true for its nonlinear counterpart.

Lemma 3.29 *Let $T \in \mathcal{C}(X)$, then $\sigma_a(T), \sigma_q(T)$, and $\sigma_\pi(T)$ are closed.*

Proof: Let $(\lambda_n)_{n\in\mathbb{N}} \subseteq \sigma_q(T)$ converge to a λ. By Proposition 2.2,(iii), we get

$$[\lambda I - T]_q = |[\lambda I - T]_q - [\lambda_n I - T]_q| \leq [\lambda I - T - \lambda_n I + T]_Q = |\lambda_n - \lambda| \to 0.$$

Hence, $\sigma_q(T)$ is closed. Similarly, $\sigma_a(T)$ is closed and, therefore, also $\sigma_\pi(T) = \sigma_a(T) \cup \sigma_q(T)$. $\square$

Theorem 3.30 *Let $T \in \mathcal{C}(X)$. Then*

$$\partial\sigma_{FMV}(T) \subseteq \sigma_\pi(T).$$

Proof: First, we want to show, that $\sigma_{FMV}(T)\backslash\sigma_\pi(T)$ is an open subset of $\mathbb{K}$. So fix a $\lambda \in \sigma_{FMV}(T)\backslash\sigma_\pi(T)$. It suffices to show that there exists a $\delta > 0$, such that $\mu I - T$ is not stably solvable for $|\mu - \lambda| < \delta$.
If this is not true, we find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \to \lambda$ such that $\lambda_n I - T$ is stably solvable for all n. Since $\lambda I - T$ is not stably solvable, there exists a compact operator S with $[S]_Q = 0$ such that $\lambda x - T(x) \neq S(x)$ for all $x \in X$. On the other hand, since all $\lambda_n I - T$ are stably solvable, we get a sequence $(x_n)_{n\in\mathbb{N}}$ with $\lambda_n x_n - T(x_n) = S(x_n)$.
In case $\|x_n\| \to \infty$ we get

$$\frac{\|\lambda x_n - T(x_n)\|}{\|x_n\|} \leq \frac{\|\lambda x_n - \lambda_n x_n\|}{\|x_n\|} + \frac{\|\lambda_n x_n - T(x_n)\|}{\|x_n\|} = |\lambda - \lambda_n| + \frac{\|S(x_n)\|}{\|x_n\|} \to 0.$$

But this means $[\lambda I - T]_q = 0$, contradicting $\lambda \notin \sigma_\pi$. Hence, there exists a bounded subsequence $(x_{n_k})_{k\in\mathbb{N}}$. This implies

$$\|(\lambda - T - S)(x_{n_k})\| = \|\lambda x_{n_k} - (T - S)(x_{n_k})\| = \|\lambda x_{n_k} - \lambda_{n_k} x_{n_k}\| \leq |\lambda - \lambda_{n_k}|\, \|x_{n_k}\| \to 0.$$

Furthermore, by Proposition 2.4,(v), $[\lambda - T - S]_a = [\lambda - T]_a > 0$. Thus, by Proposition 2.7,(ii), the preimage of the compact set $\{(\lambda - T - S)(x_{n_k})\} \cup \{0\}$, which in particular includes $(x_{n_k})_{k\in\mathbb{N}}$, is compact. We can therefore assume without loss of generality, that $x_{n_k} \to x$ for some $x \in X$. By continuity, $\lambda x - T(x) - S(x) = 0$. This contradiction shows that our assertion was true.
Now let $\lambda \in \partial\sigma_{FMV}(T)$, and assume $\lambda \notin \sigma_\pi(T)$. Then $\lambda \in \sigma_{FMV}(T)\backslash\sigma_\pi(T)$ which is an in $\mathbb{K}$ open subset of $\sigma_{FMV}(T)$, contradicting $\lambda \in \partial\sigma_{FMV}(T)$. $\square$

Corollary 3.31 *Let $T \in \mathcal{C}(X)$ and let $C \subseteq \mathbb{K}\backslash\sigma_\pi(T)$ be a connected set. Then either $C \subseteq \sigma_{FMV}(T)$ or $C \subseteq \rho_{FMV}(T)$.*

Proof: Let K be the connected component of $\mathbb{K}\backslash\sigma_\pi(T)$ which contains C. Set

$$K_0 = \{\nu \in K : \nu I - T \text{ is FMV-regular}\} = K \cap \rho_{FMV}(T)$$

and assume K_0 is not empty. By Theorem 3.30, the boundary of K_0 relative to K is empty. Therefore, K_0 is open and closed in K. Since K is connected, it follows, that $K_0 = K$. □

3.3.1 Special classes of operators

We now look at two special classes of operators for which we can give more detailed information about their FMV-spectrum. First, we look at compact operators in infinite dimensional Banach spaces. The restriction to infinite dimensional spaces is to be expected, since in finite dimensional spaces every continuous operator is compact. Hence, compactness only yields additional properties in infinite dimensions. In the following proposition, for a closed set $\Sigma \subseteq \mathbb{K}$ with $0 \notin \Sigma$, we denote by $c_0[\Sigma]$ the connected component of $\mathbb{K}\backslash\Sigma$ containing 0.

Proposition 3.32 *Let X be an infinite dimensional Banach space and suppose $T \in \mathcal{C}(X)$ is compact. Then the following is true:*

(i) $\sigma_a(T) = \{0\}$, *hence* $\sigma_\pi = \{0\} \cup \sigma_q(T)$.

(ii) *T is not surjective. In particular,* $0 \in \sigma_\delta(T)$

(iii) *Either* $0 \in \sigma_q(T)$, *or* $c_0[\sigma_q(T)] \subseteq \sigma_\delta(T)$.

(iv) *If* $0 \notin \sigma_q(T)$ *and* $\sigma_{FMV}(T)$ *is bounded, then* $\sigma_q(T)$ *contains a positive and a negative value.*

(v) *If* $\sigma_{FMV}(T) \neq \mathbb{K}$, *then* $\sigma_q(T) \neq \emptyset$.

Proof: Assertion (i) simply follows from the fact, that $[\lambda I - T]_a = |\lambda|$, which is a consequence of Proposition 2.4,(v). For (ii), assume that T is surjective. For each $y \in X$ we get $y \in T(B_n(0))$ for some $n \in \mathbb{N}$. We therefore have the representation

$$X = \bigcup_{n=1}^{\infty} T(B_n(0)) = \bigcup_{n=1}^{\infty} \overline{T(B_n(0))}.$$

By Baire's catgegory theorem, the countable union of closed sets without interior points has no interior points. Since X does have an interior point, at least one of the compact sets $\overline{T(B_n(0))}$ contains an interior point. In particular, $\overline{T(B_n(0))}$ contains a closed ball, which in turn is compact. Since closed balls in a space X are compact if and only if X is finite dimensional, this contradicts our choice of X as an infinite dimensional Banach space. $0 \in \delta_\delta(T)$ now follows

from Lemma 3.2.
To see that (iii) holds true, first observe that the condition $0 \notin \sigma_q(T)$ together with (i) implies that 0 is an isolated point of $\sigma_\pi(T)$ since $\sigma_q(T)$ is closed. Below we will show that $\lambda I - T$ is not surjective for $|\lambda|$ small enough. By Lemma 3.2, this implies that 0 is an interior point of $\sigma_\delta(T)$ and thus also of the FMV-spectrum. Theorem 3.30 and (i) then tell us that $\partial\sigma_{FMV}(T) \subseteq \sigma_q(T)$. Hence, $\sigma_\delta(T)$ has no boundary in $c_0[\sigma_q(T)]$. $\sigma_\delta(T) \cap c_0[\sigma_q(T)]$ is therefore both open and closed in the connected set $c_0[\sigma_q(T)]$, i.e. $\sigma_\delta(T) \cap c_0[\sigma_q(T)]$ is either empty or coincides with $c_0[\sigma_q(T)]$. Since $0 \in \sigma_\delta(T)$ by (ii), the latter must be true, which proves (iii).
To show that $\lambda I - T$ is not surjective for $|\lambda|$ small enough if $0 \notin \sigma_q(T)$, assume that this is not the case. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ converging to zero, such that $\lambda_n I - T$ is surjective for all n. Fix $a \in (0, \frac{1}{2}[T]_q)$, and choose $R > 0$ such that $\|T(x)\| \geq 2a\|x\|$ for $\|x\| > R$. Taking $b = 2aR$ we then have $\|T(x)\| \geq 2a\|x\| - b$ for all $x \in X$. Consequently, for $|\lambda| \leq a$ we have $\|\lambda x - T(x)\| \geq a\|x\| - b$.
Now fix $y \in B_1(0)$. By assumption, we find a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lambda_n x_n - T(x_n) = y$ for all n. Without loss of generality we may assume that $|\lambda_n| \leq a$ for all n, and thus

$$1 > \|y\| = \|\lambda_n x_n - T(x_n)\| \geq a\|x_n\| - b.$$

Hence, $x_n \in B_r(0)$ with $r = (1+b)/a$. Since $\lambda_n x_n \to 0$, we conclude that $T(x_n) \to -y$. But since $y \in B_1(0)$ was arbitrary, we see that $B_1(0) \subseteq \overline{T(B_r(0))}$, which is impossible as $T(B_r(0))$ is precompact.
(iv) is an immediate consequence of (iii), since for instance $\mathbb{R}^+ \cap \sigma_q(T) = \emptyset$ yields the unboundedness of $\sigma_\delta(T) \subseteq \sigma_{FMV}(T)$. Finally, if $\sigma_q(T) = \emptyset$, then $\sigma_\delta(T) = \mathbb{K}$ by (iii). Thus, (v) holds true. □

Definition 3.33 *An operator $T \in \mathcal{C}(X)$ is called asymptotically linear if there exists an operator $T' \in \mathcal{B}(X)$ with $[T - T']_Q = 0$. T' is called the asymptotic derivative of T.*

If an operator is asymptotically linear, its asymptotic derivate is uniquely defined, since for two asymptotic derivates T_1' and T_2' we get

$$\|T_1' - T_2'\| = [T_1' - T_2']_Q \leq [T_1' - T]_Q + [T - T_2']_Q = 0.$$

For asymptotically linear operators the spectrum of its asymptotic derivate can be used to gain information about its FMV-spectrum.

Proposition 3.34 *Let $T \in \mathcal{C}(X)$ be an asymptotically linear operator with asymptotic derivate T'. Then the following is true:*

(i) $\sigma_q(T) = \sigma_q(T')$.

(ii) $\partial\sigma(T') \subseteq \sigma_q(T)$, *in particular,* $\sigma_q(T)$ *is not empty for* $\mathbb{K} = \mathbb{C}$.

(iii) *If $T - T'$ is compact, then $\sigma_{FMV}(T) = \sigma(T')$, $\sigma_a(T) = \sigma_a(T')$, $\sigma_q(T) = \sigma_q(T')$, and $\sigma_\delta(T) = \sigma_\delta(T')$.*

Proof: First, from the definition of the asymptotic derivative and Proposition 2.2,(iii) we get

$$\left|[\lambda I - T]_q - [\lambda I - T']_q\right| \leq [\lambda I - T - \lambda I + T']_Q = [T' - T]_Q = 0,$$

from which (i) follows immediately . To see that (ii) holds, use Proposition 3.18, Theorem 3.30, Lemma 3.27, and (i) to get

$$\partial\sigma(T') = \partial\sigma_{FMV}(T') \subseteq \sigma_\pi(T') = \sigma_q(T') = \sigma_q(T).$$

Finally, if $T-T'$ is compact, i.e. $[T-T']_A = 0$, then $p_{AQ}(T-T') = \max\{[T-T']_A, [T-T']_Q\} = 0$. Hence, (iii) holds by Lemma 3.25. □

3.4 Eigenvalues and approximate eigenvalues

Definition 3.35 *For $T \in \mathcal{C}(X)$ define the point spectrum*

$$\sigma_p(T) := \{\lambda \in \mathbb{K} : \exists x \in X,\, x \neq 0 \text{ with } \lambda x - T(x) = 0\}.$$

Every $\lambda \in \sigma_p(T)$ is called an eigenvalue of T.

Eigenvalues play an important role in classical spectral theory. Therefore, an important property of the classical spectrum is the fact that $\sigma_p(T) \subseteq \sigma(T)$ for $T \in \mathcal{B}(X)$. However, the FMV-spectrum does not have this property. For example, for an operator $T \in \mathcal{C}(X)$ with $p_{AQ}(T) = 0$ we get $\sigma_{FMV}(T) \subseteq \{0\}$, since $r_{FMV}(T) \leq p_{AQ}(T)$. So if $T(x) \neq 0$ for all $x \in X$, we get $\sigma_p(T) \subseteq \mathbb{K}\backslash\{0\}$, i.e. $\sigma_p(T) \cap \sigma_{FMV}(T) = \emptyset$.
At first, this seems to be a rather big flaw in the theory of the FMV-spectrum, given the importance of the point spectrum in linear theory. However, the question arises, whether the role of eigenvalues is as important in the nonlinear theory as it is in the linear theory, or whether their importance is due to the linearity of the operators. In fact, for nonlinear operators many different notions of eigenvalues have been studied, and many of them seem to be more useful for nonlinear operators than the point spectrum. The type of eigenvalues that fits particularly well with the FMV-spectrum are approximate eigenvalues.

Definition 3.36 *For $T \in \mathcal{C}(X)$, $\sigma_q(T)$ is called the approximate point spectrum of T. Every $\lambda \in \sigma_q(T)$ is called an approximate eigenvalue.*

We have already defined an approximate point spectrum for linear operators. However, we have also shown that these two definitions coincide for linear operators, see Lemma 3.27 and

Theorem 3.28. The name approximate eigenvalue stems from the fact that $\lambda \in \sigma_q(T)$ if and only if there exists an unbouded sequence $(x_n)_{n\in\mathbb{N}}$ in X with

$$\lim_{n\to\infty} \frac{\|\lambda x_n - T(x_n)\|}{\|x_n\|} = 0.$$

For a linear operator $T \in \mathcal{B}(X)$ we obviously have $\sigma_p(T) \subseteq \sigma_q(T)$. However, this inclusion may be strict. For compact linear operators the relation between these two sets is very close.

Lemma 3.37 *Let $T \in \mathcal{B}(X)$ be compact. Then $\sigma_p\backslash\{0\} = \sigma_q\backslash\{0\}$.*

Proof: Let $\lambda \in \sigma_q\backslash\{0\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with

$$\lim_{n\to\infty} \frac{\|\lambda x_n - T(x_n)\|}{\|x_n\|} = 0,$$

and define $y_n = \frac{x_n}{\|x_n\|}$. Then $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence, and, since T is compact, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that $T(y_{n_k}) \to x$. Because of

$$\|\lambda y_{n_k} - T(y_{n_k})\| = \|\lambda \frac{x_{n_k}}{\|x_{n_k}\|} - T\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right)\| = \frac{\|\lambda x_{n_k} - T(x_{n_k})\|}{\|x_{n_k}\|} \to 0,$$

we get $y_{n_k} \to x/\lambda$. If we define $y = x/\lambda$, then we get

$$T(y) = x = \lambda y$$

by continuity reasons. Since $\|y\| = 1$, we have $\lambda \in \sigma_p(T)$. □

Furthermore, there is also a very good topological reason why the approximate point spectrum is a reasonable choice to substitute the point spectrum. To show this, we first need yet another notion of eigenvalue.

Definition 3.38 *For $T \in \mathcal{C}(X)$, we call $\lambda \in \mathbb{K}$ an unbounded eigenvalue of T, if there exists an unbounded sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lambda x_n - T(x_n) = 0$ for all n. The set*

$$\sigma_p^0(T) := \{\lambda \in \mathbb{K} : \lambda \textit{ unbounded eigenvalue of } T\}$$

is called the unbounded point spectrum of T.

Obviously, we have the two inclusions $\sigma_p^0(T) \subseteq \sigma_p(T)$ and $\sigma_p^0(T) \subseteq \sigma_q(T)$ for all $T \in \mathcal{C}(X)$. For $T \in \mathcal{B}(X)$ however, we even get $\sigma_p^0(T) = \sigma_p(T)$. So the unbounded point spectrum is always included in the FMV-spectrum and for linear operators it coincides with the point spectrum. However, the following proposition will show that the approximate point spectrum has a topological property the unbounded point spectrum lacks. It also shows that the unbounded point spectrum is never 'too far away' from the approximate point spectrum.

Proposition 3.39 *The multivalued map $\sigma_q : \mathfrak{AQ}(X) \to \mathbb{K}$, $T \mapsto \sigma_q(T)$ is the closure of the multivalued map $\sigma_p^0 : \mathfrak{AQ}(X) \to \mathbb{K}$, $T \mapsto \sigma_p^0(T)$.*

In particular, for $T \in \mathfrak{AQ}(X)$ we have

$$\begin{aligned}\sigma_q(T) &= \{\lambda \in \mathbb{K} | \, \exists (T_n)_{n\in\mathbb{N}} \subseteq \mathfrak{AQ}(X), \, (\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \lambda_n \in \sigma_p^0(T_n), p_{AQ}(T_n - T) \to 0, \lambda_n \to \lambda\} \\ &= \{\lambda \in \mathbb{K} | \, \exists S \in \mathfrak{AQ}(X) : p_{AQ}(T - S) = 0, \lambda \in \sigma_p^0(S)\}.\end{aligned}$$

Proof: Fix $\lambda \in \sigma_q(T)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence satisfying the conditions $\|x_n\| \to \infty$ and $\|\lambda x_n - T(x_n)\|/\|x_n\| \to 0$. Without loss of generality we may assume, that $\|x_m - x_n\| > 2$ for $m \neq n$. Put

$$\eta_n(x) = \max\{1 - \|x - x_n\|, 0\},$$

so that $\eta_n(x_m) = \delta_{mn}$. Now define $S : X \to X$ by

$$S(x) = T(x) + \sum_{n=1}^{\infty} \eta_n(x)(\lambda x_n - T(x_n)).$$

By definition, $\eta_n(x) \neq 0$ for only one n, because if $\eta_n(x) \neq 0$, then $\|x - x_n\| < 1$. So $\|x - x_m\| \geq |\|x - x_n\| - \|x_n - x_m\|| > 1$ and, hence, $\eta_m(x) = 0$ for $m \neq n$, i.e. S is well defined. Furthermore, $[S]_Q < \infty$. In fact, because of $[T]_Q < \infty$ and our choice of $(x_n)_{n\in\mathbb{N}}$, we get

$$\begin{aligned}[S]_Q = \limsup_{\|y\|\to\infty} \frac{\|S(y)\|}{\|y\|} &\leq \limsup_{\|y\|\to\infty} \frac{\|T(y)\|}{\|y\|} + \limsup_{\|y\|\to\infty} \frac{\|\sum_{n=1}^{\infty} \eta_n(y)(\lambda x_n - T(x_n))\|}{\|y\|} \\ &= [T]_Q + \limsup_{\|y\|\to\infty} \frac{\eta_{n(y)}(y)\|\lambda x_{n(y)} - T(x_{n(y)})\|}{\|y\|} \leq [T]_Q + \lim_{n\to\infty} \frac{\|\lambda x_n - T(x_n)\|}{\|x_n\| - 1} = [T]_Q,\end{aligned}$$

where $n(y)$ is the one n where $\eta_n(y) \neq 0$, if such an n exists. Next, let $(y_k)_{k\in\mathbb{N}}$ be a bounded sequence. Then $\|x_n - y_k\| < 1$ can only be true for finitely many n. Hence, $(S - T)(y_k) \in \operatorname{span}\{\lambda x_{n_1} - T(x_{n_1}), \cdots, \lambda x_{n_m} - T(x_{n_m})\}$ for some $m \in \mathbb{N}$. It therefore has a convergent subsequence, i.e. $S - T$ is compact. Thus, we get

$$[S]_A = \sup_{\infty>\alpha(M)>0} \frac{\alpha(S(M))}{\alpha(M)} \leq \sup_{\infty>\alpha(M)>0} \left(\frac{\alpha(T(M))}{\alpha(M)} + \frac{\alpha((S-T)(M))}{\alpha(M)} \right) = [T]_A.$$

In summary, $S \in \mathfrak{AQ}(X)$.
By definition, $S(x_n) = \lambda x_n$ for all n, i.e. $\lambda \in \sigma_p^0(S)$. Further, $p_{AQ}(S - T) = 0$. In fact,

$$[T - S]_Q \leq \limsup_{\|y\|\to\infty} \frac{\|\sum_{n=1}^{\infty} \eta_n(y)(\lambda x_n - T(x_n))\|}{\|y\|} = 0,$$

as has been shown above, and $[T - S]_A = 0$ since $S - T$ is compact. Thus, we have proven the inclusion

$$\sigma_q(T) \subseteq \{\lambda \in \mathbb{K} | \, \exists S \in \mathfrak{AQ}(X) : p_{AQ}(T - S) = 0, \lambda \in \sigma_p^0(S)\}.$$

The inclusion

$$\{\lambda \in \mathbb{K} | \exists S \in \mathfrak{AQ}(X) : p_{AQ}(T-S) = 0, \lambda \in \sigma_p^0(S)\} \subseteq$$
$$\{\lambda \in \mathbb{K} | \exists (T_n)_{n\in\mathbb{N}} \subseteq \mathfrak{AQ}(X),\, (\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \lambda_n \in \sigma_p^0(T_n) \text{ for all } n,\, p_{AQ}(T_n - T) \to 0, \lambda_n \to \lambda\}$$

is trivial.
Now we show that the map $T \mapsto \sigma_q(T)$ is closed. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ be sequences, such that $\lambda_n \to \lambda$ in $\mathbb{K}$, $p_{AQ}(T - T_n) \to 0$, and $\lambda_n \in \sigma_q(T_n)$ for all n. Since $[\lambda_n I - T_n]_q = 0$ for all n, we can find a sequence $(y_n)_{n\in\mathbb{N}}$ with $\|y_n\| > n$ and

$$\frac{\|\lambda_n y_n - T_n(y_n)\|}{\|y_n\|} < \frac{1}{n}.$$

Since, without loss of generality, $[T-T_n]_Q < 1/n$, we can choose y_n so that they fulfil additionally

$$\frac{\|(T - T_n)(y_n)\|}{\|y_n\|} < \frac{1}{n}.$$

We conclude

$$\begin{aligned}
[\lambda I - T]_q &\leq \lim_{n\to\infty} \frac{\|\lambda y_n - T(y_n)\|}{\|y_n\|} \\
&\leq \lim_{n\to\infty} \left(\frac{\|(\lambda - \lambda_n)(y_n)\|}{\|y_n\|} + \frac{\|\lambda_n y_n - T_n(y_n)\|}{\|y_n\|} + \frac{\|(T - T_n)(y_n)\|}{\|y_n\|} \right) \\
&\leq \lim_{n\to\infty} \left(|\lambda - \lambda_n| + \frac{1}{n} + \frac{1}{n} \right) = 0.
\end{aligned}$$

Hence, $\lambda \in \sigma_q(T)$. Since $\sigma_p^0(T) \subseteq \sigma_q(T)$, the graph of σ_q thus contains the closure of the graph of $\sigma_p^0(T)$. This yields the remaining inclusion. □

Finally, we bring another theorem that shows that the approximate point spectrum is a good substitute of the point spectrum.

Definition 3.40 *An operator $T \in \mathcal{C}(X)$ is called odd if $T(-x) = -T(x)$ for all $x \in X$.*
An operator $T \in \mathcal{C}(X)$ is called asymptotically odd if there exists an odd operator $\tilde{T} \in \mathcal{C}(X)$ with $[T - \tilde{T}]_Q = 0$.

Obviously, every $T \in \mathcal{B}(X)$ is odd.

Theorem 3.41 *Let $T \in \mathcal{C}(X)$ be asymptotically odd and compact. Then every $\lambda \in \sigma_{FMV}(T)$ is an asymptotic eigenvalue of T.*

Proof: First, fix $\lambda \in \sigma_{FMV}(T)$, $\lambda \neq 0$. From $[\lambda I - T]_a \geq |\lambda| - [T]_A = |\lambda| > 0$ we get that $\lambda \in \sigma_\delta(T)$ or $\lambda \in \sigma_q(T)$. We have to exclude the case that solely the first possibilty holds true, i.e. we have to show that $\lambda I - T$ is stably solvable if $\lambda \notin \sigma_q(T)$. Assume further that T itself is odd.
Since $[\lambda I - T]_q > 0$, by Lemma 3.5 we can restrict ourselves to compact operators $S \in \mathcal{C}(X)$

where $S(x) = 0$ for $\|x\| \geq R$ with suitable $R > 0$ in order to prove stable solvability. We may further assume that $\lambda x - T(x) \neq 0$ for $\|x\| > R$, because otherwise we would have $[\lambda I - T]_q = 0$. Define $H : \overline{B_R(0)} \to X$ by $H(x) = (T(x) + S(x))/\lambda$. Then H coincides with T/λ on $\partial B_R(0)$ and is odd there. By Theorem 2.36, H has a fixed point $\hat{x} \in X$. So $\hat{x} = H(\hat{x}) = (T(\hat{x}) + S(\hat{x}))/\lambda$, i.e. $\lambda\hat{x} - T(\hat{x}) = S(\hat{x})$, and $\lambda I - T$ is stably solvable.
Now assume that T is asymptotically odd. Let G be an odd operator with $[T - G]_Q = 0$. Define $\tilde{T} \in \mathcal{C}(X)$ by $\tilde{T}(x) = (T(x) - T(-x))/2$. Then $\tilde{T}$ is odd, compact, and

$$\frac{\|T(x) - \tilde{T}(x)\|}{\|x\|} = \frac{\|T(x) - \frac{T(x)-T(-x)}{2}\|}{\|x\|} = \frac{\frac{1}{2}\|T(x) + T(-x)\|}{\|x\|}$$
$$\leq \frac{1}{2}\frac{\|T(x) - G(x) + G(x) + T(-x)\|}{\|x\|} \leq \frac{1}{2}\frac{\|T(x) - G(x)\|}{\|x\|} + \frac{1}{2}\frac{\|T(-x) - G(-x)\|}{\|-x\|}.$$

Hence, $[T - \tilde{T}]_Q = 0$. Since $T - \tilde{T}$ is compact, we have $p_{AQ}(T - \tilde{T})$, and so $\sigma_{FMV}(T) = \sigma_{FMV}(\tilde{T})$ and $\sigma_q(T) = \sigma_q(\tilde{T})$ by Lemma 3.25. By what we have proven so far we may conclude that

$$\sigma_{FMV}(T)\backslash 0 = \sigma_{FMV}(\tilde{T})\backslash 0 \subseteq \sigma_q(\tilde{T}) = \sigma_q(T).$$

It remains to deal with the case $\lambda = 0$. In finite dimensional spaces, we always have $\sigma_a = \emptyset$, so the above argument also works for $\lambda = 0$. In infinite dimensional spaces, we have $0 \in \sigma_{FMV}(T)$ by Proposition 3.32. But 0 is also in $\sigma_q(T)$, because if this were not the case, then by Proposition 3.32,(iii) we get

$$c_0[\sigma_q(T)] \subseteq \sigma_\delta(T) \subseteq \sigma_{FMV}(T) \subseteq \sigma_q(T) \cup \{0\},$$

i.e. $c_0[\sigma_q(T)] \subseteq \{0\}$. But this is impossible since $c_0[\sigma_q(T)]$ is both open and nonempty. □

Corollary 3.42 *Let $T \in \mathcal{B}(X)$ be compact. Then every $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of T.*

Proof: As T is linear, we can apply Theorem 3.41 and Lemma 3.37 to show that

$$\sigma(T)\backslash\{0\} = \sigma_{FMV}(T)\backslash\{0\} = \sigma_q(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}.$$

□

Corollary 3.42 is an important result of linear spectral theory. However, when viewed from the perspective of the aproximate point spectrum, we see that it is actually a result about a larger class of operators, which are in general not linear. Only in the linear case this results conveniently coincides with one about eigenvalues. Furthermore, even in the linear case, when talking about eigenvalues we have to exclude the value 0, whereas if we talk about aproximate eigenvalues it describes the complete spectrum of the operator.

3.5 Numerical range of operators

Definition 3.43 *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $T \in \mathcal{B}(H)$. Denote by $S_1(H)$ the unit sphere in H. Then the set*

$$W(T) := \{\langle Tx, x\rangle : x \in S_1(H)\} = \left\{ \frac{\langle Tx, x\rangle}{\|x\|^2} : x \in H, x \neq 0 \right\}$$

is called the numerical range of T.

In linear theory, the numerical range has some very interesting properties. Unfortunately, there is no definition of a numerical range for nonlinear operators that encompasses all these properties. However, the property $\sigma(T) \subseteq \overline{W(T)}$ can, in some sense, be transfered to nonlinear operators. We already showed that for any operator the inclusion $\partial\sigma_{FMV}(T) \subseteq \sigma_\pi(T)$ holds true. In the case of linear operators this reduces to $\partial\sigma(T) \subseteq \sigma_q(T)$. Since $W(T)$ is convex, the statements $\sigma(T) \subseteq \overline{W(T)}$ and $\sigma_q(T) \subseteq \overline{W(T)}$ are therefore equivalent. We want to define a numerical range for nonlinear operators that retains the latter of these properties.
We need the following construction.

Definition 3.44 *For $T \in \mathcal{C}(H)$ define the operator $T^\perp : H\backslash\{0\} \to H$ by*

$$T^\perp(x) := \phi_T(x)x$$

where

$$\phi_T(x) = \frac{\langle T(x), x\rangle}{\|x\|^2}.$$

$T^\perp$ can be extended continuously to 0 if $T(0) = 0$. However, since we are only interested in the behaviour of $T^\perp$ outside of bounded sets, we will not restrict ourselves to such operators. Instead we will treat $T^\perp$ as a continuous operator on the whole space H. This can be realised by defining $T^\perp$ on $B_1(0)$ to be a continuous extension of $T^\perp$ from $S_1(H)$ to $\overline{B_1(0)}$. So if for two operators T and S the operators $T^\perp$ and $S^\perp$ coincide on $H\backslash B_1(0)$, we write $T^\perp = S^\perp$ by using the same extension to $\overline{B_1(0)}$. We gather a few properties of the operator $T^\perp$.

Proposition 3.45 *Let $T, S \in \mathcal{C}(H)$ and $\lambda \in \mathbb{K}$. Then the following is true*

(i) $T^{\perp\perp} = T^\perp$.

(ii) $(T + S)^\perp = T^\perp + S^\perp$.

(iii) $(\lambda T)^\perp = \lambda T^\perp$.

(iv) $[T^\perp]_Q \leq [T]_Q$.

(v) $[T^\perp]_q \leq [T]_q$.

Proof: The properties (i)-(iii) follow directly from the definition of $T^\perp$. The remaining two properties immediately follow from the estimate

$$\frac{\|T^\perp(x)\|}{\|x\|} = |\phi_T(x)| \leq \frac{\|T(x)\|}{\|x\|}.$$

□

Definition 3.46 *For $T \in \mathcal{C}(H)$ we call*

$$W_{FMV}(T) := \sigma_q(T^\perp)$$

the numerical range of T.

We gather some properties of $W_{FMV}(T)$.

Proposition 3.47 *Let $T, S \in \mathcal{C}(H)$, $z \in H$, and $\mu \in \mathbb{K}$. Then $W_{FMV}(T)$ has the following properties.*

(i) $W_{FMV}(T + S) \subseteq W_{FMV}(T) + W_{FMV}(S)$ *if* $[T^\perp]_Q < \infty$.

(ii) $W_{FMV}(\mu T) = \mu W_{FMV}(T)$.

(iii) $W_{FMV}(T + z) = W_{FMV}(T)$.

(iv) $W_{FMV}(\mu I - T) = \{\mu\} - W_{FMV}(T)$.

(v) $W_{FMV}(T) = \overline{W(T)}$ *if* $T \in \mathcal{B}(H)$.

(vi) $\sigma_q(T) \subseteq W_{FMV}(T)$.

Proof: If $\lambda \in W_{FMV}(T+S) = \sigma_q(T^\perp + S^\perp)$, then there exists an unbouded sequence $(x_n)_{n\in\mathbb{N}}$ in H such that $\phi_T(x_n) + \phi_S(x_n)$ converges to λ. Since $[T^\perp]_Q < \infty$, the sequence $\phi_T(x_n)$ is bounded. So without loss of generality we may assume that $\phi_T(x_n)$ converges to some $\mu \in \mathbb{K}$. Consequently, $\phi_S(x_n)$ converges to $\lambda - \mu$. We can conclude $\mu \in \sigma_q(T^\perp) = W_{FMV}(T)$ and $\lambda - \mu \in \sigma_q(S^\perp) = W_{FMV}(S)$. Thus, (i) holds true.
The assertion (ii) follows from the obvious equality $\phi_{\mu T} = \mu\phi_T$. Similarly, (iii) follows from the equality

$$\phi_{T+z}(x) = \frac{\langle T(x) + z, x\rangle}{\|x\|^2} = \phi_T(x) + \frac{\langle z, x\rangle}{\|x\|^2},$$

where the last term tends to zero as $\|x\| \to \infty$. Further, the equality $\phi_{\mu I - T} = \mu - \phi_T$ implies (iv).

To prove (v), let T be linear. Then we get the chain of equalities

$$W_{FMV}(T) = \sigma_q(T^\perp) = \left\{\lambda \in \mathbb{K} : \liminf_{\|x\|\to\infty} \frac{\|\lambda x - T^\perp(x)\|}{\|x\|} = 0\right\}$$
$$= \left\{\lambda \in \mathbb{K} : \inf_{\|x\|=1} \|\lambda x - T^\perp(x)\| = 0\right\} = \left\{\lambda \in \mathbb{K} : \inf_{\|x\|=1} |\lambda - \langle Tx, x\rangle| = 0\right\} = \overline{W(T)}.$$

Finally, Proposition 3.45,(v) gives the implication $\lambda \in \sigma_q(T) \Rightarrow \lambda \in \sigma_q(T^\perp)$. □

The assertion (v) in the proposition above shows that the definition of the numerical range for nonlinear operators yields nearly the same as the classical definition when applied to linear operators. Note that even though we have proven that $\sigma_q(T) \subseteq W_{FMV}(T)$, this does not give us any further information on $\sigma_a(T)$ or $\sigma_\delta(T)$. However, having an additional tool to localize $\sigma_q(T)$ can be usefull, as we shall see in Section 4.3.
Next, we want to give some topological information about the numerical range.

Theorem 3.48 *Let H be a Hilbert space. $W_{FMV}(T)$ allows the representation*

$$W_{FMV}(T) = \bigcap_{n\in\mathbb{N}} \overline{\phi_T(H\backslash B_n(0))}. \tag{3.1}$$

Suppose $T \in \mathcal{C}(H)$ satisfies $[T^\perp]_Q < \infty$. Then the set $W_{FMV}(T)$ is nonempty and compact. In particular $W_{FMV}(T) \subseteq \overline{B_{[T^\perp]_Q}(0)}$. If $H \neq \mathbb{R}$ then $W_{FMV}(T)$ is also connected.

Proof: First, we show that (3.1) holds true. So let $\lambda \in W_{FMV}(T)$. Then we may choose a sequence $(x_n)_{n\in\mathbb{N}}$ in H with $\|x_n\| > n$ and $\phi_T(x_n) \to \lambda$. If $\lambda \notin \overline{\phi_T(H\backslash B_m(0))}$ for some $m \in \mathbb{N}$, we can find a neighbourhood U_λ of λ such that $U_\lambda \cap \phi_T(H\backslash B_n(0)) = \emptyset$ for $n \geq m$. However, $\phi_T(x_n) \in \phi_T(H\backslash B_n(0))$ since $\|x_n\| > n$. Thus $\phi_T(x_n) \notin U_\lambda$, contradicting the fact that $\phi_T(x_n) \to \lambda$. Therefore, the inclusion '$\subseteq$' holds in (3.1).
Conversely, if λ belongs to the intersection in (3.1), we may find a sequence $(x_n)_{n\in\mathbb{N}}$ in H with $\|x_n\| > n$ and $\phi_T(x_n) \to \lambda$. This implies that $\lambda \in \sigma_q(T^\perp)$ and the second inclusion in (3.1) is proven.
The assumption $[T^\perp]_Q < \infty$ implies that the set $\phi_T(H\backslash B_n(0))$ is bounded for sufficiently large n. Moreover, $H\backslash B_n(0)$ is nonempty (and in the case of $H \neq \mathbb{R}$ also connected). Since ϕ_T is continuous there, $\phi_T(H\backslash B_n(0))$ is also nonempty (and connected). Since the right hand side in (3.1) is now the intersection of a decreasing sequence of nonempty, compact (and connected) sets, $W_{FMV}(T)$ is also nonempty, compact (and connected).
For the final assertion we again use the fact, that if $\lambda \in W_{FMV}(T)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H such that $\|x_n\| > n$ and $\phi_T(x_n) \to \lambda$. Therefore,

$$[T^\perp]_Q = \limsup_{\|x\|\to\infty} |\phi_T(x)| \geq \lim_{n\to\infty} |\phi_T(x_n)| = |\lambda|.$$

□

Theorem 3.48 allows us to give a stricter estimate on the FMV spectral radius for operators on Hilbert spaces.

Theorem 3.49 *For $T \in \mathcal{C}(H)$ we have $r_{FMV}(T) \leq \max\{[T]_A, [T^\perp]_Q\}$.*

Proof: Let $|\lambda| > \max\{[T]_A, [T^\perp]_Q\}$. Then $[\lambda I - T]_a \geq |\lambda| - [T]_A > 0$ by Proposition 2.7,(iv). Moreover, $\lambda \notin \sigma_q(T)$ since $\sigma_q(T) \subseteq W_{FMV}(T) \subseteq \overline{B_{[T^\perp]_Q}(0)}$ by Proposition 3.47,(vi) and Theorem 3.48. So it remains to show that $\lambda I - T$ is stably solvable. Let $S \in \mathcal{C}(H)$ with $[S]_A = [S]_Q = 0$. By Lemma 3.5 we can also assume that $\{x \in H : S(x) \neq 0\}$ is bounded. The stable solvability is obviously equivalent to the existence of fixed point for the map $G(x) = \lambda^{-1}(T(x) + S(x))$. Define $\pi_n \in \mathcal{C}(H)$ by

$$\pi_n(x) = \min\{1, \frac{n}{\|x\|}\}x.$$

Obviously, π_n maps H into $\overline{B_n(0)}$. We now need an estimate for $\alpha(\pi_n(M))$ for a bounded set $M \subseteq X$. If $M \subseteq \overline{B_n(0)}$, then $\pi_n(M) = M$, hence, $\alpha(\pi_n(M)) = \alpha(M)$. Let $M \subseteq X \backslash \overline{B_n(0)}$ and let $\{z_1, \cdots, z_m\}$ be a finite ϵ-net for M with $\epsilon > \alpha(M)$. Then for every $x \in M$ there is a z_i with $\|z_i - x\| < \epsilon$. First, assume that $z_i \notin \overline{B_n(0)}$. Obviously, $\pi_n(tx) = \pi_n(x)$ for every $t > 1$. We conclude

$$\begin{aligned} \|\pi_n(x) - \pi_n(z_i)\| = \|\pi_n(x\|z_i\|) - \pi_n(z_i\|x\|)\| &= \left\| \frac{n}{\|x\|\|z_i\|}x - \frac{n}{\|x\|\|z_i\|}z_i \right\| \\ &= \frac{n}{\|x\|\|z_i\|}\|x - z_i\| < \frac{1}{n}\epsilon \leq \epsilon. \end{aligned}$$

Also, if $z_i \in \overline{B_n(0)}$, then

$$\|\pi_n(x) - \pi_n(z_i)\| = \|\pi_n(x) - z_i\| \leq \|x - z_i\| < \epsilon.$$

Hence, $\{\pi_n(z_i) : i = 1, \cdots, m\}$ is a finite ϵ-net for $\pi_n(M)$ and $\alpha(\pi_n(M)) \leq \alpha(M)$. For arbitrary $M \subseteq X$ we therefore get

$$\begin{aligned} \alpha\left(\pi_n(M)\right) &= \alpha\left(\pi_n\left(M \cap \overline{B_n(0)}\right) \cup \pi_n\left(M \backslash \overline{B_n(0)}\right)\right) \\ &= \max\left\{\alpha\left(\pi_n\left(M \cap \overline{B_n(0)}\right)\right), \alpha\left(\pi_n\left(M \backslash \overline{B_n(0)}\right)\right)\right\} \leq \alpha(M) \end{aligned}$$

Using this, we are able to see that

$$\begin{aligned} [\pi_n \circ G]_A &= \sup_{\infty > \alpha(M) > 0} \frac{\alpha(\pi_n \circ G(M))}{\alpha(M)} \leq \sup_{\infty > \alpha(M) > 0} \frac{\alpha(G(M))}{\alpha(M)} = [G]_A \\ &= [\lambda^{-1}(T(x) + (S(x))]_A \leq |\lambda|^{-1}[T(x)]_A < 1. \end{aligned}$$

By the Darbo fixed point theorem, Corollary 3.8, $\pi_n \circ G$ has a fixed point $x_n \in \overline{B_n(0)}$. If $\|x_n\| = \|\pi_n \circ G(x_n)\| < n$, then we have $x_n = \pi_n \circ G(x_n) = G(x_n)$ and we are done.

Assume that $\|\pi_n \circ G(x_n)\| = \|x_n\| \geq n$ for all n. Then for each n, there exists a $c_n \leq 1$ such that $x_n = c_n G(x_n)$, i.e. $\lambda x_n = c_n(T(x_n) + S(x_n))$. Since $\{x \in H : S(x) \neq 0\}$ is bounded, we have $[T^\perp]_Q = [T^\perp + S^\perp]_Q$, because this seminorm only takes the behaviour of operators outside bounded sets into account. We end up with the contradiction

$$\begin{aligned} |\lambda| \; > \; [T^\perp]_Q = [T^\perp + S^\perp]_Q &= \limsup_{\|x\|\to\infty} \frac{\|(T^\perp + S^\perp)(x)\|}{\|x\|} \\ \geq \; \limsup_{n\to\infty} \frac{|\langle T(x_n) + S(x_n), x_n\rangle|}{\|x_n\|^2} &\geq \limsup_{n\to\infty} \frac{|\langle \lambda x_n, x_n\rangle|}{\|x_n\|^2} = |\lambda|. \end{aligned}$$

□

Chapter 4

Applications

4.1 Maps on spheres

As a first example, we want to show how the FMV-spectrum can be used to analyze maps on spheres.

Definition 4.1 *Let X be a Banach space. Then for $r > 0$*

$$S_r(X) := \{x \in X : \|x\| = r\}$$

is the sphere with radius r in X. For a map $T : S_r(X) \to X$ we define the map $\tilde{T} : X \to X$ by

$$\tilde{T}(x) := \begin{cases} \|x\| T\left(r\frac{x}{\|x\|}\right) & if \quad \|x\| \neq 0 \\ 0 & if \quad \|x\| = 0. \end{cases}$$

Finally, a map $T : X \to X$ is called positively homogenous if $T(\lambda x) = \lambda T(x)$ for all $\lambda \geq 0$.

It is obvious that $\tilde{T}$ is continous if T is, and that $\tilde{T}$ is positively homogenous. Clearly, any positively homogenous map fulfils $T(0) = 0$. In some situations, this is the only root of T.

Lemma 4.2 *Let $T \in \mathcal{C}(X)$ be positively homogenous and assume that $[T]_a > 0$. Then the equation $T(x) = 0$ has only the trivial solution $x = 0$ if and only if $[T]_q > 0$.*

Proof: First, let $[T]_q > 0$. Assume $0 \neq x \in X$ with $T(x) = 0$ and take any sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$. Then we get the contradiction

$$[T]_q = \liminf_{\|y\|\to\infty} \frac{\|T(y)\|}{\|y\|} \leq \lim_{n\to\infty} \frac{\|T(nx_n)\|}{\|nx_n\|} = \lim_{n\to\infty} \frac{n\|T(x_n)\|}{n\|x_n\|} = \frac{\|T(x)\|}{\|x\|} = 0.$$

Therefore, $T(x) = 0$ if and only if $x = 0$.
Now let $[T]_q = 0$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} \|T(x_n)\|/\|x_n\| = 0$. For $y_n = x_n/\|x_n\|$ we have $\|y_n\| = 1$ and

$$\|T(y_n)\| = \left\|T\left(\frac{x_n}{\|x_n\|}\right)\right\| = \left\|\frac{1}{\|x_n\|}T(x_n)\right\| = \frac{\|T(x_n)\|}{\|x_n\|} \to 0.$$

For $A = \{y_n : n \in \mathbb{N}\}$ we have $0 = \alpha(T(A)) \geq [T]_a \alpha(A)$. Hence, A is precompact and we can assume without loss of generality that y_n converges to some $y \in X$ with $\|y\| = 1$. The continuity of T implies that $T(y) = 0$. □

The next lemma follows easily from Lemma 4.2.

Lemma 4.3 *Let $T : S_r(X) \to X$ be continuous. Then*

$$[\tilde{T}]_q = \inf\{\|T(x)\| : x \in S_r(X)\},$$

and

$$[\tilde{T}]_Q = \sup\{\|T(x)\| : x \in S_r(X)\}.$$

Furthermore, if $[\lambda I - \tilde{T}]_a > 0$, then $\lambda \in \sigma_q(\tilde{T}) \Leftrightarrow \frac{\lambda}{r} \in \sigma_p(T)$.

Proof: The representation of $[\tilde{T}]_q$ and $[\tilde{T}]_Q$ follow directly from the definition of $\tilde{T}$. The last assertion follows from Lemma 4.2, which asserts that $\lambda \in \sigma_q(\tilde{T}) \Leftrightarrow \lambda \in \sigma_p(\tilde{T}) \Leftrightarrow \exists x \in X\backslash\{0\} : \tilde{T}(x) = \lambda x \Leftrightarrow \exists x \in X\backslash\{0\} : T(r\frac{x}{\|x\|}) = \frac{\lambda}{r} r \frac{x}{\|x\|}$. □

First, we deal with spheres in a finite dimensional spaces. We give a fixed point theorem and a condition for the surjectivity of an operator.

Theorem 4.4 *Let $T : S_r(\mathbb{K}^n) \to S_r(\mathbb{K}^n)$ be continuous and assume that T is not surjective. Then T has a fixed point and an antipodal point (i.e. a point $x \in \mathbb{K}^n$ with $T(x) = -x$).*

Proof: Since $p_{AQ}(\tilde{T}) = [\tilde{T}]_Q = r$, we have that $\lambda \in \rho_{FMV}(\tilde{T})$ for all $\lambda > r$ by Corollary 3.22. Moreover, if $\lambda \in \sigma_q(\tilde{T})$, then there is an $x \in S_r(\mathbb{R}^n)$ with $\frac{\lambda}{r}x = T(x)$ by Lemma 4.3. In particular, $r = \|T(x)\| = \|\frac{\lambda}{r}x\| = \frac{|\lambda|}{r}\|x\| = |\lambda|$. Therefore, $\sigma_q(\tilde{T}) \subseteq S_r(\mathbb{K})$. But, because T is not surjective, $0 \in \sigma_{FMV}(\tilde{T})$. Since the FMV-spectrum is closed and $\sigma_a(\tilde{T}) = \emptyset$, we get $\partial\sigma_{FMV}(\tilde{T}) \subseteq \sigma_\pi(\tilde{T}) = \sigma_q(\tilde{T}) \subseteq S_r(\mathbb{K})$ by Theorem 3.30. This yields $\sigma_{FMV}(\tilde{T}) = \overline{B_r(0)}$ and hence, $\partial\sigma_{FMV}(\tilde{T}) = \sigma_q(\tilde{T}) = S_r(\mathbb{K})$. In particular $-r$ and r both lie in $\sigma_q(\tilde{T})$ and, hence, -1 and 1 lie in $\sigma_p(T)$ by Lemma 4.3. □

Theorem 4.5 *If $T : S_r(\mathbb{K}^n) \to S_R(\mathbb{K}^n)$ is continuous and odd, then T is surjective.*

Proof: Using the same technique as in the proof of the previous theorem, we achieve $\sigma_q(\tilde{T}) \subseteq S_{\frac{R}{r}}(\mathbb{K})$. Since $\tilde{T}$ is odd, we have $\sigma_{FMV}(\tilde{T}) = \sigma_q(\tilde{T})$ by Theorem 3.41. Hence, $0 \in \rho_{FMV}(\tilde{T})$. In particular, $\tilde{T}$ is surjective. Now let $y \in S_R(\mathbb{K}^n)$. Then there is an $x \in X$ with $\tilde{T}(x) = y$. But this means $T(r\frac{x}{\|x\|}) = \frac{y}{\|x\|}$, i.e. $R = \|T(r\frac{x}{\|x\|})\| = \frac{\|y\|}{\|x\|} = \frac{R}{\|x\|}$. Hence, $\|x\| = 1$ and we get $T(rx) = y$. So T is surjective. □

Next, we want to consider infinite dimensional spaces. There, the above fixed point theorem can be extended to the following result.

Theorem 4.6 (Birkoff-Kellog Theorem) *Let X be an infinite dimensional Banach space and let $T : S_r(X) \to X$ be compact such that*

$$\inf_{x \in S_r(x)} \|T(x)\| > 0.$$

Then T has a positive and a negative eigenvalue.

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Because T is compact, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $T(r\frac{x_{n_k}}{\|x_{n_k}\|})$ converges. Since the sequence is bounded, we can assume without loss of generality, that $\|x_{n_k}\|$ also converges. Therefore $\tilde{T}(x_n)$ has a convergent subsequence and $\tilde{T}$ is compact. Since $T(S_1(X))$ is bounded, we have $[\tilde{T}]_Q < \infty$ by Lemma 4.3, and, therefore, $\tilde{T} \in \mathfrak{A}\mathfrak{Q}(X)$. Hence, $\sigma_{FMV}(\tilde{T})$ is bounded. But by assumption and by Lemma 4.3 $[\tilde{T}]_q > 0$, i.e. $0 \notin \sigma_q(\tilde{T})$. So by Proposition 3.32,(iv), $\sigma_q(\tilde{T})$ contains both a positive and a negative value. But since $[\lambda I - \tilde{T}]_a = |\lambda| > 0$ for all $\lambda \neq 0$, by Lemma 4.3 these values when divided by r are in fact eigenvalues of T. □

Corollary 4.7 *Let X be an infinite dimensional Banach space and let $T : S_r(X) \to S_r(X)$ be compact. Then T has a fixed point and an antipodal point.*

Proof: Let $\lambda_\pm$ be the positive or negative eigenvalue of T which exists by Theorem 4.6, and $x_\pm$ an eigenvector to $\lambda_\pm$. Then $r = \|T(x_\pm)\| = \|\lambda_\pm x_\pm\| = |\lambda_\pm| \, \|x_\pm\| = |\lambda| r$, i.e. $|\lambda| = 1$. Hence, $\lambda_\pm = +1$ or -1 respectively. □

The next theorem shows that in contrast to finite dimensional spaces, for an operator between spheres in an infinite dimensional space compactness and oddity are mutually exclusive.

Theorem 4.8 *Let X be an infinite dimensional Banach space. Then $T : S_r(X) \to S_R(X)$ can not be both compact and odd.*

Proof: Assume, that T is both compact and odd. Then $\tilde{T}$ is also compact and odd as seen in the beginning of the proof Theorem 4.6. Since T is bounded away from 0, we get $0 \notin \sigma_q(\tilde{T})$. But because $\tilde{T}$ is compact we have $0 \in \sigma_{FMV}(\tilde{T})$ by Prosposition 3.32. Thus, because of Theorem 3.41, we get the contradiction $0 \in \sigma_{FMV}(\tilde{T}) = \sigma_q(\tilde{T})$. □

The following theorem is one of the most useful results in topology. Here, our proof of the theorem is of special interest. Because even though we state this theorem in its original form for a sphere, the proof can be used verbatim for any point symmetric set, i.e. a set M where $x \in M$ implies $-x \in M$, for which $c_0[M]$ or $c_0[\mathbb{K}\backslash M]$ (depending on whether $0 \in M$ or not) is bounded, and that allows a positively homogenous extension of the considered function $T : M \subseteq \mathbb{K}^{n+1} \to \mathbb{K}^n$.

Theorem 4.9 (Borsuk-Ulam Theorem) *Let $T : S_r(\mathbb{K}^{n+1}) \to \mathbb{K}^n$ be continuous. Then there exists an $x \in S_r(\mathbb{K}^{n+1})$ with $T(x) = T(-x)$.*

Proof: Identify $\mathbb{K}^n$ with a subspace of $\mathbb{K}^{n+1}$. Define $S : S_r(\mathbb{K}^{n+1}) \to \mathbb{K}^{n+1}$ by $S(x) = T(x) - T(-x)$. Since we are in a finite dimensional space, we have $\sigma_a(\tilde{S}) = \emptyset$, in particular $0 \notin \sigma_a(\tilde{S})$. On the other hand, $\tilde{S}$ is not surjective because its values lie in a proper subspace of $\mathbb{K}^{n+1}$. So $0 \in \sigma_\delta(\tilde{S})$ by Lemma 3.2, and by Theorem 3.41 $0 \in \sigma_q(\tilde{S})$. Hence, $0 \in \sigma_p(S)$ by Lemma 4.3, i.e. there is an $x \in S_r(\mathbb{K}^{n+1})$ with $T(x) - T(-x) = 0$. $\square$

4.2 Differential equations

We want to show through an example how some basic properties of the FMV-spectrum can be used to deal with nonlinearities in differential equations. We consider the three point boundary value problem

$$\left.\begin{array}{l} \ddot{x}(t) = \mu f(t, x(t), \dot{x}(t)) + y(t), \\ x(0) = 0, \quad x(1) = \alpha x(\eta). \end{array}\right\} \tag{4.1}$$

Here, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $y \in L^1[0,1]$, $\eta \in (0,1)$, and $\alpha \geq 0$. This problem is called a three point boundary problem, because the values at three different points are used in the boundary conditions.

We are looking for solutions x of (4.1) in the Sobolev space $W^{2,1}[0,1]$, i.e. the set of all absolutely contiuous functions on $[0,1]$, such that $\dot{x}$ is also absolutely continuous and $\ddot{x} \in L^1[0,1]$.

In order to find this solution, consider the Banach space $X = C^1[0,1]$ of all continuously differentiable function on $[0,1]$ equipped with the norm $\|x\|_X = \max\{\|x\|_\infty, \|\dot{x}\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the supremum norm on $C[0,1]$. Further, set $Y = L^1[0,1]$ and let $\|\cdot\|_1$ denote its usual norm. Finally, define a linear operator $L : D(L) \subseteq X \to Y$, where

$$D(L) = \{x \in W^{2,1}[0,1] : x(0) = 0,\, x(1) = \alpha x(\eta)\},$$

by $Lx = \ddot{x}$ and a real valued function $c(\alpha, \eta)$ by

$$c(\alpha, \eta) = 1 + \frac{\alpha\eta + 1}{|1 - \alpha\eta|} = \begin{cases} \frac{2}{1-\alpha\eta} & \text{if } \alpha\eta < 1, \\ \frac{2\alpha\eta}{\alpha\eta - 1} & \text{if } \alpha\eta > 1. \end{cases}$$

Lemma 4.10 *If $\alpha\eta \neq 1$, the operator $L : D(L) \to Y$ is bijective. In this case its inverse satisfies*

$$\|L^{-1}\| \leq c(\alpha, \eta),$$

and is compact as an operator from Y into X.

Proof: Let $x, y \in D(L)$ so that $Lx = Ly$. Then, $L(x-y) = 0$, i.e. $x(t) - y(t) = ct + d$ with $c, d \in \mathbb{R}$. Since $x - y \in D(L)$, we get $(x-y)(0) = 0$, which yields $d = 0$ and $(x-y)(1) = \alpha(x-y)(\eta)$, which yields $c = \alpha\eta c$, and therefore $c = 0$ since $\alpha\eta \neq 1$. Hence, L is injective. Next, define a linear operator $K : Y \to D(L)$ by

$$Ky(t) = \int_0^t (t-s)y(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)y(s)ds - \frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)ds.$$

Easy calculations show that $LKy = y$, $Ky(0) = 0$, and $Ky(1) = \alpha Ky(\eta)$. Therefore, L is bijective and K is its inverse. We also easily see that for $x = Ky$ we get

$$\|x\|_\infty \leq 1\|y\|_1 + \frac{\alpha\eta}{|1-\alpha\eta|}\|y\|_1 + \frac{1}{|1-\alpha\eta|}\|y\|_1 = \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\|y\|_1,$$

and similarly

$$\|\dot{x}\|_\infty \leq \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right)\|y\|_1.$$

Hence,

$$\|L^{-1}\| = \|K\| \leq c(\alpha, \eta).$$

Finally, let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence in Y, i.e. $\|y_n\|_1 < M$ for an $M > 0$. Since K is a bounded operator, $(Ky_n)_{n\in\mathbb{N}}$ is bounded in X. In particular, $(Ky_n)_{n\in\mathbb{N}}$ is uniformly bounded. The above representation of K shows that

$$\begin{aligned}
\|Ky_n(t_1) \; - \; Ky_n(t_2)\|_\infty &\leq \left\|\int_0^{t_1} |(t_1 - t_2)y_n(s)|ds\right\|_\infty + \left\|\int_{t_1}^{t_2} |(t_2 - s)y_n(s)|ds\right\|_\infty \\
&+ |t_1 - t_2|\left\|\frac{\alpha}{1-\alpha\eta}\int_0^\eta |(\eta - s)y_n(s)|ds\right\|_\infty + |t_1 - t_2|\left\|\frac{1}{1-\alpha\eta}\int_0^1 |(1-s)y_n(s)|ds\right\|_\infty \\
&\leq |t_1 - t_2|\left\|\int_0^{t_1} |y_n(s)|ds\right\|_\infty + |t_2 - t_1|\left\|\int_{t_1}^{t_2} |y_n(s)|ds\right\|_\infty \\
&+ |t_1 - t_2|\left|\frac{\alpha\eta}{1-\alpha\eta}\right|\left\|\int_0^\eta |y_n(s)|ds\right\|_\infty + |t_1 - t_2|\left|\frac{1}{1-\alpha\eta}\right|\left\|\int_0^1 |y_n(s)|ds\right\|_\infty \\
&\leq |t_1 - t_2|\left(M + M + \left|\frac{\alpha\eta}{1-\alpha\eta}\right|M + \left|\frac{1}{1-\alpha\eta}M\right|\right) \\
&= |t_1 - t_2|M\left(2 + \left|\frac{\alpha\eta}{1-\alpha\eta}\right| + \left|\frac{1}{1-\alpha\eta}\right|\right).
\end{aligned}$$

Thus, $\|Ky_n(t_1) - Ky_n(t_2)\|_\infty < \epsilon$ for $|t_1 - t_2| < \delta$ with a small enough $\delta > 0$, i.e. $(Ky_n)_{n\in\mathbb{N}}$ is equicontinuous. By the Arzelà-Ascoli Theorem, $(Ky_n)_{n\in\mathbb{N}}$ has an in $C[0,1]$ convergent subsequence. Using the same argument for the sequence of their derivatives, we conclude that $(Ky_n)_{n\in\mathbb{N}}$ has an in X convergent subsequence. This shows that K is compact. □

In particular, Lemma 4.10 shows that the linear version of our equation (4.1), where $\mu = 0$, has a unique solution for every $y \in L^1[0,1]$, if only $\alpha\eta \neq 1$. It should be noted that the biggest difficulty in the proof of Lemma 4.10 lies in actually finding the explicit representation of L^{-1}, which we simply presented as given.
However, we want to focus on the question how one can deduce the existence of solutions to the nonlinear problem (4.1) from this. Using only some basic properties of the FMV-spectrum, this can be achieved by a simple growth condition on f.

Theorem 4.11 *Let $\alpha\eta \neq 1$. Suppose $|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t)$ for suitable functions $p, q, r \in L^1[0,1]$. Then the boundary value problem (4.1) has a solution for every function $y \in L^1[0,1]$, provided that*

$$|\mu|(\|p\|_1 + \|q\|_1) < \frac{1}{c(\alpha,\eta)}.$$

Proof: Define an operator $F : X \to Y$ by

$$F(x)(t) = f(t, x(t), \dot{x}(t)).$$

Clearly, F maps bounded sets into bounded sets. Hence, $L^{-1}F : X \to X$ is compact, i.e. $[L^{-1}F]_A = 0$. For the seminorm $p_{AQ}(L^{-1}F)$ we get the estimate

$$\begin{aligned} p_{AQ}(L^{-1}F) &= [L^{-1}F]_Q \leq \|L^{-1}\| \, [F]_Q \\ &\leq c(\alpha,\eta) \limsup_{\|x\|_X \to \infty} \frac{\|p\|_1\|x\|_\infty + \|q\|_1\|\dot{x}\|_\infty + \|r\|_1}{\|x\|_X} \\ &\leq c(\alpha,\eta) \limsup_{\|x\|_X \to \infty} \frac{(\|p\|_1 + \|q\|_1)\|x\|_X + \|r\|_1}{\|x\|_X} \\ &\leq c(\alpha,\eta) \, (\|p\|_1 + \|q\|_1) \\ &< \frac{1}{|\mu|}. \end{aligned}$$

By Corollary 3.22, $1/\mu \notin \sigma_{FMV}(L^{-1}F)$. Hence, $1/\mu - L^{-1}F$ is FMV-regular and, in particular, surjective. Thus, there exists an $x \in X$ such that

$$\frac{1}{\mu}x - L^{-1}F(x) = \frac{1}{\mu}L^{-1}y,$$

which in particular implies $x \in D(L)$.
Applying the operator μL we therefore get $Lx - \mu F(x) = y$, i.e.

$$\ddot{x}(t) = \mu f(t, x(t), \dot{x}(t)) + y(t).$$

□

We now turn to ordinary differential equations in finite dimensional spaces. The following theorem extends a well known result of the linear theory, as we will explain afterwards.

Theorem 4.12 *Let* $T \in \mathcal{C}(\mathbb{K}^n)$*. Assume that there exists a linear bijection* $A \in \mathcal{B}(\mathbb{K}^n)$ *such that* $[(A \circ T \circ A^{-1})^{\perp}]_Q < \infty$ *and* $W_{FMV}(A \circ T \circ A^{-1}) \subseteq \mathbb{K}_- = \{\lambda \in \mathbb{K} : \mathrm{Re}(\lambda) < 0\}$*. Then all solutions of the autonomous differential equation*

$$\dot{z}(t) = T(z(t))$$

are bounded as $t \to \infty$

Proof: Put $w = Az$. Then the differential equation $\dot{z} = T(z)$ becomes $\dot{w} = A \circ T \circ A^{-1}(w) = S(w)$. So it is enough to show that all solutions of the differential equation $\dot{w} = S(w)$ are bounded for $t \to \infty$. Since $[S^{\perp}]_Q < \infty$, $W_{FMV}(S)$ is compact by Theorem 3.48. Because of this and the fact that $W_{FMV}(S) \subseteq \mathbb{K}_-$, there exists a $\delta > 0$ so that $\mathrm{Re}(\lambda) < -\delta$ for all $\lambda \in W_{FMV}(S)$. Now choose an $\epsilon > 0$ with $\epsilon < -\mathrm{Re}(\lambda) - \delta$ for all $\lambda \in W_{FMV}(S)$.
Using the representation (3.1) for the numerical range of S, we see that

$$W_{FMV}(S) = \bigcap_{n=1}^{\infty} \overline{\phi_S(H \backslash B_n(0))} = \lim_{N \to \infty} \bigcap_{n=1}^{N} \overline{\phi_S(H \backslash B_n(0))}.$$

This implies $\sup_{\lambda \in W_{FMV}(S)} \left(\mathrm{dist}\left(\lambda, \bigcap_{n=1}^{N} \overline{\phi_S(H \backslash B_n(0))}\right)\right) < \epsilon$ for N sufficiently large. In particular, $\sup_{\lambda \in W_{FMV}(S)} |\mathrm{Re}(\phi_S(w)) - \mathrm{Re}(\lambda)| < \epsilon$ for $\|w\| > N$, which in turn implies that $\mathrm{Re}(\langle S(w), w \rangle) < -\delta \|w\|^2$ whenever $\|w\| > N$.
Finally, let $w(\cdot)$ be a solution of $\dot{w}(t) = S(w(t))$. Because of the above considerations we get

$$\frac{d}{dt}\|w(t)\|^2 = \langle \dot{w}(t), w(t) \rangle + \langle w(t), \dot{w}(t) \rangle = 2\,\mathrm{Re}\langle S(w(t)), w(t) \rangle < -2\delta \|w(t)\|^2 < 0$$

if $\|w(t)\| > N$. This implies that $w(t)$ is bounded. □

Note that the condition $[(A \circ T \circ A^{-1})^{\perp}]_Q < \infty$ is in particular met if $[T]_Q < 0$. Because in this case we can set $y = A^{-1}x$ and get

$$\begin{aligned} \frac{[A \circ T \circ A^{-1}]_Q}{\|A^{-1}\|} &= \frac{1}{\|A^{-1}\|} \limsup_{\|x\| \to \infty} \frac{\|(A \circ T \circ A^{-1})(x)\|}{\|x\|} = \limsup_{\|y\| \to \infty} \frac{\|(A \circ T)(y)\|}{\|A^{-1}\| \|Ay\|} \\ &\leq \limsup_{\|y\| \to \infty} \frac{\|A\| \, \|T(y)\|}{\|y\|} = \|A\| [T]_Q < \infty, \end{aligned}$$

since $\|x\| \to \infty$ if and only if $\|A^{-1}x\| \to \infty$. Lemma 3.45,(iv) then gives $[(A \circ T \circ A^{-1})^{\perp}]_Q \leq [A \circ T \circ A^{-1}]_Q \leq \|A^{-1}\| \|A\| [T]_Q < \infty$.
If T is linear, it is a well known fact that the condition $\sigma(T) \subseteq \mathbb{K}_-$ implies that every solution to the differential equation $\dot{z} = Tz$ is bounded. But $\sigma(T) \subseteq \mathbb{K}_-$ does not necessarily imply that $W(T) \subseteq \mathbb{K}_-$. However, it can be shown that there exists a linear ismorphism A, such that $W_{FMV}(A \circ T \circ A^{-1}) = \overline{W(A \circ T \circ A^{-1})} \subseteq \mathbb{K}_-$. Since also $[T]_Q < \infty$, this shows that Theorem 4.12 is indeed an extension of this assertion about linear differential equations.

4.3 The nonlinear Fredholm alternative

The Fredholm alternative is an important and very useful theorem in the theory of linear operators on a Banach space X. Its proof is in essence contained in Corollary 3.42.

Theorem 4.13 (Fredholm alternative) *Let $T \in \mathcal{B}(X)$ be compact and $0 \neq \lambda \in \mathbb{K}$. Then either*

- *$\lambda \in \sigma(T)$, i.e. λ is an eigenvalue of T,*

or

- *$\lambda I - T$ is bijective.*

Proof: If λ is not an eigenvalue of T, then $\lambda \in \rho(T)$ by Corollary 3.42. Hence $\lambda I - T$ is bijective. □

The interesting implication here is, of course, that if λ is not an eigenvalue, then the equation $\lambda x - Tx = y$ has a unique solution for each $y \in X$.
In our theory the role of bijectivity has been taken over by FMV-regularity. The property of FMV-regular operators, which can be used to guarantee solutions to certain equations, is stable solvability. This motivates the following definition.

Definition 4.14 *An operator $T \in \mathcal{C}(X)$ is called alternative if*

$$\sigma_{FMV}(T) = \sigma_\pi(T).$$

Again, the interesting implication here is that if $\lambda \notin \sigma_\pi(T)$, then $\lambda I - T$ is FMV-regular and, in particular, stably solvable. In order to assert that $\lambda \notin \sigma_\pi(T)$, we have to check that both $\lambda \notin \sigma_q(T)$ and $\lambda \notin \sigma_a(T)$.
The first of these conditions is the one that is easier to handle. By definition, $\lambda \in \sigma_q(T)$ if and only if λ is an asymptotic eigenvalue. For $\lambda \in \sigma_q(T)$, we have to check whether there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\|x_n\| \to \infty$ such that

$$\lim_{n\to\infty} \frac{\|\lambda x_n - T(x_n)\|}{\|x_n\|} = 0.$$

Furthermore, according to Proposition 3.47,(vi), $\sigma_q(T) \subseteq W_{FMV}(T)$. Thus, the implication

$$\lambda \notin W_{FMV}(T) \Rightarrow \lambda \notin \sigma_q(T)$$

holds true and gives us a further tool to see whether $\lambda \in \sigma_q(T)$ or not.
The set $\sigma_a(T)$ is more complicated to handle. It is difficult to calculate $[T]_a$, as even the calculation of the quotient $\alpha(T(M))/\alpha(M)$ for fixed M can be very challenging. We give some conditions which indicate that $\lambda \notin \sigma_a(T)$.

Lemma 4.15 *Let $T \in \mathcal{C}(X)$ fulfil $[T]_a > 0$. Then the following holds true* *(i) There is a $k > 0$ such that $\alpha(T(M)) \geq k\alpha(M)$ for all bounded sets $M \subseteq X$.*

(ii) T is proper on closed bounded sets.

(iii) If $(x_n)_{n\in\mathbb{N}}$ is bounded and $(T(x_n))_{n\in\mathbb{N}}$ converges, then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

In particular, if $\lambda I - T$ does not meet one of these three conditions, then $\lambda \notin \sigma_a(T)$.

Proof: (i) follows directly from the definition of $[T]_a$. (ii) has been proven in Proposition 2.7. We show that $(ii) \Rightarrow (iii)$. So let $(x_n)_{n\in\mathbb{N}}$ be bounded by $R > 0$ and assume that $(T(x_n))_{n\in\mathbb{N}}$ converges to some $y \in X$. Then the set $\{T(x_n) : n \in \mathbb{N}\} \cup \{y\}$ is compact. Since T is proper on $\overline{B_R(0)}$ by assumption, the preimage of $\{T(x_n) : n \in \mathbb{N}\} \cup \{y\}$ is also compact. In particular, $(x_n)_{n\in\mathbb{N}}$ lies in a compact set and, therefore, has a convergent subsequence. □

Furthermore, we want to identify some classes of alternative operators for which there is no need to be concerned with the subspectrum $\sigma_a(T)$ at all.

Proposition 4.16 *Let $T \in \mathcal{C}(X)$ be alternative. Assume that one of the conditions*

(i) X is finite dimensional,

(ii) T is compact,

(iii) T is asymptotically linear with asymptotic derivate T' such that $T - T'$ is compact,

holds true. Then $\sigma_{FMV}(T) = \sigma_q(T)$. In particular, for any $\lambda \in \mathbb{K}$ either

- *$\lambda \in \sigma_q(T)$, i.e. λ is an asymptotic eigenvalue of T,*

or

- *$\lambda I - T$ is FMV-regular.*

Proof: Since T is alternative we have $\sigma_{FMV}(T) = \sigma_\pi(T) = \sigma_q(T) \cup \sigma_a(T)$. Thus, we have to show that $\sigma_a(T) \subseteq \sigma_q(T)$.
If X is finite dimensional then $\sigma_a(T) = \emptyset$ and we are done. If T is compact, then $\sigma_a(T) = \{0\}$ by Proposition 3.32,(i), so we have to show that $0 \in \sigma_q(T)$. But if this were not the case, then Proposition 3.32,(iii) would give

$$c_0[\sigma_q(T)] \subseteq \sigma_\delta(T) \subseteq \sigma_{FMV}(T) \subseteq \sigma_q(T) \cup \{0\},$$

i.e. $c_0[\sigma_q(T)] \subseteq \{0\}$. But this is impossible since $c_0[\sigma_q(T)]$ is both open and nonempty.
Finally, if T fulfills (iii), then Proposition 3.34,(iii) and Lemma 3.27 show that

$$\sigma_a(T) = \sigma_a(T') \subseteq \sigma_q(T') = \sigma_q(T).$$

□

Proposition 4.17 *Let $T \in \mathcal{B}(X)$ be alternative. Then $\sigma(T) = \sigma_q(T)$. In particular, either*

- $\lambda \in \sigma_q(T)$*, i.e.* λ *is an asymptotic eigenvalue of* T*,*

or

- $\lambda I - T$ *is bijective.*

Proof: By Lemma 3.27 we have $\sigma_a(T) \subseteq \sigma_q(T)$. Since T is linear and alternative, this yields $\sigma(T) = \sigma_{FMV}(T) = \sigma_q(T)$. □

Now that we have explored the usefulness of the notion of alternative operators, we want to identify some classes of operators as alternative. One such class has essentially already been identified.

Theorem 4.18 *Let $T \in \mathcal{C}(X)$ be compact and asymptotically odd. Then T is alternative.*

Proof: Theorem 3.41 shows that $\sigma_{FMV}(T) = \sigma_q(T)$. □

We will identify some other classes of alternative operators. However, we will be able to do so only in Hilbert spaces. So Theorem 4.18 is of special interest since it is also applicable in Banach spaces.
For the rest of this section we will deal with operators on a complex Hilbert space H. In Hilbert spaces, the previous theorem can be seen as a special case of a larger class of alternative operators. In order to identify this class of operators, we will use some concepts of Fredholm theory.

Definition 4.19 *Let $T \in \mathcal{B}(H)$. We define the kernel of T by*

$$\ker(T) := \{x \in H : Tx = 0\}$$

and the range of T by

$$\mathrm{ran}(T) := \{Tx : x \in H\}.$$

The operator T is called semi-Fredholm, if $\mathrm{ran}(T)$ *is closed and either* $\dim\ker(T) < \infty$ *or* $\dim(\mathrm{ran}(T))^\perp < \infty$.
For a semi-Fredholm operator the number

$$\mathrm{ind}(T) := \dim\ker(T) - \dim(\mathrm{ran}(T))^\perp \in \mathbb{Z} \cup \{\pm\infty\}$$

is called the Fredholm index of T. If $|\mathrm{ind}(T)| < \infty$, T is called Fredholm.
We denote by $\mathcal{F}(H)$ the set of all semi-Fredholm operators on H.

Out of the theory of Fredholm operators, we want to import the following two results. A proof of these theorems can for example be found in [C].

Theorem 4.20 *Let $T \in \mathcal{B}(H)$ be Fredholm. Then the following is equivalent.*

(i) $\mathrm{ind}(T) = 0$.

(ii) There exists a compact operator $K \in \mathcal{B}(H)$ such that $T + K$ is invertible.

Theorem 4.21 *Let $\mathcal{F}(H)$ be equipped with the subspace topology of $\mathcal{B}(H)$. Then the map* $\mathrm{ind} : \mathcal{F}(H) \to \mathbb{Z} \cup \{\pm\infty\}$ *that assigns to each semi-Fredholm operator its Fredholm index is continuous.*

We are now able to identify another class of alternative operators.

Definition 4.22 *For $T \in \mathcal{B}(H)$ we define the essential spectrum by*

$$\sigma_{es}(T) := \{\lambda \in \mathbb{K} : \lambda I - T \textit{ is not semi-Fredholm or } \mathrm{ind}(\lambda I - T) \neq 0\}.$$

T is called balanced if $\sigma_a(T) = \sigma_{es}(T)$.

Theorem 4.23 *Let $T \in \mathcal{B}(H)$ be balanced and $S \in \mathcal{C}(H)$ be compact and asymptotically odd. Then the operator $T + S$ is alternative.*

Proof: Choose $\lambda \notin \sigma_\pi(T + S)$. We have to show that $\lambda I - (T + S)$ is FMV-regular.
Since S is compact we get $[\lambda I - T]_a = [\lambda I - (T + S)]_a > 0$ because $\lambda \notin \sigma_a(T + S)$. Hence, $\lambda \notin \sigma_{es}(T)$, since T is balanced. This means that $\lambda I - T$ is Fredholm of index 0. By Theorem 4.20, there exists a compact operator $K \in \mathcal{B}(H)$ such that $\lambda I - (T + K) =: L$ is an isomorphism. By Theorem 3.15 it is enough to show that the operator $L^{-1} \circ (\lambda I - (T + S)) = 1 - L^{-1} \circ (S + K)$ is FMV-regular.
Since S and K are both compact, $L^{-1} \circ (S + K)$ is compact and, therefore,

$$[1 - L^{-1} \circ (S + K)]_a = 1.$$

Furthermore, we can use Proposition 2.2,(v) to show

$$[1 - L^{-1} \circ (S + K)]_q = [L^{-1} \circ (\lambda I - (T + S))]_q \geq [L^{-1}]_q[\lambda I - (T + S)]_q > 0.$$

The last inequality holds true since $\lambda \notin \sigma_q(T + S)$ and $[L^{-1}]_q > 0$ by Lemma 2.8,(iv). Now, $L^{-1} \circ (S + K)$ is not only compact, but also asymptotically odd. Thus, it is alternative by Theorem 4.18. Hence, $1 - L^{-1} \circ (S + K)$ is FMV-regular. □

We are now interested in identifying classes of balanced operators
Every linear operator T on a finite dimensional Hilbert space is balanced, since it can be easily seen that $\sigma_a(T) = \sigma_{es}(T) = \emptyset$. In infinite dimensional Hilbert spaces the operators λI for $\lambda \in \mathbb{K}$ are simple examples of balanced operators, since $\sigma_a(\lambda I) = \sigma_{es}(\lambda I) = \{\lambda\}$.
To identify more interesting examples of balanced operators the following result will be very useful.

Lemma 4.24 *Let $T \in \mathcal{B}(H)$. Then $[T]_a > 0$ if and only if* $\operatorname{ran}(T)$ *is closed and fulfils* $\dim\ker(T) < \infty$.

Proof: Suppose $[T]_a > 0$ and let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\overline{B_1(0)} \cap \ker(T)$. Then $T(x_n) = 0$ for all n. In particular $(T(x_n))_{n\in\mathbb{N}}$ converges and by Lemma 4.15,(iii), the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Consequently, the closed unit ball in the subspace $\ker(T)$ is compact. Thus, $\ker(T)$ is finite dimensional.
This in turn also implies that there exists a closed subspace H_1 of H such that $H = H_1 \oplus \ker(T)$. As $T(H) = T(H_1)$ we only need to show that $T(H_1)$ is closed. Note that the restriction $T|_{H_1} : H_1 \to T(H_1)$ is bijective. It is obviously surjective, but also injective, since $T|_{H_1}(x) = T|_{H_1}(y)$ for $x \neq y$ would imply $T|_{H_1}(x-y) = 0$, which leads to the contradiction $x - y \in \ker(T)$. Thus, $T|_{H_1}$ has an inverse.
If we can show that this inverse is bounded, i.e. $T|_{H_1}$ fulfils $\|T|_{H_1}x\| \geq m\|x\|$ for all $x \in H_1$ with $\|x\| = 1$ and some $m > 0$, then $T(H_1) = T(H)$ is closed as the preimage of a closed set under a continuous map. Assume the contrary. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H_1 with $\|x_n\| = 1$ such that $T|_{H_1}x_n \to 0$. Using Lemma 4.15,(iii), we can assume without loss of generality that $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in H_1$ with $\|x\| = 1$. Obviously $Tx = 0$, which contradicts $H_1 \cap \ker(T) = \{0\}$.
Conversely, we have to show that if $\operatorname{ran}(T)$ is closed and $\dim\ker(T) < \infty$, then $[T]_a > 0$. Since $\dim\ker(T) < \infty$, there exists a closed subspace H_1 of H such that $H = H_1 \oplus \ker(T)$. Let $P_1 : H \to H_1$ be the projection along $\ker(T)$ onto H_1 and set $P_2 = I - P_1$. Let $T|_{H_1}$ be as above.
Since $\operatorname{ran}(T)$ is closed, $T|_{H_1}$ is an isomorphism of H_1 onto $\operatorname{ran}(T) = \operatorname{ran}(T|_{H_1})$ by the open mapping theorem. Since $T = T|_{H_1} \circ P_1$ we have $[T]_a \geq [T|_{H_1}]_a[P_1]_a$ by Proposition 2.7,(vii). Since P_2 maps H into the finite dimensional space $\ker(T)$, it is compact. Thus, $[P_1]_a = [I - P_2]_a = [I]_a = 1$. Furthermore, $[T|_{H_1}]_a > 0$ by Lemma 2.8,(iv). Hence, $[T]_a \geq [T|_{H_1}]_a[P_1]_a = [T|_{H_1}]_a > 0$. □

Corollary 4.25 *For $T \in \mathcal{B}(H)$ we have $\sigma_a(T) \subseteq \sigma_{es}(T) \subseteq \sigma(T)$.*

Proof: If $\lambda \notin \sigma(T)$ then $\lambda I - T$ is bijective. So $\operatorname{ran}(\lambda I - T) = H$ is closed, $\dim\ker(\lambda I - T) = 0 < \infty$, and $\operatorname{ind}(\lambda I - T) = \dim\ker(\lambda I - T) - \dim(\operatorname{ran}(\lambda I - T))^\perp = 0$, i.e. $\lambda \notin \sigma_{es}(T)$. Hence, $\sigma_{es}(T) \subseteq \sigma(T)$.
If $\lambda \notin \sigma_{es}(T)$ then in particular $\operatorname{ran}(\lambda I - T)$ is closed and $\dim\ker(\lambda I - T) < \infty$. By Lemma 4.24 this implies $\lambda \notin \sigma_a(T)$. Hence, $\sigma_a(T) \subseteq \sigma_{es}(T)$. □

In order to exploit the properties of Hilbert spaces to a fuller extent, we introduce the notion of adjoint operators.

Definition 4.26 *Let $T \in \mathcal{B}(H)$. By T^* we denote the bounded linear operator that is uniquely defined by the equation*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x, y \in H.$$

T^* is called the adjoint operator of T.
T is called normal, if $TT^* = T^*T$, and selfadjoint, if $T = T^*$.

We will use the following well known results about adjoint operators.

Theorem 4.27 *Let $T \in \mathcal{B}(H)$.*

(i) T is compact if and only if T^* is compact.

(ii) If T is normal, then $\ker(T^*) = (\mathrm{ran}(T))^\perp$.

(iii) If T is normal, then $H = \ker(T) \oplus \overline{\mathrm{ran}(T)}$.

(iv) T is selfadjoint if an only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

Proof: *See [R2] and [W].* □

Using Theorem 4.27, we are able to identify some classes of balanced operators.

Proposition 4.28 *Let $T \in \mathcal{B}(H)$ fulfil one of the following conditions.*

(i) $r(T) = 0$.

(ii) There is an $n \in \mathbb{N}$ such that T^n is compact.

(iii) T is normal.

Then T is balanced.

Proof: Since every linear operator on a finite dimensional Hilbert space is balanced, we may assume that H is infinite dimensional.
(i): Using Proposition 2.7,(vii) and Lemma 2.8,(ii) we get

$$[T]_a = \sqrt[n]{[T]_a^n} \leq \sqrt[n]{[T^n]_a} \leq \sqrt[n]{[T^n]_A} \leq \sqrt[n]{\|T^n\|} \to r(T) = 0.$$

Thus, $0 \in \sigma_a(T)$. Corollary 4.25 shows that $\{0\} \subseteq \sigma_a(T) \subseteq \sigma_{es}(T) \subseteq \sigma(T) = \{0\}$. Hence, $\sigma_a(T) = \sigma_{es}(T)$.
(ii): Let n be the smallest integer such that T^n is compact. As above, we get $[T]_a^n \leq [T^n]_a \leq [T^n]_A = 0$ and thus $0 \in \sigma_a(T)$. Since $\sigma_a(T) \subseteq \sigma_{es}(T)$ by Corollary 4.25, we are finished once we show that $\lambda \neq 0$ implies $\lambda \notin \sigma_{es}(T)$.
If $n > 1$, we can write $\lambda^n I - T^n = A \circ (\lambda I - T)$, where

$$A = \lambda^{n-1} I + \lambda^{n-2} T + \cdots + T^{n-1}.$$

So if $\lambda \neq 0$, we get $0 < |\lambda|^n = [\lambda^n I - T^n]_a \leq [A]_A [\lambda - T]_a$ by Proposition 2.7,(vii). This yields $[\lambda I - T]_a > 0$, which is also true if $n = 1$. Since T^{*n} is also compact, we get $[\overline{\lambda} I - T^*]_a > 0$.

By Lemma 4.24 and Theorem 4.27,(ii), $\operatorname{ran}(\lambda I - T)$ is closed and both $\dim\ker(\lambda I - T)$ and $\dim(\operatorname{ran}(\lambda I - T))^{\perp} = \dim\ker(\overline{\lambda} I - T^*)$ are finite. Thus, $\lambda I - T$ is Fredholm.
Since $(cT)^n$ is compact for any $c \in \mathbb{R}$, we get that $\lambda I - cT$ is Fredholm for any $c \in \mathbb{R}$ and $\lambda \neq 0$. Theorem 4.21 tells us that the Fredholm index is continuous. Hence, $\operatorname{ind}(\lambda I - T) = \operatorname{ind}(\lambda I) = 0$ and $\lambda \notin \sigma_{es}(T)$.
(iii): Let $\lambda \notin \sigma_a(T)$. By Lemma 4.24, $\dim\ker(\lambda I - T) < \infty$ and $\operatorname{ran}(\lambda I - T)$ is closed. As $\lambda I - T$ is normal by assumption, Theorem 4.27,(iii) gives $H = \ker(\lambda I - T) \oplus \operatorname{ran}(\lambda I - T)$. For the Fredholm index we get

$$\operatorname{ind}(\lambda I - T) = \dim\ker(\lambda I - T) - \dim(\operatorname{ran}(\lambda I - T))^{\perp} = \dim\ker(\lambda I - T) - \dim\ker(\lambda I - T) = 0.$$

So we have $\sigma_{es}(T) \subseteq \sigma_a(T)$ and by Corollary 4.25, $\sigma_{es}(T) = \sigma_a(T)$. □

Next, we want to show a third class of alternative operators. Since every linear selfadjoint operator is obviously normal, the previous theorem shows that they are alternative. We can, in a certain manner, extend the notion of selfadjoint operators to nonlinear operators.

Definition 4.29 *An operator $T \in \mathcal{C}(H)$ is called selfadjoint, if $\langle T(x), x\rangle \in \mathbb{R}$ for all $x \in H$. By $\mathcal{SA}(H)$ we denote the set of all selfadjoint operators on H.*

By Theorem 4.27,(iv), this coincides with the usual definition of selfadjoint linear operators. Note that the equation $\langle T(x), y\rangle = \langle x, T(y)\rangle$ can not be used for nonlinear operators, as its validity for all $x, y \in H$ already implies the linearity of T.
We will now give a condition under which selfadjoint operators are alternative.

Lemma 4.30 *If $T \in \mathcal{SA}(H)$, then $W_{FMV}(T) \subseteq \mathbb{R}$*

Proof: For $\lambda \in W_{FMV}(T) = \sigma_q(T^{\perp})$ we have

$$0 = \liminf_{\|x\|\to\infty} \frac{\left\|\lambda x - \frac{\langle T(x),x\rangle}{\|x\|^2} x\right\|}{\|x\|} = \liminf_{\|x\|\to\infty} |\lambda - \frac{\langle T(x), x\rangle}{\|x\|^2}|.$$

Since $\frac{\langle T(x),x\rangle}{\|x\|^2} \subseteq \mathbb{R}$, λ must also lie in $\mathbb{R}$. □

Theorem 4.31 *Let $T \in \mathcal{SA}(H)$. Suppose T fulfils the following two conditions.*

(i) $\sigma_a(T) \subseteq \mathbb{R}$.

(ii) There exists a λ_1 with $\operatorname{Im}\lambda_1 < 0$ and a λ_2 with $\operatorname{Im}\lambda_2 > 0$ such that $\lambda_1 I - T$ and $\lambda_2 I - T$ are stably solvable.

Then T is alternative.

Proof: Because T is selfadjoint, we have $W_{FMV}(T) \subseteq \mathbb{R}$ by Lemma 4.30. Therefore, $\sigma_\pi(T) = \sigma_q(T) \cup \sigma_a(T) \subseteq W_{FMV}(T) \cup \sigma_a(T) \subseteq \mathbb{R}$ by Proposition 3.47,(vi). Since $\partial\sigma_{FMV}(T) \subseteq \sigma_\pi(T) \subseteq \mathbb{R}$ by Theorem 3.30, this leaves only four possibilites: $\sigma_{FMV}(T) = \mathbb{C}$, $\sigma_{FMV}(T) = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq 0\}$, $\sigma_{FMV}(T) = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \leq 0\}$, and $\sigma_{FMV}(T) \subseteq \mathbb{R}$. The first three cases are impossible by assumption (ii). Thus, $\sigma_{FMV}(T) \subseteq \mathbb{R}$. Since $\mathbb{R}$ has no interior points in $\mathbb{C}$, every point of the FMV-spectrum must lie in its boundary. Hence, $\sigma_{FMV}(T) \subseteq \partial\sigma_{FMV}(T) \subseteq \sigma_\pi(T)$. □

The final two results give special cases in which the conditions of the previous theorem are met.

Proposition 4.32 *Let $T \in \mathcal{SA}(H)$. Suppose T fulfils one of the following two conditions.*

(i) T is assymptotically linear with asymptotic derivate T' and $T - T'$ is compact.

(ii) $T = R + S$ where $R \in \mathcal{B}(H)$ is selfadjoint and $S \in \mathcal{C}(H)$ is compact.

Then T fulfils condition (i) in Theorem 4.31.

Proof: In the first case, we can use Proposition 3.34,(iii), Lemma 3.27, Proposition 3.47,(vi), and Lemma 4.30 to see that

$$\sigma_a(T) = \sigma_a(T') \subseteq \sigma_q(T') = \sigma_q(T) \subseteq W_{FMV}(T) \subseteq \mathbb{R}.$$

In the second case, we get $\sigma_a(T) = \sigma_a(R + S) = \sigma_a(R)$ since S is compact. Using the linearity of R we get

$$\sigma_a(T) = \sigma_a(R) \subseteq \sigma(R) \subseteq \overline{W(R)} \subseteq \mathbb{R}.$$

□

Proposition 4.33 *Let $T \in \mathcal{SA}(H)$. Suppose T fulfils one of the following two conditions.*

(i) There exists a $\lambda \in \mathbb{R}$ such that $\lambda I - T$ is FMV-regular.

(ii) $\max\{[T]_A, [T^\perp]_Q\} < \infty$.

Then T fulfils condition (ii) in Theorem 4.31.

Proof: In the first case, we can use the fact that $\rho_{FMV}(T)$ is open to see that there exists an $\epsilon > 0$ such that $B_{2\epsilon}(\lambda) \subseteq \rho_{FMV}(T)$. Hence, $\lambda \pm i\epsilon$ are FMV-regular.
In the second case, Theorem 3.49 shows that $\sigma_{FMV}(T)$ is bounded. This obviously implies the assumption. □

Bibliography

[APV] J.Appell, E. De Pascale, A.Vignoli: *Nonlinear Spectral Theory*, Walter de Gruyter, 2004

[C] J.B.Conway: *A Course in Functional Analysis*, Springer, 1985

[D] J.Dugundji: *An extension of Tietze's theorem*, Pacific J. Math. 1 (1951), 333-367

[FW] W.Feng, J.R.L.Webb: *Surjectivity results for nonlinear mappings without oddness conditions*, Comment. Math. Univ. Carolinae 38,1 (1997), 15-28

[FMV] M.Furi, M.Martelli, A.Vignoli: *Contributions to spectral theory for nonlinear operators in Banach spaces*, Ann. Mat. Pura Appl. 118 (1978), 229-294

[MA] J.Matoušek: *Using the Borsuk-Ulam theorem*, Springer, 2003

[MI] E.Michael: *Continuous selections I*, Ann. of Math. 63 (1956), 361-382

[Q] B.v.Querenburg: *Mengentheoretische Topologie*, Springer, 1973

[R1] W.Rudin: *Real and Complex Analysis*, Third Edition, McGraw-Hill, Inc., 1987

[R2] W.Rudin: *Functional Analysis*, Second Edition, McGraw-Hill, Inc., 1991

[R3] W.Rudin: *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, Inc., 1976

[ST] K.Schmitt, R.C.Thompson: *Nonlinear Analysis and Differential Equations - An Introduction*, http://www.math.utah.edu/~schmitt/ode1.pdf

[W] D.Werner: *Funktionalanalysis*, 6., korrigierte Auflage, Springer, 2007

[Z] E.Zeidler: *Nonlinear Functional Analysis and its Applications I (Fixed-Point Theorems)*, Springer, 1986