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Generalisations of Semigroups of Operators in the View of Linear Relations

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Preface

The theory of semigroups of operators was established by Kōsaku Yosida and Einar Hille in the 1940ies. Such a semigroup is a function T defined on the right half axis $[0, \infty)$ with values in the Banach algebra of bounded linear operators on a Banach space X, which fulfils

$$T(0) = \mathrm{I}, \tag{1}$$
$$T(s+t) = T(s)T(t) \qquad \text{for all } s, t \geqslant 0 \tag{2}$$

and where I denotes the identity operator on X. In addition, T is assumed to be strongly continuous, that is, the function $T(.)x$ is continuous for every $x \in X$. Then, the so-called *infinitesimal generator* A is studied and the famous *Hille-Yosida theorem* states which properties a closed, densely defined operator needs to have to be the generator of a semigroup. The pure theory as well as its application has been investigated intensively from that time on. Semigroups are used in many fields, ranging from evolution equations over stochastics to control theory.
Beside this, there have been different developments in generalising the classic situation described above. For example, an adaption to non-linear operators $T(t)$ has been discussed by Isao Miyadera in [Miy92]. Another idea, first mentioned by Giuseppe da Prato in 1966, is an algebraic generalisation: In [DP66] the operator $T(0)$ only has to be injective and furthermore, the semigroup property (2) is weakened:

$$T(0)T(s+t) = T(s)T(t) \qquad s, t \geqslant 0.$$

Independently, E. Davies and M. Pang introduced a similar notion in 1980. There exist different names for this generalised form, in this work, the term *Pre-semigroup* will be used. The generator is defined analogously as in the common case, using the injectivity of $T(0)$ additionally. Also due to injectivity, many results from semigroup theory can be obtained analogously.

The goal of this Master's thesis is to weaken the notion of a Pre-semigroup even more. We are going to neglect any further assumption on the linear operator $T(0)$. Such functions of operators will be called *G-Semigroups*. As a consequence of the loss of injectivity of $T(0)$, it will be shown that the generator does not need to be a single-valued operator any more. At this point, the theory of *Linear Relations* comes into play. In 1961, Richard Arens started the discussion of linear subsets in a product space $X \times Y$ in the context of Functional Analysis, see [Are61]. By identification with its graph, every operator from X to Y can be interpreted as a linear relation. As for operators, operations like the sum and product with a scalar can be defined. Moreover, the resolvent set and the spectrum of a linear relation, consistent with the notions for operators, are introduced. Using this fundament, it is possible to obtain generalised results of classic semigroup theory for G-semigroups.

In another context, linear relations have already been used in connection with semigroups of operators. We mention [Bas08],[Bas04], where a semigroup is only defined on the open half line $(0, \infty)$ and (2) only holds for $s, t > 0$. The theory of linear relations is used to define and investigate proper generators for such objects. Furthermore, also in the investigation of nonlinear semigroups, [Miy92], the notion of a *multi-valued operator* occurs. Such an object is precisely a linear relation.

This work has the theory of Pre-semigroups as its basis. An extensive collection of results for this generalisation is the book by Ralph deLaubenfels, [deL94]. Nevertheless, a brief introduction is given in *Preliminaries*, Chapter 0. Moreover, basic notation and fundamental results from Functional Analysis that we are going to need are also stated there. A rather detailed guide about linear relations is provided, since the reader might not be familiar with this field and because it will be used heavily throughout the work.

Chapter 1 is the main part of this Master's thesis. In section 1.1, the basic theory about G-semigroups is established, in particular, the definition and the properties of the generator. Since the idea is not only to generalise the situation, but also to have a different point of view on known issues, some results on semigroup and Pre-semigroup theory are stated as corollaries. Section 1.2 deals with a class of G-semigroups that can be constructed from other G-semigroups by multiplying a commuting operator C. This immediately gives a broad class of examples. In sections 1.3, 1.4 and 1.5 the focus is on a special type of G-semigroups. Degenerate G-semigroups can be seen as semigroups where $T(0)$ is a projection instead of the identity. Decomposing the underlying space, such G-semigroups can be understand as the *product* of a classic semigroup and the trivial G-semigroup $P \equiv 0$. *Products* of G-semigroups on Banach spaces X, Y are discussed in general in section 1.4. Finally, we conclude chapter 1 by considering the quotient space $X/\ker P$, where $\ker P$ denotes the intersection of the kernels of the operators $P(s)$.

Contrary to semigroups, exponential boundedness is not guaranteed for G-semigroups. In chapter 2, this is assumed in addition and therefore, it is ensured that the Laplace transform $L_\lambda x$ of $P(.)x$ can be defined for suitable λ. Similar to Pre-semigroup theory, this leads to the definition of a generalised resolvent set.

The last chapter gives an introduction to the *Differential Inclusion*

$$(u, \dot{u}) \in A, \qquad u(0) = x,$$

where A is a linear relation. This is the corresponding generalisation of the Abstract Cauchy Problem in the view of linear relations. First, fundamental terminology and properties are stated. The introduction of *existence families* is motivated by the theory of Pre-semigroups done in [deL94]. Finally, the connection to G-semigroups is made.

At this point I would like to thank Prof. Dr. Michael Kaltenbäck who introduced me to the theory of Semigroups of Operators in his Functional Analysis lectures in my third year of Bachelor studies. From that time on, this field has fascinated me and I have enjoyed deepening my knowledge in it. This experience has been very helpful during the work on this Master's thesis.
Furthermore, I would like to say thank you to everyone who has encouraged me in my studies during the last five years. In particular, I want to express my deepest gratitude to my parents who have always believed in me and supported all my interests.

Felix Schwenninger Vienna, 11th May 2011

Chapter 0

Preliminaries

Let us introduce some basic results and notation which will be used throughout this work. Furthermore, a brief collection of facts in the theory of semigroups of operators will be made as well as a 'not to short' introduction to linear relations. We start with an overview of fundamental things we will need from Functional Analysis.

0.1 Notation and Basic Results

For proofs and further information we refer to any basic book about Functional Analysis, like [Wer00],[WK08], [Yos95], [Kat95]. Throughout the entire thesis, X will denote a Banach space over $\mathbb{C}$, normed by $\|.\|$. Furthermore, operators will always be linear mappings, however we use the term *linear operator* sometimes. The space of linear operators $L : \operatorname{dom} L \subseteq X \to Y$ for a Banach space Y will be denoted by $\mathcal{L}(X, Y)$ and we shall write $\mathcal{L}(X)$ in case of $X = Y$. Here, $\operatorname{dom} L \subseteq X$ is the domain and $\operatorname{ran} L \subseteq Y$ denotes the range of the operator $L \in \mathcal{L}(X, Y)$. $\mathcal{B}(X, Y)$ will be the Banach space of bounded operators $B : X \to Y$ in $\mathcal{L}(X, Y)$ and will be equipped with the operator norm which also shall be denoted by $\|.\|$. Analogously, $\mathcal{B}(X)$ will refer to the situation where $X = Y$. Furthermore, $\mathrm{I} : X \to X$ will denote the identity operator on X. An operator will be **closed**, iff its graph is closed in the product topology on $X \times Y$(for more see subsection 0.4). $L \in \mathcal{L}(X, Y)$ is said to be densely defined, iff the domain of L is dense in X. As usual, for a closed operator $L \in \mathcal{L}(X)$ one can define the **resolvent set** $\rho(L)$ and the **spectrum** $\sigma(L)$ as

$$\rho(L) = \left\{\lambda \in \mathbb{C} : (\lambda\,\mathrm{I} - L) \text{ is invertible and } (\lambda\,\mathrm{I} - L)^{-1} \in \mathcal{B}(X)\right\},$$
$$\sigma(L) = \mathbb{C} \setminus \rho(L).$$

If the **resolvent** $R(.)$ is defined as $R(\lambda) := (\lambda\,\mathrm{I} - L)^{-1}$ for $\lambda \in \rho(L)$, then the mapping $R : \rho(L) \to \mathcal{B}(X), \lambda \mapsto R(\lambda)$, is holomorphic and fulfils the resolvent identity

$$R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$$

for $\lambda, \mu \in \rho(L)$. Using the closedness of L, the resolvent set can be written equivalently as

$$\rho(L) = \{\lambda \in \mathbb{C} : (\lambda - L) : \operatorname{dom} L \to X \text{ is bijective}\}.$$

This follows from the following fundamental

THEOREM 0.1 ***Closed Graph Theorem***. *For Banach spaces X, Y and an operator L with closed domain,*

L is continuous iff it is closed.

Another important theorem which we are going to need is the

THEOREM 0.2 ***Principle of Uniform Boundedness***. *Let X, Y be Banach spaces and $\{B_i : i \in I\}$ be a family of operators $B_i \in \mathcal{B}(X, Y)$. If the family is bounded pointwisely, i.e. if for any $x \in X$ there exists a constant $M_x \geqslant 0$ such that*

$$\sup_{i \in I} \|B_i x\| \leqslant M_x \|x\|,$$

then the family is bounded uniformly,

$$\sup_{i \in I} \|B_i\| \leqslant M < \infty,$$

for a bound $M \geqslant 0$.

We turn now to some facts about integration and differentiation of Banach space valued functions. For details the reader is referred to [Kal08b], [Kal08a] for instance.

DEFINITION 0.3 *For an interval $I \subseteq \mathbb{R}$, the space $C(I, X)$ refers to the space of continuous functions $f : I \to X$. For compact I, $C(I, X)$ is a Banach space equipped with the supremum norm, $\|f\|_\infty := \sup_{t \in I} \|f(t)\|$.*

DEFINITION 0.4 *Let $f : I \to X$ be a function and t be in the interior of the interval I. If the limit*

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

exists, f is called ***strongly differentiable*** *and the limit is called the* ***strong derivative*** *at t, $f'(t)$. The notions of* ***strongly right/left differentiable*** *and* ***strong right/left derivative****, $f'^{+}(t)$, $f'^{-}(t)$, are defined in analogy to the common theory. If t is a boundary point ($\neq \pm\infty$) of the interval, we will*

identify $f'(t)$ with its strong right or left derivative, respectively. f is called ***continuously differentiable****, if f is strongly differentiable and the strong derivative $f' : I \to X$ is continuous.*
The space of continuously differentiable functions is denoted by $C^1(I, X)$.

REMARK 0.5 Similarly as in the case for $\mathbb{R}$-valued functions one sees that $f' \equiv 0$ on I implies that f is constant (see [Kal08b]). Later, we will see that $C^1(I, X)$ equipped with the norm $\|f\|_{C^1} := \|f(t)\|_\infty + \|f'(t)\|_\infty$ is a Banach space if I is a compact interval (see Remark 0.36).

In the following let a, b be real numbers with $a < b$. Analogously to real-valued functions, the **Riemann integral** of a function $f : [a, b] \to X$ can be defined by using Riemann sums,

$$\int_a^b f(s)\ ds.$$

This integral is then an element in X. Basically, many proofs and results for the $\mathbb{R}$-valued case are transformed to this situation by replacing the modulus by the norm on X. See [Kal08b] for details. In particular, it follows completely analogously that continuous functions $f : I = [a, b] \to X$ are integrable. This implies that also $\int f$ can be defined,

$$\int f : \begin{cases} [a, b] \to X, \\ t \mapsto \int_a^t f(s)ds, \end{cases}$$

and the Fundamental Theorem of Calculus holds. The following version is sufficient for our application.

THEOREM 0.6 ***Fundamental theorem of Calculus****. Let $f : [a, b] \to X$ be a continuous function. Then, $F = \int f \in C^1([a, b], X)$ is an antiderivative of f, i.e.*

$$F'(s) = f(s),$$

for all $s \in [a, b]$.

Another important property of this Banach space valued integral is that for $B \in \mathcal{B}(X)$

$$B \int_a^b f(s)\ ds = \int_a^b Bf(s)\ ds,$$

which follows directly from the definition of the integral as limit of Riemann sums. Furthermore,

$$\left\| \int_a^b f(s)\ ds \right\| \leqslant \int_a^b \|f(s)\|\ ds, \tag{0.1}$$

holds, where the integral on the right hand side denotes a classic Riemann integral for $\mathbb{R}$-valued functions. Next, we discuss the existence of **improper integrals** in this context. Let $f : [a, \infty) \to X$ be continuous. As usual, $\int_a^\infty f(s)ds$ is interpreted as

$$\lim_{\beta\to\infty} \int_a^\beta f(s)\ ds, \tag{0.2}$$

if this limit exists with respect to $\|.\|$. By (0.1) the existence of the limit

$$\lim_{\beta\to\infty} \int_a^\beta \|f(s)\|\ ds = \int_a^\infty \|f(s)\|\ ds,$$

in $\mathbb{R}$ yields a sufficient condition for the convergence of the limit in (0.2).

We want to finish this subsection by defining some special function spaces that we will need. For proofs see for instance [Sch09]

- $C_0(\mathbb{R})$ denotes the space of complex-valued continuous functions on $\mathbb{R}$ satisfying $\lim_{t\to\pm\infty} f(t) = 0$. $C_0(\mathbb{R})$ becomes a Banach space if provided with $\|.\|_\infty$.
- $C_{00}(\mathbb{R})$ is the space of continuous $f : \mathbb{R} \to \mathbb{C}$ with compact support. Again, we consider the norm $\|.\|_\infty$. It can be seen easily that $C_{00}(\mathbb{R})$ is densely contained in $C_0(\mathbb{R})$.

0.2 Classic Semigroups

We briefly state basic information about classic semigroup theory. The proof of following theorems and lemmata can be found in every book containing an introduction to semigroup theory. In this context we mention the books by Engel and Nagel [EN00] and the one by Jerome Goldstein [Gol85].

DEFINITION 0.7 *An operator valued function $T : [0, \infty) \to \mathcal{B}(X)$ is called a **(classic) semigroup**, if following conditions are satisfied*

1. *T is strongly continuous at 0,*
2. *$T(0) = \mathrm{I}$,*
3. *$T(t + s) = T(t)T(s)$ for all $s, t \geqslant 0$.*

The definition directly implies characteristic properties of the semigroup,

LEMMA 0.8 *A semigroup T is strongly continuous on $[0, \infty)$ and there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leqslant Me^{\omega t} \tag{0.3}$$

for every $t \geqslant 0$. A characteristic object related to a semigroup is its *generator.*

DEFINITION 0.9 *For a semigroup T, one defines the* ***infinitesimal generator*** A_0 *as the linear operator acting on*

$$\operatorname{dom} A_0 = \left\{x \in X : [P(.)x]'^{+}(0) \text{ exists}\right\},$$

and defined as

$$A_0 x = [P(.)x]'^{+}(0).$$

In general, this linear operator A_0 is not bounded.

LEMMA 0.10 *The following properties hold true for A_0 as generator of the semigroup T satisfying (0.3):*

- A_0 *is closed and densely defined;*
- *for $x \in \operatorname{dom} A_0$ it holds that $P(t)x \in \operatorname{dom} A_0$ and*

  $$P(t)A_0 x = P(t)A_0 x \qquad \forall t \geqslant 0;$$

- *for $x \in \operatorname{dom} A_0$ the function $T(.)x$ is continuously differentiable and*

  $$[T(.)x]'(s) = A_0 T(s)x \qquad \forall s \geqslant 0;$$

- $(\omega, \infty) \subset \rho(A_0)$ *and the resolvent can be written via the Laplace tranform*

  $$R(\lambda) = (\lambda - A_0)^{-1} x = \int_0^\infty e^{-\lambda s} T(s)x \, ds, \tag{0.4}$$

  for all $x \in X$ and $\lambda > \omega$,

- $\operatorname{dom} A_0^n$ *is a* ***core*** *for A_0,*
  i.e. $\operatorname{dom} A_0^n$ *is a dense subspace of* $\operatorname{dom} A_0$ *with respect to the graph norm (see Lemma 0.34),*
  for all positive integers n.

From Lemma 0.10 the connection to differential equations can be seen.

COROLLARY 0.11 *For $x \in \operatorname{dom} A_0$, the function $u \equiv T(.)x$ belongs to $C^1([0, \infty), X)$ and is the unique solution of the* ***Abstract Cauchy Problem***

$$\frac{d}{dt} u = A_0 u, \qquad u(0) = x.$$

The Abstract Cauchy Problem motivates a central question about semigroups of operators: Is there a bijective connection between the generators and the corresponding semigroups. The well known Hille-Yosida Theorem gives an answer to this.

THEOREM 0.12 ***Hille-Yosida.*** *A linear operator A_0 is the generator of a classic semigroup, for which (0.3) holds with constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ if and only if*

- *A_0 is closed and densely defined,*
- *$(\omega, \infty) \subset \rho(A_0)$ and the resolvents satisfy following estimate,*

$$\left\|(\lambda - A_0)^{-n}\right\| \leqslant \frac{M}{(\lambda - \omega)^n}, \tag{0.5}$$

  for all $\lambda > \omega$ and all $n \in \mathbb{N}$.

Additionally, for a generator A_0 even the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > \omega\}$ is subset of $\rho(A_0)$.

0.3 Pre-Semigroups

The idea of this work is to present generalisations of the notion *semigroup of operators*. There have been several developments in order to do that. The one directly connected to the theory with which we will deal in this work is the following.

DEFINITION 0.13 *An operator valued function $P : [0, \infty) \to \mathcal{B}(X)$, which satisfies*

1. *P is strongly continuous,*
   i.e. $P(.)x : [0, \infty) \to X, t \mapsto P(t)x$ is continuous for all $x \in X$,
2. *$P(0)$ is injective,*
3. *$P(0)P(t + s) = P(t)P(s)$ for all $t, s \geqslant 0$,*

is called a ***Pre-semigroup*** *of operators.*

In literature there exist many different names for this notion. For example, these functions are called *C-regularized semigroups* in deLaubenfels [deL94] or sometimes *C-semigroups* where C denotes the operator $P(0)$ in order to emphasise its paticular role. It was mentioned first in the 1960ies by G. DaPrato as *Semigruppi regolarizzabili* [DP66]. Independently, Davies and Pang introduced *C-Semigroups* in 1980 [DP87]. Among others, Miyadera and Tanaka worked with this concept. A book covering the theory and applications is the one by Ralph deLaubenfels [deL94]. Our terminology is adapted from S. Kantorovitz [Kan95]. Clearly, this definition generalises the classic situation.
The generator of a Pre-semigroup is defined similarly to classic semigroups.

DEFINITION 0.14 *For a Pre-semigroup, the linear operator A_0 is defined as*

$$\operatorname{dom}(A_0) = \left\{x \in X : [P(.)x]'^{+}(0) \text{ exists and is element of } \operatorname{ran} P(0)\right\},$$
$$A_0 x = P(0)^{-1}[P(.)x]'(0),$$

*and is called the **generator** of P.*

Obviously, this definition coincides with the classical infinitesimal generator in case of $P(0) = \mathrm{I}$. We state some basic results which, for instance, can be found proved directly in [deL94] or, alternatively, will follow from the results in chapter 1 as corollaries. The assumed injectivity of $P(0)$ is the key in order to translate many results from classic semigroups into the present setting.

THEOREM 0.15 *For a Pre-semigroup P and its generator A_0, the following assertions hold true:*

- *A_0 is closed and $\overline{\operatorname{dom} A_0} \supseteq \operatorname{ran} P(0)$;*
- *$A_0 P(t)x = P(t)A_0 x$ for $x \in \operatorname{dom} A_0$;*
- *$\int_0^t P(s)x\, ds \in \operatorname{dom} A_0$ and*
  $$A_0 \int_0^t P(s)x\, ds = P(t)x - P(0)x,$$
  for all $x \in X$ and all $t \geqslant 0$;
- *for $x \in \operatorname{dom} A_0$ the function $P(.)x$ is continuously differentiable and*
  $$[P(.)x]'(s) = A_0 P(s)x,$$
  for all $s \geqslant 0$.

0.4 Linear Relations

In the following, we want to give an introduction to the theory of linear relations which arise as a generalisation of linear operators. In this context, the notion *linear relation* first appeared in [Are61] in 1961. A good introduction in the topic is given in [Kal10b]. In this section, X, Y will always denote Banach spaces over $\mathbb{C}$ and we will regard the product space $X \times Y$ equipped, for instance, with the sum norm.

DEFINITION 0.16 *A **linear relation** R in $X \times Y$ (between X and Y) is nothing else but a linear subspace of $X \times Y$. We will denote this by $R \leqslant X \times Y$. We will say that R is a **linear relation on** X if $X = Y$.*

Obviously, every linear operator from a subspace in X to Y can be seen as a linear relation by considering its graph in $X \times Y$. On the other hand, not every linear relation has to be the graph of a linear operator, as the example $R = X \times Y$ shows. Therefore, in the following linear operators will always be identified with their graphs. Note that in general linear relations can be defined as linear subsets of any product $X \times Y$ of linear spaces over a field Φ analogously. Since in our case, we are interested in generalisations of linear operators $L(X)$ for a Banach space X, this definition is sufficient.

DEFINITION 0.17 *For a linear relation $R \leqslant X \times Y$, the following sets are defined:*

- *the* ***domain of*** *R,* $\operatorname{dom} R = \{x \in X : \exists y \in Y : (x, y) \in R\}$,
- *the* ***range of*** *R,* $\operatorname{ran} R = \{y \in Y : \exists x \in X : (x, y) \in R\}$,
- *the* ***kernel of*** *R,* $\ker R = \{x \in X : (x, 0) \in R\}$,
- *and the* ***multi-value part of*** *R,* $\operatorname{mul} R = \{y \in Y : (0, y) \in R\}$.

For every $x \in \operatorname{dom} R$, we set

$$Rx := \{y \in Y : (x, y) \in R\}.$$

Clearly, the sets $\operatorname{dom} R$, $\operatorname{ran} R$, $\ker R$ and $\operatorname{mul} R$ in Definition 0.17 are linear subspaces of X or Y, respectively. Furthermore, in case of a linear operator R, the spaces $\operatorname{dom} R$, $\operatorname{ran} R$ and $\ker R$ coincide with the conventional definitions. The multi-value part consists only of 0 for a linear operator. The next lemma shows that this space is indeed the key to see *how far away* we are from a linear operator and that linear relations can be seen as multi-valued operators.

LEMMA 0.18 *Let $R \leqslant X \times Y$ be a linear relation. Then, for all $(x, y) \in R$,*

$$Rx = y + \operatorname{mul} R = \{y + z \in Y : z \in \operatorname{mul} R\}$$

Proof: Let $p \in Y$ be in Rx. Therefore, $(x, p) \in R$ and since $(x, y) \in R$, linearity implies $(x - x, p - y) = (0, p - y) \in R$. By definition, $p - y \in \operatorname{mul} R$ and hence, $p \in y + \operatorname{mul} R$. Conversely, consider $z \in \operatorname{mul} R$. Then $(0, z) \in R$ and again by linearity one deduces $(x, y + p) \in R$ because $(x, y) \in R$. Therefore, $y + p \in Rx$ and the proof is completed. ■

Next, we introduce operations on linear relations.

DEFINITION 0.19 *Let Z be a Banach space and let $R, T \leqslant X \times Y$, $S \leqslant Y \times Z$ be linear relations. Furthermore, let $\lambda \in \mathbb{C}$. Then, we define*

- *the **inverse** of* R, $R^{-1} = \{(y,x) \in Y \times X : (x,y) \in R\}$,
- $\lambda R = \{(x, \lambda y) \in X \times Y : (x,y) \in R\}$,
- *the **composition** of* R *and* S,
  $RS = \{(x,y) \in X \times Z : \exists z \in Y : (x,z) \in S, (z,y) \in R\}$,
- *the **sum** of* R *and* T,
  $R+T = \{(x,y) \in X \times Y : \exists \tilde{y}, \hat{y} \in Y, y = \tilde{y} + \hat{y}, (x,\tilde{y}) \in R, (x,\hat{y}) \in T\}$.

REMARK 0.20 One can see that the sets defined in Definition 0.19 are indeed linear relations. Clearly, the notions coincide with the ones for linear operators if we identify them with their graphs. We point out that the inverse of a relation always exists and is very easily written down. By considering the inverse of non-injective operators, one easily deduces examples of linear relations which are not linear operators:
We point out that $\operatorname{mul} R^{-1} = \ker R$, $\operatorname{dom} R^{-1} = \operatorname{ran} R$ since $(R^{-1})^{-1} = R$ and also $\operatorname{mul} R = \ker R^{-1}$, $\operatorname{dom} R = \operatorname{ran} R^{-1}$. In particular, a linear relation has an inverse that is an operator, if and only if the kernel is trivial.

LEMMA 0.21 *Let* $R, S \leqslant X \times Y$ *be linear relations with* $R \subseteq S$. *Then*

$$\operatorname{dom} R \subseteq \operatorname{dom} S, \ker R \subseteq \ker S, \operatorname{ran} R \subseteq \operatorname{ran} S, \operatorname{mul} R \subseteq \operatorname{mul} S.$$

If in addition $\operatorname{dom} R = \operatorname{dom} S$, *then*

$$R = S \quad \textit{iff} \quad \operatorname{mul} R = \operatorname{mul} S.$$

Proof: Since $(x,y) \in R$ implies $(x,y) \in S$ the assorted inclusions must hold. To see the equivalence, we use Lemma 0.18. Indeed, if the multi-value parts are equal, the sets Rx and Sx coincide for all x in $\operatorname{dom} R = \operatorname{dom} S$ and therefore, $A = B$. The conversion clearly holds because of the inclusions above. ■

In this work we will sometimes consider the sum of linear relations in the sense of the sum of linear subspaces. To emphasise the difference to the sum of linear relations as defined in Definition 0.19, we introduce the following notation.

DEFINITION 0.22 *The sum of two linear subspaces* $M, N \subset X \times Y$ *in the Linear Algebra sense will be denoted by* $M \boxplus N$*:*

$$M \boxplus N = \{(x_1 + x_2, y_1 + y_2) \in X \times Y : (x_1, y_1) \in M, (x_2, y_2) \in N\}.$$

If, in addition, the intersection of M *and* N *is trivial, i.e.* $M \cap N = \{0\}$, *the sum of* M *and* N *will be denoted by* $M \oplus N$.

We state some elementary facts about linear relations. Hereby, I denotes the identity operator/relation in $X \times X$.

LEMMA 0.23 *Let $R, S, V \leqslant X \times Y$, $T \leqslant Y \times Z$ and $U \leqslant Z \times X$ be linear relations. Then, the following assertion hold true.*

1. $R \subseteq S \;\Leftrightarrow\; R^{-1} \subseteq S^{-1}$,
2. $R \subseteq S \;\Rightarrow\; RU \subseteq SU,\;\; TR \subseteq TS$,
3. $R \subseteq S \;\Rightarrow\; R + V \subseteq S + V$,
4. $R(UT) = (RU)T$,
5. $(RT)^{-1} = T^{-1}R^{-1}$.

Furthermore, we have the following identities:

$$RR^{-1} = \mathrm{I}_{\operatorname{ran} R} \boxplus (\{0\} \times \operatorname{mul} R), \qquad R^{-1}R = \mathrm{I}_{\operatorname{dom} R} \boxplus (\{0\} \times \ker R),$$

where $\mathrm{I}_{\operatorname{ran} R}$, $\mathrm{I}_{\operatorname{dom} R}$ *denote the restrictions of the identity to the subspaces* $\operatorname{ran} R$ *and* $\operatorname{dom} R$, *respectively.*

Proof: 1. This follows directly from definition.
2. Let (z, y) be in RU. This means that there exists an $x \in X$ such that $(z, x) \in T$ and $(x, y) \in R$. From that we have that $(x, y) \in S$ by assumption. This implies $(z, y) \in ST$. For the second assertion consider $(x, z) \in TR$. Then, we have a $y \in Y$ such that $(x, y) \in R$ and $(y, z) \in T$. By assumption, $(x, y) \in S$ and hence, $(x, z) \in TS$.
3. For $(x, y) \in R+V$, there exist $y_1, y_2 \in Y$ such that $y = y_1+y_2$, $(x, y_1) \in R$ and $(x, y_2) \in V$. By assumption, $(x, y_1) \in S$. Hence, $(x, y) \in S + V$.
4. Let $(y_1, y_2) \in R(UT)$. This means, $(y_1, x) \in UT$ and $(x, y_2) \in R$ for some $x \in X$ and further, $(y_1, z) \in T$ and $(z, x) \in U$ for some $z \in Z$. But this implies already $(z, y_2) \in RU$ and $(y_1, z) \in T$ which gives $(y_1, y_2) \in (RU)T$. The other inclusion follows analogously.
5. From $(x, z) \in (RT)^{-1}$ it follows that $(z, x) \in RT$. Hence, $(z, y) \in T$ and $(y, x) \in R$ for some $y \in Y$. This yields $(x, y) \in R^{-1}$ and $(y, z) \in T^{-1}$ which implies $(x, z) \in T^{-1}R^{-1}$. Doing the argumentation backwards we get the other direction.
To see $RR^{-1} \subseteq \mathrm{I}_{\operatorname{ran} R} \boxplus (\{0\} \times \operatorname{mul} R)$, consider $(x_1, x_2) \in RR^{-1}$. Therefore, there exists some $y \in Y$ such that (y, x_1),$(y, x_2) \in R$. By linearity, $(0, x_2 - x_1) \in R$ which means $x_2 - x_1 \in \operatorname{mul} R$. Therefore, (x_1, x_2) can be written as

$$(x_1, x_2) = (x_1, x_1) + (0, x_2 - x_1) \in \mathrm{I}_{\operatorname{ran} R} \boxplus (\{0\} \times \operatorname{mul} R).$$

For $y \in \operatorname{ran} R$ there exists some $x \in X$ such that $(x, y) \in R$ and $(y, x) \in R^{-1}$. Thus, $(y, y) \in RR^{-1}$. Furthermore, for $y \in \operatorname{mul} R$, $(0, y) \in R$ it follows

$(0, y) \in RR^{-1}$ since trivially $(0, 0) \in R^{-1}$.
The second identity follows from the first by considering the inverse of T and using $(R^{-1})^{-1} = R$. ■

Until now, we have stated only algebraic properties of linear relations. Like for operators, we can consider closed relations.

DEFINITION 0.24 *A linear relation $R \leqslant X \times Y$ is closed, if R is* ***closed*** *in the product topology in $X \times Y$.*

REMARK 0.25 Since X and Y are Banach spaces the closedness of a relation $R \leqslant X \times Y$ is equivalent to the following property:
For sequences (x_n, y_n) in R and $(x_n, y_n) \to (x, y)$ it follows that $(x, y) \in R$.
We remark that for a closed relation, the kernel and the multi-value part are always closed in X. Indeed, let x_n be a sequence in $\ker R$ converging to $x \in X$. Then, the sequence $(x_n, 0)$ is in R and converges to $(x, 0)$ in $X \times Y$. Closedness of R yields $(x, 0) \in R$. Hence, $\ker R$ is closed. The assertion for $\operatorname{mul} R$ follows analogously.

The next lemma shows that closedness is preserved under some operations.

LEMMA 0.26 *For a linear relation $T \leqslant X \times Y$, a bounded linear operator $B \in \mathcal{B}(X, Y)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, the following assertions are equivalent*

- *T is closed,*
- *T^{-1} is closed,*
- *λT is closed,*
- *$T + B$ is closed.*

Proof: Considering the sum norm on $X \times Y$ it is obvious that a sequence (x_n, y_n) converges to (x, y) if and only if (y_n, x_n) or $(x_n, \lambda y_n)$ converges to (y, x) or $(x, \lambda y)$, respectively. This proves the equivalence of the first three assertions.
By definition of the sum of linear relations and the assumption that $B \in \mathcal{B}(X, Y)$, we have

$$T + B = \{(x, y + Bx) : (x, y) \in T\}.$$

Assume that T is closed and let $(x_n, y_n + Bx_n)$ be a sequence in $T + B$ converging to (x, z) in $X \times Y$. Thus, $x_n \to x$ in X and, hence, $Bx_n \to Bx$ since B is continuous. This implies $y_n \to y := z - Bx$. By the closedness of T and since $(x_n, y_n) \in T$ for all $n \in \mathbb{N}$, we get $(x, y) \in T$. Therefore, $(x, y + Bx) = (x, z) \in T + B$. The conversion holds since T can be written as $T = (T + B) + (-B)$. ■

REMARK 0.27 We want to point out that we only give an introduction on the special case that relations are linear subspaces of $X \times Y$ where X,Y are Banach spaces. All results so far can be formulated in the general setting of linear subspaces of $V \times W$ where V, W are topological vector spaces. See [Kal10b] for details.

As for operators, the spectrum and the resolvent set can be defined for linear relations on X.

DEFINITION 0.28 *For a linear relation $R \leqslant X \times X$, the following sets in $\mathbb{C} \cup \{\infty\}$ are defined:*

$$\rho(R) = \left\{\lambda \in \mathbb{C} \cup \{\infty\} : (\lambda\,\mathrm{I} - R)^{-1} \in \mathcal{B}(X)\right\}$$

$\rho(R)$ is called the ***resolvent set*** *of R, and*

$$\sigma(R) = (\mathbb{C} \cup \{\infty\}) \setminus \rho(R)$$

is called the ***spectrum*** *of R. Here we define*

$$(\infty\,\mathrm{I} - R)^{-1} = R$$

with $\operatorname{ran}(\infty\,\mathrm{I} - T) = \operatorname{dom} R$. *Furthermore, we define the set of eigenvalues as the* ***pointspectrum***

$$\sigma_p(R) = \{\lambda \in \sigma(R) : \ker(\lambda\,\mathrm{I} - R) \neq \{0\}\}\,.$$

Instead of $\lambda\,\mathrm{I} - R$ we will often write $\lambda - R$ and as for operators, let $R(\lambda)$ denote $(\lambda - R)^{-1}$.

REMARK 0.29 We emphasise that for $\lambda \in \rho(R)$ and $\lambda \neq \infty$, the inverse of $\lambda - R$ has to be an operator which is bounded and has domain X. For a closed relation T, this is fulfilled precisely if

$$\ker(\lambda - T) = \{0\} \quad \text{and} \quad \operatorname{ran}(\lambda - T) = X. \tag{0.6}$$

This can be seen by using Remark 0.20 and the Closed Graph theorem. Indeed, $\lambda - T$ is closed since T is closed and therefore, $(\lambda - T)^{-1}$ is closed. Thus, using (0.6), $(\lambda - T)^{-1}$ is a closed operator defined on X. By the Closed Graph theorem, $(\lambda - T)^{-1}$ is bounded.
$\infty \in \rho(R)$ simply means that R itself belongs to $\mathcal{B}(X)$.

Next, we state some examples of linear relations that we will use in the following.

DEFINITION 0.30 *For $a, b \in \mathbb{R}$, $a < b$, we define the relations*

- $I_a^b(X) = \left\{(f, g) \in C([a, b], X)^2 : g(t) = \int_a^t f(s)ds,\ t \in [a, b]\right\}$,

- $D_a^b(X) = \{(f,g) \in C([a,b],X)^2 : f \in C^1([a,b],X), f' \equiv g\}$,

on $C([a,b],X)$, *where* $C([a,b],X)^2 = C([a,b],X) \times C([a,b],X)$. *We will write* I_a^b *and* D_a^b *if the space* X *is clear from the context.*

LEMMA 0.31 *For* $a,b \in \mathbb{R}$, $a < b$ *the following assertions hold true.*

1. I_a^b *is an injective and bounded operator with* $I_a^b \in \mathcal{B}(C([a,b],X))$
2. D_a^b *is a closed operator and can be written as*
$$D_a^b = (I_a^b)^{-1} \boxplus \{(k,0) \in C([a,b],X) \times C([a,b],X) : k \equiv c, c \in X\} \tag{0.7}$$

Proof: 1. Clearly, $(0,g) \in I_a^b$ implies $g \equiv 0$. Thus, $\operatorname{mul} I_a^b = \{0\}$ and, hence, I_a^b is indeed an operator. Therefore,

$$I_a^b : C([a,b],X) \to C([a,b],X), f \mapsto (t \mapsto \int_a^t f(s)\, ds).$$

From

$$\left\| I_a^b f \right\|_\infty = \max_{s\in[a,b]} \left| \int_a^s f(s) ds \right| \leqslant \int_a^b \|f(s)\|\, ds \leqslant |b-a| \, \|f\|_\infty \,,$$

we conclude the boundedness of I_a^b. To see the injectivity consider $(f,0) \in I_a^b$. Therefore,

$$\int_a^t f(s)\, ds = 0,$$

for all $t \in [a,b]$. Since f is continuous, we get that $f \equiv 0$ by the Fundamental theorem of Calculus.
2. We show (0.7) first. For every $(f,g) \in D_a^b$, i.e. $f' \equiv g$, we have

$$\int_a^t g(s)\, ds = f(t) - f(a),$$

by the Fundamental theorem of Calculus and Remark 0.5. Thus, $(g, f+k)$ for $k \equiv -f(a) \in X$. The other inclusion holds since $(g,f) \in (I_a^b)^{-1}$ and $k \equiv c$ for $c \in X$ implies $g \equiv (f+k)'$, hence $(f+k,g) \in D_a^b$. D_a^b is closed because it is the sum of a closed and a finite dimensional subspace as shown in line (0.7). ■

We remark that the set $\operatorname{dom} D_a^b$ coincides with $C^1([a,b],X)$. The next elementary observation will be used many times throughout the thesis.

LEMMA 0.32 *Let* X, Y *be Banach spaces. For a function* $f : [a,b] \to X$ *which is strongly differentiable at* $t \in [a,b]$ *and a linear operator* $B \in \mathcal{B}(X,Y)$, *the function* Bf

$$Bf : [a,b] \to Y : t \mapsto Bf(t),$$

is strongly differentiable at t *and* $[Bf]'(t) = B[f'(t)]$.
In particular, for a pair (f,g) *in* $D_a^b(X)$, *the pair* (Bf, Bg) *belongs to* $D_a^b(Y)$.

Proof: Consider $t, t+h \in [a,b]$. We have

$$\left\| \frac{Bf(t+h) - Bf(t)}{h} - Bf'(t) \right\|_Y \leqslant \|B\| \left\| \frac{f(t+h) - f(t)}{h} - f'(t) \right\|_X .$$

Since the term on the right hand side tends to 0 as $h \to 0$, the function Bf is strong differentiable at t and $[Bf]'(t) = B[f'(t)]$. For (f,g) in $D_a^b(X)$, Bf and Bg are continuous, because they are compositions of continuous functions. ■

For the following, we want to recall the quotient space X/M for a subspace $M \subseteq X$ which consists of equivalence classes

$$\tilde{x} = x + M = \{x + z : z \in M\},$$

for $x \in X$, and which is equipped with the norm

$$\|\tilde{x}\|_\sim := \inf\{\|z\| : z \in \tilde{x}\} = \inf\{\|x - z\| : z \in M\}. \tag{0.8}$$

Obviously, $\|\tilde{x}\|_\sim \leqslant \|x\|$ for all $x \in X$.

LEMMA 0.33 *If A is a closed linear relation in $X \times X$, then $\tilde{A} \leqslant X \times (X/\operatorname{mul} A)$ is closed, where (see Lemma 0.18)*

$$\tilde{A} = \{(x, Ax) \in X \times (X/\operatorname{mul} A) : x \in \operatorname{dom} A\},$$

and where $X/\operatorname{mul} A$ denotes the quotient space. Furthermore, $\tilde{A}$ is a linear operator from $\operatorname{dom} A \subseteq X$ to $X/\operatorname{mul} A$.

Proof: $X/\operatorname{mul} A$ is well defined because $\operatorname{mul} A$ is closed, see Remark 0.25. $\tilde{A}$ is well defined since $Ax \in X/\operatorname{mul} A$ by Lemma 0.18. Since X is a Banach space, $X/\operatorname{mul} A$ is a Banach space equipped with the norm $\|\tilde{y}\|_\sim := \inf\{\|z\| : z \in \tilde{y}\}$. Let $(x_n, \tilde{y}_n)$ be a sequence in $\tilde{A}$ which converges in $X \times (X/\operatorname{mul} A)$. Clearly, $x_n \to x$ in X. Since $\tilde{y}_n \to \tilde{y}$ in $X/\operatorname{mul} A$, there exist a sequence $z_n \in (\tilde{y}_n - \tilde{y})$ such that

$$\|z_n\| \leqslant \|\tilde{y}_n - \tilde{y}\|_\sim + \frac{1}{n},$$

for all $n \in \mathbb{N}$. Thus, z_n tends to 0 for $n \to \infty$. Since $z_n \in (\tilde{y}_n - \tilde{y})$, there exist $y_n \in \tilde{y}_n$ and $y \in \tilde{y}$ so that $z_n = y_n - y$ and, hence, $y_n \to y$ in X. By the construction of $\tilde{A}$, $\tilde{y}_n = Ax_n$ and, therefore, $(x_n, y_n) \in A$. Since A is closed, we deduce that $(x, y) \in A$. This implies $y \in Ax$, and together with $y \in \tilde{y}$ this yields $Ax = \tilde{y}$. Thus, $(x, \tilde{y}) \in \tilde{A}$ which proves the claim. The assertion that $\tilde{A}$ is an operator, i.e. $\operatorname{mul} A = \{0\}$, follows from Lemma 0.18. ■

The closedness of a linear relation A establishes a nice way of providing $\operatorname{dom} A$ with a Banach space structure.

LEMMA 0.34 *Let Y be a Banach space and $A : \operatorname{dom} A \subseteq X \to Y$ be a closed operator. The space $\operatorname{dom} A$ equipped with the graph norm $\|.\|_A$*

$$\|x\|_A := \|x\| + \|Ax\|_Y \,,$$

is a Banach space $[\operatorname{dom} A]$, and

$$\|.\| \leqslant \|.\|_A \,. \tag{0.9}$$

Proof: Clearly, $\|.\|_A$ is a norm on $\operatorname{dom} A$. In fact, the mapping $x \mapsto (x, Ax)$ is an isometric bijection from $(\operatorname{dom} A, \|.\|_A)$ onto A equipped with the sum norm, $\|.\|_1$. As $A \leqslant X \times Y$ is closed, $(A, \|.\|_1)$ and hence, $(\operatorname{dom} A, \|.\|_A)$ is complete. ■

COROLLARY 0.35 *Let A be a closed linear relation and let $\tilde{A}$ denote the linear relation defined in Lemma 0.33. The space $\operatorname{dom} A$ equipped with the graph norm $\|.\|_{\tilde{A}}$*

$$\|x\|_{\tilde{A}} := \|x\| + \left\|\tilde{A}x\right\|_{\sim} \,,$$

is a Banach space, and

$$\|.\| \leqslant \|.\|_{\tilde{A}} \,. \tag{0.10}$$

If A is a linear operator, the space coincides with the space $[\operatorname{dom} A]$. Therefore, we will use the same notation, $[\operatorname{dom} A]$, and write $\|.\|_A$ for $\|.\|_{\tilde{A}}$.

Proof: By Lemma 0.33 we know that $\tilde{A} : X \to (X/\operatorname{mul} A)$ is a closed operator with $\operatorname{dom} \tilde{A} = \operatorname{dom} A$. Therefore, the assertion follows from Lemma 0.34. If A is a linear operator, $\operatorname{mul} A = \{0\}$ and consequently the quotientspace equals X. ■

REMARK 0.36 Another application of Lemma 0.34 is given when we want to show that $C^1([a, b], X)$, equipped with $\|.\|_\infty$, is a Banach space. Indeed, we have already remarked that the set of functions in $C^1([a, b], X)$ coincides with the domain of D_a^b. Now the norm on $C^1([a, b], X)$ acts precisely like the graph norm in $\operatorname{dom} A$. Therefore, $C^1([a, b], X) = [\operatorname{dom} A]$ and, hence, $C^1([a, b], X)$ is a Banach space by lemma 0.34.

Finally, we introduce a norm on the space $\operatorname{ran} C$.

COROLLARY 0.37 *The range of a linear operator $B \in \mathcal{B}(X)$ is made into a Banach space $[\operatorname{ran} B]$ by the norm*

$$\|y\|_{\operatorname{ran} B} = \inf \{\|x\| : Bx = y\} \,,$$

for $y \in \operatorname{ran} B$.

Proof: In terms of linear relations we have $\operatorname{ran} B = \operatorname{dom} B^{-1}$ by Remark 0.20. Furthermore, B is closed by the Closed Graph theorem. Hence, B^{-1} is a closed linear relation by Lemma 0.26. Therefore, we can apply Corollary 0.35 to B^{-1}. From that, we get that $[\operatorname{dom} B^{-1}]$ is a Banach space equipped with the norm

$$\|y\|_{B^{-1}} = \|y\| + \left\|B^{-1}y\right\|_{\sim},$$

where $B^{-1}y$ is an element of $X/\operatorname{mul} B^{-1}$ as defined in 0.18 (see also Lemma 0.33). Note that

$$\left\|B^{-1}y\right\|_{\sim} = \inf\left\{\|x\| : x \in B^{-1}y\right\} = \inf\left\{\|x\| : Bx = y\right\} = \|y\|_{\operatorname{ran} B}.$$

for $y \in \operatorname{ran} B$. Furthermore, $x \in B^{-1}y$ is equivalent to $Bx = y$, and for such x, $\|B\|\,\|x\| \geqslant \|y\|$. Thus,

$$\left\|B^{-1}y\right\|_{\sim} = \|y\|_{\operatorname{ran} B} \geqslant \frac{1}{\|B\|}\,\|y\|.$$

Hence, $\|.\|_{\operatorname{ran} B}$ and $\|.\|_{B^{-1}}$ are equivalent on $\operatorname{ran} B$. Therefore, $[\operatorname{ran} B]$ is a Banach space. ∎

Chapter 1

G-Semigroups

1.1 G-Semigroups and the Generator

DEFINITION 1.1 *An operator valued function* $P : [0, \infty) \to \mathcal{B}(X)$ *is called a generalised semigroup or* ***G-semigroup****, if*

1. *P is strongly continuous, i.e.*

$$P(.)x \in C([0, \infty), X) \qquad \textit{for all} \quad x \in X,$$

2. *P fulfils the* ***additivity****, i.e.*

$$P(0)P(s + t) = P(s)P(t) \qquad \textit{for all} \quad s, t \geqslant 0.$$

For $P(0)$ *injective,* P *is a* ***Pre-semigroup*** *and for* $P(0) = \mathrm{I}$*,* P *is called a (classic)* ***semigroup****.*

REMARK 1.2

1. We point out that in comparison to classic semigroups (where $P(0) = \mathrm{I}$) and Pre-semigroups ($P(0)$ is injective), there are no restrictions on the operator $P(0)$. Therefore, this definition is a generalisation of these notions. See *Preliminaries*, Chapter 0.

2. Clearly, the second part of the definition is equivalent to the fact that

$$P(s - t)P(t) \text{ is independent of } t \text{ for } 0 \leqslant t \leqslant s.$$

In the following lemma we state some direct consequences of the strong continuity and the additivity of a G-semigroup. These results are nearly the same as for classic semigroups.

LEMMA 1.3 *The following properties hold for a G-semigroup $P(.)$.*

1. ***Commutativity*** *of P:*

$$P(s)P(t) = P(t)P(s) \quad \textit{for all } s, t \geqslant 0.$$

2. *P is uniformly bounded on bounded subsets of $[0, \infty)$.*

3. *For $b > a \geqslant 0$, the linear operator,*

$$\beta_{a,b} : X \to C([a, b], X), x \mapsto P(.)x,$$

is bounded.

4. $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} P(s)x \, ds = P(t)x$ *for all $x \in X$ and $t \geqslant 0$.*

Proof: The proof is mainly based on the Principle of Uniform Boundedness and the Fundamental theorem of Calculus.
1. Commutativity follows directly from the definition of a G-semigroup.
2. Let $\overline{L}$ be the closure of the bounded subset L of $[0, \infty)$. By strong continuity and continuity of the norm, $\|P(.)x\| : [0, \infty) \to [0, \infty)$ is continuous for fixed $x \in X$ and, hence, bounded by $M_x \geqslant 0$ on $\overline{L}$. Therefore, by the Principle of Uniform Boundedness $P(t)$ is uniformly bounded on $\overline{L}$. Hence on L we have $\|P(t)\| \leqslant M$ for some $M \geqslant 0$, $t \in L$.
3. follows from 2. Indeed, let $M \geqslant 0$ be the bound for $\|P(t)\|$ for $t \in [a, b]$. Then $\|P(.)x\|_\infty \leqslant M \|x\|$.
4. Since $P(.)x$ is continuous on $[0, \infty)$, $F : t \mapsto \int_0^t P(s)x \, ds$ is differentiable and $F'(t) = P(t)x$ for $t \geqslant 0$ by the Fundamental theorem of Calculus (where the derivative at 0 has to be understood as strong right derivative). ■

Now, having classic and Pre-semigroups in mind, we want to define an infinitesimal generator for G-semigroups. In the known situations, this was done via the derivative at zero. Then, it can be be shown that the function $t \mapsto P(t)x$ is continuously differentiable for $x \in \operatorname{dom} A$. To preserve this property for general semigroups, we give a somewhat different definition.

DEFINITION 1.4 *Let $P(.)$ be a G-semigroup. Then, the set of pairs*

$$A = \left\{(x, y) \in X \times X : P(s)y = [P(.)x]'(s) \ \forall s \geqslant 0\right\},$$

*is called the **generator** of $P(.)$.*

This definition has to be understood in the sense that a pair (x, y) is in the set A precisely if $[P(.)x]'(s)$ exists for all $s \geqslant 0$ and the above equality holds. We bring some simple examples.

***Example* 1.5** As we will see later, there is a canonical way in constructing generalised semigroups from classic ones.

- Clearly, the trivial (classic) semigroup, $T(t) := \mathrm{I}$ for all $t \geqslant 0$, is a G-semigroup. As this semigroup is a special case of the G-semigroup in the next example, it follows from (1.1), that the generator of the present G-semigroup is the zero operator $X \times \{0\}$

- For $B \in \mathcal{B}(X)$ the operator valued function $P(t) := B$, $t \geqslant 0$ is a G-semigroup, since strong continuity and the additivity are trivially fulfilled. Because
$$[P(.)x]'(s) = 0$$
for all $s \geqslant 0$ and $x \in X$, we see that (x, y) is in the generator A of P if and only if $P(t)y = By = 0$ for all $t \geqslant 0$. Therefore, we get
$$A = \{(x, y) \in X \times X : y \in \ker B\} = X \times (\ker B). \tag{1.1}$$

- If we have the special case $B \equiv 0$ in the example above, then we call this G-semigroup the **trivial G-semigroup**. The generator is then
$$A = X \times X.$$

- For commuting bounded operators $C, D \in \mathcal{B}(X)$ consider the classic semigroup $T(t) := e^{tD}$, $t \geqslant 0$, that is defined via the power series of the exponential function
$$e^{tD} = \sum_{n=0}^{\infty} \frac{t^n D^n}{n!},$$
and which converges in the operator norm topology for every $t \in \mathbb{C}$. One can show that the functional equation of the exponential function is preserved:
$$e^{(t+s)D} = e^{tD} e^{sD}.$$
The family of operators $P(t) := Ce^{tD}$ is a G-semigroup. Again, strong continuity can easily be seen by the fact that C is element of $\mathcal{B}(X)$. Since C and D commute and since C is continuous and linear, it follows that
$$Ce^{tD} = C \sum_{n=0}^{\infty} \frac{t^n D^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n C D^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n D^n C}{n!} \sum_{n=0}^{\infty} \frac{t^n D^n}{n!} C = e^{tD} C.$$
Hence, C and e^{tD} commute for each $t \geqslant 0$. From that, we get
$$\begin{aligned} P(0)P(t+s) &= Ce^{0D} Ce^{(t+s)D} = C^2 e^{(t+s)D} \\ &= C^2 e^{tD} e^{sD} = Ce^{tD} Ce^{sD} \\ &= P(t)P(s). \end{aligned}$$
Therefore, P is a G-semigroup.
In general, for a classic semigroup T and an operator $C \in \mathcal{B}(X)$, the composition
$$CT(.) : [0, \infty) \to \mathcal{B}(X), t \mapsto CT(t),$$

is a G-semigroup if we assume that $CT(t) = T(t)C$ for all $t \geqslant 0$. This will be seen in theorem 1.28.

For the sake of simplicity, we introduce the following notion,

DEFINITION 1.6 *We say that a G-semigroup P* ***commutes with an operator*** *$B \in \mathcal{B}(X)$ if*

$$P(t)B = BP(t) \qquad \text{for all } t \geqslant 0.$$

As already mentioned, in the special cases of classic semigroups or Pre-semigroups the generator was defined differently, namely, taking into account only the strong (right) derivative at zero. To investigate the correspondence between our generator in the general situation and this special choice, we introduce following linear relations.

DEFINITION 1.7 *For a G-semigroup $P(.)$, we define the following sets.*

- $A_0 = \{(x, y) \in X \times X : P(0)y = [P(.)x]'^{+}(0)\}$,
- $A_r = \{(x, y) \in X \times X : P(s)y = [P(.)x]'(s) \text{ for all } s \in [0, r]\}$, $r > 0$.

We point out that in the definition of A_0 one considers the strong right derivative, whereas for A_r and $r > 0$, we have the strong derivative for $s \in (0, r)$ and the strong right/left derivatives at the endpoints.
In the following lemmata basic properties of the generator A and the sets A_0, A_r of a G-semigroup are discussed.

LEMMA 1.8 *Let P be a G-semigroup. For $r > 0$, A_0, A_r and A are linear relations and*

$$A_0 \supseteq A_r \supseteq A_s \supseteq A, \tag{1.2}$$

for all $r, s \in (0, \infty)$, $r < s$. Furthermore,

$$A = \bigcap_{r \geqslant t} A_r, \tag{1.3}$$

for all $t \geqslant 0$. For the multi-value part, we have

$$\operatorname{mul} A_0 = \ker P(0),$$
$$\operatorname{mul} A_r = \bigcap_{s \in [0,r]} \ker P(s),$$
$$\operatorname{mul} A = \bigcap_{s \geqslant 0} \ker P(s) =: \ker P.$$

Proof: Clearly, A, A_r and A_0 are linear relation. The inclusions (1.2) and (1.3) obviously follow from the definitions of A and A_r. To calculate the multi-value parts assume that $(0, y)$ is a pair in A for $y \in X$. Then, $P(s)y = 0$ for all $s \geqslant 0$. Conversely, if $y \in \ker P$, then $P(s)y = 0$ for all $s \geqslant 0$ and hence, $(0, y) \in A$. The assertions for A_r, $r \geqslant 0$, follow analogously. ■

When handling with the relations A and A_r the following characterisation will be useful.

LEMMA 1.9 *For a G-semigroup P and $r > 0$, the equivalence*

$$(x, y) \in A_r \quad \Leftrightarrow \quad (\beta_{0,r}x, \beta_{0,r}y) \in D_0^r, \tag{1.4}$$

holds true, where $\beta_{0,r}x : [0, r] \to X, t \mapsto P(t)x$ is defined as in Lemma 1.3. Hence,

$$(x, y) \in A \quad \Leftrightarrow \quad (\beta_{0,r}x, \beta_{0,r}y) \in D_0^r \quad \forall r > 0. \tag{1.5}$$

Proof: By definition of A_r, (x, y) belongs to A_r precisely if the function $P(.)x$ is continuously differentiable with strong derivative $P(t)y$ for all $t \in [0, r]$. Writing this using $\beta_{0,r}$ and the linear relation D_0^r, yields (1.4). (1.5) is deduced from (1.3). ■

REMARK 1.10 In the following we will often write

$$(P(.)x, P(.)y) \in D_0^r$$

instead of $(\beta_{0,r}x, \beta_{0,r}y) \in D_0^r$. This notation is not completely exact, but it often will make the context more understandable and clear.

As for the infinitesimal generator of Pre-semigroups, A is closed, as shown in the next lemma.

LEMMA 1.11 *Let P be a G-semigroup. Then, A_r and A are closed for $r > 0$.*

Proof: By lemma 1.9, A_r can be written via the mapping $\beta_{0,r}$ and the differential operator D_0^r,

$$A_r = \{(x, y) \in X^2 : (\beta_{0,r}x, \beta_{0,r}y) \in D_0^r\} = (\beta_{0,r} \times \beta_{0,r})^{-1}(D_0^r).$$

Since D_0^r is a closed relation (see Lemma 0.31), A_r is closed by the continuity of $\beta_{0,r} \times \beta_{0,r} : X^2 \to C([0, r], X)^2$.
From (1.3), we conclude that A is closed, too. ■

REMARK 1.12 We want to point out that Lemma 1.11 does not discuss the linear relation A_0. Although we do not have an example, it is expected that A_0 is not closed in general.

We want to recall the following elementary fact.

LEMMA 1.13 *For a linear relation R and an operator $L \in \mathcal{B}(X)$, the following assertions are equivalent:*

1. $(x, y) \in R$ *implies* $(Lx, Ly) \in R$,
2. $RL \supseteq LR$,
3. $L^{-1}RL \supseteq R$.
   In this case, in fact, we have

$$L^{-1}RL \supseteq R \boxplus (\ker R \times \ker L) \supseteq R. \tag{1.6}$$

Proof: 1. $\Rightarrow$ 2.: Let $(x, y) \in LR$, that is, there exists a $z \in X$ such that $(x, z) \in R$ and $Lz = y$. But we assumed that with (x, z) also $(Lx, Lz) = (Lx, y)$ has to be in R. This yields $(x, y) \in RL$.
2. $\Rightarrow$ (1.6).: By lemma 0.23, we can multiply the assumed inclusion by L^{-1} from the left and get

$$L^{-1}RL \supseteq L^{-1}LR.$$

Again by Lemma 0.23, we have that $L^{-1}L = \mathrm{I} \boxplus (\{0\} \times \ker L) \supseteq \mathrm{I}$ and hence,

$$L^{-1}RL \supseteq (\mathrm{I} \boxplus (\{0\} \times \ker L))R \supseteq R.$$

Consider a pair $(x, y) \in (\{0\} \times \ker L)R$. This is equivalent to $(x, 0) \in R$ and $y \in \ker L$ and, therefore, equivalent to $(x, y) \in \ker R \times \ker L$. Thus,

$$R \boxplus (\ker R \times \ker L) = (\mathrm{I} \boxplus (\{0\} \times \ker L))R.$$

3. $\Rightarrow$ 1.: Let $(x, y) \in R$. By assumption, $(x, y) \in L^{-1}RL$ and, hence, $(Lx, z) \in R$ and $(z, y) \in L^{-1}$ for some $z \in X$. Therefore, $Ly = z$ and, thus, $(Lx, Ly) \in R$. ■

Next, we show that A and A_r, $r \geqslant 0$ are invariant under the mapping

$$(P(t) \times P(t)) : X^2 \to X^2.$$

LEMMA 1.14 *Let P be a G-semigroup. Then, for $t \geqslant 0$,*

$$(x, y) \in A \quad \textit{implies} \quad (P(t)x, P(t)y) \in A.$$

This is equivalent to

$$AP(t) \supseteq P(t)A \quad \textit{for all} \quad t \geqslant 0. \tag{1.7}$$

The same assertions hold true for A_r, $r \geqslant 0$.

Proof: Fix $t \geqslant 0$, $r > 0$ and let (x, y) be in A_r. By Lemma 1.5 and Remark 1.10, we have

$$(P(.)x, P(.)y) \in D_0^r.$$

Since $P(t) \in \mathcal{B}(X)$, Lemma 0.32 yields

$$(P(t)P(.)x, P(t)P(.)y) \in D_0^r,$$

where $P(t)P(.)x$ is meant to be the mapping

$$[0, r] \to X, s \mapsto P(t)P(s)x.$$

Commutativity of P leads to

$$(P(.)P(t)x, P(.)P(t)y) \in D_0^r.$$

Again using Lemma 1.5, $(P(t)x, P(t)y) \in A_r$. For $(x, y) \in A_0$, $P(.)x$ has strong right derivative $P(0)y$ at zero. Thus, Lemma 0.32 yields

$$(P(t)x, P(t)y) \in A_0.$$

The assertion for A holds since $(x, y) \in A$ is equivalent to $(x, y) \in A_r$ for all $r \geqslant 0$.
(1.7) holds due to Lemma 1.13. ■

The next lemmata give a feeling how big A_r and A are.

LEMMA 1.15 *Let P be a G-semigroup. Then,*

$$(\int_{\vartheta}^{\rho} P(s)x\ ds, P(\rho)x - P(\vartheta)x) \in A, \tag{1.8}$$

for all $\rho > \vartheta \geqslant 0$ and $x \in X$. Hence, for any $r \geqslant 0$,

$$(\int_{\vartheta}^{\rho} P(s)x\ ds, P(\rho)x - P(\vartheta)x) \in A_r, \tag{1.9}$$

for all $\rho > \vartheta \geqslant 0$.

Proof: Fix $\rho > \vartheta \geqslant 0$. We want to calculate the strong derivative of

$$P(.) \int_{\vartheta}^{\rho} P(s)x\ ds$$

at the point $t \geqslant 0$. For that, we consider

$$\begin{aligned}(P(t+h) - P(t)) \int_{\vartheta}^{\rho} P(s)x\ ds &= P(t)(\int_{\vartheta}^{\rho} P(s+h)x\ ds - \int_{\vartheta}^{\rho} P(s)x\ ds) \\ &= P(t)(\int_{\vartheta+h}^{\rho+h} P(s)x\ ds - \int_{\vartheta}^{\rho} P(s)x\ ds) \\ &= P(t)(\int_{\rho}^{\rho+h} P(s)x\ ds - \int_{\vartheta}^{\vartheta+h} P(s)x\ ds),\end{aligned}$$

where we have used the additivity of G-semigroups. Dividing by h and letting $h \to 0$ the Fundamental theorem of Calculus gives

$$[P(.)\int_{\vartheta}^{\rho} P(s)x\ ds]'(t) = P(t)(P(\rho) - P(\vartheta))x,$$

where we used that $P(t) \in \mathcal{B}(X)$. This proves (1.8) and (1.9) follows from (1.2). ■

As the last lemma of this series of basic lemmata for the generator we prove the following property of the domains of A_r and A.

LEMMA 1.16 *For the linear relations A_r, $r \geqslant 0$, and A of a G-semigroup P, we have*

$$\overline{\mathrm{dom}(A_r)} \supseteq \overline{\mathrm{dom}(A)} \supseteq \bigcup_{s \geqslant 0} \mathrm{ran}\, P(s) =: \mathrm{ran}\, P.$$

Proof: Fix $\rho \geqslant 0$ and $x \in X$. By Lemma 1.15 we have

$$\frac{1}{h}\int_{\rho}^{\rho+h} P(s)x\ ds \in \mathrm{dom}\, A,$$

for all $h > 0$. Hence $P(\rho)x \in \overline{\mathrm{dom}\, A}$ (see Lemma 1.3). The inclusion

$$\mathrm{dom}\, A_r \supseteq \mathrm{dom}\, A$$

holds by (1.2). Therefore, also

$$\overline{\mathrm{dom}(A_r)} \supseteq \overline{\mathrm{dom}(A)}.$$

■

REMARK 1.17 Lemma 1.8 indicates that the generator need not be a linear operator anymore. Indeed, this depends on the kernels of the operators $P(s)$. For classic semigroups, A is clearly an operator since $P(0) = \mathrm{I}$ and, hence, $\ker P(0) = \{0\}$. More general, A is an operator if P is a Pre-semigroup.

In some of the previous proofs, we already used that $(f, g) \in D_0^r$ implies

$$(Bf, Bg) \in D_0^r$$

if $B \in \mathcal{B}(X)$; see Lemma 0.32. The next technical result deals also with the conversion of this fact.

LEMMA 1.18 *For an operator $C \in \mathcal{B}(X)$ and a number $r > 0$ the following assertions are equivalent*

1. *C is injective.*

2. *$(Cf, Cg) \in D_0^r \Rightarrow (f,g) \in D_0^r$ for all $(f,g) \in C([0,r],X)^2$.*

Proof: 1.$\Rightarrow$ 2.: Let $(Cf, Cg) \in D_0^r$, i.e. $[Cf]'(s) = Cg(s)$ for all $s \in [0,r]$. By the Fundamental theorem of Calculus and since C is bounded,

$$C\int_t^{t+h} g(s)\ ds = \int_t^{t+h} Cg(s)\ ds = Cf(t+h) - Cf(t)$$

for $t, t+h \in [0,r]$. The injectivity of C yields

$$\int_t^{t+h} g(s)\ ds = f(t+h) - f(t).$$

Dividing by h and letting $h \to 0$, leads to $(f,g) \in D_0^r$.
2. $\Rightarrow$ 1.: We assume that there exists a non-trivial element $c \in \ker C$. Then, we consider the constant function $f \equiv c$ which is clearly in $C([0,r],X)$. The pair $(Cf, Cf) = (0,0)$ is in D_0^r. Hence, we conclude that (f,f) has to be in D_0^r by assumption. Since $f' \equiv 0 \in X$ and f was chosen to be equal $c \neq 0$, this is a contradiction. ■

Since the multi-value part of A equals the intersection of the kernels of the operators $P(t)$, $t \geqslant 0$, an idea to ensure that A is an operator, is to assume that there is an s_0 such that $P(s_0)$ is injective. The following proposition shows what happens if $s_0 > 0$.

PROPOSITION 1.19 *Let $P(.)$ be a G-semigroup. If there exists an $s_0 > 0$ such that $P(s_0)$ is injective, then A is an operator and*

$$A = A_r = A_{s_0}^p,$$

for all $r > s_0$, where

$$A_{s_0}^p = \left\{(x,y) \in X \times X : P(s_0)y = [P(.)x]'(s_0)\right\}.$$

Equivalently, one can write $A_{s_0}^p x = P(s_0)^{-1}[P(.)x]'(s_0)$ defined on

$$\mathrm{dom}(A_{s_0}^p) = \left\{x \in X : [P(.)x]'(s_0) \text{ exists and is element of } \operatorname{ran} P(s_0)\right\}.$$

Proof: Let $(x,y) \in A_{s_0}^p$. We show that $(P(s_0)P(.)x, P(s_0)P(.)y) \in D_0^r$ for all $r > 0$. As a consequence of the additivity of a G-semigroup

$$P(s_0)P(s+t) = P(0)P(s_0+s+t) = P(s_0+s)P(t) \tag{1.10}$$

for all $t \geqslant 0$ and $s \in \mathbb{R}$ such that $\min\{s_0+s, s+t\} \geqslant 0$.
Using this for fixed $t \geqslant 0$ and $(x,y) \in A_{s_0}^p$ we see that

$$\frac{1}{h}[P(s_0)P(t+h)x - P(s_0)P(t)x] = \frac{1}{h}[P(t)(P(s_0+h)x - P(s_0)x)] \tag{1.11}$$

for $h \in \mathbb{R}$ with $\min\{s_0 + h, t + h\} \geqslant 0$. Letting $h \to 0$, the limit exists and equals $P(t)P(s_0)y$ by the continuity of $P(t)$. Together with the commutativity, this shows

$$(P(s_0)P(.)x, P(s_0)P(.)y) \in D_0^r$$

for all $r > 0$. Now we make use of Lemma 1.18 to conclude that

$$(P(.)x, P(.)y) \in D_0^r$$

for all $r > 0$. Hence,

$$A_{s_0}^p \subseteq A \subseteq A_r$$

for $r \geqslant 0$ by (1.5) and (1.2). The other inclusion, $A_{s_0}^p \subseteq A_r$ for $r > s_0$ follows obviously by definition. Thus, $A_{s_0}^p = A = A_r$ for $r > s_0$. ■

REMARK 1.20 In the proof of the previous proposition we used the assumption $s_0 > 0$ strongly. In fact, by definition of $A_{s_0}^p$, the limit $h \to 0$ in line (1.11) exists as limit from the right hand side as well as from the left hand side. Contrary to this, for $s_0 = 0$ we only have the strong right derivative with this argumentation. We will see in Lemma 1.22 that also in this case, differentiability can be shown. But for that, some more work is needed.

Proposition 1.19 shows that we have actually the same situation as for Pre-semigroups if one of the operators of the semigroup is injective. This looks surprising, but the following theorem states that in this case, as an algebraic consequence, we already have a Pre-semigroup.

THEOREM 1.21 *Let $P(.)$ be a G-semigroup for which an $s_0 > 0$ exists such that $P(s_0)$ is injective/surjective. Then $P(s)$ is injective/surjective for all $s \geqslant 0$.*

Proof: Let $P(s_0)$ be injective. Hence, $P(s_0)P(s_0)$ is injective, too. By the commutativity and the additivity

$$P(0)P(2s_0) = P(2s_0)P(0) = P(s_0)P(s_0).$$

We conclude that both $P(0)$ and $P(2s_0)$ must be injective. In fact, for operators $C, D \in \mathcal{B}(X)$ such CD is injective, D has to be injective. Again by the additivity and the commutativity we have

$$P(0)P(2s_0) = P(2s_0 - t)P(t) = P(t)P(2s_0 - t), \qquad t \in [0, 2s_0].$$

By the same argument as above, $P(t)$ is injective for $0 \leqslant t \leqslant 2s_0$. For $4s_0 \geqslant t > 2s_0$, we get the injectivity by considering, $P(0)P(t) = P(t/2)P(t/2)$. Now, the hypothesis follows via induction.

For surjective $P(s_0)$, we use the fact that for $C, D \in \mathcal{B}(X)$ such that CD is surjective, C has to be surjective. The proof is nearly the same as for the injectivity. ■

The following lemma will be crucial for the connection between A and A_r. The proof is very similar to the one for Pre-semigroups ([Sch09], [deL94]).

LEMMA 1.22 *For a G-semigroup P and $(x, y) \in A_0$ the function*

$$P(0)P(.)x : [0, \infty) \to X, t \mapsto P(0)P(t)x, \tag{1.12}$$

is continuously differentiable, i.e. $P(0)P(.)x \in C^1([0, \infty), X)$ and we have

$$[P(0)P(.)x]'(s) = P(0)P(s)y, \qquad s \in [0, \infty)$$

Proof: First we show that $P(0)P(.)x$ is strongly right differentiable on $[0, \infty)$. Fix $t \geqslant 0$ and let $h > 0$. By the commutativity and the additivity of P we obtain,

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}(P(0)P(t+h)x - P(0)P(t)x) &= \lim_{h\to 0} \frac{1}{h}(P(t)(P(h)x - P(0)x)) \\ &= P(t)P(0)y,\end{aligned}$$

where the last equality holds since $P(t) \in \mathcal{B}(X)$ and $(x, y) \in A_0$. Hence, $[P(0)P(.)x]'^+(t)$ exists and equals $P(0)P(t)y$.
Now, we are going to show that the strong left derivative exists and also equals $P(0)P(t)y$ for $t > 0$. For that consider $h > 0$ (and let $h \to 0^+$ later). By the commutativity and the triangle inequality we obtain

$$\begin{aligned}&\left\|\frac{1}{h}[P(0)P(t) - P(0)P(t-h)]x - P(0)P(t)y\right\| \leqslant \\ &\leqslant \left\|[P(t) - P(t-h)]\frac{1}{h}[P(0) - P(h)]x\right\| + \\ &\quad + \left\|\frac{1}{h}[P(t) - P(t-h)]P(h)x - P(0)P(t)y\right\| \leqslant \\ &\leqslant \underbrace{\left\|[P(t) - P(t-h)](\frac{1}{h}[P(0)x - P(h)x] + P(0)y)\right\|}_{=:\alpha} + \\ &\quad + \underbrace{\left\| [P(t) - P(t-h)]P(0)y\right\|}_{=:\beta} + \\ &\quad + \underbrace{\left\|\frac{1}{h}[P(t) - P(t-h)]P(h)x - P(0)P(t)y\right\|}_{=:\gamma}.\end{aligned}$$

We consider the terms α, β, γ separately. By Lemma 1.3 we know that

$$\|P(t) - P(t-h)\| \leqslant M,$$

for $h \in [0,t]$ and some bound $M \in [0,\infty)$. Therefore, α can be estimated,

$$\alpha \leqslant M \left\| \frac{1}{h}[P(h)x - P(0)x] - P(0)y \right\|,$$

which converges to zero as $h \to 0^+$ since $(x,y) \in A_0$. The term β tends to 0 as $h \to 0^+$ because of strong continuity of P. We rewrite γ using the additivity as

$$\gamma = \left\| \frac{1}{h}[P(0)P(t+h)x - P(0)P(t)x] - P(0)P(t)y \right\|.$$

This converges to zero as $h \to 0^+$ since the strong right derivative of $P(0)P(.)x$ exists and equals $P(0)P(.)y$ as shown above. Hence, differentiability of

$$P(0)P(.)x : [0,\infty) \to X, t \mapsto P(0)P(t)x$$

from the left hand side is shown and the proof is completed. ■

Using the differential relation D_0^r, the previous lemma can be rewritten as follows:
For $(x,y) \in A_0$ one always has

$$(P(0)P(.)x, P(0)P(.)y) \in D_0^r \qquad \text{for all } r > 0.$$

With Lemma 1.18, we are able to proof that our definition of the generator A is consistent with the notions of generators for Pre-semigroups (and hence with the one from classic theory).

THEOREM 1.23 *Let $P(.)$ be a G-semigroup. Then following relation holds,*

$$A_0 \supseteq A_r \supseteq A_t \supseteq A.$$

for $0 < r < t$. In case that $P(.)$ is a Pre-semigroup, equality holds, i.e.

$$A_0 = A_r = A, \tag{1.13}$$

and A is an operator.

Proof: The inclusion was already seen (1.2). Let $P(.)$ be a Pre-semigroup. We show that $A \supseteq A_0$. Let $(x,y) \in A_0$. From Lemma 1.22 we know that $(P(0)P(.)x, P(0)P(.)y) \in D_0^r$ for all $r > 0$. Since $P(0)$ is injective, Lemma 1.18 gives $(P(.)x, P(.)y) \in D_0^r$ for all $r > 0$ and, hence, $(x,y) \in A$. By Lemma 1.8 the injectivity of $P(0)$ implies $\operatorname{mul} A = \{0\}$. ■

REMARK 1.24 Theorem 1.23 shows that in case of a Pre-semigroup, all relations A, A_r, $r \geqslant 0$, coincide. Moreover, in view of Proposition 1.19 and Theorem 1.21 we can conclude that

$$A = A_r = A_r^p$$

for all $r \geqslant 0$ ($A_0^p := A_0$) if there exists an $s_0 > 0$ such that $P(s_0)$ is injective.

Some of the basic properties of classic semigroups mentioned in *Preliminaries*, Chapter 0, can be deduced as a corollary of the recent results.

COROLLARY 1.25 *Let T be a classic semigroup with generator A_0 as defined in chapter 0.2. Then*

1. *A_0 is a closed and densely defined operator,*
2. *for $x \in \operatorname{dom} A_0$ it holds that $P(t)x \in \operatorname{dom} A_0$ and*
$$P(t)A_0x = P(t)A_0x, \qquad t \geqslant 0,$$
3. *for $x \in \operatorname{dom} A_0$ the function $T(.)x$ is continuously differentiable and*
$$[T(.)x]'(s) = A_0T(s)x, \qquad s \geqslant 0.$$

Proof: It follows from Theorem 1.23, that A_0 equals A, where A denotes the generator of the G-semigroup T, and that A is an operator. Hence, for $(x, y) \in A$ we can write $Ax = y$.
1. Closedness follows by Lemma 1.11. By Lemma 1.16, we have that $\operatorname{ran} T$ lies in the closure of $\operatorname{dom} A_0$. Since $T(0) = \mathrm{I}$, $\operatorname{ran} T = X$ and thus the assertion follows.
2. follows by Lemma 1.14.
3. is clear since $A_0 = A$ and the definition of A. ■

***Example* 1.26** Consider the space $X = C_0(\mathbb{R})$ and a function $h \in X$ which will be specified later on. The regarded G-semigroup will be

$$P(t)f = (x \mapsto h(x)e^{tx}f(x)).$$

Of course, h has to be chosen properly, such the operators $P(t)$ are well defined on X and bounded. We see that $P(0)f(x) = h(x)f(x)$. Consider the following possible choices for h:

- $h(x) \in C_{00}(\mathbb{R})$:
  Since h lives only on a compact set, it is easy to see that P is indeed a G-semigroup: Let f be in $C_0(\mathbb{R})$ and $t, t+\tau \geqslant 0$. Clearly, there exists a constant $C_t \geqslant 0$ such that
  $$\|P(t)f\|_\infty \leqslant C_t \|f\|_\infty ,$$
  since the support of h is compact. Furthermore,
  $$\begin{aligned}\|P(t+\tau)f - P(t)f\|_\infty &= \sup_{x\in\mathbb{R}} \left|h(x)f(x)(e^{(t+\tau)x} - e^{tx})\right| \\ &= \sup_{x\in\operatorname{supp} h} \left|h(x)f(x)e^{tx}(e^{\tau x} - 1)\right| \\ &\leqslant M \sup_{x\in\operatorname{supp} h} |e^{\tau x} - 1| \\ &\leqslant M \max\left\{\left|e^{|\tau|m_h} - 1\right|, \left|e^{-|\tau|m_h} - 1\right|\right\} \overset{\tau\to 0}{\longrightarrow} 0,\end{aligned}$$

where $m_h := \max\{|x| : x \in \operatorname{supp} h\}$ and M denotes the maximum of

$$x \mapsto \left|h(x)f(x)e^{tx}\right|$$

for $x \in \operatorname{supp} h$. Thus, P is strongly continuous. Let $s \geqslant 0$. The additivity follows by

$$\begin{aligned} P(0)P(t+s)f(x) &= h(x)h(x)e^{(t+s)x}f(x) \\ &= h(x)e^{tx}h(x)e^{sx}f(x) = P(s)P(t)f(x). \end{aligned} \tag{1.14}$$

Hence, P is a G-semigroup. For $f, g \in X$ we see that $P(s)f = P(s)g$ implies $h(x)f(x) = h(x)g(x)$, hence $f(x) = g(x)$ on the support of h, which does not depend on s. In other words,

$$\ker P(s) = \ker P(0) = \{f \in X : f|_{\operatorname{supp} h} = 0\}.$$

Hence $\operatorname{mul} A = \ker P(0) = \{f \in X : f|_{\operatorname{supp} h} = 0\}$. In oder to obtain the generator of the G-semigroup, we consider first a pair $(f, g) \in A$. Since point evaluation in $C_0(\mathbb{R})$ is continuous, we have

$$\begin{aligned} [P(.)f]'(s)(x) &= (\lim_{\tau\to 0}\frac{1}{\tau}(P(s+\tau)f - P(s)f))(x) \\ &= h(x)e^{sx}f(x)\lim_{\tau\to 0}\frac{e^{\tau x}-1}{\tau} \\ &= h(x)e^{sx}f(x)x, \end{aligned}$$

and therefore, $h(x)e^{sx}g(x) = h(x)e^{sx}xf(x)$ or

$$h(x)g(x) = h(x)xf(x), \tag{1.15}$$

for $x \in \mathbb{R}$. Thus, $g(x) = xf(x)$ for all $x \in \operatorname{supp} h$. Conversely, let $(f, g) \in X^2$ such that

$$g(x) = xf(x)$$

holds for $x \in \operatorname{supp} h$. Then, for all x the equation (1.15) holds.
For $t, t+\tau \geqslant 0$,

$$\begin{aligned} \left\|\frac{P(t+\tau)f - P(t)f}{\tau} - P(t)g\right\|_\infty &= \sup_{x\in\operatorname{supp} h}\left\|\left(\frac{e^{(t+\tau)x} - e^{tx}}{\tau} - e^{tx}x\right)f(x)\right\|_\infty \qquad (1.16) \\ &\leqslant M\left\|\frac{e^{\tau x}-1}{\tau} - x\right\|_{\operatorname{supp} h,\infty}, \end{aligned}$$

where M denotes the maximum of $\left|h(x)e^{tx}f(x)\right|$ on $\operatorname{supp} h$. Moreover,

$$\begin{aligned} \left\|\frac{e^{\tau x}-1}{\tau} - x\right\|_{\operatorname{supp} h,\infty} &\leqslant \sup_{|x|\leqslant m_h}\left|\int_0^x (e^{\tau s}-1)\, ds\right| \qquad (1.17) \\ &\leqslant m_h \sup_{|s|\leqslant m_h}|e^{\tau s}-1| \\ &= k\max(\left|e^{-\tau m_h}-1\right|, |e^{\tau m_h}-1|) \overset{\tau\to 0}{\longrightarrow} 0, \end{aligned}$$

Altogether we have shown that

$$A = \left\{(f,g) \in X^2 : g(x) = xf(x) \text{ for } x \in \operatorname{supp} h\right\}.$$

Since supp(h) is compact, we have dom $A = X$.

- $h(x) = e^{-x^2}$ for $x \in K^c$ for a compact interval K and $h(x) = 0$ for $x \in K' \subset K$ (K' compact). Let $f \in C_0(\mathbb{R})$ and $t, t+\tau \geqslant 0$. Clearly, $P(t)f \in C_0(\mathbb{R})$ since

$$\lim_{x\to\pm\infty} P(t)f(x) = \lim_{x\to\pm\infty} e^{-x^2} e^{tx} f(x) = 0.$$

The parabola $-x^2 + tx$ has its maximum $\frac{t^2}{4}$ at $\frac{t}{2}$. Thus,

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|h(x)e^{tx}\right| &\leqslant \max\left\{e^{\frac{t^2}{4}}, \|h\|_\infty e^{tm}\right\} \\ &\leqslant e^{\frac{t^2}{4}} + Ce^{tm} =: C_t, \end{aligned} \tag{1.18}$$

where $m := \max\{|x| : x \in K\}$. Thus, $P(t)$ is bounded with

$$\|P(t)f\|_\infty \leqslant C_t \|f\|_\infty.$$

For $f \in C_{00}(\mathbb{R})$, the continuity of $P(.)f$ follows similarly as in the case for $h \in C_{00}(\mathbb{R})$. For arbitrary $f \in C_0(\mathbb{R})$, we indicate that $C_{00}(\mathbb{R})$ lies densely in $C_0(\mathbb{R})$. Therefore, consider a sequence $f_n \in C_0(\mathbb{R})$ which approximates f:

$$\|f - f_n\|_\infty \overset{n\to\infty}{\longrightarrow} 0.$$

By triangle inequality, we get

$$\begin{aligned}&\|P(t+\tau)f - P(t)f\| = \\ &\leqslant \|(P(t+\tau) + P(t))\| \, \|(f - f_n)\|_\infty + \|P(t+\tau)f_n - P(t)f_n\|_\infty. \end{aligned} \tag{1.19}$$

By (1.18),

$$\begin{aligned}\|(P(t+\tau) + P(t))\| &\leqslant e^{\frac{(t+\tau)^2}{4}} + Ce^{(t+\tau)m} + e^{\frac{t^2}{4}} + Ce^{tm} \\ &\leqslant 2e^{t^2} + 2Ce^{2tm} =: C_{2,t},\end{aligned}$$

for $|\tau| \leqslant t$. Hence, (1.19) is arbitrarily small for sufficiently large n and sufficiently small $|\tau| > 0$. This proves that P is strongly continuous. The additivity follows by (1.14). As before

$$\operatorname{mul} A = \ker P(0) = \ker P(s) = \{f \in X : f|_{\operatorname{supp} h} = 0\}.$$

Analogously as above, for $(f,g) \in A$, we deduce

$$xf(x) = g(x) \qquad \text{for all } x \in \operatorname{supp} h.$$

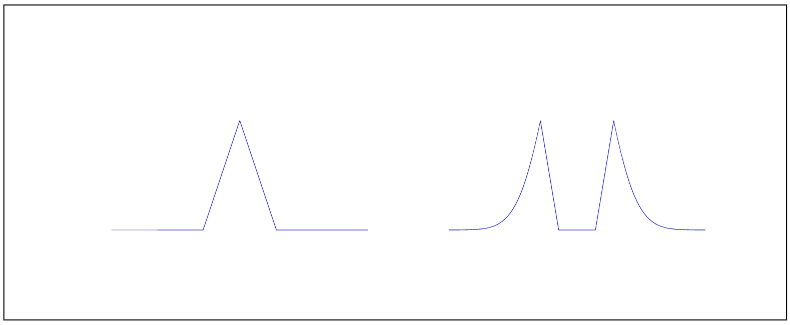

Figure 1.1: left: h with compact support, right: $h \in \mathcal{O}(e^{-x^2})$

Conversely, let $(f, g) \in C_0(\mathbb{R})^2$ with $xf(x) = g(x)$ on $\operatorname{supp} h$. We fix $t \geqslant 0$. Elementary calculations show that for all $|\tau| < 1$,

$$\left|\frac{e^{\tau x} - 1}{\tau}\right| = \left|\sum_{n=1}^{\infty} \frac{x^n \tau^{n-1}}{n!}\right| \leqslant \sum_{n=1}^{\infty} \frac{|x|^n}{n!} \leqslant e^{|x|}.$$

Hence, $\sup_{|x|>k} \left|h(x)e^{tx} f(x) \frac{e^{\tau x}-1}{\tau}\right| \to 0$ for $k \to \pm\infty$. Furthermore,

$$|h(x)xf(x)| \overset{x \to \pm\infty}{\longrightarrow} 0$$

since $g \in C_0(\mathbb{R})$ and $g(x) = xf(x)$ for $|x|$ sufficiently large. Hence, for every $\epsilon > 0$ there exists a $k_\epsilon > 0$ such that

$$\sup_{|x|>k_\epsilon} \left|\frac{1}{\tau}(P(t+\tau)f(x) - P(t)f(x)) - P(t)g(x)\right| =$$
$$= \sup_{|x|>k_\epsilon} \left|h(x)e^{tx} f(x)(\frac{e^{\tau x} - 1}{\tau} - x)\right| < \epsilon/2,$$

for every $|\tau| < 1$ with $t + \tau \geqslant 0$. For $|x| < k_\epsilon$ the same argumentation as in lines (1.16), (1.17) can be applied. Altogether, we obtain

$$\left\|\frac{1}{\tau}(P(t+\tau)f - P(t)f) - P(t)g\right\|_\infty < \epsilon$$

for τ sufficiently close to zero. This shows that (f, g) is indeed an element of A. Hence, again

$$A = \left\{(f, g) \in X^2 : g(x) = xf(x) \text{ for } x \in \operatorname{supp} h\right\}.$$

Contrary to the previous choice of h, here $\operatorname{dom} A \neq X$, since $xf(x)$ does not converge to 0 as $x \to \pm\infty$ for all $f \in C_0(\mathbb{R})$. For instance regard $f \in C_0(\mathbb{R})$ such that $f(x) = \frac{1}{x}$ for $|x| > 1$. Then, $(x \mapsto xf(x)) \notin C_0(\mathbb{R})$ since $xf(x) \to 1$ for $x \to \infty$. Figure 1.1 visualises the shape of the functions h one should have in mind. Note that h does not have to be differentiable, just continuous.

1.2 Constructions of G-Semigroups

We are now going to discuss in general how examples of generalised semigroups can be obtained from classic/Pre- semigroups and how the generators are related. The idea is to multiply the given G-semigroup with an operator C.

LEMMA 1.27 *If P is a G-semigroup that commutes with an operator $C \in \mathcal{B}(X)$, then*

$$AC \supseteq CA,$$
$$A_r C \supseteq CA_r,$$

for $r \geqslant 0$.

Proof: By Lemma 1.13, it suffices to show that A and A_r are invariant under

$$(C \times C) : X^2 \to X^2 : (x, y) \mapsto (Cx, Cy).$$

Let $(x, y) \in A_r$ and $r > 0$. Because of Lemma 1.9, $(P(.)x, P(.)y) \in D_0^r$. By Lemma 0.32, $(CP(.)x, CP(.)y) \in D_0^r$ and by the assumed commutativity $(Cx, Cy) \in A_r$. The assertion for A follows directly from Lemma 1.9. For $r = 0$, also Lemma 0.32 and commutativity gives $[P(.)Cx]'(0) = P(0)Cy$, hence $(Cx, Cy) \in A_0$. ■

THEOREM 1.28 *If P is a G-semigroup with generator A and $C \in \mathcal{B}(X)$ which commutes with P, i.e. $P(t)C = CP(t)$ for all $t \geqslant 0$, then $\tilde{P} := CP$ is a G-semigroup with $\tilde{P}(0) = P(0)C$. Denote by B the generator of the of $\tilde{P}$, and define the linear relations B_r analogously. Then*

$$B_r = C^{-1} A_r C \supseteq A_r \tag{1.20}$$

for $r \geqslant 0$ and

$$B = C^{-1} AC \supseteq A. \tag{1.21}$$

Proof: The strong continuity of $\tilde{P}$ is clear from the strong continuity of P and since C is bounded. Obviously, $\tilde{P}(0) = P(0)C$. The additivity follows from the commutativity assumption and the additivity of P:

$$\tilde{P}(0)\tilde{P}(t+s) = CP(0)CP(t+s) = CCP(t)P(s) = CP(t)CP(s) = \tilde{P}(t)\tilde{P}(s).$$

To show (1.20) consider $r > 0$ first. Let (x, y) be in B_r which is equivalent to

$$(CP(.)x, CP(.)y) \in D_0^r$$

by Lemma 1.9. By Commutativity of P and C, this is equivalent to

$$(P(.)Cx, P(.)Cy) \in D_0^r,$$

which holds precisely if (Cx, Cy) is in A_r. Using this and (1.3), we have

$$(x, y) \in B \quad \Leftrightarrow \quad (Cx, Cy) \in A.$$

Hence, B_r equals $C^{-1}A_rC$ and $B = C^{-1}AC$. For $r = 0$, the hypothesis follows analogously, if one considers the definition of A_0. The inclusions are clear by Lemma 1.27. ∎

As we know by Theorem 1.23 for Pre-semigroups the generator coincides with A_0. The next corollary shows that this property is preserved under the construction from Theorem 1.28.

COROLLARY 1.29 *Let T be a Pre-semigroup, i.e. $T(0)$ is injective, with generator A. Let $C \in \mathcal{B}(X)$ an operator which commutes with P. Then $P := CT$ is a G-semigroup with generator A_P and*

$$A_P = A_{P,r}$$

for all $r \geqslant 0$ where $A_{P,r}$ denotes the linear relation A_r of P.

Proof: By the theorem above, P is a G-semigroup. By (1.2), it remains to show that $A_{P,0} \subseteq A_P$. Let (x, y) be in $A_{P,0}$. By (1.20), $(Cx, Cy) \in A_0$. Since T is a Pre-semigroup, A_0 equals A by Theorem 1.23 and thus, $(Cx, Cy) \in A$. By (1.21), this means $(x, y) \in A_P$. ∎

Theorem 1.28 states that the generator of the G-semigroup $\tilde{P} = CP$ includes the generator of the initial G-semigroup. Indeed, example 1.33 will demonstrate that in general we cannot expect equality for A_P and $A_{\tilde{P}}$. However, in case that C is injective, this is guaranteed as shown in the following corollary.

COROLLARY 1.30 *Let P be a G-semigroup and let $C \in \mathcal{B}(X)$ commute with P. Furthermore, let C be injective. Then, the generator B of $\tilde{P} := CP$ coincides with the generator A of P. Moreover,*

$$B_r = A_r$$

for $r > 0$. In particular, if, in addition, P is a Pre-semigroup, then $\tilde{P}$ is also a Pre-semigroup.

Proof: It remains to show that $C^{-1}A_rC \subseteq A$ and $C^{-1}AC \subseteq A$ by Theorem 1.28. This follows directly from Lemma 1.18. In fact, for $(x, y) \in C^{-1}A_rC$, we have $(Cx, Cy) \in A_r$. Using Lemma 1.9, the commutativity of P and C and the assumption that C is injective, Lemma 1.18 yields $(x, y) \in A_r$.

From that, $C^{-1}AC \subseteq A$ follows by (1.3).
The second assertion from $\tilde{P}(0) = CP(0)$. ∎

REMARK 1.31 We point out that, except for the case that P is a Pre-semigroup, it is not clear whether $A_0 = B_0$ in the situation of Corollary 1.30.

Corollary 1.29 tells us that in the situation of a G-semigroup P that can be written as CT for a Pre-semigroup T, the generators $A_{P,0}$ and A_P coincide. The following examples shall bring a closer look to this fact.

***Example* 1.32** If we consider Example 1.26, we can ask ourselves whether this G-semigroup is of the form CT for an operator C and a Pre-semigroup T. Due to Corollary 1.30, such a C cannot be injective since P is not a Pre-semigroup. One is tempted to set $C : C_0(\mathbb{R}) \to C_0(\mathbb{R}), f \mapsto (x \mapsto h(x)f(x))$ since $f \mapsto (x \mapsto e^{tx}f(x))$ makes the impression to be a semigroup. But the problem is that this supposed semigroup is not well defined on $C_0(\mathbb{R})$. For instance, let $f(x) = \frac{1}{|x|+1}$. Then, the term $e^{tx}f(x)$ does not converge to zero as $x \to \pm\infty$. However, we can define C differently

$$C : C_0(\mathbb{R}) \to C_0(\mathbb{R}), f \mapsto (x \mapsto \frac{h(x)}{e^{-x^2}}f(x)),$$

which is bounded by the choice of h (see Example 1.26). Then, the Pre-semigroup T has to be defined as

$$T(t)f(x) = e^{-x^2}e^{tx}f(x),$$

where $T(0)$ is injective since $T(0)f = T(0)g$ implies

$$\begin{aligned} T(0)f(x) = T(0)g(x) \quad &\text{for all } x \in \mathbb{R} \\ \Leftrightarrow e^{-x^2}f(x) = e^{-x^2}g(x) \quad &\text{for all } x \in \mathbb{R} \\ \Leftrightarrow f = g, \end{aligned}$$

for all $f, g \in C_0(\mathbb{R})$. Hence, the G-semigroup P can indeed be written as product CT for a Pre-semigroup T. One can observe that this decomposition is obviously not unique.

From Example 1.32 a trivial question arises: *Can we always rewrite a G-semigroup as a product of a bounded operator and a Pre-semigroup?* Intuitively, the answer should be *No*. The following example discusses such a G-semigroup.

***Example* 1.33** We consider the product space $X \times X$. Define the function of operators $P : [0, \infty) \to \mathcal{B}(X \times X)$ as block operators acting on $X \times X$,

$$P(t) = \begin{pmatrix} 0 & t\,\mathrm{I}_X \\ 0 & 0 \end{pmatrix}.$$

Clearly, P is strongly continuous. By the structure of the block operator we have

$$P(t)P(s) = \begin{pmatrix} 0 & t\,\mathrm{I}_X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & s\,\mathrm{I}_X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad t, s \geqslant 0.$$

Therefore, the additivity is trivially fulfilled. Hence, P is a G-semigroup. It follows immediately that $\ker P(0) = X \times X$ and $\ker P(s) = X \times \{0\}$ for $s > 0$. Therefore, $A_0 \supsetneq A$, because

$$\operatorname{mul} A_0 = \ker P(0) = X \times X$$

and

$$\operatorname{mul} A = \bigcap_{s \geqslant 0} \ker P(s) = X \times \{0\}\,,$$

by Lemma 1.8. By Corollary 1.29, P cannot be written as CT for a Pre-semigroup T and some commuting operator C.
To calculate the linear relations A_r and A of P, we consider $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X^2$ and calculate

$$\frac{1}{h}(P(t+h)x - P(t)x) = \frac{1}{h}\left(\begin{pmatrix} (t+h)x_2 \\ 0 \end{pmatrix} - \begin{pmatrix} tx_2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}. \tag{1.22}$$

Hence, $[P(.)x]'(t) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ for all $t \geqslant 0$. In particular, a pair (x, y) is in A_0 if and only if $\begin{pmatrix} x_2 \\ 0 \end{pmatrix} = P(0)y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore,

$$A_0 = (X \times \{0\}) \times (X \times X). \tag{1.23}$$

For $r > 0$, a pair (x, y) is in A_r precisely if $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ equals $P(t)y = \begin{pmatrix} ty_2 \\ 0 \end{pmatrix}$ for all $t \in [0, r]$. For that, $x_2 = y_2 = 0$ must hold. Thus,

$$A = A_r = (X \times \{0\}) \times (X \times \{0\}),$$

for all $r > 0$.

***Example* 1.34** Next, we want to investigate the question what happens if we multiply the G-semigroup from Example 1.32 with an operator $C \in \mathcal{B}(X^2)$. Writing C in block operator form $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ we get

$$CP(t) = \begin{pmatrix} 0 & tC_{11} \\ 0 & tC_{21} \end{pmatrix},$$

and

$$P(t)C = \begin{pmatrix} tC_{21} & tC_{22} \\ 0 & 0 \end{pmatrix}.$$

Hence, C commutes with P iff $C_{21} = 0$ and $C_{22} = C_{11}$. In this situation $\tilde{P} = CP$ is a G-semigroup by Theorem 1.28. The same considerations as in lines (1.23), (1.22) lead to conditions for a pair (x, y) to be in $A_{\tilde{P}}$ or $A_{\tilde{P},0}$, respectively. Indeed, $(x, y) \in A_{\tilde{P},0}$ iff $(Cx, Cy) \in A_{P,0}$ which is equivalent to

$$C_{11}x_2 = 0.$$

Furthermore, $(x, y) \in A_{\tilde{P},r}$ precisely if

$$C_{11}x_2 = tC_{11}y_2 \quad \text{for all } t \in [0, r].$$

Consequently, $A_{\tilde{P}} = A_{\tilde{P},r}$ for $r > 0$ and

$$\begin{aligned} A_{\tilde{P},0} &= (X \times \ker C_{11}) \times (X \times X) \\ A_{\tilde{P}} &= (X \times \ker C_{11}) \times (X \times \ker C_{11}). \end{aligned}$$

Therefore, for a C_{11} with non trivial kernel, we have $A_{\tilde{P}} \supsetneq A_P$.
From theorem 1.28 we know that $A_{\tilde{P}} \supseteq A_P$ and this example shows that in general we cannot expect equality for the generators if C is not injective.

The following lemma deals the question *When the sum of two G-semigroups is again a G-semigroup?*

LEMMA 1.35 *For G-semigroups P, Q the operator valued function*

$$S : [0, \infty) \to \mathcal{B}(X), t \mapsto P(t) + Q(t)$$

is a G-semigroup precisely if

$$P(0)Q(t + s) + Q(0)P(t + s) = P(t)Q(s) + Q(t)P(s), \qquad s, t \geqslant 0.$$

Proof: P and Q are strongly continuous. Thus, since the mapping

$$+ : X \times X \to X, (x, y) \mapsto x + y$$

is continuous, $S(.)x = P(.)x + Q(.)x$ is continuous for all $x \in X$.
Therefore, S is strongly continuous. Let t, s be in $[0, \infty)$. For the additivity of S the following identity must hold true.

$$(P(0) + Q(0))(P(t + s) + Q(t + s)) = (P(t) + Q(t))(P(s) + Q(s))$$

Expanding the terms shows that (1.2) holds true precisely if

$$\begin{aligned} &P(0)P(t + s) + Q(0)Q(t + s) + P(0)Q(t + s) + Q(0)P(t + s) = \\ &= P(t)P(s) + Q(t)Q(s) + Q(t)P(s) + P(t)Q(s). \end{aligned}$$

Using the additivity of P and Q this is equivalent to

$$P(0)Q(t+s) + Q(0)P(t+s) = Q(t)P(s) + P(t)Q(s).$$

■

We finish this section of basic information about G-semigroups by studying if or in which sense a generator determines a G-semigroup uniquely.

REMARK 1.36 Contrary to classic semigroup theory, already for Pre-semigroups a one-to-one relation between Pre-semigroup and generator does not hold. Indeed, for generalised semigroups this can be seen by considering Corollary 1.30: For a given G-semigroup P and an injective operator C that commutes with P, the G-semigroup CP has the same generator as P. For this reason, one considers G-semigroups with the same operator at zero when discussing uniqueness.

We need the following product rule for differentiation of Banach space valued functions.

LEMMA 1.37 ***Product Rule***. *Let $a, b \in \mathbb{R}$ with $a < b$ and $s \in [a, b]$. Let $W(.) : [a, b] \to \mathcal{B}(X)$ be a strongly continuous function, i.e.*

$$W(.)x \in C([a, b], X) \qquad \text{for all } x \in X.$$

Assume that $W(.)x$ is strongly differentiable at s for some $x \in X$. Furthermore, let $v : [a, b] \to X$ be strongly differentiable at s with $v(s) = x$. Then, the function

$$W(.)v(.) : [a, b] \to X : t \mapsto W(t)v(t)$$

is strongly differentiable at s and

$$\big(W(.)v(.)\big)'(s) = W(s)v'(t) + W'(s)v(s), \tag{1.24}$$

where $W'(s)x := [W(.)x]'(s)$. The corresponding assertion holds true for the strong left or right derivative.
If $W(.)x \in C^1([a, b], X)$ for all x in a linear subspace $U \subset X$ and $v : [a, b] \to U$ belongs to $C^1([a, b], X)$, then $W(.)v(.)$ is continuously differentiable on $[a, b]$ and (1.24) holds true for all $s \in [a, b]$.

Proof: Set $g(.) = W(.)v(.)$. We consider the differential quotient of g at s. With elementary rearrangements we get

$$\begin{aligned}\frac{1}{h}(g(s+h) - g(s)) &= \frac{1}{h}[W(s+h)v(s+h) - W(s)v(s)] \\ &= W(s+h)\frac{v(s+h)-v(s)}{h} + \frac{1}{h}[W(s+h) - W(s)]v(s) \\ &= \underbrace{W(s+h)\Big[\frac{v(s+h)-v(s)}{h} - v'(s)\Big]}_{=:\alpha} + \underbrace{W(s+h)v'(s)}_{=:\beta} \\ &\quad + \underbrace{\frac{1}{h}[W(s+h) - W(s)]v(s)}_{=:\gamma}\end{aligned}$$

for $s+h \in [a,b]$. As a consequence of he Principle of the Uniform Boundedness theorem, $\|W(s+h)\|$ is bounded by a constant M for h in a compact interval. Therefore, we can write

$$\alpha \leqslant M \left\| \frac{v(s+h)-v(s)}{h} - v'(s) \right\|,$$

where the right hand side tends to 0 for $h \to 0$, since v is strongly differentiable at s. For β we have

$$\beta \overset{h \to 0}{\to} W(s)v'(s),$$

since $W(.)$ is strongly continuous. Finally,

$$\gamma \overset{h \to 0}{\to} W'(s)v(s),$$

because $v(s) = x$ and $W(.)x$ is strongly differentiable. Altogether, this shows that g is strongly differentiable at s and for the derivative we have

$$g'(s) = W(s)v'(s) + W'(s)v(s).$$

The second part follows directly from first assertion and using that g' is continuous by the assumptions on W and v. ■

We point out that $W'(t)v(t)$ is not the composition of the operators "$W'(t)$" and $v(t)$.

THEOREM 1.38 *Let the relation A be the generator of G-semigroups P and Q with $P(0) = Q(0)$. Then*

$$P(t)x = Q(t)x \qquad \textit{for all } x \in P(0) \operatorname{dom} A$$

and for all $t \geqslant 0$. If, in addition, P and Q are Pre-semigroups, then

$$P(t)x = Q(t)x \qquad x \in \operatorname{dom} A. \tag{1.25}$$

Proof: The technique of the proof is similar to the one done for Pre-semigroups [Sch09]. We fix $(x, y) \in A$ and $t > 0$ and show that the function

$$f_t : [0, t] \to X, s \mapsto P(t - s)Q(s)x$$

is constant. Since $(Q(.)x, Q(.)y) \in D_0^t$, it follows by the Product Rule, Lemma 1.37, that f_t is continuously differentiable. Note that the assumptions of Lemma 1.37 are satisfied since $Q(s)x \in \operatorname{dom} A$ for all $s \in [0, t]$. Moreover,

$$\frac{d}{ds} f_t(s) = P(t - s)[Q(.)x]'(s) + [P(. - t)Q(s)x]'(s)$$

By Lemma 1.14, $(Q(s)x, Q(s)y) \in A$. Hence,

$$[P(. - t)Q(s)x]'(s) = -P(s - t)Q(s)y.$$

Together with $[Q(.)x]'(s) = Q(s)y$ this yields,

$$\frac{d}{ds} f_t(s) = P(t - s)Q(s)y - P(t - s)Q(s)y = 0.$$

for all $s \in [0, t]$. Therefore, f_t is constant and evaluating $s = 0$ and $s = t$ gives

$$P(t)Q(0)x = P(0)Q(t)x.$$

Since $P(0) = Q(0)$ by assumption, we have by the commutativity

$$P(t)P(0)x = Q(t)P(0)x \tag{1.26}$$

for any $t > 0$.

For the assertion about the Pre-semigroups case consider

$$P(0)P(t)x = P(0)Q(t)x,$$

which is equivalent to (1.26) by the commutativity of a G-semigroup. By the injectivity of $P(0)$, (1.25) follows. ∎

The result for classic semigroups can be formulated as a corollary,

COROLLARY 1.39 *Let P and Q be classic semigroups with generator A. Then*

$$P \equiv Q \qquad \text{on } X.$$

Proof: Clearly, $P(0) = Q(0) = \mathrm{I}$. Thus, by Theorem 1.38, P and Q coincide on $\operatorname{dom} A$ and therefore on $\overline{\operatorname{dom} A}$ by continuity. By Lemma 1.16 and since $\operatorname{ran} P(0) = X$, the domain of A lies dense in X. ∎

1.3 Degenerate Semigroups

In this section the focus is on a special class of G-semigroups. Degenerate Semigroups have already been used in the theory of differential equations, [FY99]. We further refer to [BC02], where they were used in a context with linear relations. Our defintion of a *degenerate semigroup* is chosen to fit in with our point of view.

DEFINITION 1.40 *A G-semigroup P, where*

- *$P(0)$ is a projection, i.e. $P(0)^2 = P(0)$,*
- *$\ker P(0) \subset \ker P(s)$ for all $s \in (0, \infty)$,*

*is called **degenerate semigroup**.*

REMARK 1.41 We point out that the second property from above, i.e. the inclusion of the kernel of $P(0)$, is not naturally given for G-semigroups. For instance, consider Example 1.33. There,

$$\ker P(0) = X \times X \supset X \times \{0\} = \ker P(s)$$

holds for all $s \geqslant 0$. However, for a G-semigroup $P = CT$, where T is a Pre-semigroup and $C \in \mathcal{B}(X)$ such that C and T commute, we have

$$\ker P(0) = \ker CT(0) = \ker T(0)C = \ker C \subset \ker T(s)C = \ker P(s), \qquad s > 0$$

Here we used that $T(0)$ is injective. Therefore, for this class of G-semigroups, this property is already guaranteed.

As indicated above, in literature the notion of a degenerate semigroup occurs with a different definition. The following lemma shows equivalence.

LEMMA 1.42 *Let P be a G-semigroup. Then following assertions are equivalent:*

1. *P is a degenerate semigroup.*
2. *$P(s+t) = P(s)P(t)$ for all $s, t \geqslant 0$.*

Proof: 1.⇒2.: Since $P(0)$ is a bounded projection, we can write X as $X = \ker P(0) \oplus \operatorname{ran} P(0)$. For every $x \in X$ there exist $x_r \in \operatorname{ran} P(0)$ and $x_k \in \ker P(0)$ such that $x = x_r + x_k$. Furthermore, $P(t)x_k = 0$. Hence $\ker P(0) \subset$

$\ker P(t)$ for all $t \geqslant 0$ by definition of a degenerate semigroup. By additivity, commutativity and the fact that $P(0)$ is a projection, we deduce

$$\begin{aligned} P(t+s)x &= P(t+s)x_r + P(t+s)x_k = P(t+s)P(0)x_r \\ &= P(0)P(t+s)x_r = P(t)P(s)x_r \\ &= P(t)P(s)x \end{aligned}$$

for $t, s \geqslant 0$.
2.⇒1.: Evaluating at $s = t = 0$ in the assumed identity, it follows that $P(0)^2 = P(0)$, i.e. $P(0)$ is a projection. For $x \in \ker P(0)$ we obtain

$$P(t+0)x = P(t)P(0)x = 0.$$

Hence,

$$\ker P(0) \subseteq \ker P(s) \qquad \text{for all } s \in (0, \infty).$$

■

The previous lemma indicates a tight connection between this class of G-semigroups and classic semigroups. We note that degenerate semigroups are precisely the objects that emerge if one neglects the condition $P(0) = I$ for classic semigroups.

THEOREM 1.43 *For a degenerate semigroup P the following properties hold true.*

1. $\ker P(0)$ *and* $\operatorname{ran} P(0)$ *are closed and* $P(t)$ *invariant for all* $t \geqslant 0$.
2. *Considering the decomposition* $X = \operatorname{ran} P(0) \oplus \ker P(0)$, P *can be written as*
$$P = \begin{pmatrix} P_r & 0 \\ 0 & 0_{\ker P(0)} \end{pmatrix},$$
where $P_r : [0,\infty) \to \mathcal{B}(\operatorname{ran} P(0)), t \mapsto P(t)|_{\operatorname{ran} P(0)}$ *is a semigroup.*
3. P *can be written as* $P = P(0)T = TP(0)$ *for a semigroup* T *on* X.

Proof: 1. $\ker P(0)$ and $\operatorname{ran} P(0) = \ker(\mathrm{I} - P(0))$ are closed since $P(0)$ is a projection. For the rest of the assertion we use Lemma 1.42. By the commutativity it follows for $x_r \in \operatorname{ran} P(0)$, $x_k \in \ker P(0)$ that

$$P(t)x_r = P(t)P(0)x_r = P(0)P(t)x_r, \qquad P(0)P(t)x_k = P(t)P(0)x_k = 0$$

for $t \geqslant 0$. This shows the invariance.
2. Since $\operatorname{ran} P(0)$ is a closed subspace of X, it is a Banach space. P_r is well defined by point 1. The strong continuity of P_r holds due to the strong continuity of P because the norms of X and $\operatorname{ran} P(0)$ coincide. Clearly,

$P(0)x = x$ for $x \in \operatorname{ran} P(0)$ and the classic additivity is fulfilled by Lemma 1.42. Thus, P_r is a classic semigroup. Clearly,

$$P|_{\ker P(0)} \equiv 0_{\ker P(0)}.$$

3. Set

$$T(t) = \begin{pmatrix} P_r(t) & 0 \\ 0 & \mathrm{I}_{\ker P(0)} \end{pmatrix}.$$

Since $T_2 := \mathrm{I}_{\ker P(0)}$ and P_r are semigroups, it can easily be shown that T is a semigroup, too. Obviously, $P(0)T = TP(0) = P$. ■

REMARK 1.44 Theorem 1.43 shows that we can view the situation from the classic semigroup approach. The special situation that P can be written as composition of a classic semigroup T and a bounded operator that commutes with T can be used to adapt many of the results from the classic theory. For instance, the property that strong continuity at zero implies already strong continuity on the whole interval $[0, \infty)$ (see [EN00]), is handed on to degenerate semigroups. Indeed, if $Q : [0, \infty) \to \mathcal{B}(X)$ is a function which is strongly continuous at 0 and satisfies,

$$Q(t + s) = Q(t)Q(s) \qquad \text{for all } t, s \geqslant 0,$$

then the version of Theorem 1.43 with continuity just assumed at 0 shows that Q can be written as $Q = Q(0)T$ where T is continuous at 0 and has the algebraic properties of a classic semigroup. By classic theory, this implies that T is already strongly continuous on $[0, \infty)$, hence, a semigroup. Therefore, also Q is strongly continuous on $[0, \infty)$.
Furthermore, degenerate semigroups are always exponentially bounded, but this is not the case for all G-semigroups as we will see later.

Next, we are going to use the decomposition from Theorem 1.43 for calculating the generator.

THEOREM 1.45 *Let P be a degenerate semigroup. For the generator A we have*

$$A = A_s = A_{P_r} \boxplus (\ker P(0) \times \ker P(0)),$$

for $s \geqslant 0$, where A_{P_r} denotes the generator of $P_r = P|_{\operatorname{ran} P(0)}$ (see Theorem 1.43).

Proof: By Theorem 1.43 we know that $P = P(0)T$ for a classic semigroup T. Thus, Corollary 1.29 implies $A_s = A$ for all $s \geqslant 0$. Let $(x, y) \in A_0$, i.e. $P(0)y = [P(.)x]'(0)$. Set $(x_r, y_r) = (P(0)x, P(0)y)$. Again by Theorem

1.43, $P(t)x = P(t)x_r$ for all $t \geqslant 0$ and thus, $P(0)y_r = [P(.)x_r]'(0)$. Hence, $(x_r, y_r) \in A_{P_r}$. This gives,

$$(x, y) \in A_{P_r} \boxplus (\ker P(0) \times \ker P(0)).$$

Conversely, consider $x = x_r + x_k$, $y = y_r + y_k$ with $(x_r, y_r) \in A_{P_r}$ and $x_k, y_k \in \ker P(0)$. This yields

$$P(0)y = P(0)y_r = [P(.)x_r]'(0) = [P(.)x]'(0),$$

which implies $(x, y) \in A_0$. ■

Examples of degenerate semigroups can easily be constructed from classic semigroups.

***Example* 1.46** Let Y, Z be closed subspaces of X, such that $X = Y \oplus Z$.

$$C = \begin{pmatrix} \mathrm{I}_Y & 0 \\ 0 & 0_Z \end{pmatrix}, \quad T(t) = e^{tD} \in \mathcal{B}(X),$$

for an operator $D \in \mathcal{B}(X)$ of the form

$$D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}.$$

Obviously, C commutes with D and hence with T, as shown in example 1.5. Elementarily, we have

$$\ker CP(t) = \ker P(t)C \supset \ker C = Z$$

for $t \geqslant 0$. Hence, CT is a degenerate semigroup since C is projection. Moreover using that C is a projection, we obtain

$$Ce^{tD} = C \sum_{n=0}^{\infty} \frac{t^n D^n}{n!} = C \sum_{n=0}^{\infty} \frac{t^n (CD)^n}{n!} = Ce^{tCD},$$

where CD can be seen as the restriction of D to $\operatorname{ran} C = Y$. As shown in Theorem 1.45, the generator then is

$$A = CD \boxplus (Z \times Z).$$

1.4 G-Semigroups on Product Spaces

In the previous section, the space X was decomposed into a direct product via the projection $P(0)$. Consequently, the semigroup could be split up into its parts on the subspaces $\operatorname{ran} P(0)$ and $\ker P(0)$. In general, if one considers G-semigroups P_X and P_Y on Banach spaces X and Y, it is straight forward to define a semigroup on the product space $X \times Y$.

DEFINITION 1.47 *Let X, Y be Banach spaces and denote P_X and P_Y G-semigroups on X and Y, respectively. The G-semigroup P on the product space $X \times Y$ is defined as a block operator*

$$P(t)\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} P_X(t) & 0 \\ 0 & P_Y(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for all } t \geqslant 0$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y$.

Although we called P a *G-semigroup* in definition 1.47, this fact has to be shown. In the upcomming lemma the following mapping will be used.

DEFINITION 1.48 *For Banach spaces X and Y, let μ denote the mapping*

$$\mu : X^2 \times Y^2 \to (X \times Y)^2 : (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_2 \\ y_2 \end{pmatrix}) \mapsto (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}).$$

Here X^2 and Y^2 denote the product spaces $X \times X$ and $Y \times Y$, respectively.

LEMMA 1.49 *The family of operators P introduced in definition 1.47 is indeed a G-semigroup. The relations $A_{X,r}$, $A_{Y,r}$ and A_r are related as follows*

$$\begin{aligned} A_r &= \mu(A_{X,r} \times A_{Y,r}) \\ &= \left\{ (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \in (X \times Y)^2 : (x_1, x_2) \in A_{X,r}, (y_1, y_2) \in A_{Y,r} \right\}, \end{aligned}$$

for $r \geqslant 0$. For the generator A of P we have

$$\begin{aligned} A &= \mu(A_X \times A_Y) \\ &= \left\{ (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \in (X \times Y)^2 : (x_1, x_2) \in A_X, (y_1, y_2) \in A_Y \right\}. \end{aligned}$$

Proof: To show that P is strongly continuous, one directly uses the strong continuity on X and Y. Fix $\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y$. Then

$$P(.)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P_X(.)x \\ P_Y(.)y \end{pmatrix},$$

which is clearly continuous as a function from $[0, \infty)$ into $X \times Y$ equipped with the sum norm. The additivity of P is clear by the form of P and the additivity of P_X and P_Y. Furthermore, consider $(x, y) = (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \in$

$(X \times Y)^2$ and $t, t+h \geqslant 0$. Then,

$$\frac{1}{h}[P(t+h)x - P(t)x] - P(t)y = \begin{pmatrix} \frac{1}{h}[P_X(t+h)x_1 - P_X(t)x_1] - P_X(t)y_1 \\ \\ \frac{1}{h}[P_Y(t+h)x_2 - P_Y(t)x_2] - P_Y(t)y_2 \end{pmatrix}.$$

Let $r \geqslant 0$. By considering the sum norm, it is clear that this term converges to 0 for all $t \in [0, r]$ as $h \to 0$ if and only if $(x_1, x_2) \in A_{X,r}$ and $(y_1, y_2) \in A_{Y,r}$. The assertion for A follows by (1.3). ■

COROLLARY 1.50 *If P_X and P_Y are Pre-semigroups, then P is a Pre-semigroup. The generator, A equals $A_X \times A_Y$. In particular, if P_X and P_Y are classic semigroups, then P is a classic semigroup.*

Proof: This follows by Lemma 1.49 and the fact that,

$$P(0) = \begin{pmatrix} P_X(0) & 0 \\ 0 & P_Y(0) \end{pmatrix},$$

is injective or equals the identity on $X \times Y$ precisely if both $P_X(0)$ and $P_Y(0)$ are injective or both equal the identities on X, Y, respectively. ■

Example **1.51** Let us consider degenerate semigroups. By Theorem 1.43, such a G-semigroup P is of the form

$$P = \begin{pmatrix} P_r & 0 \\ 0 & P_k \end{pmatrix},$$

where $P_k \equiv 0_{\ker P(0)}$. Since we have for the trivial G-semigroup on the space $\ker P(0)$

$$A = A_s = \ker P(0) \times \ker P(0)$$

for all $s \geqslant 0$, we conclude from lemma 1.49 that

$$\begin{aligned} A_P &= \mu(A_{P_r} \times A_{P_k}) \\ &= \Bigg\{ \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \in (\operatorname{ran} P(0) \oplus \ker P(0))^2 : \\ &\qquad (x_1, x_2) \in A_{P_r}, (y_1, y_2) \in (\ker P(0) \times \ker P(0)) \Bigg\} \\ &= A_r \boxplus (\ker P(0) \times \ker P(0)). \end{aligned}$$

We see that this is just our result from Theorem 1.45.

1.5 Factorised G-Semigroups

The difference of general G-semigroups and Pre-semigroups is reflected in the kernel of G-semigroups. Since $\ker P$ is closed, a straight forward idea is to consider the quotient space $X/\ker P$. Then, one can investigate P on $X/\ker P$. We recall that the quotient space $\tilde{X}$ is equipped with the norm $\|.\|_\sim$ (see (0.8) in *Preliminaries*, Chapter 0). For $\tilde{x} \in \tilde{X}$ we have

$$\|\tilde{x}\|_\sim \leqslant \|x\| \tag{1.27}$$

for $x \in \tilde{x}$.

DEFINITION 1.52 *For a G-semigroup P, we define the operator valued function*

$$\tilde{P} : [0, \infty) \to \mathcal{B}(X/\ker P)$$

with

$$\tilde{P}(t)(x + \ker P) = P(t)x + \ker P$$

for $x + \ker P \in (X/\ker P)$ and $t \geqslant 0$.

LEMMA 1.53 *The operator valued function $\tilde{P}$ defined in (1.52) is well defined and a G-semigroup. We will call this G-semigroup the **factorised G-semigroup**.*

Proof: Let $x + \ker P \in (X/\ker P)$ and $t \geqslant 0$. Consider $y \in X$ such that

$$x + \ker P = y + \ker P.$$

Therefore, $x - y \in \ker P$ and, hence, $P(t)x = P(t)y$ and further

$$P(t)x + \ker P = \ker P(t)y + \ker P$$

for all $t \geqslant 0$. Thus, $\tilde{P}(t)(x + \ker P) = \tilde{P}(t)(y + \ker P)$, and we see that $\tilde{P}(t)$ is well-defined. Furthermore, it follows from (1.27) that

$$\left\|\tilde{P}(t)(x + \ker P)\right\|_\sim = \|P(t)(x + z) + \ker P\|_\sim \leqslant \|P(t)(x + z)\| \leqslant \|P(t)\| \, \|x + z\|$$

for all $\in \ker P$. Taking the infimum over $z \in \ker P$, we get

$$\left\|\tilde{P}(t)(x + \ker P)\right\|_\sim \leqslant \|P(t)\| \, \|x + \ker P\|_\sim .$$

Again by (1.27), for $t, t + h \geqslant 0$ we get

$$\left\|(\tilde{P}(t + h) - \tilde{P}(t))(x + \ker P)\right\|_\sim \leqslant \|P(t + h)x - P(t)x\| .$$

By the strong continuity of P the strong continuity of $\tilde{P}$ follows. The additivity of $\tilde{P}$ follows from

$$\begin{aligned}\tilde{P}(t)\tilde{P}(s)(x + \ker P) &= \tilde{P}(t)(P(s)x + \ker P) = P(t)P(s) + \ker P \\ &= P(0)P(t+s)x + \ker P = \tilde{P}(0)(P(t+s)x + \ker P) \\ &= \tilde{P}(0)\tilde{P}(t+s)(x + \ker P),\end{aligned}$$

where we used the additivity of P. ■

The next question is what happens to the kernel of the G-semigroup when one considers the factorised G-semigroup $\tilde{P}$.

LEMMA 1.54 *The kernel,* $\ker \tilde{P}$ *of the factorised G-semigroup is given by*

$$\ker \tilde{P} = \{x + \ker P : P(t)x \in \ker P \quad \textit{for all } t \geqslant 0\}\,.$$

Proof: $x \in \ker \tilde{P}$ is equivalent to $\tilde{P}(t)(x + \ker P) = \tilde{0}$ which, in turn, holds true precisely if $P(t)x \in \ker P$. ■

From the last lemma we see that the kernel of $\tilde{P}$ and, therefore, since $\ker \tilde{P} = \operatorname{mul} \tilde{A}$, the multi-value part of the generator $\tilde{A}$ in general is not trivial.

REMARK 1.55 Let $\pi : X \to (X/\ker P), x \mapsto x + \ker P$ denote the continuous embedding of X in $X/\ker P$. Then, for $t \geqslant 0$, the above considerations can be written in following scheme

$$\begin{array}{ccc} X & \stackrel{P(t)}{\longrightarrow} & X \\ \pi \downarrow & & \downarrow \pi \\ X/\ker P & \stackrel{\tilde{P}(t)}{\longrightarrow} & X/\ker P \end{array}$$

The diagram means that

$$\tilde{P}(t) \circ \pi = \pi \circ P(t). \tag{1.28}$$

Note that $\tilde{P}$ can be written as,

$$\tilde{P}(t) = \pi \circ P(t) \circ \pi^{-1},$$

where π^{-1} is the inverse of π in terms of linear relations.

Finally, we consider the generator of $\tilde{P}$.

THEOREM 1.56 *Let* $\tilde{A} \leqslant (X/\ker P) \times (X/\ker P)$ *be the generator of* $\tilde{P}$ *and let* $\tilde{A}_r$*,* $r \geqslant 0$*, be the linear relations associated with* $\tilde{P}$ *as defined before. Then*

$$(x, y) \in A_r \quad \textit{implies} \quad (\pi(x), \pi(y)) \in \tilde{A}_r. \tag{1.29}$$

and

$$(x,y) \in A \quad \textit{implies} \quad (\pi(x), \pi(y)) \in \tilde{A}. \tag{1.30}$$

Hence, we have

$$(\pi \times \pi)(A_r) \subseteq \tilde{A}_r,$$
$$(\pi \times \pi)(A) \subseteq \tilde{A}.$$

Proof: Let us consider a pair $(x,y) \in A_r$ for $r > 0$. Then,

$$(P(.)x, P(.)y) \in D_0^r(X).$$

Since $\pi \in \mathcal{B}(X, X/\ker P)$, it follows by Lemma 0.32 that

$$(\pi P(.)x, \pi P(.)y) \in D_0^r(X/\ker P),$$

where $\pi P(.)x$ denotes the function $t \mapsto \pi(P(t)x)$. By (1.28),

$$\pi(P(t)x) = \tilde{P}(t)\pi(x)$$

and, hence,

$$(\tilde{P}(.)\pi(x), \tilde{P}(.)\pi(y)) \in D_0^r,$$

Thus, $(\pi(x), \pi(y)) \in \tilde{A}_r$. The argumenation for $r = 0$ is nearly the same (also uses Lemma 0.32). The assertion for A follows from (1.3). ∎

Chapter 2

Exponential Boundedness and Spectral Properties

2.1 Exponential Boundedness

DEFINITION 2.1 *A function of operators $P : [0, \infty) \to \mathcal{B}(X)$ is called **exponentially bounded**, if there exist constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0. \tag{2.1}$$

This property naturally occurs in the case of classic semigroups, see for instance [EN00]. In this case, the bound M in (2.1) has to be greater or equal 1 since $\|P(0)\| = \|I\| = 1$. Clearly, in general, we do not have this restriction on M for G-semigroups any more. Moreover, in the general situation, exponential boundedness does not hold for a G-semigroup. To see this, we bring an example.

***Example* 2.2** Consider the G-semigroup from example 1.26. Let h be

$$h(x) = \begin{cases} 0, & x \in [-1, 1] \\ e^{-4}(|x| - 1), & |x| \in [1, 2] \\ e^{-x^2}, & |x| \geqslant 2 \end{cases}$$

The shape of h can be seen in figure 1.1 (the graph on the right hand side). Obviously, $P(t)$ satisfies

$$\|P(t)\| \leqslant \sup_{x \in \mathbb{R}} \left| h(x)e^{tx} \right|.$$

We calculate $\max_{|x| \geqslant 2} \left| h(x)e^{tx} \right|$ by considering the parabola

$$p : \mathbb{R} \to \mathbb{R}, x \mapsto (-x^2 + tx).$$

For the latter function we get $\frac{t^2}{4}$ for the maximum, which is attained at $\frac{t}{2}$. Depending on whether $\frac{t}{2} \geqslant 2$ or not, the maximum x_m of $p|_{|x|\geqslant 2}$ is attained at $\frac{t}{2}$ or at the boundary $|x| = 2$. Since $p(2) > p(-2)$,

$$\max_{|x|\geqslant 2} |p(x)| = \begin{cases} 2t-4, & 0 \leqslant t < 4, \\ t^2/4, & t \geqslant 4. \end{cases}$$

Hence, by monotony of the exponential function,

$$\max_{|x|\geqslant 2} \left|h(x)e^{tx}\right| = \begin{cases} e^{2t-4}, & 0 \leqslant t < 4, \\ e^{t^2/4}, & t \geqslant 4, \end{cases}$$

which is also attained at x_m. For $x \in [-2, 2]$, $e^{-4} \geqslant |h(x)|$ and, therefore,

$$\max_{|x|\leqslant 2} |h(x)e^{tx}| \leqslant e^{2t-4}.$$

Since $e^{2t-4} \leqslant e^{t^2/4}$, it follows that

$$\sup_{x\in\mathbb{R}} \left|h(x)e^{tx}\right| = \begin{cases} e^{2t-4}, & 0 \leqslant t < 4, \\ e^{t^2/4}, & t \geqslant 4. \end{cases}$$

Now, we can choose a positive function $f \in C_0(\mathbb{R})$ such that f also attains its maximum at x_m. In this case, we have

$$\|P(t)f\|_\infty = \|f\|_\infty \sup_{x\in\mathbb{R}} \left|h(x)e^{tx}\right|.$$

Hence,

$$\|P(t)\| = \begin{cases} e^{2t-4}, & 0 \leqslant t < 4, \\ e^{t^2/4}, & t \geqslant 4. \end{cases}$$

This example shows that we cannot expect a G-semigroup to be exponentially bounded. Moreover, if we modify example 2.2 in the sense that $h(x)$ equals e^{-x^2} on the whole real axis, we deduce that $\|P(t)\| = e^{t^2/4}$ for all $t \geqslant 0$. Since this h has no zeros, $P(0)$ is injective. Hence, we can observe that even for Pre-semigroups we do not necessarily have exponential boundedness. But at least some canonical constructions of G-semigroups preserve this property.

LEMMA 2.3 *Let P be an exponentially bounded G-semigroup with*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \textit{for all } t \geqslant 0,$$

for constants $M \geqslant 0$, $\omega \in \mathbb{R}$. Let $C \in \mathcal{B}(X)$ be an operator that commutes with P. Then $P = CT$ is an exponentially bounded G-semigroup with

$$\|P(t)\| \leqslant \|C\| \, Me^{\omega t}$$

for all $t \geqslant 0$.

Proof: P is a G-semigroup by Theorem 1.28. Since

$$\|T(t)\| \leqslant Me^{\omega t},$$

for all $t \geqslant 0$, the hypothesis follows from the inequality

$$\|CT(t)\| \leqslant \|C\| \, \|T(t)\| \, .$$

■

For a bijective operator $P(0)$, the situation is very similar to the one for classic semigroups.

COROLLARY 2.4 *A G-semigroup P with a bijective $P(0)$ (hence P is already a Pre-semigroup) is exponentially bounded.*

Proof: Since $P(0)^{-1} \in \mathcal{B}(X)$ by the Closed Graph Theorem, we can write

$$P(t) = P(0)P(0)^{-1}P(t),$$

and define $T(t) = P(0)^{-1}P(t)$ for all $t \geqslant 0$. Furthermore, from the additivity we deduce,

$$P(t+s) = P(0)^{-1}P(t)P(s).$$

From this, we see that T is a classic semigroup, indeed, for $s, t \geqslant 0$ we have

$$\begin{aligned} T(t+s) &= P(0)^{-1}P(t+s) = P(0)^{-1}P(0)^{-1}P(s)P(t) \\ &= P(0)^{-1}P(t)P(0)^{-1}P(s) = T(t)T(s), \end{aligned}$$

since $P(0)^{-1}$ as the inverse of $P(0)$ commutes with $P(t)$ for all $t \geqslant 0$. Strong continuity of T is preserved by the continuity of $P(0)^{-1}$. Since classic semigroups are exponentially bounded, lemma 2.3 can be applied. ■

The next lemma is also known from the classic theory and is useful for many proofs where one wants to reduce the situation to a bounded semigroup, i.e.

$$\|T(t)\| \leqslant M,$$

for all $t \geqslant 0$. We will see, that the result is similar for G-semigroups.

LEMMA 2.5 *Let P be a G-semigroup with its generator A and the linear relations A_r, $r \geqslant 0$, defined in chapter 1. For a complex number λ the operator valued function*

$$P_\lambda : [0, \infty) \to \mathcal{B}(X) : t \mapsto e^{\lambda t}P(t)$$

is a G-semigroup with generator

$$A_{P_\lambda} = A + \lambda \mathrm{I}$$

and

$$A_{P_\lambda, r} = A_r + \lambda \mathrm{I}$$

for $r \geqslant 0$.

Proof: Clearly, P_λ is well-defined as a mapping to $\mathcal{B}(X)$. For strong continuity, we argue that $P(.)x$ as well as $(t \mapsto e^{\lambda t})$ are continuous for all $x \in X$. Furthermore, the product mapping

$$\zeta : \mathbb{R} \times X \to X, (c, x) \mapsto cx,$$

is continuous and, hence, so is the composition $P_\lambda x = \zeta(e^{\lambda \cdot}, P(.)x)$. Since $e^{\lambda t}$ is a scalar, the additivity follows directly from the additivity of P and the well-known functional equation of the exponential function.
It remains to show the assertions about the generator and the relations A_r. Let $r \geqslant 0$ and $(x, y) \in A_r + \lambda\,\mathrm{I}$ which is equivalent to $(x, y - \lambda x) \in A_r$. Clearly,

$$(e^{\lambda t}x, e^{\lambda t}(y - \lambda x)) \in A_r \qquad \text{for all } t \geqslant 0.$$

Therefore, by the definition of A_r, we can use the Product Rule 1.37 for differentiating $e^{\lambda \cdot}P(.)x = P(.)e^{\lambda \cdot}x$:

$$[P(.)e^{\lambda \cdot}x]'(s) = P(s)e^{\lambda t}(y - \lambda x) + P(s)\lambda e^{\lambda t}x = P(s)e^{\lambda t}y \qquad \text{for all } s \in [0, r].$$

As before, the derivative has to be understood as the strong right/left derivative for $s = 0$ and $s = r$. This implies $(x, y) \in A_{P_\lambda, r}$. Hence, $A_r + \lambda\,\mathrm{I} \subseteq A_{P_\lambda, r}$. The same argumentation can be applied for the operator $A_{P_\lambda, r} - \lambda\,\mathrm{I}$ and the associated G-semigroup $e^{-\lambda \cdot}P_\lambda(.) = P(.)$, in order to deduce $A_{P_\lambda, r} - \lambda\,\mathrm{I} \subseteq A_r$. Hence, $A_{P_\lambda, r} \subseteq A_r + \lambda\,\mathrm{I}$ by the definition of the sum of linear relations. Altogether we have

$$A_r + \lambda\,\mathrm{I} = A_{P_\lambda, r}.$$

The assertion for A follows analogously by (1.3). ■

REMARK 2.6 For an exponentially bounded G-semigroup P with

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0,$$

for constants $M \geqslant 0$, $\omega \in \mathbb{R}$, the last lemma can be used to construct G-semigroups that are bounded. In fact,

$$\left\| e^{-\lambda t}P(t) \right\| \leqslant Me^{(\omega - \lambda)t} \leqslant M \qquad t \geqslant 0,$$

if $\lambda \geqslant \omega$. For $\lambda > \omega$, P_λ is even decaying to zero as $t \to \infty$.

In the previous remark, we introduced decaying G-semigroups. Here are some nice properties of this special class:

LEMMA 2.7 *Let P be a G-semigroup with*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \textit{for all } t \geqslant 0,$$

where $\omega < 0$ and $M \geqslant 0$. If A denotes the generator, then

$$\ker A = \ker P. \tag{2.2}$$

Proof: For $(x, 0)$ in A we have

$$[P(.)x]'(s) = 0, \qquad \text{for all } s \geqslant 0.$$

Therefore, the mapping $(s \mapsto P(s)x)$ is constant on $[0, \infty)$. Since $\omega < 0$, we have that

$$\|P(s)x\| \leqslant Me^{\omega s} \to 0$$

as $s \to \infty$. Thus, $P(s)x$ has to be 0 for all $s \geqslant 0$. This means, $x \in \ker P$. Conversely, if $x \in \ker P$, then, clearly, the strong right derivative of $P(.)x \equiv 0$ exists and equals zero. Hence, $(x, 0) \in A$. ■

2.2 A Generalised Resolvent

One of the assertions of the famous Hille-Yosida Theorem is that the resolvent set $\rho(A)$ of the generator A of a classic semigroup always contains an open half plane $\mathbb{C}_\omega = \{z \in \mathbb{C} : \mathrm{Re} > \omega\}$ (see Preliminaries, Theorem 0.12). The real number ω is chosen such that exponential boundedness (2.1) is satisfied. This result leads to the question whether the same is true for G-semigroups. The answer is *No*.

We recall following technical lemma.

LEMMA 2.8 *Let $R \leqslant X \times X$ be a linear relation and $T \in \mathcal{B}(X)$ operator. Assume that*

$$RT \supseteq TR. \tag{2.3}$$

Then T commutes with the resolvents of R, i.e.

$$(\lambda - R)^{-1}T = T(\lambda - R)^{-1}$$

for all $\lambda \in \rho(R)$ and $t \geqslant 0$.

Proof: Let λ be in $\rho(R)$ with $\lambda \neq \infty$. From the assumption (2.3) we get

$$(\lambda - R)T \supseteq T(\lambda - R) \tag{2.4}$$

by Lemma 0.23. We multiply (2.4) with the resolvent $(\lambda - R)^{-1}$ from the left and the right hand side and get

$$(\lambda - R)^{-1}(\lambda - R)T(\lambda - R)^{-1} \supseteq (\lambda - R)^{-1}T(\lambda - R)(\lambda - R)^{-1}. \tag{2.5}$$

Lemma 0.23 yields

$$(\lambda - R)^{-1}(\lambda - R) = \mathrm{I}_{\mathrm{dom}\, R} \quad \text{and} \quad (\lambda - R)(\lambda - R)^{-1} = \mathrm{I}, \tag{2.6}$$

since $\operatorname{ran}(\lambda - R)^{-1} = \operatorname{dom} R$. By (2.3) and Lemma 1.13, $\operatorname{dom} R$ is invariant under the operator T and thus,

$$I_{\operatorname{dom} R}\, T(\lambda - R)^{-1} = T(\lambda - R)^{-1}.$$

Using this and (2.6), (2.5) reads

$$T(\lambda - R)^{-1} \supseteq (\lambda - R)^{-1} T. \tag{2.7}$$

Since both $T(\lambda - R)^{-1}$ and $(\lambda - R)^{-1}T$ are operators defined on X, in fact, we have equality in (2.7).
For $\lambda = \infty$, $\lambda \in \rho(R)$ means that $R \in \mathcal{B}(X)$. Therefore, the operators in (2.3) are defined on X and again, equality must hold. ■

LEMMA 2.9 *Let P be a G-semigroup with generator A. Then P commutes with the resolvents of A, i.e.*

$$(\lambda - A)^{-1} P(t) = P(t)(\lambda - A)^{-1}$$

for all $\lambda \in \rho(A)$ and $t \geqslant 0$. The same assertions hold true for the linear relations A_r, $r \geqslant 0$.

Proof: By lemma 1.14 the assumptions of lemma 2.8 are satisfied. ■

Already for Pre-semigroups the resolvent set can be empty as the following example (from [deL94]) shows.

***Example* 2.10** Compared to [deL94], the setting for the following example will be slightly different.
In this example, X will be a Hilbert space. Let B be a densely defined, unbounded operator with $0 \in \rho(B)$. This implies that the powers

$$B^{-n} : X \to \operatorname{dom} B^n \subseteq X$$

exist and are bounded for $n \in \mathbb{N}$. Furthermore, assume that iB is selfadjoint which implies that B generates a bounded classic semigroup T (in fact, T is even a unitary group) by Stone's Theorem (which, for instance, can be found in [EN00], Theorem 3.24). Now consider the product space $X \times X$ and define

$$P(t) = \begin{pmatrix} T(t)B^{-n} & tT(t) \\ 0 & T(t)B^{-n} \end{pmatrix}$$

for all $t \geqslant 0$. One easily sees that P is a G-semigroup. In fact, P can be written as

$$P(t) = \underbrace{\begin{pmatrix} T(t)B^{-n} & 0 \\ 0 & T(t)B^{-n} \end{pmatrix}}_{=:Q_1} + \underbrace{\begin{pmatrix} 0 & tT(t) \\ 0 & 0 \end{pmatrix}}_{=:Q_2}.$$

It follows from Lemma 2.9 that T and B^{-n} commute and hence, $B^{-n}T$ is a G-semigroup by Theorem 1.28. Therefore, by Theorem 1.49, Q_1 is a G-semigroup. Similar as in Example 1.33, it can be seen that also Q_2 is a G-semigroup. To show that the sum of Q_1 and Q_2 is a G-semigroup, we use Lemma 1.35: For $s, t \geqslant 0$ we calculate

$$\begin{aligned} Q_1(0)Q_2(s+t) &= \begin{pmatrix} 0 & (s+t)B^{-n}T(s+t) \\ 0 & 0 \end{pmatrix}, \\ Q_2(0)Q_1(s+t) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ Q_1(s)Q_2(t) &= \begin{pmatrix} 0 & tT(s)B^{-n}T(t) \\ 0 & 0 \end{pmatrix}, \\ Q_2(s)Q_1(t) &= \begin{pmatrix} 0 & sT(t)T(s)B^{-n} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus,

$$Q_1(0)Q_2(t+s) + Q_2(0)Q_1(t+s) = Q_1(t)Q_2(s) + Q_1(t)Q_2(s)$$

and by lemma 1.35, $P = Q_1 + Q_2$ is indeed a G-semigroup. Because B^n is injective,

$$P(0) = \begin{pmatrix} B^{-n} & 0 \\ 0 & B^{-n} \end{pmatrix}$$

is injective as well. Therefore, P is even a Pre-semigroup with

$$\operatorname{ran} P(0) = \operatorname{dom} B^n \times \operatorname{dom} B^n.$$

Let us further consider the operator

$$\begin{aligned} &D : \operatorname{dom} B \times \operatorname{dom} B^n \to X \times X, \\ &D = \begin{pmatrix} B & B^n \\ 0 & B \end{pmatrix}, \end{aligned}$$

for which we will show that its graph is contained in the generator A of P. For that, let $x = (x_1, x_2)^\top \in \operatorname{dom} D$. Since the generator of a Pre-semigroup equals A_0, it suffices to consider the strong right derivative at zero:

$$\begin{aligned} \frac{1}{h}[P(h) - P(0)]x - P(0)Dx = \frac{1}{h} &\begin{pmatrix} (T(h) - \mathrm{I})B^{-n}x_1 + hT(h)x_2 \\ \\ (T(h) - \mathrm{I})B^{-n}x_2 \end{pmatrix} + \\ - &\begin{pmatrix} B^{-n+1}x_1 + x_2 \\ \\ B^{-n+1}x_2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{1}{h}(T(h)-\mathrm{I})B^{-n}x_1 - BB^{-n}x_1 \\ \\ \frac{1}{h}(T(h)-\mathrm{I})B^{-n}x_2 - BB^{-n}x_2 \end{pmatrix} + $$
$$+ \begin{pmatrix} T(h)x_2 - x_2 \\ \\ 0 \end{pmatrix}.$$

Let $h \to 0^+$. The second term tends to zero by the strong continuity of T. Since $B^{-n}x_1, B^{-n}x_2 \in \operatorname{dom} B$, the first term converges to zero because B is the generator of the semigroup T. Therefore, $D \subseteq A$.

We show that for $n > 2$ the generator A has an empty resolvent set. For that, assume first that $\ker(\lambda - B) = \{0\}$. Then, by the definition of D, $\ker(\lambda - D)$ is trivial, hence $(\lambda - D)^{-1}$ is an operator. Define

$$L := \begin{pmatrix} (\lambda - B)^{-1} & -B^n(\lambda - B)^{-2} \\ 0 & (\lambda - B)^{-1} \end{pmatrix}.$$

We can calculate formally

$$(\lambda - D)L =$$
$$\begin{pmatrix} (\lambda - B)(\lambda - B)^{-1} & -(\lambda - B)B^n(\lambda - B)^{-2} + B^n(\lambda - B)^{-1} \\ \\ 0 & (\lambda - B)(\lambda - B)^{-1} \end{pmatrix}$$

and

$$L(\lambda - D) =$$
$$\begin{pmatrix} (\lambda - B)^{-1}(\lambda - B) & -(\lambda - B)^{-1}B^n + B^n(\lambda - B)^{-2}(\lambda - B) \\ \\ 0 & (\lambda - B)^{-1}(\lambda - B) \end{pmatrix}$$

One sees that both $L(\lambda - D)$ and $(\lambda - D)L$ are at least defined on

$$M = (\operatorname{dom} B \cap \operatorname{ran}(\lambda - B)) \times ((\lambda - B)^2 \operatorname{dom} B^{n+2}).$$

Since

$$(\lambda - B)B^n y = B^n(\lambda - B)y, \qquad y \in \operatorname{dom} B^{n+1},$$

it follows that

$$(\lambda - B)^{-1}B^n y = B^n(\lambda - B)^{-1}y, \qquad y \in (\lambda - B)^2 \operatorname{dom} B^{n+2}.$$

Using this commutativity we get

$$L(\lambda - D)x = x = (\lambda - D)Lx, \qquad x \in M.$$

Therefore, at least on the subspace M, $(\lambda - D)^{-1}$ equals L. Hence,

$$L|_M \subseteq (\lambda - D)^{-1} \subseteq (\lambda - A)^{-1}.$$

By results of the classic theory, lemma 0.10, we know that $\operatorname{dom} B^{n+2}$ is a dense subset of X, hence, M a is non-empty subspace of $X \times X$. Since iB is selfadjoint, using the corresponding functional calculus it can be shown that $B^n(\lambda - B)^{-2}$ is unbounded for $n > 2$ on $(\lambda - B)^2 \operatorname{dom} B^{n+2}$. Hence, there exists a sequence x_l in $(\lambda - B)^2 \operatorname{dom} B^{n+2}$ such that

$$\left\|B^n(\lambda - B)^{-2} x_l\right\| \geqslant l \left\|x_l\right\|, \qquad l \in \mathbb{N}.$$

Thus, since $z_l = (0, x_l)^\top \in M$,

$$\begin{aligned}\|L z_l\| &= \left\|B^n(\lambda - B)^{-2} x_l\right\| + \left\|(\lambda - B)^{-1} x_l\right\| \\ &\geqslant l \left\|x_l\right\| = l \left\|z_l\right\|\end{aligned}$$

for all $l \in \mathbb{N}$. Hence, L is unbounded. Therefore, if $(\lambda - A)^{-1}$ is an operator, $(\lambda - A)^{-1}$ is unbounded as an extension of L. Hence, $\lambda \in \sigma(A)$.
For a $\lambda \in \mathbb{C}$ such that $(\lambda - B)$ is not injective, there exists a non-trivial $x_1 \in \operatorname{dom} B$ such that $(\lambda - B)x_1 = 0$. By definition of D, it follows that,

$$(\lambda - D)\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Therefore, $(\lambda - A)$ is not injective, which yields $\lambda \in \sigma(A)$.
Altogether, we can conclude that

$$\sigma(A) = \mathbb{C}.$$

The previous example shows that we cannot expect a non-empty resolvent set of the generator. What makes the example even more interesting, is that the constructed Pre-semigroup is bounded (see Remark 2.6). Having the results from the classic theory in mind, we know that the resolvent $(\lambda - A)^{-1}$ is calculated via the Laplace transform L_λ (see Lemma 0.10). Obviously by definition, we cannot guarantee that L_λ exists if we have no further information like exponential boundedness. Therefore, we will consider exponentially bounded G-semigroups in the following. For exponentially bounded Pre-semigroups, this problem has already been tackled. In [deL94] a weakened resolvent, which is depending on the operator $P(0)$, was introduced. We choose a similar approach for G-semigroups. From this point on we will focus on the generator A of a G-semigroup alone, instead of considering the sets A_r, $r \geqslant 0$ simultaneously.

DEFINITION 2.11 *For $\omega \in \mathbb{R}$, we define the open half plane $\mathbb{C}_\omega$ of complex numbers*

$$\mathbb{C}_\omega = \{z \in \mathbb{C} : \operatorname{Re} z > \omega\}.$$

The following lemma is an elementary consequence of the exponential boundedness.

LEMMA 2.12 *Let P be an exponentially bounded G-semigroup with constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0.$$

Then, the Laplace transform L_λ,

$$L_\lambda x = \int_0^\infty e^{-\lambda s} P(s)x \; ds, \tag{2.8}$$

is a bounded linear operator for all $\lambda \in \mathbb{C}_\omega$. Furthermore,

$$L_\lambda P(t) = P(t) L_\lambda \tag{2.9}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{C}_\omega$.

Proof: Fix $\lambda \in \mathbb{C}_\omega$ and $x \in X$. The improper Riemann integral (2.8) exists since

$$\int_0^\infty \left\| e^{-\lambda s} P(s)x \right\| \; ds \leqslant M \|x\| \int_0^\infty e^{(-\operatorname{Re}\lambda + \omega)s} \; ds = \frac{M}{\operatorname{Re}\lambda - \omega} \|x\| ,$$

where we used the exponential boundedness of P. This inequality also shows that L_λ is a bounded operator. By the commutativity of P and since $P(t) \in \mathcal{B}$,

$$\begin{aligned} L_\lambda P(t)x &= \int_0^\infty e^{-\lambda s} P(s)P(t)x \; ds \\ &= \lim_{n\to\infty} P(t) \int_0^n e^{\lambda s} P(s)x \; ds \\ &= P(t)L_\lambda x \end{aligned}$$

for all $x \in X$, $t \geqslant 0$ and $\lambda \in \mathbb{C}_\omega$. ■

LEMMA 2.13 *Let P be an exponentially bounded G-semigroup with constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0$$

and let A be its generator. Then

$$P(0)L_\lambda^{-1} \subseteq (\lambda - A) \tag{2.10}$$

for any $\lambda \in \mathbb{C}_\omega$.

Proof: Fix $\lambda \in \mathbb{C}_\omega$. We consider the G-semigroup $P_\lambda(t) = e^{\lambda t}P(t)$. By the choice of λ this G-semigroup is exponentially decaying (see Remark 2.6) and its generator A_λ equals $A-\lambda$ by Lemma (2.5). Since $P(0)$ and L_λ are linear operators defined on X, the inclusion 2.10 is equivalent to

$$(L_\lambda x, P(0)x) \in (\lambda - A) \qquad \text{for all } x \in X. \tag{2.11}$$

For every positiv integer n consider the pair

$$(\int_0^n P_\lambda(s)x\, ds, P_\lambda(n)x - P_\lambda(0)x),$$

which is an element of $A-\lambda$ by Lemma 1.15. By the choice of λ we see that

$$\int_0^n P_\lambda(s)x\, ds = \int_0^n e^{\lambda s}P(s)x\, ds \to L_\lambda x,$$

as $n \to \infty$. Furthermore, $P_\lambda(n)x$ converges to 0, because P_λ is exponentially decaying:

$$\left\| e^{\lambda n}P(n)x \right\| \leqslant e^{(\omega - \operatorname{Re}\lambda)n} \|x\| \overset{n\to 0}{\to} 0.$$

Therefore, $P_\lambda(n)x - P_\lambda(0)x \to -P(0)x$ as $n \to \infty$. The closedness of A, Lemma 1.11, yields

$$(L_\lambda x, -P(0)x) \in A - \lambda.$$

Finally, by definiton of the scalar multiplication of linear relations

$$(L_\lambda x, P(0)x) \in \lambda - A$$

■

Elementary rules for calculating with linear relations give the following consequences.

LEMMA 2.14 *Let P be an exponentially bounded G-semigroup with constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \textit{for all } t \geqslant 0.$$

For the generator A, the following inclusions hold for all $\lambda \in \mathbb{C}_\omega$:

$$(\lambda - A)^{-1}P(0) \supseteq L_\lambda \boxplus (\{0\} \times L_\lambda \ker P(0)), \tag{2.12}$$
$$P(0)(\lambda - A)^{-1} \supseteq L_\lambda|_{\operatorname{ran} P(0)}. \tag{2.13}$$

Proof: We use (2.10). Since the inclusion is preserved under the inversion of linear relations, (2.10) is equivalent to

$$(P(0)L_\lambda^{-1})^{-1} \subseteq (\lambda - A)^{-1}.$$

By elementary rules for calculating with relations (see Preliminaries, Lemma 0.23), we have

$$L_\lambda P(0)^{-1} \subseteq (\lambda - A)^{-1}.$$

Multiplying by $P(0)$ from the left or from the right, respectively, leads to

$$L_\lambda \boxplus (\{0\} \times (L_\lambda \ker P(0))) \subseteq (\lambda - A)^{-1} P(0),$$
$$L_\lambda|_{\operatorname{ran} P(0)} \subseteq P(0)(\lambda - A)^{-1},$$

where we used (2.9) and Lemma 0.23 again. ■

REMARK 2.15 *When does equality hold in (2.12)?* To answer this question, we make following observations:
The linear relation $L_\lambda \boxplus (\{0\} \times (L_\lambda \ker P(0)))$ has domain X. Therefore, the inclusion (2.12) is only reflected in the multi-value parts by Lemma 0.18. That means, by lemma 0.21, equality in (2.12) holds precisely if the multi-value parts coincide. In general,

$$\operatorname{mul}((\lambda - A)^{-1} P(0)) = \ker(\lambda - A)$$

by definition of the composition of linear relations. Furthermore, $\ker(\lambda - A) = \ker P$ by Lemma 2.7, Lemma 2.5 and the fact that $\ker(\lambda - A) = \ker(A - \lambda)$. Altogether we see that (2.12) is equivalent to

$$\ker P \supseteq L_\lambda \ker P(0). \tag{2.14}$$

for all $\lambda \in \mathbb{C}_\omega$. As already mentioned, if equality holds in (2.14), then equality holds in (2.12). In particular, in case of a Pre-semigroup, $\ker P(0)$ and hence $\ker P$ equal $\{0\}$. Thus, the multi-value parts are trivial and, therefore,

$$(\lambda - A)^{-1} P(0) = L_\lambda.$$

Hence, $(\lambda - A)^{-1} P(0)$ is a bounded operator. For a classic semigroup P this shows that $\lambda \in \rho(A)$ and $R(\lambda) = L_\lambda$ for $\lambda \in \mathbb{C}_\omega$.

The previous remark motivates the following definition.

DEFINITION 2.16 *For an exponentially bounded G-semigroup P with generator A, we define the **generalised resolvent set** of A as*

$$\rho_{P(0)}(A) = \{\lambda \in \mathbb{C} : \operatorname{ran}(\lambda - A) \supseteq \operatorname{ran} P(0), \ker(\lambda - A) = \{0\}\}.$$

REMARK 2.17 Definition 2.16 can be seen as a generalisation of the situation for Pre-semigroups, where the so-called *C-resolvent set*

$$\rho_C(A) = \{\lambda \in \mathbb{C} : \operatorname{ran}(\lambda - A) \supseteq \operatorname{ran} C, (\lambda - A) \text{ is injective}\}$$

was introduced by R. deLaubenfels (see [deL94]) and where C denoted the operator $P(0)$.

The definition of $\rho_{P(0)}(A)$ implies that $(\lambda - A)^{-1}P(0)$ is an operator for $\lambda \in \rho_{P(0)}(A)$. In fact, we have:

LEMMA 2.18 *Let P be an exponentially bounded G-semigroup P with constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0.$$

Let A be the generator of P. Then

$$(\lambda - A)^{-1}P(0) \in \mathcal{B}(X)$$

for all $\lambda \in \rho_{P(0)}(A)$.

Proof: Let $\lambda \in \rho_{P(0)}(A)$. Since $\ker(\lambda - A) = \{0\}$, $(\lambda - A)^{-1}$ is an operator and by $\operatorname{dom}(\lambda - A)^{-1} = \operatorname{ran}(\lambda - A) \supseteq \operatorname{ran} P(0)$,

$$(\lambda - A)^{-1}P(0)$$

is an operator defined on X. A is closed by Lemma 1.11, therefore, $(\lambda - A)^{-1}$ is closed by Lemma 0.26. Let x_n be a sequence in X with

$$x_n \overset{n\to\infty}{\longrightarrow} x \in X \qquad \text{and} \qquad (\lambda - A)^{-1}P(0)x_n \overset{n\to\infty}{\longrightarrow} y \in X.$$

By the continuity of $P(0)$, $P(0)x_n$ converges to $P(0)x$. The closedness of the operator $(\lambda - A)^{-1}$ yields

$$(\lambda - A)^{-1}(P(0)x_n) \longrightarrow (\lambda - A)^{-1}P(0)x$$

as $n \to \infty$. Since the limit is unique, $y = P(0)x$. Hence, $(\lambda - A)^{-1}P(0)$ is closed and, therefore,

$$(\lambda - A)^{-1}P(0) \in \mathcal{B}(X)$$

by the Closed Graph theorem. ■

LEMMA 2.19 *Let P be an exponentially bounded G-semigroup P with constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that*

$$\|P(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0.$$

Let A be the generator of P and assume that $\ker P = \{0\}$. *Then,*

$$\rho_{P(0)}(A) \supseteq \mathbb{C}_\omega$$

and, hence, the mapping

$$R : \mathbb{C}_\omega \to \mathcal{B}(X), \lambda \mapsto (\lambda - A)^{-1}P(0)$$

is well-defined.

Proof: By remark 2.15, in particular (2.14), we have

$$(\lambda - A)^{-1}P(0) = L_\lambda$$

for all $\lambda \in \mathbb{C}_\omega$. By lemma 2.8, $L(\lambda)$ belongs to $\mathcal{B}(X)$. ■

Chapter 3

Differential Inclusions

Now we are going to use the theory of G-semigroups in the context of differential inclusions. This is a generalisation of the *Abstract Cauchy Problem* which is tightly connected to classic semigroups. The first section covers a short introduction on differential inclusions and some general properties. More detailed information can, for example, be found in [FY99].
In this chapter, A will always denote a linear relation on X.

3.1 The Differential Inclusion

Maybe the most important application of semigroups is the study of the *Abstract Cauchy Problem*

$$\dot{u} = Au, \qquad u(0) = x, \tag{3.1}$$

for a linear operator A. Here, A is in general not bounded. If A is the generator of a semigroup T, the classic theory provides the unique solution

$$u(.) \equiv T(.)x.$$

In case that A is the generator of a Pre-semigroup P, the solution is defined analogously. Of course, x has to be in $P(0)\operatorname{dom} A$ so that $u(0) = x$ can be satisfied. For classic semigroups this condition reduces to $x \in \operatorname{dom} A$.
If we consider a G-semigroup P, we are automatically lead to the fact that

$$(P(t)x, [P(.)x]'(t)) \in A$$

for all $t \geqslant 0$ by using the definition of the generator and Lemma 1.14. This motivates the following generalisation of (3.1).

DEFINITION 3.1 *For a linear relation A and $x \in X$, we consider the following task:*
Find a function $u : [0, \infty) \to X$ such that

$$(u(t), \dot{u}(t)) \in A \qquad \textit{for all } t \geqslant 0, \qquad u(0) = x. \tag{3.2}$$

We call this the ***Differential Inclusion*** *for A with the initial value x. For short, this will be written as*

$$(u, \dot{u}) \in A, \qquad u(0) = x. \tag{3.3}$$

A (classic) ***solution*** *of (3.3) is a function $u \in C^1([0,\infty), X)$ which satisfies (3.2). This implies that $u(t) \in \operatorname{dom} A$ for all $t \geqslant 0$. Clearly, a classic solution can only exist for $x \in \operatorname{dom} A$.*

Trivially, every Abstract Cauchy Problem (3.1) can be interpreted as a special case of a Differential Inclusion, where $\operatorname{mul} A = \{0\}$.
Non-trivial examples of differential inclusions naturally occur in the following setting.

***Example* 3.2** Consider the following differential equation for linear operators F, G:

$$F\dot{u} = Gu, \qquad u(0) = x, \tag{3.4}$$

and a function $u : [0, \infty) \to X$. Furthermore, assume that F is not invertible, i.e. $\ker F \neq \{0\}$. Hence F^{-1} and, therefore, $F^{-1}G$ is a linear relations with non-trivial multi-value part. In terms of linear relations (3.4) can be written as

$$(u, F\dot{u}) \in G, \qquad u(0) = x,$$

which is equivalent to

$$(u, \dot{u}) \in F^{-1}G, \qquad u(0) = x. \tag{3.5}$$

Therefore, we can write (3.4) in the form (3.3).

Next we introduce the notion of a mild solution.

DEFINITION 3.3 *A function $u \in C([0,\infty), X)$ is called a* ***mild solution*** *of the Differential Inclusion (3.3), if for all $t \geqslant 0$*

$$\left(\int_0^t u(s)\ ds, u(t) - x\right) \in A.$$

REMARK 3.4 We remark that the term of a mild solution arises by integrating (3.3) for closed A. Indeed, let u be a classic solution of the Differential Inclusion for A with initial value x, and consider the pair

$$\left(\int_0^t u(s) ds, u(t) - x\right)$$

for $t \geqslant 0$. By the Fundamental theorem of Calculus and since $u(0) = x$,

$$u(t) - x = \int_0^t \dot{u}(s)\ ds. \tag{3.6}$$

Since u is a classic solution of the Differential Inclusion (3.3), we have

$$(u(s), \dot{u}(s)) \in A \qquad \text{for all } s \geqslant 0. \tag{3.7}$$

Consider the Riemann sums

$$S(\mathcal{R}, u) = \sum_{j=1}^{n(\mathcal{R})} (\xi_j - \xi_{j-1}) u(s_j), \qquad S(\mathcal{R}, \dot{u}) = \sum_{j=1}^{n(\mathcal{R})} (\xi_j - \xi_{j-1}) \dot{u}(s_j),$$

where

$$\mathcal{R}: \quad 0 = \xi_1 < \ldots < \xi_{n(\mathcal{R})} = t, \qquad s_j \in [\xi_{j-1}, \xi_j],$$

denotes the partition of the interval $[0, t]$. Clearly, by linearity of A and (3.7),

$$(S(\mathcal{R}, u), S(\mathcal{R}, \dot{u})) \in A$$

for all partitions $\mathcal{R}$. By refining the partitions, the net of pairs converges

$$(S(\mathcal{R}, u), S(\mathcal{R}, \dot{u})) \overset{|\mathcal{R}| \to 0}{\longrightarrow} (\int_0^t u(s)\ ds, \int_0^t \dot{u}(s)\ ds),$$

since $u, \dot{u}$ are continuous, hence, Riemann integrable. By the closedness of A and (3.6),

$$(\int_0^t u(s) ds, u(t) - x) \in A$$

Hence, u is a mild solution of (3.3).
We further point out that, in contrast to the classic solutions, a mild solution of (3.3) can exist for all $x \in X$.

We will see that if A is the generator of a G-semigroup P, we will be able to construct mild and classic solutions for a certain set of initial values x. The following notions cover these properties in a more general form.

DEFINITION 3.5 *Let C be in $\mathcal{B}(X)$. An operator valued function*

$$Q : [0, \infty) \to \mathcal{B}(X)$$

is called a ***mild C-existence family*** *of A, if*

- *Q is strongly continuous,*
- *$(\int_0^t Q(s)x\ ds, Q(t)x - Cx) \in A$ for all $t \geqslant 0$ and for all $x \in X$.*

DEFINITION 3.6 *A function of operators $Q : [0, \infty) \to \mathcal{B}(X)$ is called a* ***C-existence family*** *of A, if*

- *Q is strongly continuous,*
- *$Q(.)x$ is in $C^1([0, \infty), X)$ for all $x \in \operatorname{dom} A$,*

- *For* $x \in \operatorname{dom} A$ *we have* $(Q(t)x, Q'(t)x) \in A$ *for all* $t \geqslant 0$,
- $Q(0)x = Cx$.

REMARK 3.7 The notion of existence families is adapted from [deL94], where a similar definition is given for a classic Abstract Cauchy Problem. The main difference is in the definition of the C-existence family: In [deL94] a C-existence family maps $[0, \infty)$ to $\mathcal{B}([\operatorname{dom} A])$, where $[\operatorname{dom} A]$ denotes the space $\operatorname{dom} A$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|$$

for a closed operator A. We remark that the same notion can be introduced for linear relations where the norm on $\operatorname{dom} A$ is the graph norm on the product space $X \times X_\sim$. Here $X_\sim$ denotes the quotient space $X/\operatorname{mul} A$. See *Preliminaries*, Corollary 0.35.
In further consequence, a *strong C-existence family* is defined in [deL94], which combines the notion of a mild C-existence family and this different notion of a C-existence family: In fact, if a mild C-existence family restricted to $[\operatorname{dom} A]$ is a C-existence family, then it is called a strong C-existence family.

Because of the continuity assumption, (mild) C-existence families have the following basic properties.

LEMMA 3.8 *If* Q *is a mild C-existence family of* A, *then the Differential Inclusion (3.3) has a mild solution* $u(., z)$ *for all initial values* $z \in \operatorname{ran} C$ *given by*

$$u(.) \equiv Q(.)x$$

where $x \in X$ *is chosen such that* $z = Cx$.
For a sequence $x_n \in X$ *with* $x_n \to 0$ *it follows that* $u(., Cx_n) \equiv Q(.)x_n$ *converges to zero uniformly on compact subsets of* $[0, \infty)$.

Proof: Let $z \in \operatorname{ran} C$. By definition of a mild C-existence family $Q(.)x$ is a mild solution of the Differential Inclusion for A and the initial value $z = Cx$. Also by definition, Q is strongly continuous and, therefore, $\|Q(t)\|$ is uniformly bounded on compact subsets in $[0, \infty)$ by the Principle of Uniform Boundedness. Hence,

$$\|Q(t)x_n\| \leqslant M_K \|x_n\|$$

for t in a compact set K and a suitable constant $M_K \geqslant 0$. ■

LEMMA 3.9 *If Q is a C-existence family of A, then the Differential Inclusion (3.3) has a classic solution $u(.,z)$ for all initial values $z \in C(\operatorname{dom} A)$ given by*

$$u(.) \equiv Q(.)x$$

where $x \in \operatorname{dom} A$ is chosen such that $z = Cx$.
For a sequence $x_n \in \operatorname{dom} A$ with $\|x_n\| \to 0$ it follows that $u(., Cx_n) \equiv Q(.)x_n$ converges to zero uniformly on compact subsets of $[0, \infty)$.

Proof: Let $z \in C(\operatorname{dom} A)$ and let $x \in \operatorname{dom} A$ such that $Cx = z$. By definition of a C-existence family of A, $Q(.)x$ is a classic solution of the Differential Inclusion for A and the initial value $z = Cx$. As in the previous proof, strong continuity of Q and the Principle of Uniform Boundedness yield the uniform boundedness of $\|Q(t)\|$ on compact sets and hence the desired convergence of $Q(.)x_n$ ■

REMARK 3.10 Lemma 3.8 gives a first answer to the question if the Differential Inclusion problem is *well-posed*. Although there exist different notions of *well-posedness*, we will focus on the typical property that the solutions are arbitrarily close in some sense, if the initial values are sufficiently close.
By lemma 3.8, a mild C-existence family gives us the mild solutions of the Differential Inclusion (3.3) for all initial values in the range of the operator C. This motivates to consider the Banach space $[\operatorname{ran} C]$ equipped with the norm

$$\|y\|_{\operatorname{ran} C} = \inf \{\|x\| : Cx = y\},$$

see *Preliminaries*, Corollary 0.37. Let us consider a sequence $z_n \to z$ in $[\operatorname{ran} C]$. Then there exists a sequence $x_n \to x$ in X such that $Cx_n = z_n$ and $Cx = z$. For instance, such a sequence x_n can be constructed as follows: By definition of the norm $\|.\|_{\operatorname{ran} C}$, we always find a sequence $\tilde{x}_n \in X$, such that $C\tilde{x}_n = z_n - z$ and

$$\|z_n - z\|_{\operatorname{ran} C} \leqslant \|\tilde{x}_n\| \leqslant \|z_n - z\|_{\operatorname{ran} C} + \frac{1}{n}.$$

Hence, $\tilde{x}_n$ converges to 0. Now choose an $x \in X$ fulfilling $Cx = z$. Then, $x_n := \tilde{x}_n - x$ is of the desired form.
Lemma 3.8 shows that the mild solutions $u(., z_n) = Q(.)x_n$ converge to the mild solution $u(., z) = Q(.)x$ uniformly on compact subsets of $[0, \infty)$. Therefore, if one has a mild C-existence family, the Differential Inclusion is well-posed in this sense.

We have not yet discussed uniqueness of the solution and the mild solution. For a (mild) C-existence family this is also depending on the operator C.

Under some assumption about the point spectrum of the linear relation A, we can achieve at least uniqueness for exponentially bounded solutions. To proof that, we need an elementary lemma first.

LEMMA 3.11 *Let A be closed and let $u \in C([0,\infty), X)$ be a mild solution of the Differential Inclusion (3.3) for A with initial value $x \in X$. Assume that u is exponentially bounded, i.e.*

$$\|u(t)\| \leqslant Me^{\omega t} \qquad t \geqslant 0,$$

for constants $M \geqslant 0$ and $\omega \in \mathbb{R}$. Then, for $z \in \mathbb{C}$ with $\operatorname{Re} z > \omega$ the Laplace transform $L_z u$,

$$L_z u = \int_0^\infty e^{-zs} u(s)\ ds,$$

exists and the inclusion

$$(L_z u, x) \in (z\,\mathrm{I} - A)$$

holds true.

Proof: Fix $z \in \mathbb{C}$ with $\operatorname{Re} z > \omega$. First, we show that the Laplace transform, $L_z u$, exists. Because of the exponential boundedness of u

$$\begin{aligned} \int_0^\infty \left\|e^{-zs} u(s)\right\|\ ds &= \int_0^\infty e^{-s \operatorname{Re} z} \left\|u(s)\right\|\ ds \\ &\leqslant M \int_0^\infty e^{(\omega - \operatorname{Re} z)s}\ ds \\ &= \frac{M}{\operatorname{Re} z - \omega}. \end{aligned} \tag{3.8}$$

As X is Banach space this implies that

$$L_z u = \int_0^\infty e^{-zs} u(s)\ ds$$

exists. Integrating by parts, we see that

$$\int_0^N e^{-zs} u(s)\ ds = e^{-zN} \int_0^N u(t)\ dt + z \int_0^N e^{-zs} \int_0^s u(s)\ dt\ ds. \tag{3.9}$$

Since u is a mild solution and by the linearity of A, we have

$$\left(ze^{-zs} \int_0^s u(t)dt, ze^{-zs}(u(s) - x)\right) \in A$$

for all $s \geqslant 0$. By the closedness of A, we deduce

$$\Big(\int_0^N ze^{-zs} \int_0^s u(t)\ dt\ ds, \int_0^N ze^{-zs}(u(s) - x)\ ds\Big) \in A,$$

because the integrals exist and can be interpreted as limits of Riemann sums. By (3.9),

$$\Big(\int_0^N e^{-zs} u(s)\ ds - e^{-zN} \int_0^N u(t)\ dt, \int_0^N ze^{-zs} u(s)\ ds - \int_0^N ze^{-zs} x\ ds\Big) \in A.$$

By the exponential boundedness of u and $\operatorname{Re} z > \omega$ we have

$$e^{-zN} \int_0^N u(t)dt \to 0$$

as $N \to \infty$. This and $\int_0^N ze^{-zs}xds \to x$ as $N \to \infty$ imply

$$\Big(\int_0^\infty e^{-zs}u(s)\ ds, z \int_0^\infty e^{-zs}u(s)\ ds - x \Big) \in A,$$

because A is closed. This means $(L_z u, zL_z u - x) \in A$ which is equivalent to

$$(L_z u, x) \in (z\,\mathrm{I} - A).$$

■

Now we are able to prove uniqueness of the Differential Inclusion in some sense. For the definition of the point spectrum of a linear relation, we refer to *Preliminaries*, Chapter 0. We remark that the following theorem as well as Lemma 3.11 were deduced by generalising the corresponding results in [deL94].

THEOREM 3.12 *Let A be closed such that $\mathbb{C}_\omega \cap (\sigma_p(A))^c$ has accumulation points in the half-plane*

$$z \in \mathbb{C}_\omega = \{z \in \mathbb{C} : \operatorname{Re} z > \omega\}$$

for all real numbers $\omega > \omega_0$, where $\omega_0 \in \mathbb{R}$ is given. Then all exponentially bounded mild solutions as well as exponentially bounded solutions of the Differential Inclusion (3.3) for A and initial value x are unique.

Proof: Let v, w be exponentially bounded mild solutions of (3.3). We consider $u := v - w$, which is clearly a mild solution of the Differential Inclusion for initial value $x = 0$. Furthermore, u is exponentially bounded, say

$$\|u(t)\| \leqslant Me^{\omega t} \qquad \text{for all } t \geqslant 0.$$

Clearly, we can choose $\omega > \omega_0$. From Lemma 3.11 follows that for $z \in \mathbb{C}_\omega$ the Laplace transform $L_z u$ exists and that

$$(L_z u, 0) \in (z\,\mathrm{I} - A).$$

The assumption about the point spectrum of A implies

$$L_z u = 0 \qquad \text{for all } z \in \mathbb{C}_\omega \cap (\sigma_p(A))^c$$

The set $\mathbb{C}_\omega \cap (\sigma_p(A))^c$ has an accumulation point in $\mathbb{C}_\omega$ and, therefore, the properties of the Laplace transform imply that $u \equiv 0$ (see [Kal10a], where this result is proved for complex-valued functions. The generalisation to Banach space-valued functions can be shown by applying all x' in the dual space X'). Hence, $v \equiv w$, which proves the uniqueness. The assertion holds for classic solutions because the closedness of A implies that a classic solution is also a mild solution as shown in remark 3.4. ■

3.2 G-Semigroups vs. Existence Families

In the following we establish a connection between G-semigroups and the theory developed in section 3.1. The main observation used in the proof of following theorem is contained in Lemma 1.14. ped in section 3.1. The main observation used in the proof of following theorem is contained in Lemma 1.14.

THEOREM 3.13 *Let A be the generator of a G-semigroup P. Then*

- *P is a mild $P(0)$-existence family for A and*
- *P is a $P(0)$-existence for A.*

Proof: By definition of a G-semigroup, P is strongly continuous on X. By Lemma 1.15

$$(\int_0^t P(s)xds, P(t)x - P(0)x) \in A$$

for all $x \in X$ and $t \geqslant 0$. Hence, P is a mild $P(0)$-existence family.
Let $x \in \operatorname{dom} A$. By definition of A as the generator of P, $P(.)x$ belongs to $C^1([0,\infty), X)$. We have

$$[P(.)x]'(t) = P(t)y \qquad \text{for all } t \geqslant 0 \quad \text{and } y \in Ax.$$

Lemma 1.14 shows that $(P(t)x, P(t)y) \in A$. Thus,

$$(P(t)x, [P(.)x]'(t)) \in A \qquad \text{for all } t \geqslant 0.$$

■

In other words, if we know that A is the generator of a G-semigroup P, then we have a solution. The next lemma shows that if we are interested in a solution of a particular Differential Inclusion (3.3), then it suffices that A has an extension which generates a G-semigroup.

LEMMA 3.14 *Let B be the generator of a G-semigroup P and B is an extension of the linear relation A. If* $\operatorname{dom} A$ *is P invariant, i.e.*

$$P(t) \operatorname{dom} A \subseteq \operatorname{dom} A \qquad \textit{for all } t \geqslant 0 \tag{3.10}$$

and if

$$\operatorname{mul} A = \operatorname{mul} B, \tag{3.11}$$

then

$$u \equiv P(.)x$$

is a classic solution of the Differential Inclusion

$$(u, \dot{u}) \in A, \qquad u(0) = P(0)x$$

for $x \in \operatorname{dom} A$.

Proof: Since B is an extension of A and since both have the same multi-value part by assumption, it follows from Lemma 0.18 that

$$Az = Bz \qquad \text{for all } z \in \operatorname{dom} A. \tag{3.12}$$

Let $x \in \operatorname{dom} A$. By Theorem 3.13, $u \equiv P(.)x$ is a classic solution of the Differential Inclusion

$$(u, \dot{u}) \in B, \qquad u(0) = P(0)x.$$

By (3.10), $u(t) \in \operatorname{dom} A$ for all $t \geqslant 0$. Therefore, line (3.12) implies that

$$(u(t), \dot{u}(t)) \in A$$

for all $t \geqslant 0$. ∎

Finally, we give a finite dimensional example.

***Example* 3.15** Consider the differential equation,

$$\dot{v} = v + w, \tag{3.13}$$

$$0 = 2v + w, \tag{3.14}$$

with initial values $v(0) = v_0$ and $w(0) = w_0$. With the notation of Example 3.2, this reads as

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=:F} \dot{u} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{=:G} u, \qquad u(0) = \underbrace{\begin{pmatrix} v_0 \\ w_0 \end{pmatrix}}_{=:u_0} \tag{3.15}$$

where $u := \begin{pmatrix} v \\ w \end{pmatrix}$. As done in Example 3.2, we consider the Differential Inclusion

$$(u, \dot{u}) \in F^{-1}G, \qquad u(0) = u_0, \tag{3.16}$$

which is a proper differential inlusion since F is not injective. Let us have a closer look on the linear relation $F^{-1}G$:

$$\begin{aligned} F^{-1}G &= \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : Gx = Fy\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_2 = -2x_1 \wedge y_1 = -x_1\right\}. \end{aligned} \tag{3.17}$$

Obviously, the relation B

$$B := \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_1 = -x_1\right\}$$

is an extension of $F^{-1}G$ with the same multi-value part

$$\operatorname{mul} B = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] := \left\{y \in \mathbb{R}^2 : y_1 = 0\right\}.$$

Now we show that B is the generator of the G-semigroup

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & 0 \end{pmatrix}.$$

To see this, observe that P can be written as

$$P(t) = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{=:C} \underbrace{\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}}_{=:T(t)} = T(t)C,$$

where T is clearly a classic semigroup (see Corollary 1.50) with the generator

$$-\mathrm{I} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using Theorem 1.28, one concludes that P is a G-semigroup with generator

$$-C^{-1}\,\mathrm{I}\,C = -C^{-1}C = -\mathrm{I} \boxplus (\{0\} \times \ker C),$$

where we used Lemma 0.23. Therefore,

$$-\mathrm{I} \boxplus (\{0\} \times \ker C) = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_1 = -x_1\right\} = B,$$

since $\ker C = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$, which proves that B is the generator of P.
By (3.17), we see that

$$\operatorname{dom} F^{-1}G = \left\{x \in \mathbb{R}^2 : x_2 = -2x_1\right\},$$

and by the form of P,

$$P(t) \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} e^{-t}x_1 \\ -2e^{-t}x_1 \end{pmatrix} \in \operatorname{dom} F^{-1}G.$$

Thus, $\operatorname{dom} F^{-1}G$ is invariant under $P(t)$ for all $t \geqslant 0$. Hence, by Lemma 3.14

$$u(t) = P(t)u_0 = \begin{pmatrix} e^{-t}v_0 \\ e^{-t}w_0 \end{pmatrix},$$

is a classic solution of (3.16) for $u_0 \in \operatorname{dom} F^{-1}G$. Clearly, the condition that u_0 has to be in the domain of $F^{-1}G$ can be rewritten as

$$w_0 = -2v_0.$$

Altogether,

$$v(t) = e^{-t}v_0, \tag{3.18}$$
$$w(t) = -2e^{-t}v_0, \tag{3.19}$$

is a solution for $w_0 = -2v_0$.
Apart from the previous considerations, one can solve (3.13)-(3.14) simply by using (3.14) to get $w = -2v$ and then rewrite (3.13) as

$$\dot{v} = -v \qquad v(0) = v_0.$$

Clearly, the solution of this differential equation is $v(t) = v_0 e^{-t}$ and, therefore, yields the same solution as in (3.18)-(3.19).
Finally, we want to calculate the spectrum of $F^{-1}G$. For details about the spectrum of a linear relation see *Preliminaries*, Chapter 0. Since $F^{-1}G$ is not an operator, $\infty \in \sigma_p(F^{-1}G)$. Therefore, $\infty \in \sigma(F^{-1}G)$. Since $F^{-1}G$ is closed as a finite dimensional space, a complex number λ lies in $\sigma(F^{-1}G)$ precisely if

$$\ker(\lambda - F^{-1}G) \neq \{0\} \qquad \textit{or} \qquad \operatorname{ran}(\lambda - F^{-1}G) \neq X,$$

by Remark 0.29. By elementary rules for calculating with linear relations, we have

$$\begin{aligned}
\lambda - F^{-1}G &= \left\{(x, \lambda x - y) \in \mathbb{R}^2 \times \mathbb{R}^2 : (x, y) \in F^{-1}G\right\} \\
&= \left\{(x, \lambda x - y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_2 = -2x_1 \wedge y_1 = -x_1\right\} \\
&= \left\{ (\begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix}, \begin{pmatrix} (\lambda+1)x_1 \\ -2\lambda x_1 - y_2 \end{pmatrix}) : x_1, y_2 \in \mathbb{R} \right\}.
\end{aligned}$$

Therefore, $x \in \mathbb{R}^2 \setminus \{0\}$ is in $\ker(\lambda - F^{-1}G)$ if and only if $x_2 = -2x_1$ and

$$\begin{aligned}
(\lambda+1)x_1 &= 0 \\
-2\lambda x_1 - y_2 &= 0
\end{aligned}$$

for some $y_2 \in \mathbb{R}$. This has a non-zero solution only if $\lambda = -1$. On the other hand, the range of $(\lambda - F^{-1}G)$ only equals X if $\lambda \neq -1$. Thus,

$$\sigma(F^{-1}G) = \sigma_p(F^{-1}G) = \{-1, \infty\}.$$

If one neglects the point ∞, the result is in a way familiar in view of the classic ordinary linear differential equations:
The eigenvalues of the system represent the factors in the exponents of the exponentials. With nearly the same considerations one also sees that

$$\sigma(B) = \sigma(F^{-1}G).$$

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