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D I S S E R T A T I O N

Asymptotics of Sequences Defined by Certain Recurrences

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Abstract

The aim of this thesis is to describe the asymptotic behaviour of sequences $\{f_n\}_{n\in\mathbb{N}_0}$ defined by the recurrence

$$f_n[q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1)] = a_1\alpha^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{q}{\alpha}\right)^k f_k + b_1\beta^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{1}{\beta}\right)^k f_k + c_1\gamma^n, \quad \forall n \in \mathbb{N}$$

with the initial condition

$$f_0[1 - (a_0 + a_1) - (b_0 + b_1)] = c_0 + c_1$$

or by the Taylor coefficients of the analytic solution of the functional–difference equation

$$f\left(q^2 z\right) = \left(a_0 + a_1 e^{\alpha z}\right) f(qz) + \left(b_0 + b_1 e^{\beta z}\right) f(z) + \left(c_0 + c_1 e^{\gamma z}\right).$$

The Laplace–Borel transform is used to receive a related functional equation with constant coefficients which can be solved by the conventional theory of q–difference equations of C. R. Adams. The methods of P. Flajolet and A. Odlyzko on singularity analysis of generating functions are used to calculate asymptotic expansions. The method is of general interest as it can be easily adopted to functional–differential equations of similar type.

Kurzfassung

Ziel dieser Arbeit ist es, das asymptotische Verhalten von Folgen $\{f_n\}_{n\in\mathbb{N}_0}$ zu bestimmen, welche durch die Rekursion

$$f_n[q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1)] = \\ a_1\alpha^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{q}{\alpha}\right)^k f_k + b_1\beta^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{1}{\beta}\right)^k f_k + c_1\gamma^n, \quad \forall n \in \mathbb{N}$$

mit der Anfangsbedingung

$$f_0[1 - (a_0 + a_1) - (b_0 + b_1)] = c_0 + c_1$$

beziehungsweise durch die Taylorkoeffizienten der analytischen Lösung der Funktionalgleichung

$$f\big(q^2 z\big) = (a_0 + a_1 e^{\alpha z})\, f(qz) + \big(b_0 + b_1 e^{\beta z}\big)\, f(z) + (c_0 + c_1 e^{\gamma z})$$

festgelegt sind. Es wird die Laplace–Borel Transformation benützt, um eine inhomogene q–Differenzengleichung mit konstanten Koeffizienten zu erhalten, welche dann mit Hilfe der Theorie von C. R. Adams gelöst wird. Dann werden die Methoden von P. Flajolet und A. Odlyzko über „ singularity analysis of generating functions " erweitert, um anhand der gewonnenen Lösung das asymptotische Verhalten der Folge $\{f_n\}_{n\in\mathbb{N}_0}$ zu bestimmen. Diese Methode ist allgemein interessant, da sie auf einfache Weise auf allgemeinere Rekursionen bzw. Funktional–Differentialgleichungen erweitert werden kann.

Contents

Chapter 1

Introduction

In theoretical computer science recurrences play a prominent role (cf. [DTTV], [Hof1], [KiP2], [KnP], [Kn], [P2]). If starting values and recurrence relations are known, it is possible to calculate each value of the sequence. In general it is a difficult task to determine the asymptotic behaviour of sequences defined by recurrence relations only. We will give some examples for an impression of these difficulties.

- Storing data in a small structure with low computational complexity for searching, replacing, etc. is always an important item. A possibility to store n items is the use of so-called (binary) tries (from information re*trie*val) (cf. [KiP2]). N data are stored in external nodes of a binary tree. It is assumed that each item has a related sequence $\{a_i\}_{i\in\mathbb{N}}$, $a_i \in \{0,1\}$ where all such sequences are equally likely. In the tree each left (resp. right) branch is labelled 0 (resp. 1). This yields an encoding of each external node by means of the sequence of labels describing the path from the root to this node. Each item is stored in that external node corresponding to the shortest unique prefix of its key. If we consider the related sequence (the shortest prefixes are indicated)

  A: 1 1 1 0 1 . . .
  B: 0 0 1 1 0 . . .
  C: 1 0 0 1 0 . . .
  D: 0 0 0 1 1 . . .
  E: 1 1 1 1 0 . . .

  we get the corresponding trie:

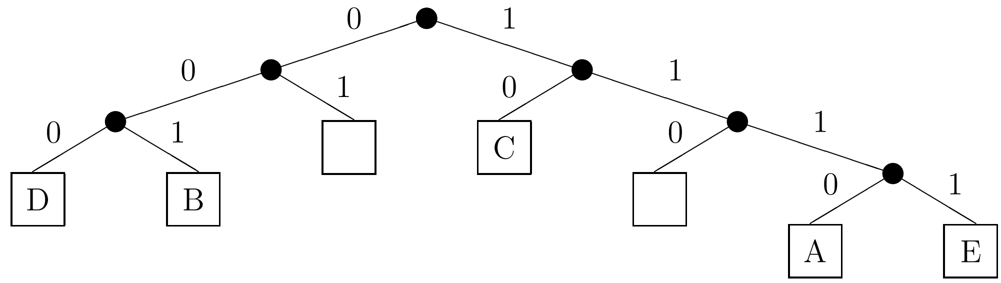


Figure 1.1: A binary trie storing $n = 5$ items

If we denote by f_n the average number of internal nodes of a trie that keeps n data it is clear that $f_0 = f_1 = 0$. If a trie stores more than one data then these items are split depending on the first values of the related sequence into the left and right subtrie. Since our assumption that all sequences are equally likely the probability that k data are put into the left trie and $n-k$ into the right is $\frac{1}{2^n}\binom{n}{k}$. Thus we receive the recurrence

$$f_n = 1 + \sum_{k=0}^{n} \frac{1}{2^n}\binom{n}{k}(f_k + f_{n-k}) \qquad n \geq 2. \tag{1.1}$$

If we multiply the last equation by $\frac{z^n}{n!}$ and sum up over all $n \geq 2$ we get

$$f(z) = e^z - 1 - z + 2e^{\frac{z}{2}} f\left(\frac{z}{2}\right) \tag{1.2}$$

where $f(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$ denotes the exponential generating function.

- Another example for a recurrence relation that comes from the problem of storing data is a tree where internal nodes can keep items as well as external nodes (cf. [FR]). A tree that should store n data is built up by the following recurrence: If there are $n \leq m$ items with a fixed integer m, then all of them are stored in the node. If there are $n > m$ items then m data are put aside into the node and the remaining $n - m$ items separate into two subtrees, each of them flipping an unbiased coin. The subgroups again split recursively by the same process. If a group has a cardinality $n \leq m$ then its recursive splitting stops. Since a fair coin is used to separate the $n - m$ items, the probability to split them into k items for the left and $n - m - k$ items for the right subtree is $\frac{1}{2^{n-m}}\binom{n-m}{k}$ If we denote by f_n the number of nodes that contain data we get the recurrence

$$f_n = 1 + \sum_{k=0}^{n-m} \frac{1}{2^{n-m}}\binom{n-m}{k}(f_k + f_{n-m-k}) \qquad n \geq m \tag{1.3}$$

Figure 1.2: A generalized tree corresponding to $m = 5$ with $n = 33$ items

with initial conditions $f_0 = 0, f_1 = f_2 = \cdots = f_m = 1$. If we denote by $f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$ the exponential generating function it can be easily checked that the recurrence relation implies the functional equation

$$\frac{d^m}{dz^m} f(z) = e^z + 2e^{\frac{z}{2}} f\left(\frac{z}{2}\right). \tag{1.4}$$

- Next we examine the minimum order statistic of the Cantor distribution (cf. [KnP]). Take a random sequence $\{a_i\}_{i\in\mathbb{N}}$, $a_i \in \{0,1\}$, where each sequence is equally likely. Given a fixed parameter ϕ with $0 < \phi \leq \frac{1}{2}$ then the random number $T(a_1, a_2, \dots)$ is considered with

$$T(a_1, a_2, \dots) = \frac{1-\phi}{\phi} \sum_{i=1}^{\infty} a_i \phi^i \in [0,1]. \tag{1.5}$$

The sequences now have a natural order from the usual ordering of the real numbers. It can be easily seen that this is equivalent to the lexicographic ordering of sequences, i.e. $\{a_i\}_{i\in\mathbb{N}} < \{b_i\}_{i\in\mathbb{N}}$ iff there is a k such that $a_i = b_i$ for $i = 1, 2, \dots, k-1$ and $a_k < b_k$. It thus makes sense to speak of order statistics for sequences. The particular choice $\phi = \frac{1}{3}$ produces real numbers $T(w)$ that belong to the Cantor set, hence the choice of name for the distribution. Suppose that we have n independent random sequences $w_1, w_2, \dots, w_n$. We denote by f_n the average values of the minimum of the n real numbers $T(w_1), T(w_2), \dots, T(w_n)$. Hosking [Hos, equation 5] derives the following recursion for the expected minimum:

$$(2^n - 2\phi) f_n = 1 - \phi + \phi \sum_{k=1}^{n-1} \binom{n}{k} f_k \qquad n \geq 1. \tag{1.6}$$

It is convenient to set $f_0 = 0$. Let $f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$ be the exponential generating function, then the recurrence leads to the functional equation

$$f(2z) - 2\phi f(z) = (1-\phi)(e^z - 1) + \phi(e^z - 1) f(z). \tag{1.7}$$

- Finally we look at the moments of the Cantor–Fibonacci distribution (cf. [P2]). Fibonacci words are sequences $\{a_i\}_{i\in\mathbb{N}}$, $a_i \in \{0,1\}$ with the restriction that two adjacent letters "1" are not allowed ($a_i = 1 \Rightarrow a_{i+1} = 0$). Given a fixed parameter ϕ with $0 < \phi \leq \frac{1}{2}$ we then consider equivalently to the last example, the random number $T(a_1, a_2, \dots)$ with

$$T(a_1, a_2, \dots) = \frac{1-\phi}{\phi} \sum_{i=1}^{\infty} a_i \phi^i \in [0,1]. \tag{1.8}$$

We denote by f_n the nth moment of the Cantor–Fibonacci distribution. Prodinger [P2] derives the following recursion: $f_0 = 1$ and for $n \geq 1$

$$f_n(\alpha^2 - \alpha\phi^n) = \sum_{k=0}^{n} \binom{n}{k} (1-\phi)^{n-k} \phi^{2k} f_k \tag{1.9}$$

with $\alpha = \frac{1+\sqrt{5}}{2}$. Let $f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$ be the exponential generating function, then the recurrence leads to the functional equation

$$\alpha^2 f(z) - \alpha f(\phi z) = e^{(1-\phi)z} f(\phi^2 z). \tag{1.10}$$

Remember that in all of these cases the recurrences lead to q–difference or functional–differential equations. The aim of this thesis is to give a classification of the asymptotics of sequences defined by the recurrence

$$f_n[q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1)] = \\ a_1\alpha^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{q}{\alpha}\right)^k f_k + b_1\beta^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{1}{\beta}\right)^k f_k + c_1\gamma^n, \quad n \in \mathbb{N}$$

or defined by the Taylor coefficients of an analytic function that satisfies the functional equation

$$f\left(q^2 z\right) = (a_0 + a_1 e^{\alpha z}) f(qz) + \left(b_0 + b_1 e^{\beta z}\right) f(z) + (c_0 + c_1 e^{\gamma z}),$$

where $f(z)$ is the exponential generating function built by the sequence $\{f_n\}_{n\in\mathbb{N}_0}$.

In Chapter 2 preliminaries introduce the reader to mathematical notations and tools which are used in the theorems later on. Basic properties of entire functions, Laplace– and Laplace–Borel transformation and theorems on singularity analysis are stated there.

Since the prove of the classification theorem in Chapter 4 contains many technical details (e.g. calculating the order of analytic solutions and expanding analytic domains) these technical parts are put aside into Chapter 3, Technical Lemmas.

In Chapter 4 the classification theorem is proved. The proof contains three important steps: Application of Laplace–Borel transformation, usage of the theory on q–difference equations and calculating asymptotics by methods from singularity analysis.
A different approach to the proof may be employed as well. It is based on the computation of the growth of $f(z)$ within a cone $S_{\frac{\pi}{2}-\epsilon} = \left\{z : |\arg z| < \frac{\pi}{2} - \varepsilon\right\}$ and the proof that outside of that cone the function is of lower exponential growth. Knowing this growth a Depoissonization Lemma of Jacquet and Szpankowski [JS2, Theorem 1] can be applied to extract the asymptotics of f_n. This method applied to the functional equation

$$f(qz) = (a_0 + a_1 e^{\alpha z})\, f(z) + \left(b_0 + b_1 e^{\beta z}\right)$$

will be published by Derfel and Vogl [DV2].

In Chapter 5 some examples are given that compare results calculated by the classification theorem with results recently published. Furthermore, it is explained how this method can be generalized to functional–differential equations.

In Chapter 6 related problems are described where these methods cannot be applied directly. Nevertheless, similar methods might lead to further classifications for these functional equations.

Chapter 2

Preliminaries

To understand the stated proofs it is necessary to be familar with complex analysis. Here some basic facts about the Laplace transformtion, Laplace-Borel transformation and singularity analysis are stated.

2.1 Order and type of entire functions

A complex valued function $f(z)$ which is analytic in the hole complex plane is called *entire function*. This function is said to be *of finite order* if there exists a number k such that

$$\max_{|z|=r} |f(z)| \leq e^{r^k} \quad \text{for } r \to \infty \tag{2.1}$$

holds. The lower limit ρ of all k in (2.1) is called the *order of the function* $f(z)$. For all $\varepsilon > 0$ there exists an r_0 such that

$$\max_{|z|=r} |f(z)| < e^{r^{\rho+\varepsilon}} \quad \text{for } r > r_0.$$

If we denote $M(r) = \max_{|z|=r} |f(z)|$ and take twice the log of (2.1) we get

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \quad (0 \leq \rho \leq \infty). \tag{2.2}$$

If $\rho = 0$ then we have a function of order zero. For example, all polynomials are of order zero. If $\rho = \infty$ then we call $f(z)$ of infinite order. Functions of finite order can be classified by a further number, the *type of the function*. If there is a function of finite order ρ and the estimation

$$M(r) \leq e^{ar^\rho} \quad \text{for } r \to \infty \tag{2.3}$$

holds with a positive a then we call the lower limit of all numbers a which satisfy (2.3) the *type* σ of the function. Equation (2.3) implies that

$$\sigma = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}. \tag{2.4}$$

We will call the function $f(z)$ of *minimal type* if $\sigma = 0$, of *normal type* if $0 < \sigma < \infty$ and of *maximal type* if $\sigma = \infty$. If $\rho = 1$ we say $f(z)$ *is of exponential type* σ.

There exists a strong relation between the coefficients of the Taylor series of a function and the order. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.5}$$

is the Taylor series of an entire function $f(z)$.

Theorem 2.1. *The entire function $f(z)$ has finite order iff*

$$\mu = \limsup_{n\to\infty} b_n \qquad \text{with } b_n = \begin{cases} -\frac{n\log n}{\log|a_n|} & \text{if } a_n \neq 0 \\ 0 & \text{else} \end{cases} \tag{2.6}$$

is finite. In that case $\rho = \mu$.

Proof. First we will show that $\rho \geq \mu$. This means that if $\mu = \infty$ then $\rho = \infty$ and $f(z)$ is of infinite order. If $\mu = 0$ then $\rho \geq \mu$ is trivially satisfied. Thus we can suppose that $0 < \mu \leq \infty$. Choose ε such that $0 < \varepsilon < \mu$. If $\mu < \infty$ then

$$\mu - \varepsilon \leq -\frac{n\log n}{\log|a_n|} \qquad \text{or} \qquad \log|a_n| \geq -\frac{n\log n}{\mu - \varepsilon} \tag{2.7}$$

is valid for infinitely many $n \in \mathbb{N}$. If $\mu = \infty$ then for infinitely many n

$$\frac{1}{\varepsilon} \leq -\frac{n\log n}{\log|a_n|} \qquad \text{or} \qquad \log|a_n| \geq -\varepsilon n \log n \tag{2.8}$$

holds. If we sum up these results we get for infinitely many n

$$\log|a_n| \geq -\frac{n\log n}{\lambda} \qquad \text{with } \lambda = \begin{cases} \mu - \varepsilon & \text{for } \mu < \infty \\ \frac{1}{\varepsilon} & \text{for } \mu = \infty. \end{cases} \tag{2.9}$$

From the Cauchy–Integral equation we get

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz \tag{2.10}$$

$$\implies \quad |a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leq \frac{1}{2\pi}\frac{M(r)}{r^{n+1}}2\pi r = \frac{M(r)}{r^n} \qquad \text{with } M(r) = \max_{|z|=r}|f(z)|. \tag{2.11}$$

Taking the log of the last inequality we receive

$$\log M(r) \geq n\log r + \log|a_n| \geq n\left(\log r - \frac{\log n}{\lambda}\right). \tag{2.12}$$

Choosing $r = (en)^{\frac{1}{\lambda}}$ we get for the infinitely many r which correspond to the infinitely many n from equation (2.9)

$$\log M(r) \geq \frac{n}{\lambda} = \frac{r^\lambda}{e\lambda} \tag{2.13}$$

and

$$\frac{\log\log M(r)}{\log r} \geq \lambda - \frac{\log(\lambda e)}{\log r}, \tag{2.14}$$

where λ has been chosen independently from r. Therefore we see that

$$\rho = \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r} \geq \lambda = \begin{cases}\mu - \varepsilon & \text{for } \mu < \infty\\ \frac{1}{\varepsilon} & \text{for } \mu = \infty.\end{cases} \tag{2.15}$$

Since ε has been chosen arbitrary this implies $\rho \geq \mu$.
Now we have to show that $\rho \leq \mu$. In the case where $\mu = \infty$ there is nothing to show. Suppose that $\mu < \infty$. For all $n > n_0$ with n_0 large enough it follows from (2.6) that

$$0 \leq -\frac{n\log n}{\log|a_n|} \leq \mu + \varepsilon \tag{2.16}$$

and therefore,

$$\sqrt[n]{|a_n|} \leq n^{-\frac{1}{\mu+\varepsilon}}. \tag{2.17}$$

Since adding of a polynomial function to an entire function does not change the order and type of the function the inequality is valid for $n \leq n_0$ too. First of all it follows that the radius of convergence of the power series is infinite. Thus $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function. Furthermore, for all positive r we can choose $n_0(r) = \lceil (2r)^{\mu+\varepsilon}\rceil$ [1] and receive

$$|a_n|r^n \leq n^{-\frac{n}{\mu+\varepsilon}}r^n \leq \frac{1}{2^n} \qquad \forall n \geq n_0(r). \tag{2.18}$$

[1] $\lceil x\rceil = \min\{n \in \mathbb{N} : n \geq x\}$

$$M(r) \leq \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{n_0(r)-1} |a_n| r^n + \sum_{n=n_0(r)}^{\infty} |a_n| r^n = S_1 + S_2. \tag{2.19}$$

For S_2 we get the estimation

$$S_2 \leq \sum_{n=n_0(r)}^{\infty} \frac{1}{2^n} = \frac{1}{2^{n_0(r)-1}}, \tag{2.20}$$

and therefore $S_2 < 1$ for r large enough. To estimate S_1 we search for the largest summand. The function $n^{-\frac{n}{\mu+\varepsilon}} r^n$ has a maximal value for $n = \frac{r^{\mu+\varepsilon}}{e}$. Therefore,

$$S_1 \leq \sum_{n=0}^{n_0(r)-1} n^{-\frac{n}{\mu+\varepsilon}} r^n \leq n_0(r) e^{\frac{1}{\mu+\varepsilon}\frac{r^{\mu+\varepsilon}}{e}} \leq [(2r)^{\mu+\varepsilon}+1] e^{\frac{1}{\mu+\varepsilon}\frac{r^{\mu+\varepsilon}}{e}} \tag{2.21}$$

and

$$S_1 \leq e^{r^{\mu+2\varepsilon}} \qquad \text{for } r \text{ large enough.} \tag{2.22}$$

Thus $M(r) \leq e^{r^{\mu+2\varepsilon}}$ implies $\rho \leq \mu + 2\varepsilon$ and since ε has been chosen arbitrary we conclude $\rho \leq \mu$. □

Using similar methods we can prove:

Theorem 2.2. *The function $f(z)$ of order ρ with $0 < \rho < \infty$ is of type σ iff*

$$\nu := \limsup_{n\to\infty} n \left(\sqrt[n]{|a_n|}\right)^{\rho} = e\rho\sigma. \tag{2.23}$$

If $\nu = 0$, then $f(z)$ is of minimal type and if $\nu = \infty$ then $f(z)$ is of maximal type.

Proof. First we will show that $\nu \geq e\rho\sigma$.
If $\nu = \infty$ then the inequality is trivially fulfilled. Suppose that $0 \leq \nu < \infty$. For all $\varepsilon > 0$ there exists an n_0 such that

$$\nu + \varepsilon \geq n|a_n|^{\frac{\rho}{n}} \qquad \text{or} \qquad |a_n| \leq \left(\frac{\nu+\varepsilon}{n}\right)^{\frac{n}{\rho}} \tag{2.24}$$

is valid. Since adding of a polynomial function to an entire function does not change either order or type of the function we can suppose that the inequality is valid for $n \leq n_0$ too. Now take n large enough such that $\left(\frac{\nu+\varepsilon}{n}\right)^{\frac{1}{\rho}} r \leq \frac{1}{2}$. For example we can take $n > n_0(r) = \lceil (2r)^{\rho}(\nu+\varepsilon) \rceil$. Similar to the last proof we get

$$M(r) = \max_{|z|=r} |f(z)| \leq \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{n_0(r)-1} |a_n| r^n + \sum_{n_0(r)}^{\infty} |a_n| r^n = S_1 + S_2 \tag{2.25}$$

with

$$S_2 = \sum_{n_0(r)}^{\infty} |a_n| r^n \le \sum_{n_0(r)}^{\infty} \frac{1}{2^n} = \frac{1}{2^{n_0(r)-1}}. \tag{2.26}$$

The function $\left(\frac{\nu+\varepsilon}{n}\right)^{\frac{n}{\rho}} r^n = \left(\frac{\nu+\varepsilon}{n} r^\rho\right)^{\frac{n}{\rho}}$ has a maximal value for $n = \frac{\nu+\varepsilon}{e} r^\rho$. Therefore,

$$S_1 = \sum_{n=0}^{n_0(r)-1} |a_n| r^n \le \sum_{n=0}^{n_0(r)-1} \left(\frac{\nu+\varepsilon}{n} r^\rho\right)^{\frac{n}{\rho}} \le n_0(r) e^{\frac{n}{\rho}} \le [(2r)^\rho(\nu+\varepsilon)+1] e^{\frac{\nu+\varepsilon}{e\rho} r^\rho} \tag{2.27}$$

and

$$M(r) \le e^{\frac{\nu+2\varepsilon}{e\rho} r^\rho} \tag{2.28}$$

for r large enough. Since ε has been chosen arbitrary, $\sigma \le \frac{\nu}{e\rho} \Leftrightarrow \nu \ge e\rho\sigma$.

Now we want to show that $\nu \le e\rho\sigma$.

If $\sigma = \infty$ then the inequality is trivially fulfilled. Suppose that $0 \le \sigma < \infty$. Together with

$$M(r) \le e^{(\sigma+\varepsilon) r^\rho} \tag{2.29}$$

for an arbitrary $\varepsilon > 0$ and $r > r_0(\varepsilon)$ we get by the Cauchy coefficient inequality (cf. (2.11))

$$|a_n| \le \frac{M(r)}{r^n} \le \frac{e^{(\sigma+\varepsilon) r^\rho}}{r^n}. \tag{2.30}$$

The right side reaches a minimal value for $r_1 = \left(\frac{n}{(\sigma+\varepsilon)\rho}\right)^{\frac{1}{\rho}}$. Since r_1 increases with n we can find an $n_0(\varepsilon)$ such that for $n > n_0(\varepsilon)$ this implies that $r_1 > r_0(\varepsilon)$. Therefore (2.30) implies

$$|a_n| \le \frac{e^{(\sigma+\varepsilon)\frac{n}{(\sigma+\varepsilon)\rho}}}{r_1^n} = \left(\frac{e}{r_1^\rho}\right)^{\frac{n}{\rho}} = \left(\frac{(\sigma+\varepsilon)\rho e}{n}\right)^{\frac{n}{\rho}} \qquad \forall n \ge n_0(\varepsilon). \tag{2.31}$$

Now we see that

$$\nu = \limsup_{n\to\infty} n \left(\sqrt[n]{|a_n|}\right)^\rho \le e\rho(\sigma+\varepsilon). \tag{2.32}$$

Since $\varepsilon > 0$ has been chosen arbitrary we get $\nu \le e\rho\sigma$. □

2.2 Basic facts about the Laplace Transformation

Suppose that the function $f(t)$ is defined a.e. on the intervall $0 \leq t < \infty$ and the integral $\int_{t_1}^{t_2} f(t)\, dt$ exists in the Lebesgue sense for all $t_1, t_2 \in \mathbb{R}^+$. If

$$\lim_{\omega \to \infty} \int_0^{\omega} e^{-st} f(t)\, dt$$

exists for any complex s then the function $f(t)$ is Laplace–transformable and is denoted by

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt = \lim_{\omega \to \infty} \int_0^{\omega} e^{-st} f(t)\, dt \tag{2.33}$$

as the Laplace transform of $f(t)$. It is well known that if the Laplace integral exists at a point s_0 then it will exist in the halfplane $\Re(s) > \Re(s_0)$ too. More precisely, we formulate (see [Do, p.141])

Theorem 2.3. *If the Laplace-Integral* (2.33) *exists for a* s_0 *then it converges uniformly in all cones* $\mathbf{A}(s_0, \varphi) := \{s \in \mathbb{C} \,|\, |\arg(s - s_0)| \leq \varphi\}$ *with* $\varphi < \frac{\pi}{2}$.

Proof. Denoting

$$f_0 = \int_0^{\infty} e^{-s_0 t} f(t)\, dt$$

$$\Phi(t) = \int_0^{t} e^{-s_0 \tau} f(\tau)\, d\tau$$

it is clear that

$$(s - s_0) \int_0^{\omega} e^{-(s-s_0)t}\, dt = 1 - e^{-(s-s_0)\omega},$$

so

$$0 = f_0 \left(1 - e^{-(s-s_0)\omega}\right) - f_0 (s - s_0) \int_0^{\omega} e^{-(s-s_0)t}\, dt. \tag{2.34}$$

Applying partial integration at the finite Laplace integral

$$\begin{aligned}\int_0^\omega e^{-st} f(t)\,dt &= \int_0^\omega e^{-(s-s_0)t} e^{-s_0 t} f(t)\,dt \\ &= \left[e^{-(s-s_0)t} \int_0^t e^{-s_0\tau} f(\tau)\,d\tau \right]_{t=0}^{\omega} + (s-s_0) \int_0^\omega e^{-(s-s_0)t} \int_0^t e^{-s_0\tau} f(\tau)\,d\tau\,dt \\ &= e^{-(s-s_0)\omega}\Phi(\omega) + (s-s_0)\int_0^\omega e^{-(s-s_0)t}\Phi(t)\,dt,\end{aligned}$$

and adding equation (2.34) we get

$$\int_0^\omega e^{-st} f(t)\,dt = f_0 + e^{-(s-s_0)\omega}(\Phi(\omega) - f_0) + (s-s_0)\int_0^\omega e^{-(s-s_0)t}(\Phi(t) - f_0)\,dt. \tag{2.35}$$

Since $\Phi(\omega) \to f_0$ holds uniformly for all s with $\omega \to \infty$ and $\left|e^{-(s-s_0)\omega}\right| < 1$ for all $\Re(s) > \Re(s_0)$ the second summand converges uniformly to zero. It remains to show that the third summand of the last equation converges uniformly too. Since $\Phi(t) \to f_0$ holds for $t \to \infty$, one can say that

$$\forall \varepsilon > 0 \quad \Rightarrow \quad \exists\, t_0 \quad \text{such that} \quad |\Phi(t) - f_0| < \varepsilon\cos\varphi \quad \forall\, t > t_0. \tag{2.36}$$

Take $t_0 \le \omega_1 < \omega_2$ and suppose that $s \neq s_0$, $s \in \mathbf{A}(s_0, \varphi)$, then

$$\begin{aligned}\left|(s-s_0)\int_{\omega_1}^{\omega_2} e^{-(s-s_0)t}(\Phi(t) - f_0)\,dt\right| &\le |s-s_0|\varepsilon\cos\varphi \int_{\omega_1}^{\omega_2} e^{-\Re(s-s_0)t}\,dt \\ &\le |s-s_0|\varepsilon\cos\varphi \int_0^\infty e^{-\Re(s-s_0)t}\,dt = \varepsilon\cos\varphi \frac{|s-s_0|}{\Re(s-s_0)} \\ &= \varepsilon\cos\varphi\,\frac{1}{\cos(\arg(s-s_0))} \le \varepsilon.\end{aligned}$$

For $s = s_0$ it follows that the third summand of (2.35) is zero. Thus the right side of equation (2.35) converges uniformly for all $s \in \mathbf{A}(s_0, \varphi)$. □

There are three cases of convergence of the Laplace integrals:

- The Laplace integral does not exist for any $s_0 \in \mathbb{C}$, e.g.:

$$\int_0^\infty e^{-sx}\, e^{e^x}\,dx. \qquad\qquad (s_0 = \infty)$$

- There exists an $s_0 \in \mathbb{C}$ such that the Laplace integral is convergent for $\Re(s) > \Re(s_0)$ and divergent for $\Re(s) < \Re(s_0)$, e.g.:

$$\int_0^\infty e^{-sx}\, e^{ax}\, dx. \qquad (s_0 = a)$$

- The Laplace integral exists for any $s_0 \in \mathbb{C}$, e.g.:

$$\int_0^\infty e^{-sx}\, e^{-x^2}\, dx. \qquad (s_0 = -\infty)$$

The inverse Laplace transformation is given by

$$\hat{f}(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{st} F(s)\, ds \quad \text{with an arbitrary } x > s_0. \tag{2.37}$$

If the function $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order s_0 then

$$\hat{f}(t) = \frac{f(t+) + f(t-)}{2}$$

holds and therefore the inverse Laplace transformation gives the original value in each point where $f(t)$ is continuous.
In Table 2.1 we state some basic relations between $f(t)$ and $F(s)$.

$F(s)$	$f(t)$
$aF_1(s) + bF_2(s)$	$af_1(t) + bf_2(t)$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$, $a > 0$
$F(s-a)$	$e^{at} f(t)$
$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$	$f^{(n)}(t)$
$(-1)^n F^{(n)}(s)$	$t^n f(t)$
$F(s) = \frac{1}{s}$	$f(t) \equiv 1$

Table 2.1: General properties of the Laplace transformation

2.3 The Laplace–Borel Transformation

Suppose that $f(z)$ is analytic at the origin. Then one can define the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = R \tag{2.38}$$

then it is well known that if $R \neq 0$ then the series represents the function $f(z)$ for $|z| < \frac{1}{R}$. If $R = 0$ then $f(z)$ is analytic in the whole complex plane. We can assign to $f(z)$ the *Borelfunction*

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n. \tag{2.39}$$

Applying Theorem 2.1 we conclude that $g(z)$ is an entire function of exponential type ($\rho = 1$). Using the Cauchy–Coefficient inequality (cf. (2.11)) we get

$$|g(z)| \leq \sum_{n=0}^{\infty} \frac{M(r)}{r^n n!} |z|^n = M(r) e^{\frac{|z|}{r}} \qquad \text{with } M(r) = \max_{|z|=r} |f(z)| \tag{2.40}$$

and

$$|g^{(n)}(z)| \leq \frac{M(r)}{r^n} e^{\frac{|z|}{r}} \qquad \forall n \in \mathbb{N}_0. \tag{2.41}$$

With Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{e^{-n} n^n \sqrt{2\pi n}} = 1 \tag{2.42}$$

we can calculate the type of $g(z)$

$$\sigma = \limsup_{n\to\infty} \frac{n}{e} \sqrt[n]{\frac{|a_n|}{n!}} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = R. \tag{2.43}$$

Furthermore, R is the radius of convergence of the Taylor series around the point ∞ of the function

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}. \tag{2.44}$$

Summing up the last results we receive (see [Ru, p. 209]):

Theorem 2.4. *The entire function*

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \tag{2.45}$$

is of exponential type σ iff σ is the radius of convergence of

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}. \tag{2.46}$$

The function $h(z)$ is usally called the Laplace–Borel transform of $g(z)$. The following theorem explains why this notation is convenient.

Theorem 2.5. *Suppose that the functions $g(z)$ and $h(z)$ have the representations* (2.45) *and* (2.46)*. Then*

$$h(z) = \int_0^{\infty} e^{-zt} g(t)\, dt \tag{2.47}$$

is the Laplace transform of the entire function $g(z)$. Conversely, one can calculate $g(z)$ via a special case of the inverse Laplace transformation

$$g(z) = \frac{1}{2\pi i} \oint_{|u|=\sigma+\varepsilon} e^{zu} h(u)\, du. \tag{2.48}$$

Remark 2.6. Equation (2.47) is only valid for $\Re(z) > z_0$ with $z_0 \leq \sigma$. To obtain an integral representation of $h(z)$ for complex numbers z with $\Re(z) \leq z_0$ we have to change the path of integration.

Proof. The integral (2.47) is convergent for $\Re(z) > \rho$ because $|g(t)| \leq e^{(\sigma+\varepsilon)t}$ for any $\varepsilon > 0$ and all t large enough. Since the integral is uniformly convergent (cf. Theorem 2.3) we may change the order of summation and integration and obtain

$$\int_0^{\infty} e^{-zt} g(t)\, dt = \int_0^{\infty} e^{-zt} \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} e^{-zt} t^n\, dt = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = h(z). \tag{2.49}$$

Conversely, we get for all complex numbers u

$$\frac{1}{2\pi i} \oint_{|u|=\sigma+\varepsilon} e^{zu} h(u)\, du = \frac{1}{2\pi i} \oint_{|u|=\sigma+\varepsilon} e^{zu} \sum_{n=0}^{\infty} \frac{a_n}{u^{n+1}}\, du = \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i} \oint_{|u|=\sigma+\varepsilon} \frac{e^{zu}}{u^{n+1}}\, du \tag{2.50}$$

and applying Cauchy's Integralequation gives

$$\frac{1}{2\pi i} \oint\limits_{|u|=\sigma+\varepsilon} e^{zu} h(u)\, du = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = g(z). \tag{2.51}$$

□

Corollary 2.7. *There exists a function $h(z)$ which satisfies*

- *$h(z)$ is analytic for $|z| > \sigma$ and for $z = \infty$,*
- *$h(\infty) = 0$ and*
- *$h(z)$ has at least one singularity on the circle $|z| = \sigma$*

iff there exists a function $g(z)$ of exponential order and type σ such that $h(z)$ has the Laplace representation

$$h(z) = \int\limits_0^{\infty} e^{-zt} g(t)\, dt \qquad \text{for } \Re(z) > \sigma. \tag{2.52}$$

Conversely, $g(z)$ satisfies

$$g(z) = \frac{1}{2\pi i} \oint\limits_{|u|=\sigma+\varepsilon} e^{zu} h(u)\, du. \tag{2.53}$$

2.4 Some theorems about singularity analysis

The main ideas of the following two theorems are from a paper of Flajolet and Odlyzko [FO]. Since in the proof later on periodic fluctuations occur the theorems here are stated and proved in a more general way than in [FO].

Theorem 2.8. *Suppose that $f(z) = \sum\limits_{n=0}^{\infty} f_n \frac{z^n}{n!}$ is analytic in $|z| < 1$. Then*

$$f(z) = O(|1-z|^{\alpha}) K\left(\log_q \frac{1}{1-z}\right) \qquad \text{for } z \to 1,\ |z| < 1 \tag{2.54}$$

with $\alpha < -1$, $q > 1$ and $K(z)$ a 1–periodic function which is analytic in the stripe $-\frac{\pi}{2} < \Im(z) \log q < \frac{\pi}{2}$ implies

$$f_n = O(n^{-(\alpha+1)}). \tag{2.55}$$

Proof. The main tool to prove this statement is Cauchy's Integralformula applied to a special path of integration.

$$f_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z^{n+1}}\,dz, \qquad \Gamma = \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 - \frac{1}{n} \right\}. \tag{2.56}$$

From (2.54) we know that there exists a constant k_1 such that

$$|f(z)| \le k_1 |1-z|^\alpha \left| K\left(\log_q \frac{1}{1-z}\right)\right| \qquad \forall z \in \mathbb{C}, |z| < 1.$$

Taking $z = \left(1 - \frac{1}{n}\right) e^{i\Theta}$, $0 \le \Theta \le 2\pi$ in (2.56) we get with $k_2 = \frac{k_1 e}{\pi}$

$$\begin{aligned}
|f_n| &\le \frac{1}{2\pi}\oint_\Gamma |f(z)||z|^{-n}\,|z|^{-1}\,|dz| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z)| 2e\,d\Theta \\
&\le k_2 \int_0^{2\pi} |1-z|^\alpha \left|K\left(\log_q \frac{1}{1-z}\right)\right| \,d\Theta \\
&= \underbrace{k_2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |1-z|^\alpha \left|K\left(\log_q \frac{1}{1-z}\right)\right| \,d\Theta}_{I_1:=} + \underbrace{k_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |1-z|^\alpha \left|K\left(\log_q \frac{1}{1-z}\right)\right| \,d\Theta}_{I_2:=}
\end{aligned}$$

- I_1:
  It is easy to see that $2 > |1-z| > 1$ and $\arg \frac{1}{1-z} \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ which is a bounded set. This implies that

$$|1-z|^\alpha < 1 \quad (\alpha < -1),$$

and there exists a constant k such that

$$\left|K\left(\log_q \frac{1}{1-z}\right)\right| \le k.$$

Therefore, with $k_3 = k k_2 \pi$

$$I_1 = k_2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \underbrace{|1-z|^\alpha}_{\le 1} \underbrace{\left|K\left(\log_q \frac{1}{1-z}\right)\right|}_{\le k} \,d\Theta \le k_3 \quad \forall n \in \mathbb{N}.$$

- I_2:
  In the case $\Theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have

$$\Re(1-z) \geq \frac{1}{n}.$$

For $n \geq 2$ we get

$$\left(1-\frac{1}{n}\right) \geq \frac{\pi}{20} \quad \Rightarrow \quad \left|\left(1-\frac{1}{n}\right)\frac{2}{\pi}\,\Theta\right| \geq \left|\frac{\Theta}{10}\right|$$

$$\Rightarrow \qquad |\Im(1-z)| = \left|\left(1-\frac{1}{n}\right)\sin\Theta\right| \geq \left|\left(1-\frac{1}{n}\right)\frac{2}{\pi}\,\Theta\right| \geq \left|\frac{\Theta}{10}\right|$$

and finally

$$|1-z| \geq \frac{1}{2}(|\Re(1-z)| + |\Im(1-z)|) \geq \frac{1}{2}\left(\frac{1}{n}+\frac{|\Theta|}{10}\right).$$

To estimate the periodic function $K(z)$ we use the estimations

$$\frac{1}{n} \leq |1-z| \leq \sqrt{2} \quad \Rightarrow \quad \frac{1}{\sqrt{2}} \leq \frac{1}{|1-z|} \leq n$$

$$\text{and} \quad \arg\frac{1}{1-z} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Therefore, with constants k_4 and k_5,

$$\begin{aligned}
I_2 &= k_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |1-z|^{\alpha} \left|K\left(\log_q \frac{1}{1-z}\right)\right| \, d\Theta \\
&\leq k_2 2^{-\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{n}+\frac{|\Theta|}{10}\right)^{\alpha} \left|K\left(\log_q \frac{1}{|1-z|} + \frac{i}{\log q}\underbrace{\arg\frac{1}{1-z}}_{\in\,\left]-\frac{\pi}{2},\frac{\pi}{2}\right[}\right)\right| \, d\Theta \\
&\leq k_4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{n}+\frac{|\Theta|}{10}\right)^{\alpha} \, d\Theta \\
&= 20k_4 \frac{1}{(\alpha+1)}\left[\left(\frac{1}{n}+\frac{\pi}{20}\right)^{\alpha+1} - n^{-(\alpha+1)}\right] \\
&\leq k_5 n^{-(\alpha+1)}.
\end{aligned}$$

Summing up these results we get

$$f_n \leq I_1 + I_2 = k_3 + k_5 n^{-(\alpha+1)} = O\big(n^{-(\alpha+1)}\big).$$

□

Theorem 2.9. *Suppose that* $f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$ *is analytic in* $\Delta(\eta, \phi) \setminus \{1\}$ *where*

$$\Delta(\eta, \phi) = \bigl\{z \in \mathbb{C} \bigr| \, |z| \le 1 + \eta, \ |\arg(z-1)| \ge \phi \bigr\} \tag{2.57}$$

with $\eta > 0$, $\phi \in (0, \frac{\pi}{2})$. *If*

$$f(z) = O(|1-z|^{\alpha}) \, K\left(\log_q \frac{1}{1-z}\right) \qquad \text{for } z \to 1, \ z \in \Delta(\eta, \phi) \tag{2.58}$$

with $\alpha < 0$, $q > 1$ *and* $K(z)$ *a 1–periodic function which is analytic in the stripe* $-2\pi < \Im(z) \log q < 2\pi$ *then the Taylor coefficients of* $f(z)$ *satisfy*

$$f_n = O(n^{-(\alpha+1)}) \, \tilde{K}(\log_q n), \quad n \to \infty \tag{2.59}$$

with a 1–periodic function $\tilde{K}(z)$ *bounded on the real axis.*

Figure 2.1: a) Domain $\Delta(\eta, \phi)$ b) Path of integration

Proof. The main tool to prove this statement is again Cauchy's Integralformula applied to a special path of integration .

$$f_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z^{n+1}} \, dz \qquad \Gamma \subset \Delta(\eta, \phi). \tag{2.60}$$

For the integration we choose (see Figure 2.1 b))

$$\begin{aligned}
&\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \quad \text{with} \\
&\Gamma_1 = \left\{ z \in \mathbb{C} \,\middle|\, |z-1| = \frac{1}{n}, \ |\arg(z-1)| \ge \phi \right\} \\
&\Gamma_2 = \left\{ z \in \mathbb{C} \,\middle|\, |z-1| > \frac{1}{n}, \ |z| \le 1+\eta, \ \arg(z-1) = \phi \right\} \\
&\Gamma_3 = \{ z \in \mathbb{C} \,|\, |z| = 1+\eta, \ |\arg(z-1)| > \phi \} \\
&\Gamma_4 = \left\{ z \in \mathbb{C} \,\middle|\, |z-1| > \frac{1}{n}, \ |z| \le 1+\eta, \ \arg(z-1) = -\phi \right\}
\end{aligned}$$

and suppose that n is large enough, such that $1+\frac{1}{n} < 1+\eta$. From (2.58) we know that for $|z| > \frac{1}{2}$, $z \in \Delta(\eta,\phi)\setminus\{1\}$ there exists a constant k_1 such that

$$|f(z)| \le k_1 |1-z|^{\alpha} \left| K\left(\log_q \frac{1}{1-z}\right)\right| \qquad \forall z \in \Delta(\eta,\phi)\setminus\{1\}, |z| > \frac{1}{2}.$$

Now it follows that

$$\begin{aligned} |f_n| &\le \frac{1}{2\pi} \oint_{\Gamma} \frac{|f(z)|}{|z|^{n+1}}\,|dz| \\ &= \frac{1}{2\pi}\int_{\Gamma_1} \frac{|f(z)|}{|z|^{n+1}}\,|dz| + \frac{1}{2\pi}\int_{\Gamma_2} \frac{|f(z)|}{|z|^{n+1}}\,|dz| + \frac{1}{2\pi}\int_{\Gamma_3} \frac{|f(z)|}{|z|^{n+1}}\,|dz| + \frac{1}{2\pi}\int_{\Gamma_4} \frac{|f(z)|}{|z|^{n+1}}\,|dz| \\ &= f_n^{(1)} + f_n^{(2)} + f_n^{(3)} + f_n^{(4)}. \end{aligned}$$

- Γ_1:
  For $z \in \Gamma_1$ the following relations hold:

$$z - 1 = \frac{1}{n} e^{i\Theta}, \qquad \Theta \in (\phi, 2\pi-\phi)$$

$$|z| \ge 1 - \frac{1}{n} \quad \Rightarrow \quad |z|^{-(n+1)} \le \left(1-\frac{1}{n}\right)^{-(n+1)} \le 8,\ n \ge 2$$

$$K\left(\log_q \frac{1}{1-z}\right) = K\left(\log_q n - i\,\frac{\Theta-\pi}{\log q}\right).$$

  Since Θ is bounded we can find a 1–periodic function $\tilde{K}(z)$ such that

$$\left|K\left(\log_q \frac{1}{1-z}\right)\right| \le \tilde{K}(\log_q n) \quad \forall z \in \Gamma_1,\ n \ge 2.$$

  Now we can estimate

$$\begin{aligned} f_n^{(1)} &= \frac{1}{2\pi}\int_{\Gamma_1} \frac{|f(z)|}{|z|^{n+1}}\,|dz| \le \frac{1}{2\pi}\int_{\Gamma_1} k_1|1-z|^{\alpha} \left|K\left(\log_q \frac{1}{1-z}\right)\right| \frac{1}{|z|^{n+1}}\,|dz| \\ &\le \frac{1}{2\pi} k_1 n^{-\alpha} \tilde{K}(\log_q n) 8 \frac{2\pi}{n} \le 8k_1\, n^{-(\alpha+1)} \tilde{K}(\log_q n). \end{aligned}$$

- Γ_2:
  If we parametrize Γ_2 by

$$z = 1 + \frac{\omega t}{n}, \quad \omega = e^{i\phi},\ t \in [1, En] \text{ with a constant } E = E(\eta,\phi)$$

we get

$$f_n^{(2)} = \frac{1}{2\pi}\int\limits_{\Gamma_2} \frac{|f(z)|}{|z|^{n+1}}\,|dz| \le \frac{1}{2\pi}\int\limits_1^{En} k_1|1-z|^\alpha \left|K\Big(\log_q \frac{1}{1-z}\Big)\right| \frac{1}{|z|^{n+1}}\,\frac{dt}{n}$$
$$\le \frac{k_1 n^{-(\alpha+1)}}{2\pi}\int\limits_1^{En} t^\alpha \left|K\Big(\log_q -\frac{n}{\omega t}\Big)\right| \left(1+\frac{t\cos\phi}{n}\right)^{-(n+1)} dt.$$

We know from [FO, p. 222, (2.13)] that

$$\int\limits_1^{En} t^\alpha \left(1+\frac{t\cos\phi}{n}\right)^{-(n+1)} dt \le J(\alpha,\phi) \quad \forall n > 2|\alpha|+4$$

with

$$J(\alpha,\phi) = \int\limits_1^{\infty} t^\alpha \left(1+\frac{t\cos\phi}{\nu}\right)^{-\nu} dt, \quad \nu = 2|\alpha|+4.$$

For $K(z)$ there exists a constant k_2 such that

$$\left|K\Big(\log_q -\frac{n}{\omega t}\Big)\right| = \left|K\Big(\log_q \frac{n}{t} + \frac{i\,\phi}{\log q}\Big)\right| \le k_2.$$

Thus we get

$$f_n^{(2)} = k_1\,k_2\,\frac{J(\alpha,\phi)}{2\pi}\,n^{-(\alpha+1)}.$$

- Γ_3:
  For $z\in\Gamma_3$ we can find a constant k_3 such that

$$\left|K\Big(\log_q \frac{1}{1-z}\Big)\right| = \big|K\big(-\log_q \underbrace{|1-z|}_{\in(\eta,2-\eta)} - \frac{i}{\log q}\underbrace{\arg(1-z)}_{\in(0,2\pi)}\big)\big| \le k_3.$$

$$f_n^{(3)} = \frac{1}{2\pi}\int\limits_{\Gamma_3} \frac{|f(z)|}{|z|^{n+1}}\,|dz| \le \frac{1}{2\pi}\int\limits_{\Gamma_3} k_1|1-z|^\alpha \left|K\Big(\log_q \frac{1}{1-z}\Big)\right| \frac{1}{|z|^{n+1}}\,|dz|$$
$$\le \frac{1}{2\pi}k_1\eta^\alpha k_3(1+\eta)^{-(n+1)}2\pi(1+\eta)$$
$$\le k_1k_3\eta^\alpha(1+\eta)^{-n}.$$

- Γ_4: For $z\in\Gamma_4$ we get the same estimation as in case $z\in\Gamma_2$.

Summing up the results we get

$$\begin{aligned} f_n &\leq 8k_1\, n^{-(\alpha+1)} \tilde{K}(\log_q n) + 2k_1\, k_2\, \frac{J(\alpha,\phi)}{2\pi}\, n^{-(\alpha+1)} + k_1 k_3 \eta^{\alpha} (1+\eta)^{-n} \\ &= O(n^{-(\alpha+1)})\, \tilde{K}(\log_q n). \end{aligned}$$

□

Chapter 3

Technical Lemmas

The aim of this chapter is to prove some technical lemmas in order to shorten the prove of the main theorem.

Lemma 3.1. *Assume that $f(x)$ is entire and the solution of the functional equation*

$$f(q^2x) = (a_0 + a_1 e^{\alpha x}) f(qx) + (b_0 + b_1 e^{\beta x}) f(x) + c_0 + c_1 e^{\gamma x} \tag{3.1}$$

with $\alpha, \beta, \gamma \geq 0$ and $a_i, b_i, c_i \in \mathbb{C}$, $i = 0, 1$, $q > 1$. For convenience we put α (resp. β, γ) zero if a_1 (resp. b_1, c_1) is zero. If $f(x) \not\equiv 0$ then $f(x)$ is of exponential type λ,

$$\lambda = \max \left\{ \frac{\alpha}{q^2 - q}, \frac{\beta}{q^2 - 1}, \frac{\gamma}{q^2} \right\}. \tag{3.2}$$

Proof. If we define

$$M(r) := \max_{|x|=r} |f(x)|$$

then equation (3.1) implies for $r > 0$

$$\begin{aligned} M(q^2 r) &= \max_{|x|=r} \left| (a_0 + a_1 e^{\alpha x}) f(qx) + (b_0 + b_1 e^{\beta x}) f(x) + c_0 + c_1 e^{\gamma x} \right| \\ &\leq M(qr) \left[|a_0 + a_1 e^{\alpha r}| + |b_0 + b_1 e^{\beta r}| \frac{M(r)}{M(qr)} + |c_0 + c_1 e^{\gamma r}| \frac{1}{M(qr)} \right]. \end{aligned} \tag{3.3}$$

Since $M(r)$ is a positive and monotone increasing function $\frac{M(r)}{M(qr)}$ and $\frac{1}{M(qr)}$ are bounded. Therefore, we can estimate the expression in brackets by an exponentially increasing function. If we substitute r by $\frac{r}{q}$ we will get the equation

$$M(qr) \leq M(r)\, e^{\omega r} \quad \forall\, r > r_0 \tag{3.4}$$

with a fitting ω and r_0. By iterating (3.4) we see that

$$\begin{aligned} M(r) &\le M\left(\frac{r}{q}\right) e^{\omega\frac{1}{q}r} \\ &\le M\left(\frac{r}{q^2}\right) e^{\omega\left(\frac{1}{q}+\frac{1}{q^2}\right)r} \\ &\le \dots \\ &\le M\left(\frac{r}{q^n}\right) e^{\omega\left(\frac{1}{q}+\frac{1}{q^2}+\cdots+\frac{1}{q^n}\right)r} \\ &< M\left(\frac{r}{q^n}\right) e^{\omega\frac{1}{q-1}r} \quad \forall\, r > q^n r_0,\, n\in\mathbb{N}. \end{aligned} \tag{3.5}$$

For all $r > qr_0$ we can find an $n \in \mathbb{N}$ such that $r_0 \le \frac{r}{q^n} < qr_0$. Denoting $M_0 = M(qr_0)$ we obtain

$$|f(re^{i\varphi})| \le M(r) \le M_0\, e^{\omega\frac{1}{q-1}r} \quad \forall r > qr_0. \tag{3.6}$$

It is clear that there exists a $\lambda \in \mathbb{R}$ such that $\forall\,\varepsilon > 0$

$$M(r) = O\big(e^{(\lambda+\varepsilon)r}\big) \quad \text{and} \quad M(r) \ne O\big(e^{(\lambda-\varepsilon)r}\big) \quad \text{for } r\to\infty,$$

or equivalently

$$M(r) = e^{\lambda r}\,\varphi(r), \tag{3.7}$$

where φ satisfies the inequality $e^{-\varepsilon r} < \varphi(r) < e^{\varepsilon r}\ \forall\,\varepsilon > 0$ for $r \to \infty$. If we insert this result into inequality (3.3) we get

$$\begin{aligned} & e^{q^2\lambda r}\varphi(q^2 r) \le |a_0 + a_1 e^{\alpha r}|\, e^{q\lambda r}\varphi(qr) + |b_0 + b_1 e^{\beta r}|\, e^{\lambda r}\varphi(r) + |c_0 + c_1 e^{\gamma r}| \\ \Rightarrow\quad & \varphi(q^2 r) \le (|a_0|e^{-\alpha r} + |a_1|)\, e^{(\alpha-(q^2-q)\lambda)r}\varphi(qr) \\ & \qquad + (|b_0|e^{-\beta r} + |b_1|)\, e^{(\beta-(q^2-1)\lambda)r}\varphi(r) \\ & \qquad + (|c_0|e^{-\gamma r} + |c_1|)\, e^{(\gamma-q^2\lambda)r} \end{aligned} \tag{3.8}$$

The right side of (3.8) cannot be exponentially small since $\varphi(q^2 r) > e^{-q^2\varepsilon r}$. Therefore, we get

$$\max\left\{\alpha-(q^2-q)\lambda,\ \beta-(q^2-1)\lambda,\ \gamma-q^2\lambda\right\} \ge 0, \tag{3.9}$$

which shows that $\lambda \le \max\left\{\frac{\alpha}{q^2-q},\ \frac{\beta}{q^2-1},\ \frac{\gamma}{q^2}\right\} =: \lambda_{max}$. If $\lambda_{max} = \frac{\gamma}{q^2}$ then equation (3.1) shows immediately the validity of (3.2). To prove the other cases we have to examine the growth of $f(x)$ along the positive real axis. It is clear that there exists a $\lambda_0 \le \lambda$ such that $\forall\,\varepsilon > 0$

$$e^{(\lambda_0-\varepsilon)x} \le |f(x)| \le e^{(\lambda_0+\varepsilon)x} \quad \text{for } x\to\infty,\ x\in\mathbb{R}_+. \tag{3.10}$$

Suppose that $\lambda_0 = \lambda_{max} - k$ with $k > 0$. If we take $0 < \varepsilon < \min\left\{k, \frac{q^2-1}{q^2+1}\right\}$ and divide (3.1) by $e^{q^2(\lambda_{max}-k+\varepsilon)x}$ then we get for $x \in \mathbb{R}_+$

$$\frac{|f(q^2x)|}{e^{q^2(\lambda_{max}-k+\varepsilon)x}} = \frac{\left|(a_0 + a_1e^{\alpha x})f(qx) + (b_0 + b_1e^{\beta x})f(x) + c_0 + c_1e^{\gamma x}\right|}{e^{q^2(\lambda_{max}-k+\varepsilon)x}}. \tag{3.11}$$

Equation (3.10) implies that the left side of (3.11) is bounded by 1 for $x \to \infty$, $x \in \mathbb{R}_+$. Let us examine the exponential growth of the summands on the right side ($x \in \mathbb{R}_+$, x large enough):

$$\begin{aligned}
\frac{|e^{\alpha x}f(qx)|}{e^{q^2(\lambda_{max}-k+\varepsilon)x}} &= e^{[\alpha - q^2\lambda_{max} + q^2k - q^2\varepsilon)]x}\,|f(qx)| \geq e^{[\alpha - q^2\lambda_{max} + q^2k - q^2\varepsilon)]x}\ e^{q(\lambda_{max}-k-\varepsilon)x} \\
&= e^{[(\alpha-(q^2-q)\lambda_{max})+(q^2-q)k-(q^2+q)\varepsilon]x} \\
\frac{\left|e^{\beta x}f(x)\right|}{e^{q^2(\lambda_{max}-k+\varepsilon)x}} &= e^{[\beta - q^2\lambda_{max} + q^2k - q^2\varepsilon)]x}\,|f(x)| \geq e^{[\beta - q^2\lambda_{max} + q^2k - q^2\varepsilon)]x}\ e^{(\lambda_{max}-k-\varepsilon)x} \\
&= e^{[(\beta-(q^2-1)\lambda_{max})+(q^2-1)k-(q^2+1)\varepsilon]x} \\
\frac{|e^{\gamma x}|}{e^{q^2(\lambda_{max}-k+\varepsilon)x}} &= e^{[(\gamma-q^2\lambda_{max})+q^2(k-\varepsilon)]x}.
\end{aligned}$$

Since at least one of the terms $(\alpha - (q^2 - q)\lambda_{max})$, $(\beta - (q^2 - 1)\lambda_{max})$ and $(\gamma - q^2\lambda_{max})$ vanishes and because of the specially chosen ε we see that the right side of equation (3.11) is unbounded for $x \to \infty$, $x \in \mathbb{R}_+$ which is a contradiction to the bounded quotient on the left side. Thus we can conclude that $k = 0$ which is equivalent to $\lambda_0 = \lambda_{max}$. Since $\lambda_0 \leq \lambda \leq \lambda_{max}$ we proved (3.2).

□

Lemma 3.2. *Suppose that the function $f(x)$ is of exponential type δ, $k \in \mathbb{N}_0$ fixed and*

$$g(x) := x^k e^{-\delta x} f(x) \tag{3.12}$$

$$G\left(\frac{1}{s}\right) := \frac{s^{k+1}}{k!}(Lg)(s), \tag{3.13}$$

where L means the Laplace-Borel transformation. Then the function $G(t)$ is analytic for $\Re(t) > \frac{-1}{2\delta}$.

Proof. Since $f(x)$ is of exponential type δ its LB-transform $(Lf)(s)$ is analytic for $|s| > \delta$. From equation (3.12) we know that

$$(Lg)(s) = (-1)^k \frac{d^k}{ds^k}(Lf)(s + \delta).$$

Therefore, $G\left(\frac{1}{s}\right)$ is analytic for

$$\begin{aligned}
|s + \delta| > \delta &\iff (a + \delta)^2 + b^2 > \delta^2,\ s = a + ib \\
&\iff \Re\left(\frac{1}{s}\right) = \frac{a}{a^2 + b^2} > -\frac{1}{2\delta}.
\end{aligned}$$

□

Lemma 3.3. *Suppose that $G(t)$ is a complex valued function, $q > 1$, $a_0, a_1, b_0, b_1, c_0, c_1 \in \mathbb{C}$, $\alpha, \beta, \gamma \geq 0$ and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. For convenience we put α (resp. β, γ) zero if a_1 (resp. b_1, c_1) is zero. Assume that depending on $\delta = \max\left\{\frac{\alpha}{q^2-q}, \frac{\beta}{q^2-1}, \frac{\gamma}{q^2}\right\}$ the function $G(t)$ satisfies one of the following functional equations:*

1. $\delta = \frac{\alpha}{q^2-q}$ *and* $\delta > \max\left\{\frac{\beta}{q^2-1}, \frac{\gamma}{q^2}\right\}$

$$G(q^2t) = a_1 G(qt) + v(t) \tag{3.14}$$

with

$$\begin{aligned} v(t) = &\frac{a_0}{(1+(q^2-q)\delta t)^{k+1}}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\ &+ \frac{b_0}{(1+(q^2-1)\delta t)^{k+1}}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\ &+ \frac{b_1}{(1+((q^2-1)\delta-\beta)\,t)^{k+1}}\, G\left(\frac{t}{1+((q^2-1)\delta-\beta)\,t}\right) \\ &+ \frac{c_0}{(1+q^2\delta t)^{k+1}} + \frac{c_1}{(1+(q^2\delta-\gamma)t)^{k+1}}. \end{aligned} \tag{3.15}$$

2. $\delta = \frac{\beta}{q^2-1}$ *and* $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\gamma}{q^2}\right\}$

$$G(q^2t) = b_1 G(t) + v(t) \tag{3.16}$$

with

$$\begin{aligned} v(t) = &\frac{a_0}{(1+(q^2-q)\delta t)^{k+1}}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\ &+ \frac{a_1}{(1+((q^2-q)\delta-\alpha)t)^{k+1}}\, G\left(\frac{qt}{1+((q^2-q)\delta-\alpha)t}\right) \\ &+ \frac{b_0}{(1+(q^2-1)\delta t)^{k+1}}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\ &+ \frac{c_0}{(1+q^2\delta t)^{k+1}} + \frac{c_1}{(1+(q^2\delta-\gamma)\,t)^{k+1}}. \end{aligned} \tag{3.17}$$

3. $\delta = \frac{\gamma}{q^2}$ *and* $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\beta}{q^2-1}\right\}$

$$G(q^2t) = c_1 + v(t) \tag{3.18}$$

with

$$
\begin{aligned}
v(t) = {} & \frac{a_0}{1+(q^2-q)\delta t}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\
& + \frac{a_1}{1+((q^2-q)\delta-\alpha)t}\, G\left(\frac{qt}{1+((q^2-q)\delta-\alpha)t}\right) \\
& + \frac{b_0}{1+(q^2-1)\delta t}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\
& + \frac{b_1}{1+((q^2-1)\delta-\beta)\,t}\, G\left(\frac{t}{1+((q^2-1)\delta-\beta)\,t}\right) \\
& + \frac{c_0}{1+q^2\delta t}.
\end{aligned} \tag{3.19}
$$

4. $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1}$ *and* $\delta > \frac{\gamma}{q^2}$

$$G(q^2t) = a_1 G(qt) + b_1 G(t) + v(t) \tag{3.20}$$

with

$$
\begin{aligned}
v(t) = {} & \frac{a_0}{(1+(q^2-q)\delta t)^{k+1}}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\
& + \frac{b_0}{(1+(q^2-1)\delta t)^{k+1}}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\
& + \frac{c_0}{(1+q^2\delta t)^{k+1}} + \frac{c_1}{(1+(q^2\delta-\gamma)t)^{k+1}}.
\end{aligned} \tag{3.21}
$$

5. $\delta = \frac{\alpha}{q^2-q} = \frac{\gamma}{q^2}$ *and* $\delta > \frac{\beta}{q^2-1}$

$$G(q^2t) = a_1 G(qt) + c_1 + v(t) \tag{3.22}$$

with

$$
\begin{aligned}
v(t) = {} & \frac{a_0}{1+(q^2-q)\delta t}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\
& + \frac{b_0}{1+(q^2-1)\delta t}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\
& + \frac{b_1}{1+((q^2-1)\delta-\beta)\,t}\, G\left(\frac{t}{1+((q^2-1)\delta-\beta)\,t}\right) + \frac{c_0}{1+q^2\delta t}
\end{aligned} \tag{3.23}
$$

6. $\delta = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$ *and* $\delta > \frac{\alpha}{q^2-q}$

$$G(q^2t) = b_1 G(t) + c_1 + v(t) \tag{3.24}$$

with

$$\begin{aligned} v(t) = &\frac{a_0}{1+(q^2-q)\delta t}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\ &+ \frac{a_1}{1+((q^2-q)\delta-\alpha)t}\, G\left(\frac{qt}{1+((q^2-q)\delta-\alpha)t}\right) \\ &+ \frac{b_0}{1+(q^2-1)\delta t}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) + \frac{c_0}{1+q^2\delta t}. \end{aligned} \tag{3.25}$$

7. $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$

$$G(q^2t) = a_1G(qt) + b_1G(t) + c_1 + v(t) \tag{3.26}$$

with

$$\begin{aligned} v(t) = &\frac{a_0}{1+(q^2-q)\delta t}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\ &+ \frac{b_0}{1+(q^2-1)\delta t}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) + \frac{c_0}{1+q^2\delta t}. \end{aligned} \tag{3.27}$$

If $G(t)$ is analytic for $\Re(t) > -\frac{1}{2\delta}$ then

(i) $G(t)$ is analytic for $\Re(t) > -\frac{1}{\delta}$

(ii) $v(t)$ is analytic for

$$t \in \begin{cases} \mathbb{C}\setminus\{K_1\cup K_3\cup\{-\frac{1}{q^2\delta-\gamma}\}\} & \text{if } \delta=\frac{\alpha}{q^2-q},\ \delta>\max\left\{\frac{\beta}{q^2-1},\ \frac{\gamma}{q^2}\right\} \\ \mathbb{C}\setminus\{K_1\cup K_2\cup\{-\frac{1}{q^2\delta-\gamma}\}\} & \text{if } \delta=\frac{\beta}{q^2-1},\ \delta>\max\left\{\frac{\alpha}{q^2-q},\ \frac{\gamma}{q^2}\right\} \\ \mathbb{C}\setminus\{K_1\cup K_2\cup K_3\} & \text{if } \delta=\frac{\gamma}{q^2},\ \delta>\max\left\{\frac{\alpha}{q^2-q},\ \frac{\beta}{q^2-1}\right\} \\ \mathbb{C}\setminus\{K_1\cup\{-\frac{1}{q^2\delta-\gamma}\}\} & \text{if } \delta=\frac{\alpha}{q^2-q}=\frac{\beta}{q^2-1},\ \delta>\frac{\gamma}{q^2} \\ \mathbb{C}\setminus\{K_1\cup K_3\} & \text{if } \delta=\frac{\alpha}{q^2-q}=\frac{\gamma}{q^2},\ \delta>\frac{\beta}{q^2-1} \\ \mathbb{C}\setminus\{K_1\cup K_2\} & \text{if } \delta=\frac{\beta}{q^2-1}=\frac{\gamma}{q^2},\ \delta>\frac{\alpha}{q^2-q} \\ \mathbb{C}\setminus K_1 & \text{if } \delta=\frac{\alpha}{q^2-q}=\frac{\beta}{q^2-1}=\frac{\gamma}{q^2}, \end{cases} \tag{3.28}$$

with

$$\begin{aligned} K_1 &= \left\{t\in\mathbb{C} :\ \left|t+\frac{2q-1}{2(q-1)q^2\delta}\right| \le \frac{1}{2(q-1)q^2\delta}\right\} \\ K_2 &= \left\{t\in\mathbb{C} :\ \left|t+\frac{q(2q-1)\delta-2\alpha}{2\left((q^2-q)\delta-\alpha\right)(q^2\delta-\alpha)}\right| \le \frac{q\delta}{2\left((q^2-q)\delta-\alpha\right)(q^2\delta-\alpha)}\right\} \\ K_3 &= \left\{t\in\mathbb{C} :\ \left|t+\frac{(2q^2-1)\delta-2\beta}{2((q^2-1)\delta-\beta)(q^2\delta-\beta)}\right| \le \frac{\delta}{2((q^2-1)\delta-\beta)(q^2\delta-\beta)}\right\} \end{aligned}$$

Proof. We have to extend the domain where we know that $G(t)$ is analytic. Obviously, it can be seen from equation (3.15) (respectively (3.17), (3.19), (3.21), (3.23), (3.25) and (3.27)) that $v(t)$ has singularities at $-\frac{1}{(q^2-q)\delta}, -\frac{1}{(q^2-1)\delta}, -\frac{1}{q^2\delta}$. In the different cases of δ occur further singularities at points which lie on the negative real axis and have a real value less than $-\frac{1}{q^2\delta}$. There might occur further singular points in $v(t)$ (and thus in $G(t)$) from $G(w)$ with $w = \frac{k_1 t}{1+k_2 t}$. Since we know that $G(t)$ is analytic in the half plane $\Re(t) > -\frac{1}{2\delta}$ we compute the area of $\mathbb{C}$ such that $\Re(w) > -\frac{1}{2\delta}$.

$$w = \frac{k_1 t}{1+k_2 t} \quad \Leftrightarrow \; t = \frac{w}{k_1 - k_2 w}.$$

$$\text{If } w = -\frac{d}{\delta}, \text{ then } t = -\frac{d}{\delta}\frac{1}{k_1 + k_2\frac{d}{\delta}} = -\frac{d}{k_1\delta + k_2 d}.$$

$$\text{If } w = \infty, \text{ then } t = -\frac{1}{k_2}.$$

$$\text{If } w = -\frac{d}{\delta}(1 \pm i), \text{ then } t = -\frac{d}{\delta}\frac{1 \pm i}{k_1 + k_2\frac{d}{\delta}(1 \pm i)} = -d\,\frac{k_1\delta + k_2 d + 1 \pm i k_1\delta}{(k_1\delta + k_2 d)^2 + k_2^2 d^2}.$$

$$\text{If } w = 0, \text{ then } t = 0.$$

Since every Möbius transform maps circles to circles it is obvious that $\Re(w) > -\frac{d}{\delta}$ is equivalent to t not lying in the circle with center $-\frac{k_1\delta + 2k_2 d}{2k_2(k_1\delta + k_2 d)}$ and radius $\frac{k_1\delta}{2k_2(k_1\delta + k_2 d)}$. If we adopt these results to all cases of Möbius transforms in (3.15), (3.17), (3.19), (3.21), (3.23), (3.25) and (3.27) we get the following relations.

Möbius transform	Corresponding areas: w–space	t–space
$w = \frac{qt}{1+(q^2-q)\delta t}$	$\Re(w) > -\frac{d}{\delta} \; \Leftrightarrow$	$\left\lvert t + \frac{1+2(q-1)d}{2\delta q(q-1)(1+(q-1)d)} \right\rvert > \frac{1}{2\delta q(q-1)(1+(q-1)d)}$
$w = \frac{qt}{1+((q^2-q)\delta-\alpha)t}$	$\Re(w) > -\frac{d}{\delta} \; \Leftrightarrow$	$\left\lvert t + \frac{q\delta + 2\left((q^2-q)\delta-\alpha\right)d}{2\left((q^2-q)\delta-\alpha\right)\left(q\delta+((q^2-q)\delta-\alpha)d\right)} \right\rvert > \frac{q\delta}{2\left((q^2-q)\delta-\alpha\right)\left(q\delta+((q^2-q)\delta-\alpha)d\right)}$
$w = \frac{t}{1+(q^2-1)\delta t}$	$\Re(w) > -\frac{d}{\delta} \; \Leftrightarrow$	$\left\lvert t + \frac{1+2(q^2-1)d}{2(q^2-1)\delta(1+(q^2-1)d)} \right\rvert > \frac{1}{2(q^2-1)\delta(1+(q^2-1)d)}$
$w = \frac{t}{1+((q^2-1)\delta-\beta)t}$	$\Re(w) > -\frac{d}{\delta} \; \Leftrightarrow$	$\left\lvert t + \frac{\delta + 2\left((q^2-1)\delta-\beta\right)d}{2\left((q^2-1)\delta-\beta\right)\left(\delta+((q^2-1)\delta-\beta)d\right)} \right\rvert > \frac{\delta}{2\left((q^2-1)\delta-\beta\right)\left(\delta+((q^2-1)\delta-\beta)d\right)}$

Thus knowing that $G(t)$ is analytic for $\Re(t) > -\frac{d}{\delta}$ with $0 < d < 1$ implies that $v(t)$ is analytic for $\Re(t) > -\frac{d}{q\delta(1+(q-1)d)}$. This again implies via (3.14) (respectively (3.16), (3.18), (3.20), (3.22), (3.24), (3.26)) that $G(t)$ is analytic for $\Re(t) > -\frac{qd}{\delta(1+(q-1)d)}$ and so we have enlarged the domain where we know that $G(t)$ is analytic. We will prove that $G(t)$ is analytic for $\Re(t) > -\frac{1}{\delta}$ by induction.
Set $d_k = \frac{q^k}{q^k+1}$.

- From our suppositions we know that $G(t)$ is analytic for $\Re(t) > -\frac{d_0}{\delta} = -\frac{1}{2\delta}$.
- Assume that $G(t)$ is analytic for $\Re(t) > -\frac{d_k}{\delta}$. This implies via the functional equations (3.14) – (3.27) that $G(t)$ is analytic for

$$\Re(t) > -\frac{q\,d_k}{\delta(1+(q-1)d_k)} = -\frac{q\frac{q^k}{q^k+1}}{\delta\left(1+(q-1)\frac{q^k}{q^k+1}\right)} = -\frac{q^{k+1}}{\delta\left(q^k+1+(q-1)q^k\right)}$$
$$= -\frac{d_{k+1}}{\delta}.$$

Since $G(t)$ is analytic for $\Re(t) > -\frac{d_{k+1}}{\delta}$ $\forall k \in \mathbb{N}_0$ and $d_k \to 1$ for $k \to \infty$, $G(t)$ is analytic for $\Re(t) > -\frac{1}{\delta}$.
If we apply $d = 1$ to the last table we get the following relations.

Möbius transform	Corresponding areas	
	w–space	t–space
$w = \frac{qt}{1+(q^2-q)\delta t}$	$\Re(w) > -\frac{1}{\delta}$ $\Leftrightarrow$	$\left\lvert t + \frac{2q-1}{2\delta q^2(q-1)}\right\rvert > \frac{1}{2\delta q^2(q-1)}$
$w = \frac{qt}{1+((q^2-q)\delta-\alpha)t}$	$\Re(w) > -\frac{1}{\delta}$ $\Leftrightarrow$	$\left\lvert t + \frac{q(2q-1)\delta-2\alpha}{2\left((q^2-q)\delta-\alpha\right)\left(q^2\delta-\alpha\right)}\right\rvert > \frac{q\delta}{2\left((q^2-q)\delta-\alpha\right)\left(q^2\delta-\alpha\right)}$
$w = \frac{t}{1+(q^2-1)\delta t}$	$\Re(w) > -\frac{1}{\delta}$ $\Leftrightarrow$	$\left\lvert t + \frac{2q^2-1}{2(q^2-1)q^2\delta}\right\rvert > \frac{1}{2(q^2-1)q^2\delta}$
$w = \frac{t}{1+((q^2-1)\delta-\beta)t}$	$\Re(w) > -\frac{1}{\delta}$ $\Leftrightarrow$	$\left\lvert t + \frac{(2q^2-1)\delta-2\beta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)}\right\rvert > \frac{\delta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)}$

In Case 1 we can see from (3.15) that the 1st, 3rd and 4th Möbius transform of the last table occur. Thus $v(t)$ is analytic if t is in each corresponding area and $t \neq -\frac{1}{q^2\delta}, t \neq -\frac{1}{q^2\delta-\gamma}$. Since the 3rd area includes the first one and $-\frac{1}{q^2\delta}$ is no point of the first area we see that

$$v(t) \quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \{K_1 \cup K_3 \cup \{-\frac{1}{q^2\delta-\gamma}\}\}$$

with

$$K_1 = \left\{ t \in \mathbb{C} : \left\lvert t + \frac{2q-1}{2(q-1)q^2\delta}\right\rvert \leq \frac{1}{2(q-1)q^2\delta}\right\}$$

$$K_3 = \left\{ t \in \mathbb{C} : \left\lvert t + \frac{(2q^2-1)\delta-2\beta}{2((q^2-1)\delta-\beta)(q^2\delta-\beta)}\right\rvert \leq \frac{\delta}{2((q^2-1)\delta-\beta)(q^2\delta-\beta)}\right\}.$$

This proves the validity of (3.28) in the first case $\left(\delta = \frac{\alpha}{q^2-q}, \delta > \frac{\beta}{q^2-1}, \delta > \frac{\gamma}{q^2}\right)$. All other cases can be treated in the same way as described above. □

Corollary 3.4. *Suppose that $f(x)$ and $g(x)$ are functions of exponential type δ and 2δ. Furthermore, suppose that the assumptions of Lemma 3.2 and 3.3 are fulfilled. Then it follows that*

$$G(t) \quad \textit{is analytic for (see Figure 3.1)}$$
$$t \in \mathbf{D}_0 = \left\{ t \in \mathbb{C} : \left| t + \frac{(\eta+1)^2}{\delta\eta(\eta+2)} \right| > \frac{\eta+1}{\delta\eta(\eta+2)}, \quad \left| \arg\left(t + \frac{1}{\delta} \right) \right| < \pi - \theta \right\} \tag{3.29}$$

$$\textit{with} \quad 0 < \eta < q-1, \quad \theta_0 < \theta < \frac{\pi}{2} \quad \textit{where } \theta_0 \in \left]0, \frac{\pi}{2}\right[\quad \textit{such that}$$

$$\sin\theta_0 = \begin{cases} \max\left\{ \frac{q}{q+2(q-1)^2}, \frac{q^2\delta^2}{q^2\delta^2+2\beta((q^2-1)\delta-\beta)} \right\} & \textit{if } \delta = \frac{\alpha}{q^2-q}, \ \delta > \frac{\beta}{q^2-1}, \ \delta \geq \frac{\gamma}{q^2} \\ \max\left\{ \frac{q}{q+2(q-1)^2}, \frac{q^3\delta^2}{q^3\delta^2+2\alpha((q^2-q)\delta-\alpha)} \right\} & \textit{if } \delta = \frac{\beta}{q^2-1}, \ \delta > \frac{\alpha}{q^2-q}, \ \delta \geq \frac{\gamma}{q^2} \\ \max\left\{ \frac{q}{q+2(q-1)^2}, \frac{q^3\delta^2}{q^3\delta^2+2\alpha((q^2-q)\delta-\alpha)}, \frac{q^2\delta^2}{q^2\delta^2+2\beta((q^2-1)\delta-\beta)} \right\} & \\ & \textit{if } \delta = \frac{\gamma}{q^2}, \ \delta > \frac{\alpha}{q^2-q}, \ \delta > \frac{\beta}{q^2-1} \\ \frac{q}{q+2(q-1)^2} & \textit{if } \delta = \frac{\alpha}{q^2-q}, \ \delta = \frac{\beta}{q^2-1}, \ \delta \geq \frac{\gamma}{q^2} \end{cases}$$

Figure 3.1: Domain where $G(t)$ is analytic: $\mathbf{D}_0 \cup \left\{ t \in \mathbb{C} : \Re(t) > -\frac{1}{\delta} \right\}$

Proof.

- Case 1: $\delta = \frac{\alpha}{q^2-q}$, $\delta > \max\left\{\frac{\beta}{q^2-1}, \frac{\gamma}{q^2}\right\}$

$$\begin{aligned} G(t) &\quad \text{is analytic for} \quad \Re(t) > -\frac{1}{\delta} \\ v(t) &\quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{K_1 \cup K_3 \cup \left\{-\frac{1}{q^2\delta - \gamma}\right\}\right\} \\ G(q^2 t) &= a_1 G(qt) + v(t) \end{aligned} \tag{3.30}$$

If $\Re(t) > -\frac{1}{\delta}$ it is obvious that $G(t)$ is analytic. If $\Re(t) \le -\frac{1}{\delta}$ we can use (3.30) n times and obtain

$$G(t) = a_1^n G\big(q^{-n}t\big) + \sum_{k=0}^{n-1} a_1^k v\big(q^{-(k+2)}t\big)\,, \quad n \in \mathbb{N}. \tag{3.31}$$

Since we can always find an $n \in \mathbb{N}$ such that $\Re(q^{-n}t) > -\frac{1}{\delta}$ the singular part of $G(t)$ can only come from the terms $v\big(q^{-(k+2)}t\big)$. This implies that

$$\begin{aligned} G(t) &\quad \text{is analytic for} \\ &\quad t \in \mathbb{C} \setminus \left\{\bigcup_{n\in\mathbb{N}} q^{n+1}K_1 \cup \bigcup_{n\in\mathbb{N}} q^{n+1}K_3 \cup \bigcup_{n\in\mathbb{N}} \left\{-\frac{q^{n+1}}{q^2\delta - \gamma}\right\}\right\} \\ &\quad \text{with} \quad q^{n+1}K_i = \left\{t \in \mathbb{C} : \frac{t}{q^{n+1}} \in K_i\right\}, \quad i = 1, 3. \end{aligned} \tag{3.32}$$

- Case 2: $\delta = \frac{\beta}{q^2-1}$, $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\gamma}{q^2}\right\}$

$$\begin{aligned} G(t) &\quad \text{is analytic for} \quad \Re(t) > -\frac{1}{\delta} \\ v(t) &\quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{K_1 \cup K_2 \cup \left\{-\frac{1}{q^2\delta - \gamma}\right\}\right\} \\ G(q^2 t) &= b_1 G(t) + v(t) \end{aligned} \tag{3.33}$$

If $\Re(t) > -\frac{1}{\delta}$ it is again obvious that $G(t)$ is analytic. If $\Re(t) \le -\frac{1}{\delta}$ we can use (3.33) n times and obtain

$$G(t) = b_1^n G\big(q^{-2n}t\big) + \sum_{k=1}^{n} b_1^{k-1} v\big(q^{-2k}t\big)\,, \quad n \in \mathbb{N}. \tag{3.34}$$

Since we can always find an $n \in \mathbb{N}$ such that $\Re(q^{-2n}t) > -\frac{1}{\delta}$ the singular part of $G(t)$ can only come from the terms $v\big(q^{-2k}t\big)$. This implies as before that

$$\begin{aligned} G(t) \quad & \text{is analytic for} \\ & t \in \mathbb{C} \setminus \left\{ \bigcup_{n\in\mathbb{N}} q^{2n}K_1 \cup \bigcup_{n\in\mathbb{N}} q^{2n}K_2 \cup \bigcup_{n\in\mathbb{N}} \left\{ -\frac{q^{2n}}{q^2\delta-\gamma} \right\} \right\} \\ & \text{with} \quad q^{2n}K_i = \left\{ t \in \mathbb{C} : \frac{t}{q^{2n}} \in K_i \right\}, \quad i = 1, 2. \end{aligned} \tag{3.35}$$

- Case 3: $\delta = \frac{\gamma}{q^2}$, $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\beta}{q^2-1}\right\}$

$$\begin{aligned} v(t) \quad & \text{is analytic for} \quad t \in \mathbb{C} \setminus \{K_1 \cup K_2 \cup K_3\} \\ G(q^2t) &= c_1 + v(t) \end{aligned} \tag{3.36}$$

This implies that

$$\begin{aligned} G(t) \quad & \text{is analytic for} \quad t \in \mathbb{C} \setminus \{q^2K_1 \cup q^2K_2 \cup q^2K_3\} \\ & \text{with} \quad q^2K_i = \left\{ t \in \mathbb{C} : \frac{t}{q^2} \in K_i \right\}, \quad i = 1, 2 \text{ and } 3. \end{aligned} \tag{3.37}$$

- Case 4: $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1}$, $\delta > \frac{\gamma}{q^2}$

$$\begin{aligned} G(t) \quad & \text{is analytic for} \quad \Re(t) > -\frac{1}{\delta} \\ v(t) \quad & \text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{ K_1 \cup \left\{ -\frac{1}{q^2\delta-\gamma} \right\} \right\} \\ G(q^2t) &= a_1G(qt) + b_1G(t) + v(t) \end{aligned} \tag{3.38}$$

For $\Re(t) > -\frac{1}{\delta}$, $G(t)$ is analytic. If $\Re(t) \le -\frac{1}{\delta}$ we can use (3.38) n times and get

$$G(t) = \lambda_{n+1}G(q^{-n}t) + \lambda_n b_1 G(q^{-(n+1)}t) + \sum_{k=1}^{n} \lambda_k v(q^{-(k+1)}t), \quad n \in \mathbb{N} \tag{3.39}$$

$$\begin{aligned} & \text{with} \quad \lambda_n = 2^{1-n} \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n}{2k+1} (a_1^2 + 4b_1)^k a_1^{n-2k-1} \\ & \text{and} \quad \left\lceil \frac{n}{2} \right\rceil = \min\left\{ k \in \mathbb{N} : \ k \ge \frac{n}{2} \right\}. \end{aligned}$$

Since we can always find an $n \in \mathbb{N}$ such that $\Re(q^{-n}t) > -\frac{1}{\delta}$ the singular part of $G(t)$ can only come from the terms $v\big(q^{-(k+1)}t\big)$. This implies that

$$G(t) \quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{ \bigcup_{n\in\mathbb{N}} q^{n+1}K_1 \cup \bigcup_{n\in\mathbb{N}} \left\{ -\frac{q^{n+1}}{q^2\delta - \gamma} \right\} \right\} \tag{3.40}$$
$$\text{with} \quad q^{n+1}K_1 = \left\{ t \in \mathbb{C} : \frac{t}{q^{n+1}} \in K_1 \right\}.$$

- Case 5: $\delta = \frac{\alpha}{q^2-q} = \frac{\gamma}{q^2}$, $\delta > \frac{\beta}{q^2-1}$

$$\begin{aligned} G(t) &\quad \text{is analytic for} \quad \Re(t) > -\frac{1}{\delta} \\ v(t) &\quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \{K_1 \cup K_3\} \\ G(q^2t) &= a_1 G(qt) + c_1 + v(t) \end{aligned} \tag{3.41}$$

If $\Re(t) > -\frac{1}{\delta}$, $G(t)$ is analytic. If $\Re(t) \leq -\frac{1}{\delta}$ we can use (3.41) n times and have

$$G(t) = a_1^n G\big(q^{-n}t\big) + c_1 \sum_{k=0}^{n-1} a_1^k + \sum_{k=0}^{n-1} a_1^k v\big(q^{-(k+2)}t\big), \quad n \in \mathbb{N}. \tag{3.42}$$

Since we can always find an $n \in \mathbb{N}$ such that $\Re(q^{-n}t) > -\frac{1}{\delta}$ the singular part of $G(t)$ can only come from the terms $v\big(q^{-(k+2)}t\big)$. Thus

$$G(t) \quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{ \bigcup_{n\in\mathbb{N}} q^{n+1}K_1 \cup \bigcup_{n\in\mathbb{N}} q^{n+1}K_3 \right\} \tag{3.43}$$
$$\text{with} \quad q^{n+1}K_i = \left\{ t \in \mathbb{C} : \frac{t}{q^{n+1}} \in K_i \right\}, \quad i = 1, 3.$$

- Case 6: $\delta = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$, $\delta > \frac{\alpha}{q^2-q}$

$$\begin{aligned} G(t) &\quad \text{is analytic for} \quad \Re(t) > -\frac{1}{\delta} \\ v(t) &\quad \text{is analytic for} \quad t \in \mathbb{C} \setminus \{K_1 \cup K_2\} \\ G(q^2t) &= b_1 G(t) + c_1 + v(t) \end{aligned} \tag{3.44}$$

For $\Re(t) > -\frac{1}{\delta}$, $G(t)$ is analytic. If $\Re(t) \leq -\frac{1}{\delta}$ we can use (3.44) n times and receive

$$G(t) = b_1^n G\big(q^{-2n}t\big) + c_1 \sum_{k=0}^{n-1} b_1^k + \sum_{k=1}^{n} b_1^{k-1} v\big(q^{-2k}t\big), \quad n \in \mathbb{N}. \tag{3.45}$$

Similar to the last case we conclude

$$\begin{aligned} G(t) \quad &\text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{ \bigcup_{n\in\mathbb{N}} q^{2n} K_1 \cup \bigcup_{n\in\mathbb{N}} q^{2n} K_2 \right\} \\ \text{with} \quad & q^{2n}K_i = \left\{ t \in \mathbb{C} : \frac{t}{q^{2n}} \in K_i \right\}, \quad i = 1, 2. \end{aligned} \tag{3.46}$$

- Case 7: $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$

$$\begin{aligned} G(t) \quad &\text{is analytic for} \quad \Re(t) > -\frac{1}{\delta} \\ v(t) \quad &\text{is analytic for} \quad t \in \mathbb{C} \setminus K_1 \\ G(q^2 t) &= a_1 G(qt) + b_1 G(t) + c_1 + v(t) \end{aligned} \tag{3.47}$$

For $\Re(t) > -\frac{1}{\delta}$, $G(t)$ is analytic. If $\Re(t) \leq -\frac{1}{\delta}$ we can use (3.47) n times and obtain

$$G(t) = \lambda_{n+1} G(q^{-n}t) + \lambda_n b_1 G(q^{-(n+1)}t) + c_1 \sum_{k=1}^{n} \lambda_k + \sum_{k=1}^{n} \lambda_k v(q^{-(k+1)}t), \tag{3.48}$$

$$\begin{aligned} n \in \mathbb{N}, \quad \text{with} \quad & \lambda_n = 2^{1-n} \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n}{2k+1} (a_1^2 + 4b_1)^k a_1^{n-2k-1} \\ \text{and} \quad & \left\lceil \frac{n}{2} \right\rceil = \min \left\{ k \in \mathbb{N} : \; k \geq \frac{n}{2} \right\}. \end{aligned}$$

Similar to the last case we conclude

$$\begin{aligned} G(t) \quad &\text{is analytic for} \quad t \in \mathbb{C} \setminus \left\{ \bigcup_{n\in\mathbb{N}} q^{n+1} K_1 \right\} \\ \text{with} \quad & q^{n+1}K_1 = \left\{ t \in \mathbb{C} : \frac{t}{q^{n+1}} \in K_1 \right\}. \end{aligned} \tag{3.49}$$

Until now we have shown that $G(t)$ is analytic for

$$t \in \begin{cases} \mathbb{C} \setminus \left\{ \bigcup\limits_{n\in\mathbb{N}} q^{n+1}K_1 \cup \bigcup\limits_{n\in\mathbb{N}} q^{n+1}K_3 \cup \bigcup\limits_{n\in\mathbb{N}} \left\{ -\frac{q^{n+1}}{q\delta-\gamma} \right\} \right\} & \\ & \text{if } \delta = \frac{\alpha}{q^2-q}, \delta > \frac{\beta}{q^2-1}, \delta > \frac{\gamma}{q^2} \\ \mathbb{C} \setminus \left\{ \bigcup\limits_{n\in\mathbb{N}} q^{2n}K_1 \cup \bigcup\limits_{n\in\mathbb{N}} q^{2n}K_2 \cup \bigcup\limits_{n\in\mathbb{N}} \left\{ -\frac{q^{2n}}{q\delta-\gamma} \right\} \right\} & \\ & \text{if } \delta = \frac{\beta}{q^2-1}, \delta > \frac{\alpha}{q^2-q}, \delta > \frac{\gamma}{q^2} \\ \mathbb{C} \setminus \{ q^2K_1 \cup q^2K_2 \cup q^2K_3 \} & \text{if } \delta = \frac{\gamma}{q^2}, \delta > \frac{\alpha}{q^2-q}, \delta > \frac{\beta}{q^2-1} \\ \mathbb{C} \setminus \left\{ \bigcup\limits_{n\in\mathbb{N}} q^{n+1}K_1 \cup \bigcup\limits_{n\in\mathbb{N}} \left\{ -\frac{q^{n+1}}{q\delta-\gamma} \right\} \right\} & \text{if } \delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1}, \delta > \frac{\gamma}{q^2} \\ \mathbb{C} \setminus \left\{ \bigcup\limits_{n\in\mathbb{N}} q^{n+1}K_1 \cup \bigcup\limits_{n\in\mathbb{N}} q^{n+1}K_3 \right\} & \text{if } \delta = \frac{\alpha}{q^2-q} = \frac{\gamma}{q^2}, \delta > \frac{\beta}{q^2-1} \\ \mathbb{C} \setminus \left\{ \bigcup\limits_{n\in\mathbb{N}} q^{2n}K_1 \cup \bigcup\limits_{n\in\mathbb{N}} q^{2n}K_2 \right\} & \text{if } \delta = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}, \delta > \frac{\alpha}{q^2-q} \\ \mathbb{C} \setminus \left\{ \bigcup\limits_{n\in\mathbb{N}} q^{n+1}K_1 \right\} & \text{if } \delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2} \end{cases} \tag{3.50}$$

Since $0 < \eta < q-1$ the circle $\left| t + \frac{(\eta+1)^2}{\delta\eta(\eta+2)} \right| = \frac{\eta+1}{\delta\eta(\eta+2)}$ covers q^2K_1. Now we will calulate the minimal distance between $q^{n+1}K_1, n \geq 2$, $q^{2n}K_2, n \in \mathbb{N}$, $q^{n+1}K_3, n \in \mathbb{N}$ and the cone $\left|\arg\left(t + \frac{1}{\delta}\right)\right| < \pi - \theta$.

- Distance between $q^{n+1}K_1, n \geq 2$ and $-\frac{1}{\delta} + re^{i(\pi-\theta)}, r \in \mathbb{R}_+$.
  If we express $-\frac{1}{\delta} + re^{i(\pi-\theta)}, r \in \mathbb{R}_+$ in Hesse form we get

$$\delta x + \frac{\delta}{\tan\theta} y + 1 = 0, \qquad x < -\frac{1}{\delta}. \tag{3.51}$$

  Now the distance to the center of the circle can be expressed by

$$dist = \left| \frac{ax_0 + by_0 + 1}{\sqrt{a^2+b^2}} \right|, \quad \text{with } a = \delta,\ b = \frac{\delta}{\tan\theta},$$

  where x_0 and y_0 are the real and imaginary parts of the center of the circle. Thus we get

$$dist = \left| \frac{-\delta q^{n+1} \frac{2q-1}{2(q-1)q^2\delta} + 1}{\sqrt{\delta^2\left(1 + \frac{1}{\tan^2\theta}\right)}} \right| = \frac{\sin\theta}{\delta} \left(q^{n+1} \frac{2q-1}{2(q-1)q^2} - 1 \right) =$$

$$
\begin{aligned}
&= q^{n+1} \frac{1}{2(q-1)q^2\delta} \sin\theta \left(2q - 1 - \frac{2(q-1)q^2}{q^{n+1}}\right) \\
&\geq q^{n+1} \frac{1}{2(q-1)q^2\delta} \sin\theta \left(1 + 2(q-1) - \frac{2(q-1)}{q}\right) \\
&= q^{n+1} \frac{1}{2(q-1)q^2\delta} \sin\theta \left(1 + 2(q-1)^2 \frac{1}{q}\right) \\
&> q^{n+1} \frac{1}{2(q-1)q^2\delta} \\
&\quad \text{for} \quad n \geq 2, \quad \text{as} \ \sin\theta > \frac{q}{q + 2(q-1)^2}.
\end{aligned}
$$

Since the radii of the circles are $q^{n+1}\frac{1}{2(q-1)q^2\delta}$ and there is a strict inequality it is obvious that $-\frac{1}{\delta} + re^{i(\pi-\theta)}, r \in \mathbb{R}_+$ does not intersect $q^{n+1}K_1, \forall n \geq 2$. Therefore, the minimal distance between $q^{n+1}K_1$ and the cone is positive for all $n \geq 2$.

- $\delta > \frac{\alpha}{q^2-q}$: Distance between $q^{2n}K_2, n \in \mathbb{N}$ and $-\frac{1}{\delta} + re^{i(\pi-\theta)}, r \in \mathbb{R}_+$. Similar to above we calulate the distance between the line and the center of the circle as

$$
\begin{aligned}
dist &= \left| \frac{-\delta q^{2n} \frac{q(2q-1)\delta - 2\alpha}{2((q^2-q)\delta - \alpha)(q^2\delta - \alpha)} + 1}{\sqrt{\delta^2 \left(1 + \frac{1}{\tan^2\theta}\right)}} \right| \\
&= \frac{\sin\theta}{\delta} \left(\delta q^{2n} \frac{q(2q-1)\delta - 2\alpha}{2\left((q^2-q)\delta - \alpha\right)\left(q^2\delta - \alpha\right)} - 1\right) \\
&= q^{2n} \frac{q\delta}{2\left((q^2-q)\delta - \alpha\right)\left(q^2\delta - \alpha\right)} \sin\theta \\
&\quad \cdot \left((2q-1) - \frac{2\alpha}{q\delta} - \frac{2}{q^{2n+1}\delta^2} \underbrace{\left((q^2-q)\delta - \alpha\right)}_{>0} \underbrace{\left(q^2\delta - \alpha\right)}_{>0}\right) \\
&\geq q^{2n} \frac{q\delta}{2\left((q^2-q)\delta - \alpha\right)\left(q^2\delta - \alpha\right)} \sin\theta \\
&\quad \cdot \left(1 + 2(q-1) - \frac{2\alpha}{q\delta} - 2\left((q-1) - \frac{\alpha}{q\delta}\right)\left(1 - \frac{\alpha}{q^2\delta}\right)\right) \\
&= q^{2n} \frac{q\delta}{2\left((q^2-q)\delta - \alpha\right)\left(q^2\delta - \alpha\right)} \sin\theta \left(1 + 2\left((q-1) - \frac{\alpha}{q\delta}\right)\frac{\alpha}{q^2\delta}\right) \\
&> q^{2n} \frac{q\delta}{2\left((q^2-q)\delta - \alpha\right)\left(q^2\delta - \alpha\right)} \\
&\quad \forall n \in \mathbb{N}, \quad \text{as} \ \sin\theta > \frac{q^3\delta^2}{q^3\delta^2 + 2\alpha\left((q^2-q)\delta - \alpha\right)}.
\end{aligned}
$$

Since the radii of the circles are $q^{2n}\frac{q\delta}{2((q^2-q)\delta-\alpha)(q^2\delta-\alpha)}$ and there is a strict inequality it is obvious that $-\frac{1}{\delta}+re^{i(\pi-\theta)}$, $r\in\mathbb{R}_+$ does not intersect $q^{2n}K_2$, $\forall n\in\mathbb{N}$.

- $\delta>\frac{\beta}{q^2-1}$: Distance between $q^{n+1}K_3, n\in\mathbb{N}$ and $-\frac{1}{\delta}+re^{i(\pi-\theta)}, r\in\mathbb{R}_+$. Similar to the last case we calulate the distance between the line and the center of the circle as

$$\begin{aligned}
dist &= \left|\frac{-\delta q^{n+1}\frac{(2q^2-1)\delta-2\beta}{2((q^2-1)\delta-\beta)(q^2\delta-\beta)}+1}{\sqrt{\delta^2\left(1+\frac{1}{\tan^2\theta}\right)}}\right| \\
&= \frac{\sin\theta}{\delta}\left(\delta q^{n+1}\frac{(2q^2-1)\delta-2\beta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)}-1\right) \\
&= q^{n+1}\frac{\delta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)}\sin\theta \\
&\quad\cdot\left((2q^2-1)-\frac{2\beta}{\delta}-\frac{1}{q^{n+1}\delta^2}2\underbrace{\left((q^2-1)\delta-\beta\right)}_{>0}\underbrace{\left(q^2\delta-\beta\right)}_{>0}\right) \\
&\geq q^{n+1}\frac{\delta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)}\sin\theta \\
&\quad\cdot\left(1+2(q^2-1)-\frac{2\beta}{\delta}-2\left((q^2-1)-\frac{\beta}{\delta}\right)\left(1-\frac{\beta}{q^2\delta}\right)\right) \\
&= q^{n+1}\frac{\delta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)}\sin\theta\left(1+2\left((q^2-1)-\frac{\beta}{\delta}\right)\frac{\beta}{q^2\delta}\right) \\
&> q^{n+1}\frac{\delta}{2\left((q^2-1)\delta-\beta\right)\left(q^2\delta-\beta\right)} \\
&\quad \forall n\in\mathbb{N},\quad \text{as } \sin\theta>\frac{q^2\delta^2}{q^2\delta^2+2\beta\left((q^2-1)\delta-\beta\right)}.
\end{aligned}$$

Since the radii of the circles are $q^{n+1}\frac{\delta}{2((q^2-1)\delta-\beta)(q^2\delta-\beta)}$ and the above inequality is strict it is clear that $-\frac{1}{\delta}+re^{i(\pi-\theta)}, r\in\mathbb{R}_+$ does not intersect $q^{n+1}K_3, \forall n\in\mathbb{N}$.

In all cases the distance between the circles and the cone is positive. Therefore $G(t)$ is analytic for all $t\in\mathbf{D}_0$. □

Corollary 3.5. *Assume that*

$$F\left(\frac{1}{\delta}t\right)=\frac{1}{(1-t)^{k+1}}G\left(\frac{t}{\delta(1-t)}\right)\quad \textit{for } |t|<1 \tag{3.52}$$

and that

$$G(t) \quad \textit{is analytic for}$$
$$t \in \mathbf{D_0} = \left\{ t \in \mathbb{C} : \left| t + \frac{(\eta+1)^2}{\delta\eta(\eta+2)} \right| > \frac{\eta+1}{\delta\eta(\eta+2)}, \quad \left| \arg\left(t + \frac{1}{\delta} \right) \right| < \pi - \theta \right\} \tag{3.53}$$

with $0 < \eta < q-1, \quad \theta_0 < \theta < \frac{\pi}{2}$ *where* $\theta_0 \in \left]0, \frac{\pi}{2}\right[$ *as in Corollary 3.4.*

Then $F(t)$ *can be analytically continued in a domain*

$$\mathbf{D} = \left\{ t \in \mathbb{C} : \ |t| < \frac{1}{\delta}(1+\eta), \left| \arg\left(t - \frac{1}{\delta} \right) \right| > \theta \right\}. \tag{3.54}$$

Proof. With $w = \frac{t}{\delta(1-t)}$, the following equivalences hold.

$$\begin{aligned}
t = 1+\eta \quad &\Longleftrightarrow \quad w = \frac{1}{\delta}\,\frac{1+\eta}{1-(1+\eta)} = -\frac{1}{\delta}\left(1+\frac{1}{\eta}\right) \\
t = -(1+\eta) \quad &\Longleftrightarrow \quad w = \frac{-1}{\delta}\,\frac{1+\eta}{1+(1+\eta)} = -\frac{1}{\delta}\left(1-\frac{1}{2+\eta}\right) \\
t = \pm i(1+\eta) \quad &\Longleftrightarrow \quad w = \frac{1}{\delta}\,\frac{\pm i(1+\eta)}{1 \mp i(1+\eta)} = -\frac{1}{\delta}\,\frac{(1+\eta)^2 \mp i(1+\eta)}{1+(1+\eta)^2} \\
t = 1+\eta_0 e^{\pm i\theta} \quad &\Longleftrightarrow \quad w = \frac{1}{\delta}\,\frac{1+\eta_0 e^{\pm i\theta}}{1-1-\eta_0 e^{\pm i\theta}} = -\frac{1}{\delta}\left(1+\frac{1}{\eta_0}\,e^{\mp i\theta}\right) = \\
&\qquad\quad = -\frac{1}{\delta} + \frac{1}{\eta_0}\,e^{i(\pi \mp \theta)} \\
t = 0 \quad &\Longleftrightarrow \quad w = 0
\end{aligned}$$

Thus we see that the circle $t = (1+\eta)e^{i\phi}$ is mapped to a circle with center

$$m = -\frac{1}{2\delta}\left(1+\frac{1}{\eta}+1-\frac{1}{2+\eta}\right) = -\frac{1}{\delta}\,\frac{(1+\eta)^2}{\eta(2+\eta)} \tag{3.55}$$

and radius

$$r = -\frac{1}{\delta}\left(1-\frac{1}{2+\eta}-\frac{(1+\eta)^2}{\eta(2+\eta)}\right) = \frac{1}{\delta}\,\frac{1+\eta}{\eta(2+\eta)}. \tag{3.56}$$

Since all $t \in \mathbf{D}$ are mapped under the Möbius transform to an element of $\mathbf{D_0}$, $F(t)$ can be continued analytically in the domain $\mathbf{D}$. ☐

Chapter 4

Main Theorem

For convenience we always take the main branches of the logarithms and the principal value of the complex solutions.

Theorem 4.1. *Assume that* $f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$ *is entire and the solution of*

$$f(q^2x) = (a_0 + a_1 e^{\alpha x}) f(qx) + (b_0 + b_1 e^{\beta x}) f(x) + c_0 + c_1 e^{\gamma x} \tag{4.1}$$

with $q > 1$ *and constants* $\alpha, \beta, \gamma \geq 0$, $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, $a_0, a_1, b_0, b_1, c_0, c_1 \in \mathbb{C}$ *such that* $q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1) \neq 0 \ \ \forall n \in \mathbb{N}_0$. *For convenience we put* α *(resp.* β, γ*) zero if* a_1 *(resp.* b_1, c_1*) is zero. Denote by*

$$\delta = \max\left\{\frac{\alpha}{q^2 - q}, \frac{\beta}{q^2 - 1}, \frac{\gamma}{q^2}\right\}. \tag{4.2}$$

Furthermore, we denote by $\rho_a, \rho_b \in \mathbb{C}$ *the solutions of* $q^\rho = a_1$ *and* $q^{2\rho} = b_1$ *and by* ρ_1 *and* ρ_2 *two complex numbers such that* q^{ρ_1} *and* q^{ρ_2} *are the two solutions of* $x^2 - a_1 x - b_1 = 0$. *In the following,* $K_1(t), K_2(t)$ *are 1-periodic functions which are analytic in the stripe* $|\Im(t)| < \nu, \ \ \nu > 0$.

Then the sequence f_n *has the following asymptotic growth for* $n \to \infty$.

Case 1: $\delta = \frac{\alpha}{q^2-q}$ *and* $\delta > \max\left\{\frac{\beta}{q^2-1}, \frac{\gamma}{q^2}\right\}$

$$\delta^{-n} f_n = n^{\rho_a} K_1(\log_q n) + O\big(n^{\rho_a - 1}\big) \tag{4.3}$$

Case 2: $\delta = \frac{\beta}{q^2-1}$ *and* $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\gamma}{q^2}\right\}$

$$\delta^{-n} f_n = n^{\rho_b} K_1(\log_q n) + O\big(n^{\rho_b - 1}\big) \tag{4.4}$$

Case 3: $\delta = \frac{\gamma}{q^2}$ *and* $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\beta}{q^2-1}\right\}$

$$\delta^{-n} f_n = c_1 + O\big(n^{-1}\big) \tag{4.5}$$

Case 4: $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1}$ *and* $\delta > \frac{\gamma}{q^2}$

4.1: $\rho_1 \neq \rho_2$ ($\Leftrightarrow a_1^2 + 4b_1 \neq 0$)

$$\delta^{-n} f_n = \big(n^{\rho_1} K_1(\log_q n) + n^{\rho_2} K_2(\log_q n)\big)\big(1 + O\big(n^{-1}\big)\big) \tag{4.6}$$

4.2: $\rho_1 = \rho_2$ ($\Leftrightarrow a_1^2 + 4b_1 = 0$)

$$\delta^{-n} f_n = n^{\rho_1}\big(K_1(\log_q n) + K_2(\log_q n)\log_q n\big)\big(1 + O\big(n^{-1}\big)\big) \tag{4.7}$$

Case 5: $\delta = \frac{\alpha}{q^2-q} = \frac{\gamma}{q^2}$ *and* $\delta > \frac{\beta}{q^2-1}$

$$\delta^{-n} f_n = \begin{cases} \dfrac{c_1}{1-a_1} + O(n^{-1}) & \textit{if } |a_1| \leq \frac{1}{q} \\ \dfrac{c_1}{1-a_1} + n^{\rho_a} K_1(\log_q n) + O(n^{-1}) & \textit{if } \frac{1}{q} < |a_1| \leq 1, a_1 \neq 1 \\ c_1 \log_q n + K_1(\log_q n) + O(n^{-1}) & \textit{if } a_1 = 1 \\ n^{\rho_a} K_1(\log_q n) + \dfrac{c_1}{1-a_1} + O\big(n^{\rho_a - 1}\big) & \textit{if } 1 < |a_1| < q \\ n^{\rho_a} K_1(\log_q n) + O\big(n^{\rho_a - 1}\big) & \textit{if } q \leq |a_1| \end{cases} \tag{4.8}$$

Case 6: $\delta = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$ *and* $\delta > \frac{\alpha}{q^2-q}$

$$\delta^{-n} f_n = \begin{cases} \dfrac{c_1}{1-b_1} + O(n^{-1}) & \textit{if } |b_1| \leq \frac{1}{q} \\ \dfrac{c_1}{1-b_1} + n^{\rho_b} K_1(\log_q n) + O(n^{-1}) & \textit{if } \frac{1}{q} < |b_1| \leq 1, b_1 \neq 1 \\ c_1 \log_q n + K_1(\log_q n) + O(n^{-1}) & \textit{if } b_1 = 1 \\ n^{\rho_b} K_1(\log_q n) + \dfrac{c_1}{1-b_1} + O\big(n^{\rho_b - 1}\big) & \textit{if } 1 < |b_1| < q \\ n^{\rho_b} K_1(\log_q n) + O\big(n^{\rho_b - 1}\big) & \textit{if } q \leq |b_1| \end{cases} \tag{4.9}$$

Case 7: $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$

7.1: $\rho_1 \neq \rho_2$ ($\Leftrightarrow a_1^2 + 4b_1 \neq 0$)

$$\delta^{-n} f_n = \begin{cases} \Big(\dfrac{c_1}{1-a_1-b_1} + n^{\rho_1} K_1(\log_q n) + \\ \quad + n^{\rho_2} K_2(\log_q n)\Big)\big(1 + O(n^{-1})\big) & \text{if } a_1 + b_1 \neq 1, \\ \Big(\dfrac{c_1}{1-a_1} \log_q n + n^{\rho_1} K_1(\log_q n) + \\ \quad + n^{\rho_2} K_2(\log_q n)\Big)\big(1 + O(n^{-1})\big) & \text{if } a_1 + b_1 = 1 \end{cases} \tag{4.10}$$

7.2: $\rho_1 = \rho_2$ ($\Leftrightarrow a_1^2 + 4b_1 = 0$)

$$\delta^{-n} f_n = \begin{cases} \Big(\dfrac{c_1}{1-a_1-b_1} + n^{\rho_1}\big(K_1(\log_q n) + \\ \quad + \log_q n \; K_2(\log_q n)\big)\Big)\big(1 + O(n^{-1})\big) & \text{if } a_1 + b_1 \neq 1, \\ \Big(\dfrac{c_1}{1-a_1} \log_q n + n^{\rho_1}\big(K_1(\log_q n) + \\ \quad + \log_q n \; K_2(\log_q n)\big)\Big)\big(1 + O(n^{-1})\big) & \text{if } a_1 + b_1 = 1 \end{cases} \tag{4.11}$$

Remark 4.2.

1. In Case 1 even
$$\binom{n+k}{n} \delta^{-n} f_n = \frac{\Gamma(n + \rho_a + k + 1)}{n!} K(\log_q n) + O\Big(\frac{1}{n}\Big) \tag{4.12}$$
can be shown, where $k \in \mathbb{N}_0$ is chosen such that $q^k |a_1| > 1$.

2. In Case 2 even
$$\binom{n+k}{n} \delta^{-n} f_n = \frac{\Gamma(n + \rho_b + k + 1)}{n!} K(\log_q n) + O\Big(\frac{1}{n}\Big) \tag{4.13}$$
can be shown, where $k \in \mathbb{N}_0$ is chosen such that $q^{2k} |b_1| > 1$.

3. In Case 4 even
$$\begin{aligned} \binom{n+k}{n} \delta^{-n} f_n = K_1(\log_q n) \frac{\Gamma(n + \rho_1 + k + 1)}{n!} \\ + K_2(\log_q n) \frac{\Gamma(n + \rho_2 + k + 1)}{n!} + O\Big(\frac{1}{n}\Big) \end{aligned} \tag{4.14}$$

can be shown for $\rho_1 \neq \rho_2$. In case of $\rho_1 = \rho_2$

$$\binom{n+k}{n}\delta^{-n} f_n = K_1(\log_q n)\,\frac{\Gamma(n+\rho_1+k+1)}{n!} + K_2(\log_q n)\,\frac{\Gamma(n+\rho_2+k+1)}{n!} + O\left(\frac{1}{n}\right) \tag{4.15}$$

can be shown. $k \in \mathbb{N}_0$ is a fixed constant such that $q^k \left|\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} + b_1}\right| > 1$.

Proof. First we will show that there exists a solution for the sequence $\{f_n\}_{n\in\mathbb{N}}$. Using the exponential series for $f(x)$ and e^x in (4.1) we get

$$\sum_{n\geq 0} q^{2n} f_n \frac{x^n}{n!} = \left(a_0 + a_1 \sum_{n\geq 0} \alpha^n \frac{x^n}{n!}\right) \sum_{n\geq 0} q^n f_n \frac{x^n}{n!} + \left(b_0 + b_1 \sum_{n\geq 0} \beta^n \frac{x^n}{n!}\right) \sum_{n\geq 0} f_n \frac{x^n}{n!} + c_0 + c_1 \sum_{n\geq 0} \gamma^n \frac{x^n}{n!}. \tag{4.16}$$

Comparing coefficients one has

$$[x^0]: \quad f_0 = (a_0 + a_1) f_0 + (b_0 + b_1) f_0 + c_0 + c_1 \tag{4.17}$$

$$[x^n]: \quad q^{2n} f_n \frac{1}{n!} = (a_0 + a_1) q^n f_n \frac{1}{n!} + a_1 \sum_{i=0}^{n-1} \frac{q^i f_i}{i!} \frac{\alpha^{n-i}}{(n-i)!} + (b_0 + b_1) f_n \frac{1}{n!} + b_1 \sum_{i=0}^{n-1} \frac{f_i}{i!} \frac{\beta^{n-i}}{(n-i)!} + c_1 \frac{\gamma^n}{n!}, \quad n \in \mathbb{N}, \tag{4.18}$$

and therefore,

$$f_0[1 - (a_0 + a_1) - (b_0 + b_1)] = c_0 + c_1 \tag{4.19}$$

$$f_n[q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1)] = a_1 \alpha^n \sum_{i=0}^{n-1} \binom{n}{i} q^i \alpha^{-i} f_i + b_1 \beta^n \sum_{i=0}^{n-1} \binom{n}{i} \beta^{-i} f_i + c_1 \gamma^n, \quad n \in \mathbb{N}. \tag{4.20}$$

These two equations explain why we had to assume that $q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1) \neq 0 \ \ \forall n \in \mathbb{N}_0$.

Case 1: $\delta = \frac{\alpha}{q^2 - q}$ and $\delta > \max\left\{\frac{\beta}{q^2-1}, \frac{\gamma}{q^2}\right\}$

Since $a_1 \neq 0$ we can choose a $k \in \mathbb{N}_0$ such that $|q^k a_1| > 1$. If we write

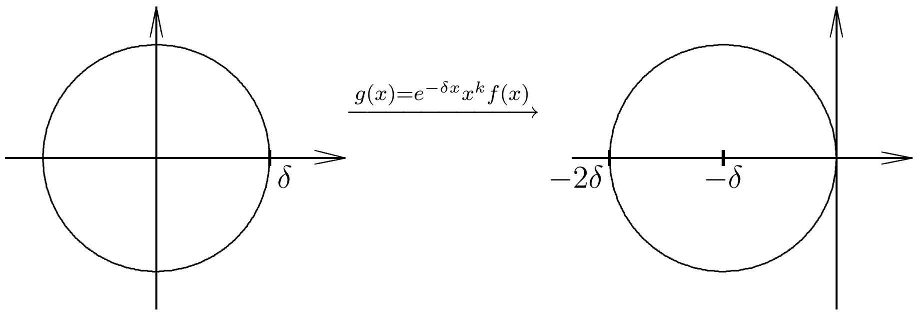
$$f(x) = e^{\delta x} x^{-k} g(x) \tag{4.21}$$

we get with (4.1) the functional equation for $g(x)$

$$\begin{aligned}\frac{e^{q^2\delta x}}{q^{2k}x^k}\,g(q^2x) &= \left(a_0 + a_1 e^{(q^2-q)\delta x}\right)\frac{e^{q\delta x}}{q^k x^k}\,g(qx) + \left(b_0 + b_1 e^{\beta x}\right)\frac{e^{\delta x}}{x^k}\,g(x) + c_0 + c_1 e^{\gamma x}\\ \Rightarrow\quad g(q^2x) &= a_1 q^k g(qx) + a_0 q^k e^{-(q^2-q)\delta x} g(qx) + b_0 q^{2k} e^{-(q^2-1)\delta x} g(x)\\ &\quad + b_1 q^{2k} e^{-\left((q^2-1)\delta-\beta\right)x} g(x) + c_0 q^{2k} e^{-q^2\delta x} x^k + c_1 q^{2k} e^{-\left(q^2\delta-\gamma\right)x} x^k.\end{aligned} \tag{4.22}$$

In Lemma 3.1 we have shown that $f(x)$ is of exponential type δ. Therefore, $g(x)$ is of exponential type less or equal to 2δ. Now we can build the LB-transform of $g(x)$. For shortening we denote by $\tilde{g}(s)$ the LB-transform of $g(x)$.

$$\begin{aligned}\frac{1}{q^2}\,\tilde{g}\left(\frac{s}{q^2}\right) &= a_1 q^{k-1}\tilde{g}\left(\frac{s}{q}\right) + a_0 q^{k-1}\tilde{g}\left(\frac{s+(q^2-q)\delta}{q}\right)\\ &\quad + b_0 q^{2k}\tilde{g}\big(s+(q^2-1)\delta\big) + b_1 q^{2k}\tilde{g}\big(s+(q^2-1)\delta-\beta\big)\\ &\quad + c_0 q^{2k}\frac{k!}{\left(s+q^2\delta\right)^{k+1}} + c_1 q^{2k}\frac{k!}{\left(s+(q^2\delta-\gamma)\right)^{k+1}}\end{aligned} \tag{4.23}$$



a) Circle covering the indicator function of $f(x)$

b) Circle covering the indicator function of $g(x)$

Figure 4.1: Areas enclosing the indicator function

The estimation of the indicator function (Figure 4.1) shows that $\tilde{g}(s)$ exists in the domain $\{s \in \mathbb{C} \,|\, \Re(s) > 0 \text{ or } |s| > 2\delta\}$. Thus equation (4.23) is at least valid for $\Re(s) > 0$ and for $|s| > 2q^2\delta$. Having a look at (4.21) we see the following relations between $f(x)$, $g(x)$ and their LB-transforms.

$$
\begin{array}{ccl}
x^k f(x) & \xrightarrow{LB} & (-1)^k \tilde{f}^{(k)}(s) = \frac{k!}{s^{k+1}} \sum_{n=0}^{\infty} \binom{n+k}{n} f_n s^{-n} =: \frac{k!}{s^{k+1}}\, F\left(\frac{1}{s}\right) \\
\Big\downarrow e^{-\delta x} & & \\
g(x) := \sum_{n=0}^{\infty} g_n \frac{x^{n+k}}{n!} & \xrightarrow{LB} & (-1)^k \tilde{f}^{(k)}(s+\delta) = \tilde{g}(s) = \frac{k!}{s^{k+1}} \sum_{n=0}^{\infty} \binom{n+k}{n} g_n s^{-n} \\
& & =: \frac{k!}{s^{k+1}}\, G\left(\frac{1}{s}\right)
\end{array}
$$

For $|s| > \delta$ it follows that

$$
\frac{k!}{s^{k+1}}\, F\left(\frac{1}{s}\right) = (-1)^k \tilde{f}^{(k)}(s) = \tilde{g}(s-\delta) = \frac{k!}{(s-\delta)^{k+1}}\, G\left(\frac{1}{s-\delta}\right).
$$

Putting $t = \frac{\delta}{s}$ we have

$$
F\left(\frac{1}{\delta}\, t\right) = \frac{1}{(1-t)^{k+1}}\, G\left(\frac{t}{\delta(1-t)}\right) \quad \text{for } |t| < 1. \tag{4.24}
$$

Since we put $\tilde{g}(s) = \frac{k!}{s^{k+1}}\, G\big(\frac{1}{s}\big)$ there follows from (4.23) that

$$
\begin{aligned}
\frac{1}{q^2}\, \frac{q^{2(k+1)} k!}{s^{k+1}}\, G\left(\frac{q^2}{s}\right) = {} & a_1 q^{k-1} \frac{q^{k+1} k!}{s^{k+1}}\, G\left(\frac{q}{s}\right) \\
& + a_0 q^{k-1} \frac{q^{k+1} k!}{(s+(q^2-q)\delta)^{k+1}}\, G\left(\frac{q}{s+(q^2-q)\delta}\right) \\
& + b_0 q^{2k} \frac{k!}{(s+(q^2-1)\delta)^{k+1}}\, G\left(\frac{1}{s+(q^2-1)\delta}\right) \\
& + b_1 q^{2k} \frac{k!}{(s+(q^2-1)\delta-\beta)^{k+1}}\, G\left(\frac{1}{s+(q^2-1)\delta-\beta}\right) \\
& + c_0 q^{2k} \frac{k!}{(s+q^2\delta)^{k+1}} + c_1 q^{2k} \frac{k!}{(s+q^2\delta-\gamma)^{k+1}}.
\end{aligned}
$$

If we put $t = \frac{1}{s}$ we get

$$
\begin{aligned}
G\big(q^2 t\big) = {} & a_1 G(qt) + a_0 \frac{1}{(1+(q^2-q)\delta t)^{k+1}}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\
& + b_0 \frac{1}{(1+(q^2-1)\delta t)^{k+1}}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\
& + b_1 \frac{1}{(1+((q^2-1)\delta-\beta)\, t)^{k+1}}\, G\left(\frac{t}{1+((q^2-1)\delta-\beta)\, t}\right) \\
& + c_0 \frac{1}{(1+q^2\delta t)^{k+1}} + c_1 \frac{1}{(1+(q^2\delta-\gamma)t)^{k+1}},
\end{aligned}
$$

or equivalently,

$$G\big(q^2t\big) = a_1G(qt) + v(t) \tag{4.25}$$

with

$$\begin{aligned} v(t) = {} & \frac{a_0}{(1+(q^2-q)\delta t)^{k+1}}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\ & + \frac{b_0}{(1+(q^2-1)\delta t)^{k+1}}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\ & + \frac{b_1}{(1+((q^2-1)\delta-\beta)\,t)^{k+1}}\, G\left(\frac{t}{1+((q^2-1)\delta-\beta)\,t}\right) \\ & + \frac{c_0}{(1+q^2\delta t)^{k+1}} + \frac{c_1}{(1+(q^2\delta-\gamma)t)^{k+1}}. \end{aligned} \tag{4.26}$$

Since $\tilde{g}(s)$ is analytic in the halfplane $\Re(s) > 0$, $G(t)$ is analytic for $\Re(t) > 0$. Therefore, $t^{k+1}v(t)$ is bounded for $|t| > r_0$, whereby r_0 is sufficiently large. As shown in Corollary 3.4 Case 1 $G(t)$ can be analytically continued in the domain $\mathbf{D}_0 = \left\{t \in \mathbb{C} : \left|t + \frac{(\eta+1)^2}{\delta\eta(\eta+2)}\right| > \frac{\eta+1}{\delta\eta(\eta+2)},\ \left|\arg\left(t+\frac{1}{\delta}\right)\right| < \pi - \theta\right\}$ with $0 < \eta < q-1$ and $\theta_0 < \theta < \frac{\pi}{2}$ where θ_0 is a positive constant. The rightmost singularity lies at $-\frac{1}{\delta}$.

If we put $G(t) = t^{\rho_a}u(t)$, where ρ_a is the solution of $q^\rho = a_1$, we get

$$u(qt) = u(t) + \frac{1}{a_1 t^{\rho_a}}\, v\left(\frac{t}{q}\right). \tag{4.27}$$

The number k has been chosen large enough such that $q^k|a_1| > 1$ which implies that $\Re(k+\rho_a+1) > 1$. Therefore, it follows that the analytic function $\frac{1}{t^{\rho_a}}\, v\left(\frac{t}{q}\right) = O(t^{-(\rho_a+k+1)})$ is bounded.

$$u(t) = K_1(\log_q t) + O\big(t^{-(\rho_a+k+1)}\big) \tag{4.28}$$

for $|t| \to \infty$, where $K_1(t)$ is a 1-periodic function. Now we get for $G(t)$

$$G(t) = t^{\rho_a}K_1(\log_q t) + O\big(t^{-(k+1)}\big) \quad \text{for } |t| \to \infty. \tag{4.29}$$

Note that iteration (4.27) and $u(t) = t^{-\rho_a}G(t)$ show that $u(t)$ is analytic and bounded in $\mathbf{D}_0 \setminus \{0\}$. Finally, with (4.24) it follows that

$$\begin{aligned} F\left(\frac{1}{\delta}\,t\right) &= \frac{1}{(1-t)^{k+1}}\left[\frac{t^{\rho_a}}{\delta^{\rho_a}(1-t)^{\rho_a}}K_1\left(\log_q \frac{t}{\delta(1-t)}\right) + O\left(\left(\frac{1-t}{t}\right)^{k+1}\right)\right] \\ &= \delta^{-\rho_a}(1-t)^{-(\rho_a+k+1)}K_1\left(\log_q \frac{1}{\delta(1-t)}\right) + O(1) \quad \text{for } t \to 1_-. \end{aligned} \tag{4.30}$$

Obviously, $F(\frac{1}{\delta}t)$ is analytic in $|t| < 1$ with a singularity in $t = 1$. Corollary 3.5 implies that $F(t)$ can be continued analytically in the domain $\mathbf{D} = \left\{t \in \mathbb{C} : |t| < \frac{1}{\delta}(1+\eta),\ \left|\arg\left(t - \frac{1}{\delta}\right)\right| > \theta\right\}$. Thus we can apply Theorem 2.9 which yields

$$\binom{n+k}{n}\delta^{-n} f_n = \delta^{-\rho_a} \frac{\Gamma(n+\rho_a+k+1)}{n!\,\Gamma(\rho_a+k+1)}\,\bar{K}_1\Big(\log_q \frac{n}{\delta}\Big) + O\Big(\frac{1}{n}\Big)$$
$$= \frac{\Gamma(n+\rho_a+k+1)}{n!\,\Gamma(\rho_a+k+1)}\,\tilde{K}_1\big(\log_q n\big) + O\Big(\frac{1}{n}\Big),$$

and using the asymptotic expansion for $\binom{n+k}{n} = \frac{n^k}{k!}\left(1 + O\left(\frac{1}{n}\right)\right)$ we obtain

$$\frac{n^k}{k!}\left(1 + O\Big(\frac{1}{n}\Big)\right)\delta^{-n} f_n = n^{\rho_a+k}\left(1 + O\Big(\frac{1}{n}\Big)\right)\tilde{K}_1\big(\log_q n\big) + O\Big(\frac{1}{n}\Big)$$
$$\delta^{-n} f_n = n^{\rho_a}\hat{K}_1\big(\log_q n\big) + O\big(n^{\rho_a-1}\big) + O\Big(\frac{1}{n^{k+1}}\Big).$$

The functions $\bar{K}_1(t)$, $\tilde{K}_1(t)$ and $\hat{K}_1(t)$ are 1-periodic functions which are analytic in a stripe $|\Im(t)| < \nu$. The number $k \in \mathbb{N}$ has been chosen large enough such that $q^k|a_1| > 1$ so $O(n^{\rho_a-1}) + O(n^{-(k+1)}) = O(n^{\rho_a-1})$. We get the following asymtotic expansion for f_n,

$$\delta^{-n} f_n = n^{\rho_a}\hat{K}_1\left(\log_q n\right) + O\left(n^{\rho_a-1}\right) \qquad \text{for } n \to \infty. \tag{4.31}$$

Case 2: $\delta = \frac{\beta}{q^2-1}$ and $\delta > \max\left\{\frac{\alpha}{q^2-q}, \frac{\gamma}{q^2}\right\}$

Since $b_1 \neq 0$ we can find a $k \in \mathbb{N}_0$ such that $q^{2k}|b_1| > 1$. If we write

$$f(x) = e^{\delta x} x^{-k} g(x) \tag{4.32}$$

we get with (4.1) the functional equation for $g(x)$

$$\frac{e^{q^2\delta x}}{q^{2k}x^k}\,g(q^2x) = (a_0 + a_1e^{\alpha x})\,\frac{e^{q\delta x}}{q^k x^k}\,g(qx) + \Big(b_0 + b_1e^{(q^2-1)\delta x}\Big)\,\frac{e^{\delta x}}{x^k}\,g(x) + c_0 + c_1e^{\gamma x}$$
$$\begin{aligned}\Rightarrow\quad g(q^2x) &= b_1q^{2k}g(x) + a_0q^k e^{-(q^2-q)\delta x}g(qx) + a_1q^k e^{-((q^2-q)\delta-\alpha)x}g(qx)\\ &\quad + b_0q^{2k}e^{-(q^2-1)\delta x}g(x) + c_0q^{2k}e^{-q^2\delta x}x^k + c_1q^{2k}e^{-\left(q^2\delta-\gamma\right)x}x^k.\end{aligned} \tag{4.33}$$

In Lemma 3.1 we have shown that $f(x)$ is of exponential type δ. Therefore, $g(x)$ is of exponential type less or equal to 2δ. Now we can build the LB-transform of $g(x)$. For shortening we denote by $\tilde{g}(s)$ the LB-transform of $g(x)$.

$$\begin{aligned}\frac{1}{q^2}\,\tilde{g}\Big(\frac{s}{q^2}\Big) &= b_1q^{2k}\tilde{g}(s) + a_0q^{k-1}\tilde{g}\Big(\frac{s+(q^2-q)\delta}{q}\Big)\\ &\quad + a_1q^{k-1}\tilde{g}\Big(\frac{s+(q^2-q)\delta-\alpha}{q}\Big) + b_0q^{2k}\tilde{g}\left(s+(q^2-1)\delta\right)\\ &\quad + c_0q^{2k}\frac{k!}{(s+q^2\delta)^{k+1}} + c_1q^{2k}\frac{k!}{(s+q^2\delta-\gamma)^{k+1}}\end{aligned} \tag{4.34}$$

If we put $\tilde{g}(s) = \frac{k!}{s^{k+1}} G\big(\frac{1}{s}\big)$ and $t = \frac{1}{s}$ then equation (4.34) implies

$$G\big(q^2t\big) = b_1 G(t) + v(t) \tag{4.35}$$

with

$$\begin{aligned} v(t) = {} & \frac{a_0}{(1+(q^2-q)\delta t)^{k+1}}\, G\bigg(\frac{qt}{1+(q^2-q)\delta t}\bigg) \\ & + \frac{a_1}{(1+((q^2-q)\delta-\alpha)t)^{k+1}}\, G\bigg(\frac{qt}{1+((q^2-q)\delta-\alpha)t}\bigg) \\ & + \frac{b_0}{(1+(q^2-1)\delta t)^{k+1}}\, G\bigg(\frac{t}{1+(q^2-1)\delta t}\bigg) \\ & + \frac{c_0}{(1+q^2\delta t)^{k+1}} + \frac{c_1}{(1+(q^2\delta-\gamma)t)^{k+1}}. \end{aligned} \tag{4.36}$$

The function $t^{k+1}v(t)$ is analytic and bounded for $|t| > r_0$, with r_0 sufficiently large since $\tilde{g}(s)$ and thus $G\big(\frac{1}{s}\big) = G(t)$ is analytic for $\Re(s) > 0$. As shown in Corollary 3.4 Case 2 $G(t)$ can be continued analytically in the domain $\mathbf{D}_0$ (see Case 1). The rightmost singularity lies at $-\frac{1}{\delta}$.
If we put $G(t) = t^{\rho_b}u(t)$, where ρ_b is a solution of $q^{2\rho} = b_1$, we get

$$u(q^2t) = u(t) + \frac{1}{b_1 t^{\rho_b}}\, v(t) = u(t) + O\big(t^{-(\rho_b+k+1)}\big)\,. \tag{4.37}$$

The number k has been chosen large enough such that $q^{2k}|b_1| > 1$ which implies that $\Re(k+\rho_b+1) > 1$. Therefore, it follows that the analytic function $\frac{1}{t^{\rho_b}}\, v(t) = O\big(t^{-(\rho_b+k+1)}\big)$ is bounded for $|t| \to \infty$.

$$u(t) = K_2(\log_q t) + O\big(t^{-(\rho_b+k+1)}\big)\,, \quad |t| \to \infty,\ K_2(t+1) = K_2(t) \tag{4.38}$$

and

$$G(t) = t^{\rho_b} K_2(\log_q t) + O\left(t^{-(k+1)}\right), \quad |t| \to \infty,\ K_2(t+1) = K_2(t). \tag{4.39}$$

Note that iteration (4.37) and $u(t) = t^{-\rho_b}G(t)$ show that $u(t)$ is analytic in $\mathbf{D}_0 \setminus \{0\}$. Similar to Case 1 we set

$$\frac{k!}{s^{k+1}}\, F\bigg(\frac{1}{s}\bigg) := (-1)^k \tilde{f}^{(k)}(s) = \tilde{g}(s-\delta) = \frac{k!}{(s-\delta)^{k+1}}\, G\bigg(\frac{1}{s-\delta}\bigg) \tag{4.40}$$

for $|s| > \delta$. If we put $t = \frac{\delta}{s}$

$$F\bigg(\frac{1}{\delta}\, t\bigg) = \frac{1}{(1-t)^{k+1}}\, G\bigg(\frac{t}{\delta(1-t)}\bigg) \quad \text{for } |t| < 1$$

and with (4.39)

$$F\Big(\frac{1}{\delta}\,t\Big) = \frac{1}{(1-t)^{k+1}}\left[\frac{t^{\rho_b}}{\delta^{\rho_b}(1-t)^{\rho_b}}\,K_2\Big(\log_q\frac{t}{\delta(1-t)}\Big) + O\bigg(\Big(\frac{1-t}{t}\Big)^{k+1}\bigg)\right]$$
$$= \delta^{-\rho_b}(1-t)^{-(\rho_b+k+1)}K_2\Big(\log_q\frac{1}{\delta(1-t)}\Big) + O(1) \quad \text{for } t\to 1_-. \tag{4.41}$$

$F(\frac{1}{\delta}t)$ is an analytic function for $|t|<1$ with a singularity at $t=1$. Corollary 3.5 implies that $F(t)$ can be continued analytically in the domain $\mathbf{D}$ (see Case 1). Thus we apply Theorem 2.9 to obtain

$$\binom{n+k}{n}\delta^{-n}f_n = \delta^{-\rho_b}\,\frac{\Gamma(n+\rho_b+k+1)}{n!\,\Gamma(\rho_b+k+1)}\,\bar{K}_2\Big(\log_q\frac{n}{\delta}\Big) + O\Big(\frac{1}{n}\Big). \tag{4.42}$$

Applying further expansions it follows that

$$\delta^{-n}f_n = n^{\rho_b}\hat{K}_2(\log_q n) + O\big(n^{\rho_b-1}\big), \quad n\to\infty, \tag{4.43}$$

with 1-periodic functions $\bar{K}_2(t)$ and $\hat{K}_2(t)$ which are analytic in a stripe $|\Im(t)| < \nu$.

Case 3: $\delta = \frac{\gamma}{q^2}$ and $\delta > \max\left\{\frac{\alpha}{q^2-q},\ \frac{\beta}{q^2-1}\right\}$

If we write

$$f(x) = e^{\delta x}g(x) \tag{4.44}$$

we get with (4.1) the functional equation for $g(x)$

$$\begin{aligned} e^{q^2\delta x}g(q^2x) &= (a_0 + a_1e^{\alpha x})\,e^{q\delta x}g(qx) + \big(b_0+b_1e^{\beta x}\big)\,e^{\delta x}g(x) + c_0 + c_1e^{q^2\delta x}\\ \Rightarrow\quad g(q^2x) &= c_1 + a_0e^{-(q^2-q)\delta x}g(qx) + a_1e^{-\big((q^2-q)\delta-\alpha\big)x}g(qx)\\ &\quad + b_0e^{-(q^2-1)\delta x}g(x) + b_1e^{-\big((q^2-1)\delta-\beta\big)x}g(x) + c_0e^{-q^2\delta x}. \end{aligned} \tag{4.45}$$

In Lemma 3.1 we have shown that $f(x)$ is of exponential type δ. Therefore, $g(x)$ is of exponential type less or equal to 2δ. Now we can build the LB-transform of $g(x)$. For shortening we denote by $\tilde{g}(s)$ the LB-transform of $g(x)$.

$$\begin{aligned} \frac{1}{q^2}\,\tilde{g}\Big(\frac{s}{q^2}\Big) &= \frac{c_1}{s} + \frac{a_0}{q}\,\tilde{g}\Big(\frac{s+(q^2-q)\delta}{q}\Big) + \frac{a_1}{q}\,\tilde{g}\Big(\frac{s+(q^2-q)\delta-\alpha}{q}\Big)\\ &\quad + b_0\,\tilde{g}\big(s+(q^2-1)\delta\big) + b_1\,\tilde{g}\big(s+(q^2-1)\delta-\beta\big) + \frac{c_0}{s+q^2\delta} \end{aligned} \tag{4.46}$$

If we put $\tilde{g}(s) = \frac{1}{s}\, G\big(\frac{1}{s}\big)$ and $t = \frac{1}{s}$ we get

$$G\big(q^2 t\big) = c_1 + v(t) \tag{4.47}$$

with

$$\begin{aligned} v(t) = {} & \frac{a_0}{1+(q^2-q)\delta t}\, G\bigg(\frac{qt}{1+(q^2-q)\delta t}\bigg) \\ & + \frac{a_1}{1+((q^2-q)\delta-\alpha)\,t}\, G\bigg(\frac{qt}{1+((q^2-q)\delta-\alpha)\,t}\bigg) \\ & + \frac{b_0}{1+(q^2-1)\delta t}\, G\bigg(\frac{t}{1+(q^2-1)\delta t}\bigg) \\ & + \frac{b_1}{1+((q^2-1)\delta-\beta)\,t}\, G\bigg(\frac{t}{1+((q^2-1)\delta-\beta)\,t}\bigg) + \frac{c_0}{1+q^2\delta t}. \end{aligned}$$

Since $\tilde{g}(s)$ is analytic for $\Re(s) > 0$, $G(t)$ is analytic for $\Re(t) > 0$. Therefore, the function $tv(t)$ is analytic and bounded for $|t| > r_0$, with r_0 sufficiently large.

$$v(t) = O\bigg(\frac{1}{t}\bigg), \quad |t| \to \infty. \tag{4.48}$$

With $\frac{1}{s}F\big(\frac{1}{s}\big) = \tilde{f}(s) = \tilde{g}(s-\delta) = \frac{1}{s-\delta}\, G\big(\frac{1}{s-\delta}\big)$ and $t = \frac{\delta}{s}$ we get

$$F\bigg(\frac{1}{\delta}\,t\bigg) = \frac{1}{1-t}\, G\bigg(\frac{t}{\delta(1-t)}\bigg) \qquad \text{for } |t| < 1$$

and finally

$$\begin{aligned} F\bigg(\frac{1}{\delta}\,t\bigg) &= \frac{c_1}{1-t} + \frac{1}{1-t}\, O\bigg(\frac{1-t}{t}\bigg) \\ &= \frac{c_1}{1-t} + O(1), \quad t \to 1_-. \end{aligned} \tag{4.49}$$

This implies by the lemma of Flajolet and Odlyzko [FO]

$$\delta^{-n} f_n = c_1 + O\bigg(\frac{1}{n}\bigg). \tag{4.50}$$

Until now we have discussed the cases where exactly one term on the right side of the equation (4.1) determines the exponential growth. In the following considerations we treat the case where more than one term have the same growth. Now the coefficients determine the growth of f_n.

Case 4: $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1}$ and $\delta > \frac{\gamma}{q^2}$.

Since $a_1 \neq 0$ we can take a $k \in \mathbb{N}_0$ such that $q^k \left| \left(\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} + b_1} \right) \right| > 1$. Similar to Case 1 we put

$$f(x) = e^{\delta x} x^{-k} g(x) \tag{4.51}$$

and get with (4.1) the functional equation for $g(x)$

$$\begin{aligned} \frac{e^{q^2\delta x}}{q^{2k}x^k} g(q^2x) &= \left(a_0 + a_1 e^{(q^2-q)\delta x}\right) \frac{e^{q\delta x}}{q^k x^k} g(qx) + \left(b_0 + b_1 e^{(q^2-1)\delta x}\right) \frac{e^{\delta x}}{x^k} g(x) \\ &\quad + c_0 + c_1 e^{\gamma x} \\ \Rightarrow \quad g(q^2x) &= a_1 q^k g(qx) + b_1 q^{2k} g(x) + a_0 q^k e^{-(q^2-q)\delta x} g(qx) \\ &\quad + b_0 q^{2k} e^{-(q^2-1)\delta x} g(x) + c_0 q^{2k} e^{-q^2\delta x} x^k + c_1 q^{2k} e^{-(q^2\delta-\gamma)x} x^k. \end{aligned} \tag{4.52}$$

In Lemma 3.1 we have shown that $f(x)$ is of exponential type δ. Therefore, $g(x)$ is of exponential type less or equal to 2δ and we can build the LB-transform of $g(x)$. For shortening we denote by $\tilde{g}(s)$ the LB-transform of $g(x)$.

$$\begin{aligned} \frac{1}{q^2}\tilde{g}\left(\frac{s}{q^2}\right) &= a_1 q^{k-1} \tilde{g}\left(\frac{s}{q}\right) + b_1 q^{2k} \tilde{g}(s) + a_0 q^{k-1} \tilde{g}\left(\frac{s + (q^2-q)\delta}{q}\right) \\ &\quad + b_0 q^{2k} \tilde{g}\left(s + (q^2-1)\delta\right) + c_0 q^{2k} \frac{k!}{(s+q^2\delta)^{k+1}} \\ &\quad + c_1 q^{2k} \frac{k!}{(s+q^2\delta-\gamma)^{k+1}}. \end{aligned} \tag{4.53}$$

If we put $\tilde{g}(s) = \frac{k!}{s^{k+1}} G\left(\frac{1}{s}\right)$ and $t = \frac{1}{s}$ then it follows that

$$\begin{aligned} \frac{1}{q^2}\frac{q^{2(k+1)}k!}{s^{k+1}} G\left(\frac{q^2}{s}\right) &= a_1 q^{k-1} \frac{q^{k+1}k!}{s^{k+1}} G\left(\frac{q}{s}\right) + b_1 q^{2k} \frac{k!}{s^{k+1}} G\left(\frac{1}{s}\right) \\ &\quad + a_0 q^{k-1} \frac{q^{k+1}k!}{(s+(q^2-q)\delta)^{k+1}} G\left(\frac{q}{s+(q^2-q)\delta}\right) \\ &\quad + b_0 q^{2k} \frac{k!}{(s+(q^2-1)\delta)^{k+1}} G\left(\frac{1}{s+(q^2-1)\delta}\right) \\ &\quad + c_0 q^{2k} \frac{k!}{(s+q^2\delta)^{k+1}} + c_1 q^{2k} \frac{k!}{(s+q^2\delta-\gamma)^{k+1}} \end{aligned}$$

and finally

$$G(q^2t) = a_1 G(qt) + b_1 G(t) + v(t) \tag{4.54}$$

with

$$\begin{aligned} v(t) = &\frac{a_0}{(1+(q^2-q)\delta t)^{k+1}}\, G\left(\frac{qt}{1+(q^2-q)\delta t}\right) \\ &+ \frac{b_0}{(1+(q^2-1)\delta)^{k+1}}\, G\left(\frac{t}{1+(q^2-1)\delta t}\right) \\ &+ c_0 \frac{1}{(1+q^2\delta t)^{k+1}} + c_1 \frac{1}{(1+(q^2\delta-\gamma)t)^{k+1}}. \end{aligned}$$

Since $\tilde{g}(s)$ is analytic in the halfplane $\Re(s) > 0$, $G(t)$ is analytic for $\Re(t) > 0$. Obviously, $t^{k+1}v(t)$ is bounded and analytic for $|t| > r_0$, whereby r_0 is sufficiently large. The characteristic function of the functional equation (4.54) is

$$q^{2\rho} - a_1 q^\rho - b_1 = 0 \tag{4.55}$$

and with $\lambda = q^\rho$

$$\lambda^2 - a_1\lambda - b_1 = 0.$$

There exist two complex numbers $\rho_1, \rho_2 \in \mathbb{C}$ such that

$$q^{\rho_1} = \frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} + b_1} \qquad \text{and} \qquad q^{\rho_2} = \frac{a_1}{2} - \sqrt{\frac{a_1^2}{4} + b_1}. \tag{4.56}$$

Case 4.1: $\rho_1 \neq \rho_2 \quad (\Leftrightarrow a_1^2 \neq -4b_1)$
As shown in Corollary 3.4 Case 4 $G(t)$ can be continued analytically in the domain $\mathbf{D}_0$ (see Case 1). The rightmost singularity lies at $-\frac{1}{\delta}$.
The general solution of equation (4.54) is

$$\begin{aligned} G(t) = t^{\rho_1} K_1(\log_q t) + t^{\rho_2} K_2(\log_q t) + O(t^{-(k+1)}), \\ K_1(t) = K_1(t+1),\ K_2(t) = K_2(t+1). \end{aligned} \tag{4.57}$$

Since $G(t)$ is analytic in $\mathbf{D}_0$ we know that the periodic fluctuations $K_1(\log_q t)$ and $K_2(\log_q t)$ are analytic for $t \in \mathbf{D}_0 \setminus \{0\}$. With $\frac{1}{s}\, F\left(\frac{1}{s}\right) := \tilde{f}(s) = \tilde{g}(s-\delta) = \frac{1}{s-\delta}\, G\left(\frac{1}{s-\delta}\right)$ and $t = \frac{\delta}{s}$ we get

$$\begin{aligned} F\left(\frac{1}{\delta}\, t\right) &= \frac{1}{(1-t)^{k+1}}\, G\left(\frac{t}{\delta(1-t)}\right) \\ &= \frac{1}{(1-t)^{k+1}} \left[\frac{t^{\rho_1}}{\delta^{\rho_1}(1-t)^{\rho_1}}\, K_1\left(\log_q \frac{t}{\delta(1-t)}\right) \right. \\ &\quad \left. + \frac{t^{\rho_2}}{\delta^{\rho_2}(1-t)^{\rho_2}}\, K_2\left(\log_q \frac{t}{\delta(1-t)}\right) + O\left(\frac{(1-t)^{k+1}}{t^{k+1}}\right) \right] = \end{aligned}$$

$$
\begin{aligned}
&= \delta^{-\rho_1}(1-t)^{-(\rho_1+k+1)} K_1\left(\log_q \frac{1}{\delta(1-t)}\right) \\
&\quad + \delta^{-\rho_2}(1-t)^{-(\rho_2+k+1)} K_2\left(\log_q \frac{1}{\delta(1-t)}\right) + O(1), \quad \text{as } t \to 1_-.
\end{aligned} \tag{4.58}
$$

$F(\frac{1}{\delta}t)$ is an analytic function for $|t|<1$ with a singularity at $t=1$. Corollary 3.5 implies that we can enlarge the space where we know that $F(t)$ is analytic such that we can apply Theorem 2.9 and obtain

$$
\begin{aligned}
\binom{n+k}{n}\delta^{-n} f_n = \delta^{-\rho_1}&\frac{\Gamma(n+\rho_1+k+1)}{n!\,\Gamma(\rho_1+k+1)}\,\bar{K}_1\left(\log_q \frac{n}{\delta}\right) \\
&+ \delta^{-\rho_2}\frac{\Gamma(n+\rho_2+k+1)}{n!\,\Gamma(\rho_2+k+1)}\,\bar{K}_2\left(\log_q \frac{n}{\delta}\right) + O\left(\frac{1}{n}\right).
\end{aligned}
$$

After asymptotic expansions for $\binom{n+k}{n}$ and the Γ-function we get

$$
\begin{gathered}
\delta^{-n} f_n = \left(n^{\rho_1} K_1(\log_q n) + n^{\rho_2} K_2(\log_q n)\right)\left(1 + O\left(\frac{1}{n}\right)\right) \\
\text{with } \hat{K}_1(t+1) = \hat{K}_1(t),\ \hat{K}_2(t+1) = \hat{K}_2(t).
\end{gathered} \tag{4.59}
$$

Remember that k has been chosen large enough that $\Re(\rho_1) - 1 > -(k+1)$ and $\Re(\rho_2) - 1 > -(k+1)$ holds.

Case 4.2: $\rho_1 = \rho_2 \quad (\Leftrightarrow a_1^2 = -4b_1)$
In this resonance case the solution of equation (4.54) is

$$
G(t) = t^{\rho_1}\left(K_1(\log_q t) + \log_q t\, K_2(\log_q t)\right) + O\left(t^{-(k+1)}\right) \tag{4.60}
$$

which is analytic in $\mathbf{D}_0$. The 1–periodic fluctuations $K_1(\log_q t)$ and $K_2(\log_q t)$ are analytic for $t \in \mathbf{D}_0 \setminus \{0\}$. With $\frac{1}{s}\,F\left(\frac{1}{s}\right) := \tilde{f}(s) = \tilde{g}(s-\delta) = \frac{1}{s-\delta}\,G\left(\frac{1}{s-\delta}\right)$ and $t = \frac{\delta}{s}$ we get

$$
\begin{aligned}
F\left(\frac{1}{\delta}t\right) &= \frac{1}{(1-t)^{k+1}}\,G\left(\frac{t}{\delta(1-t)}\right) \\
&= \delta^{-\rho_1}(1-t)^{-(\rho_1+k+1)}\left[K_1\left(\log_q\left(\frac{1}{\delta(1-t)}\right)\right)\right. \\
&\quad \left. + K_2\left(\log_q\left(\frac{1}{\delta(1-t)}\right)\right)\log_q\left(\frac{1}{\delta(1-t)}\right)\right] + O(1), \quad \text{for } t \to 1_-.
\end{aligned}
$$

$F(\frac{1}{\delta}t)$ is an analytic function for $|t|<1$ with a singularity at $t=1$. Corollary 3.5 implies that we can enlarge the space where we know that $F(t)$ is analytic such that we can apply the transfer lemma of Flajolet and Odlyzko. The result of the lemma is then

$$\binom{n+k}{n}\delta^{-n} f_n = \delta^{-\rho_1}\frac{\Gamma(n+\rho_1+k+1)}{n!\,\Gamma(\rho_1+k+1)} \cdot \Big[\bar K_1\Big(\log_q \frac{n}{\delta}\Big) + \bar K_2\Big(\log_q \frac{n}{\delta}\Big)\log_q \frac{n}{\delta}\Big] + O\Big(\frac{1}{n}\Big),$$

and after applying asymptotic expansions for the binomial coefficients and the Γ–function we get

$$\delta^{-n} f_n = n^{\rho_1}\big(K_1(\log_q n) + K_2(\log_q n)\log_q n\big)\left(1 + O\left(\frac{1}{n}\right)\right)$$
$$\text{with } \tilde K_1(t+1) = \tilde K_1(t),\ \tilde K_2(t+1) = \tilde K_2(t), \quad \text{for } n\to\infty. \tag{4.61}$$

Case 5: $\delta = \frac{\alpha}{q^2-q} = \frac{\gamma}{q^2}$ and $\delta > \frac{\beta}{q^2-1}$
If we write

$$f(x) = e^{\delta x} g(x) \tag{4.62}$$

we get with (4.1) the functional equation for $g(x)$

$$e^{q^2\delta x} g(q^2x) = \Big(a_0 + a_1 e^{(q^2-q)\delta x}\Big) e^{q\delta x} g(qx) + \big(b_0 + b_1 e^{\beta x}\big) e^{\delta x} g(x) + c_0 + c_1 e^{q^2\delta x}$$

$$\Rightarrow \quad g(q^2x) = a_1 g(qx) + c_1 + a_0 e^{-(q^2-q)\delta x} g(qx) + b_0 e^{-(q^2-1)\delta x} g(x) + b_1 e^{-\left((q^2-1)\delta-\beta\right)x} g(x) + c_0 e^{-q^2\delta x}. \tag{4.63}$$

In Lemma 3.1 we have shown that $f(x)$ is of exponential type δ. Therefore, $g(x)$ is of exponential type less or equal to 2δ and we can build the LB-transform. For shortening we denote by $\tilde g(s)$ the LB-transform of $g(x)$.

$$\frac{1}{q^2}\,\tilde g\Big(\frac{s}{q^2}\Big) = \frac{a_1}{q}\,\tilde g\Big(\frac{s}{q}\Big) + \frac{c_1}{s} + \frac{a_0}{q}\,\tilde g\bigg(\frac{s+(q^2-q)\delta}{q}\bigg) + b_0\,\tilde g\big(s+(q^2-1)\delta\big) + b_1\,\tilde g\big(s+(q^2-1)\delta-\beta\big) + \frac{c_0}{s+q^2\delta} \tag{4.64}$$

If we put $\tilde g(s) = \frac{1}{s}\,G\big(\frac{1}{s}\big)$ and $t = \frac{1}{s}$ then there follows that

$$G\big(q^2t\big) = a_1 G(qt) + c_1 + v(t) \tag{4.65}$$

with

$$v(t) = \frac{a_0}{1+(q^2-q)\delta t}\,G\bigg(\frac{qt}{1+(q^2-q)\delta t}\bigg) + \frac{b_0}{1+(q^2-1)\delta t}\,G\bigg(\frac{t}{1+(q^2-1)\delta t}\bigg) + \frac{b_1}{1+((q^2-1)\delta-\beta)\,t}\,G\bigg(\frac{t}{1+((q^2-1)\delta-\beta)\,t}\bigg) + \frac{c_0}{1+q^2\delta t}$$
$$= O\Big(\frac{1}{t}\Big) \quad \text{for } t\to\infty. \tag{4.66}$$

The growth of the function $G(t)$ depends now on the size of $|a_1|$. The solution of (4.65) is

$$G(t) = \begin{cases} \frac{c_1}{1-a_1} + t^{\rho_a} K_1(\log_q t) + O\left(\frac{1}{t}\right) & \text{if } a_1 \neq 1, \text{ for } t \to \infty \\ c_1 \log_q t + t^{\rho_a} K_1(\log_q t) + O\left(\frac{1}{t}\right) & \text{if } a_1 = 1, \text{ for } t \to \infty \end{cases} \tag{4.67}$$

with $K_1(t) = K_1(t+1)$. As shown in Corollary 3.4 Case 5 $G(t)$ can be continued analytically in the domain $\mathbf{D}_0$ (see Case 1). Therfore the periodic fluctuation $K_1(\log_q t)$ is analytic in the domain $\mathbf{D}_0 \setminus \{0\}$. Using the relation

$$\frac{1}{s} F\left(\frac{1}{s}\right) = \tilde{f}(s) = \tilde{g}(s-\delta) = \frac{1}{s-\delta} G\left(\frac{1}{s-\delta}\right) \tag{4.68}$$

we get with $t = \frac{\delta}{s}$

$$F\left(\frac{1}{\delta} t\right) = \frac{1}{1-t} G\left(\frac{t}{\delta(1-t)}\right)$$
$$= \begin{cases} (1-t)^{-1} \dfrac{c_1}{1-a_1} + \\ \quad + (1-t)^{-(1+\rho_a)} \tilde{K}_1\left(\log_q \dfrac{1}{1-t}\right) + O(1) & \text{for } t \to 1_-, \ a_1 \neq 1 \\ \dfrac{c_1}{1-t} \log_q \dfrac{1}{1-t} + \\ \quad + \dfrac{1}{1-t} \tilde{K}_1\left(\log_q \dfrac{1}{1-t}\right) + O(1) & \text{for } t \to 1_-, \ a_1 = 1. \end{cases} \tag{4.69}$$

$F(\frac{1}{\delta} t)$ is an analytic function for $|t| < 1$ with a singularity at $t = 1$. Corollary 3.5 implies that we can enlarge the area where we know that $F(t)$ is analytic such that we can apply the transfer lemma of Flajolet and Odlyzko. The result of the lemma is then

$$\delta^{-n} f_n = \begin{cases} \dfrac{c_1}{1-a_1} + O(n^{-1}) & \text{if } |a_1| \leq \frac{1}{q}, \text{ for } n \to \infty \\ \dfrac{c_1}{1-a_1} + n^{\rho_a} \tilde{K}_1(\log_q n) + O(n^{-1}) & \text{if } \frac{1}{q} < |a_1| \leq 1, a_1 \neq 1, \\ & \text{for } n \to \infty \\ c_1 \log_q n + \tilde{K}_1(\log_q n) + O(n^{-1}) & \text{if } a_1 = 1, \text{ for } n \to \infty \\ n^{\rho_a} \tilde{K}_1(\log_q n) + \dfrac{c_1}{1-a_1} + O\big(n^{\rho_a - 1}\big) & \text{if } 1 < |a_1| < q, \text{ for } n \to \infty \\ n^{\rho_a} \tilde{K}_1(\log_q n) + O\big(n^{\rho_a - 1}\big) & \text{if } q \leq |a_1|, \text{ for } n \to \infty \end{cases}$$

with $\tilde{K}_1(t+1) = \tilde{K}_1(t)$.

(4.70)

Case 6: $\delta = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$ and $\delta > \frac{\alpha}{q^2-q}$.
Similar to Case 2 with $k = 0$ we write

$$f(x) = e^{\delta x} g(x) \tag{4.71}$$

and obtain with (4.1) the functional equation for $g(x)$

$$\begin{aligned} &e^{q^2\delta x} g(q^2x) = (a_0 + a_1e^{\alpha x})e^{q\delta x} g(qx) + (b_0 + b_1 e^{(q^2-1)\delta x})e^{\delta x} g(x) + c_0 + c_1 e^{q^2\delta x} \\ \Rightarrow \quad & g(q^2x) = b_1 g(x) + c_1 + a_0 e^{-(q^2-q)\delta x} g(qx) + a_1 e^{-\left((q^2-q)\delta-\alpha\right)x} g(qx) \\ &\qquad + b_0 e^{-(q^2-1)\delta x} g(x) + c_0 e^{-q^2\delta x}, \end{aligned} \tag{4.72}$$

and with LB-transform it there follows that

$$\begin{aligned} \frac{1}{q^2}\,\tilde g\Big(\frac{s}{q^2}\Big) &= b_1\,\tilde g(s) + \frac{c_1}{s} + \frac{a_0}{q}\,\tilde g\Big(\frac{s+(q^2-q)\delta}{q}\Big) + \frac{a_1}{q}\,\tilde g\Big(\frac{s+(q^2-q)\delta-\alpha}{q}\Big) \\ &\quad + b_0\,\tilde g\big(s+(q^2-1)\delta\big) + \frac{c_0}{s+q^2\delta}. \end{aligned} \tag{4.73}$$

If we put $\tilde g(s) = \frac{1}{s}\,G\big(\frac{1}{s}\big)$ and $t = \frac{1}{s}$ we get

$$G\big(q^2t\big) = b_1 G(t) + c_1 + v(t) \tag{4.74}$$

with

$$\begin{aligned} v(t) = {} & \frac{a_0}{1+(q^2-q)\delta t}\,G\Big(\frac{qt}{1+(q^2-q)\delta t}\Big) \\ & + \frac{a_1}{1+((q^2-q)\delta-\alpha)t}\,G\Big(\frac{qt}{1+((q^2-q)\delta-\alpha)t}\Big) \\ & + \frac{b_0}{1+(q^2-1)\delta t}\,G\Big(\frac{t}{1+(q^2-1)\delta t}\Big) + \frac{c_0}{1+q^2\delta t}. \end{aligned} \tag{4.75}$$

Since $\tilde g(s)$ is analytic in the halfplane $\Re(s) > 0$, $G(t)$ is analytic for $\Re(t) > 0$. Therefore, $tv(t)$ is bounded for $|t| > r_0$, whereby r_0 is sufficiently large. The growth of the function $G(t)$ depends on the size of $|b_1|$. The solution of (4.74) is

$$G(t) = \begin{cases} \frac{c_1}{1-b_1} + t^{\rho_b} K_1(\log_q t) + O\big(\frac{1}{t}\big) & \text{if } b_1 \neq 1, \text{ for } t \to \infty \\ c_1 \log_q t + t^{\rho_b} K_1(\log_q t) + O\big(\frac{1}{t}\big) & \text{if } b_1 = 1, \text{ for } t \to \infty \end{cases} \tag{4.76}$$

with $K_1(t+1) = K_1(t)$. As shown in Corollary 3.4 Case 6 $G(t)$ can be continued analytically in the domain $\mathbf{D}_0$ (see Case 1). Therfore the periodic fluctuation $K_1(\log_q t)$ is analytic in the domain $\mathbf{D}_0 \setminus \{0\}$. Using the relation

$$\frac{1}{s}\,F\Big(\frac{1}{s}\Big) = \tilde f(s) = \tilde g\,(s-\delta) = \frac{1}{s-\delta}\,G\Big(\frac{1}{s-\delta}\Big) \tag{4.77}$$

we get with $t = \frac{\delta}{s}$

$$F\left(\frac{1}{\delta}t\right) = \frac{1}{1-t}\,G\left(\frac{t}{\delta(1-t)}\right)$$
$$= \begin{cases} (1-t)^{-1}\dfrac{c_1}{1-b_1} + \\ \quad + (1-t)^{-(1+\rho_b)}\tilde{K}_1\left(\log_q \dfrac{1}{1-t}\right) + O(1) & \text{for } t \to 1_-,\ b_1 \neq 1 \\ \dfrac{c_1}{1-t}\ \log_q \dfrac{1}{1-t} + \\ \quad + \dfrac{1}{1-t}\ \tilde{K}_1\left(\log_q \dfrac{1}{1-t}\right) + O(1) & \text{for } t \to 1_-,\ b_1 = 1. \end{cases} \tag{4.78}$$

$F(\frac{1}{\delta}\,t)$ is an analytic function for $|t| < 1$ with a singularity at $t = 1$. Corollary 3.5 implies that we can enlarge the area where we know that $F(t)$ is analytic such that we can apply Theorem 2.9 and receive

$$\delta^{-n} f_n = \begin{cases} \dfrac{c_1}{1-b_1} + O(n^{-1}) & \text{if } |b_1| \leq \frac{1}{q},\ \text{for } n \to \infty \\ \dfrac{c_1}{1-b_1} + n^{\rho_b}\tilde{K}_1(\log_q n) + O(n^{-1}) & \text{if } \frac{1}{q} < |b_1| \leq 1, b_1 \neq 1, \\ & \text{for } n \to \infty \\ c_1 \log_q n + \tilde{K}_1(\log_q n) + O(n^{-1}) & \text{if } b_1 = 1,\ \text{for } n \to \infty \\ n^{\rho_b}\tilde{K}_1(\log_q n) + \dfrac{c_1}{1-b_1} + O\left(n^{\rho_b - 1}\right) & \text{if } 1 < |b_1| < q,\ \text{for } n \to \infty \\ n^{\rho_b}\tilde{K}_1(\log_q n) + O\left(n^{\rho_b-1}\right) & \text{if } q \leq |b_1|,\ \text{for } n \to \infty \end{cases} \tag{4.79}$$

with $\tilde{K}_1(t+1) = \tilde{K}_1(t)$.

Case 7: $\delta = \frac{\alpha}{q^2-q} = \frac{\beta}{q^2-1} = \frac{\gamma}{q^2}$

Similar to Case 4 with $k = 0$ we write

$$f(x) = e^{\delta x} g(x) \tag{4.80}$$

and receive with (4.1) the functional equation for $g(x)$

$$\begin{aligned} e^{q^2\delta x} g(q^2 x) &= (a_0 + a_1 e^{(q^2-q)\delta x}) e^{q\delta x} g(qx) + (b_0 + b_1 e^{(q^2-1)\delta x}) e^{\delta x} g(x) + c_0 + c_1 e^{q^2\delta x} \\ \Rightarrow \quad g(q^2 x) &= a_1 g(qx) + b_1 g(x) + c_1 \\ &\quad + a_0 e^{-(q^2-q)\delta x} g(qx) + b_0 e^{-(q^2-1)\delta x} g(x) + c_0 e^{-q^2\delta x}, \end{aligned} \tag{4.81}$$

and with LB-transform it follows that

$$\begin{aligned}\frac{1}{q^2}\,\tilde g\Big(\frac{s}{q^2}\Big) &= \frac{a_1}{q}\,\tilde g\Big(\frac{s}{q}\Big) + b_1\tilde g(s) + \frac{c_1}{s}\\ &\quad + \frac{a_0}{q}\,\tilde g\Big(\frac{s+(q^2-q)\delta}{q}\Big) + b_0\tilde g\big(s+(q^2-1)\delta\big) + \frac{c_0}{s+q^2\delta}.\end{aligned} \tag{4.82}$$

If we put $\tilde g(s) = \frac{1}{s}G\left(\frac{1}{s}\right)$ and $t = \frac{1}{s}$ we get

$$G\big(q^2t\big) = a_1G(qt) + b_1G(t) + c_1 + v(t) \tag{4.83}$$

with

$$\begin{aligned}v(t) &= \frac{a_0}{1+(q^2-q)\delta t}\,G\Big(\frac{qt}{1+(q^2-q)\delta t}\Big)\\ &\quad + \frac{b_0}{1+(q^2-1)\delta t}\,G\Big(\frac{t}{1+(q^2-1)\delta t}\Big) + \frac{c_0}{1+q^2\delta t}.\end{aligned} \tag{4.84}$$

The function $tv(t)$ is analytic and bounded for $|t| > r_0$, with r_0 sufficiently large since $\tilde g(s)$ and thus $G\big(\frac{1}{s}\big) = G(t)$ is analytic for $\Re(s) > 0$. The characteristic function of equation (4.83) is

$$\lambda^2 - a_1\lambda - b_1 = 0.$$

We call the solutions λ_1 and λ_2 and denote $\rho_1 = \log_q \lambda_1$ and $\rho_2 = \log_q \lambda_2$.

Case 7.1: $\rho_1 \neq \rho_2 \quad (\Leftrightarrow a_1^2 \neq -4b_1)$
As shown in Corollary 3.4 Case 7 $G(t)$ can be continued analytically in the domain $\mathbf{D}_0$ (see Case 1). Therefore, the general solution of (4.83) is

$$G(t) = \begin{cases} \dfrac{c_1}{1-a_1-b_1} + t^{\rho_1}K_1(\log_q t) + \\ \qquad + t^{\rho_2}K_2(\log_q t) + O\Big(\dfrac{1}{t}\Big) & \text{if } a_1+b_1 \neq 1, \text{ for } t\to\infty \\ \dfrac{c_1}{1-a_1}\log_q t + t^{\rho_1}K_1(\log_q t) + \\ \qquad + t^{\rho_2}K_2(\log_q t) + O\Big(\dfrac{1}{t}\Big) & \text{if } a_1+b_1 = 1, \text{ for } t\to\infty \end{cases} \tag{4.85}$$

with $K_1(t+1) = K_1(t),\ K_2(t+1) = K_2(t)$.

The periodic fluctuations $K_1(\log_q t)$ and $K_2(\log_q t)$ are analytic in the domain $\mathbf{D}_0 \setminus \{0\}$. Using the relation

$$\frac{1}{s}\,F\Big(\frac{1}{s}\Big) = \tilde f(s) = \tilde g(s-\delta) = \frac{1}{s-\delta}\,G\Big(\frac{1}{s-\delta}\Big) \tag{4.86}$$

we get with $t = \frac{\delta}{s}$

$$F\left(\frac{1}{\delta}\,t\right) = \frac{1}{1-t}\,G\left(\frac{t}{\delta(1-t)}\right)$$
$$= \begin{cases} (1-t)^{-1}\dfrac{c_1}{1-a_1-b_1} + \\ \quad + (1-t)^{-(1+\rho_1)}\tilde{K}_1\left(\log_q \dfrac{1}{1-t}\right) + & \text{for } t \to 1_-, \\ \quad + (1-t)^{-(1+\rho_2)}\tilde{K}_2\left(\log_q \dfrac{1}{1-t}\right) + O(1) & a_1+b_1 \neq 1 \\ \dfrac{1}{1-t}\left[\log_q\left(\dfrac{t}{\delta(1-t)}\right)\dfrac{c_1}{1-a_1} + \right. \\ \quad + (1-t)^{-\rho_1}\tilde{K}_1\left(\log_q \dfrac{1}{1-t}\right) + & \text{for } t \to 1_-, \\ \quad \left. + (1-t)^{-\rho_2}\tilde{K}_2\left(\log_q \dfrac{1}{1-t}\right)\right] + O(1) & a_1+b_1 = 1. \end{cases} \tag{4.87}$$

$F(\frac{1}{\delta}\,t)$ is an analytic function for $|t| < 1$ with a singularity at $t = 1$. Corollary 3.5 implies that we can enlarge the area where we know that $F(t)$ is analytic such that we can apply Theorem 2.9 and receive

$$\delta^{-n} f_n = \begin{cases} \left(\dfrac{c_1}{1-a_1-b_1} + n^{\rho_1}K_1(\log_q n) + \right. & \text{if } a_1+b_1 \neq 1, \\ \quad \left. + n^{\rho_2}K_2(\log_q n)\right)\left(1+O\left(\dfrac{1}{n}\right)\right) & \text{for } n \to \infty \\ \left(\dfrac{c_1}{1-a_1}\log_q n + n^{\rho_1}K_1(\log_q n) + \right. & \text{if } a_1+b_1 = 1, \\ \quad \left. + n^{\rho_2}K_2(\log_q n)\right)\left(1+O\left(\dfrac{1}{n}\right)\right) & \text{for } n \to \infty \end{cases} \tag{4.88}$$

with $\tilde{K}_1(t+1) = \tilde{K}_1(t),\ \tilde{K}_2(t+1) = \tilde{K}_2(t)$.

Case 7.2: $\rho_1 = \rho_2 \quad (\Leftrightarrow a_1^2 = -4b_1)$

The general solution of (4.83) is

$$G(t) = \begin{cases} t^{\rho_1}\left(K_1(\log_q t) + \log_q t\; K_2(\log_q t)\right) + \\ \quad + \dfrac{c_1}{1-a_1-b_1} + O\left(\dfrac{1}{t}\right) & \text{if } a_1+b_1 \neq 1,\ \text{for } t \to \infty \\ t^{\rho_1}\left(K_1(\log_q t) + \log_q t\; K_2(\log_q t)\right) + \\ \quad + \dfrac{c_1}{1-a_1}\log_q t + O\left(\dfrac{1}{t}\right) & \text{if } a_1+b_1 = 1,\ \text{for } t \to \infty \end{cases}$$

with $K_1(t+1) = K_1(t),\ K_2(t+1) = K_2(t)$.

(4.89)

The periodic fluctuations $K_1(\log_q t)$ and $K_2(\log_q t)$ are analytic in the domain $\mathbf{D}_0 \setminus \{0\}$. Using the relation

$$\frac{1}{s}\, F\left(\frac{1}{s}\right) = \tilde{f}(s) = \tilde{g}\,(s-\delta) = \frac{1}{s-\delta}\, G\left(\frac{1}{s-\delta}\right) \tag{4.90}$$

we get with $t = \frac{\delta}{s}$

$$F\left(\frac{1}{\delta}\, t\right) = \frac{1}{1-t}\, G\left(\frac{t}{\delta(1-t)}\right)$$
$$= \begin{cases} (1-t)^{-(1+\rho_1)} \left[\tilde{K}_1\left(\log_q \frac{1}{1-t}\right) + \right. & \\ \quad \left. + \log_q \frac{1}{1-t}\, \tilde{K}_2\left(\log_q \frac{1}{1-t}\right)\right] + & \text{for } t \to 1_-, \\ \quad + (1-t)^{-1} \frac{c_1}{1-a_1-b_1} + O(1) & a_1+b_1 \neq 1 \\ (1-t)^{-(1+\rho_1)} \left[\tilde{K}_1\left(\log_q \frac{1}{1-t}\right) + \right. & \\ \quad \left. + \log_q \frac{1}{1-t}\, \tilde{K}_2\left(\log_q \frac{1}{1-t}\right)\right] + & \text{for } t \to 1_-, \\ \quad (1-t)^{-1} \frac{c_1}{1-a_1} + O(1) & a_1+b_1 = 1. \end{cases} \tag{4.91}$$

$F(\frac{1}{\delta}\, t)$ is an analytic function for $|t| < 1$ with a singularity at $t = 1$. Corollary 3.5 implies that we can enlarge the area where we know that $F(t)$ is analytic such that we can apply the transfer lemma of Flajolet and Odlyzko. The result of the lemma is that

$$\delta^{-n} f_n = \begin{cases} \left(\frac{c_1}{1-a_1-b_1} + n^{\rho_1}\left(K_1(\log_q n) + \right.\right. & \text{if } a_1+b_1 \neq 1, \\ \quad \left.\left. + \log_q n\, K_2(\log_q n)\right)\right)\left(1 + O\left(\frac{1}{n}\right)\right) & \text{for } n \to \infty \\ \left(\frac{c_1}{1-a_1} \log_q n + n^{\rho_1}\left(K_1(\log_q n) + \right.\right. & \text{if } a_1+b_1 = 1, \\ \quad \left.\left. + \log_q n\, K_2(\log_q n)\right)\right)\left(1 + O\left(\frac{1}{n}\right)\right) & \text{for } n \to \infty \end{cases} \tag{4.92}$$

with $\quad \tilde{K}_1(t+1) = \tilde{K}_1(t),\ \tilde{K}_2(t+1) = \tilde{K}_2(t).$

□

Remark 4.3. A different approach to the proof of Theorem 4.1 may be taken. It is based on the computation of the growth of $f(z)$ within a cone $S_{\frac{\pi}{2}-\epsilon} = \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2} - \varepsilon\}$ and on the proof that outside of this cone the function is of lower exponential growth (e.g. in Case 1: $f(z) = e^{\frac{\alpha}{q^2-q}z} z^{\rho_a}\big(K(\log_q z)+ O(|e^{-\omega z}|)\big)$ for $z \in S_{\frac{\pi}{2}-\epsilon}$ and $|f(z)| \leq Ae^{\kappa z}$ for $z \notin S_{\frac{\pi}{2}-\epsilon}$). Knowing this growth there can be applied a Depoissonization Lemma of Jacquet and Szpankowski [JS2, Theorem 1] to extract the asymptotics of f_n. This method will be published by Derfel and Vogl [DV2].

Corollary 4.4. *Assume that the sequence f_n satisfies the recurrence*

$$
\begin{aligned}
&f_n[q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1)] = \\
&\quad a_1\alpha^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{q}{\alpha}\right)^k f_k + b_1\beta^n \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{1}{\beta}\right)^k f_k + c_1\gamma^n, \quad \forall n \in \mathbb{N},
\end{aligned} \tag{4.93}
$$

with the initial condition

$$
f_0[1 - (a_0 + a_1) - (b_0 + b_1)] = c_0 + c_1. \tag{4.94}
$$

Here, $q > 1$, α, β and γ are nonnegative arbitrary numbers and a_0, a_1, b_0, b_1, c_0 and c_1 are complex numbers which satisfy $q^{2n} - q^n(a_0 + a_1) - (b_0 + b_1) \neq 0\ \forall n \in \mathbb{N}_0$. For convenience we put α (resp. β, γ) zero if a_1 (resp. b_1, c_1) is zero. Furthermore, we denote by $\rho_a, \rho_b \in \mathbb{C}$ the solutions of $q^\rho = a_1$ and $q^{2\rho} = b_1$. There exist two complex numbers (ρ_1 and ρ_2) such that q^{ρ_1} and q^{ρ_2} are the two solutions of $x^2 - a_1x - b_1 = 0$.
In the following results $K_1(t), K_2(t)$ are 1–periodic functions which are analytic in the stripe $|\Im(t)| < \nu$, $\nu > 0$. Finally,

$$
\delta := \max\left\{\frac{\alpha}{q^2 - q}, \frac{\beta}{q^2 - 1}, \frac{\gamma}{q^2}\right\}. \tag{4.95}
$$

Then the sequence f_n has the asymptotic bahaviour as stated in Theorem 4.1.

Proof. The aim is to show that the exponential generating function, with coefficients f_n, satisfies a q–difference equation with exponentially growing coefficients. Then we will be able to apply Theorem 4.1. Multiplying equation (4.93) by $\frac{z^n}{n!}$, summing over all $n \in \mathbb{N}$ and adding equation (4.94) gives

$$
\begin{aligned}
&\sum_{n=0}^{\infty} f_n \frac{(q^2z)^n}{n!} - (a_0 + a_1) \sum_{n=0}^{\infty} f_n \frac{(qz)^n}{n!} - (b_0 + b_1) \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} = \\
&\quad a_1 \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n}{k} q^k f_k \alpha^{n-k} \frac{z^n}{n!} + b_1 \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n}{k} f_k \beta^{n-k} \frac{z^n}{n!} + c_0 + c_1 \sum_{n=0}^{\infty} \frac{(\gamma z)^n}{n!}.
\end{aligned} \tag{4.96}
$$

If we denote $f(z) = \sum\limits_{n=0}^{\infty} f_n \frac{z^n}{n!}$ we get

$$f(q^2 z) = a_0 f(qz) + a_1 e^{\alpha z} f(qz) + b_0 f(z) + b_1 e^{\beta z} f(z) + c_0 + c_1 e^{\gamma z}. \qquad (4.97)$$

Thus we can apply Theorem 4.1 and obtain the growth of f_n as described. $\square$

Chapter 5

Applications

Since there has been created a rather general tool to calculate the asymptotic behaviour of sequences defined by a recursion or an equivalent functional equation with mostly exponential growing coefficients it is possible to apply this approach to many cases.

- Tries (cf. [KiP2])
  The recurrence can be written in the form

$$f_n = \left(1 + \frac{1}{2^{n-1} - 1}\right)\left(1 + \frac{1}{2^{n-1}}\right)\sum_{k=2}^{n-1}\binom{n}{k} f_k, \quad n \geq 2,\ f_0 = f_1 = 0$$

  and the related exponential generating function $f(z) = \sum\limits_{n=0}^{\infty} f_n \frac{z^n}{n!}$ satisfies the functional equation

$$f(4z) = 2e^{2z} f(2z) + e^{4z} - 1 - 4z. \tag{5.1}$$

  If we want to apply Theorem 4.1 we have to take $a_0 = 0, a_1 = 2, b_0 = b_1 = 0, c_0 = 1 - 4z, c_1 = 1, \alpha = 2, \beta = 0, \gamma = 4$. Since Theorem 4.1 has been stated with $c_0 \in \mathbb{C}$ the proof of Case 5 with $\delta = \frac{\alpha}{2} = \frac{\gamma}{4} = 1$ has to be adopted. If we write$f(z) = e^z g(z)$ we get with (5.1) the functional equation for $g(z)$

$$g(4z) = 2g(2z) + 1 - (1 + 4z)e^{-4z}.$$

  If we take the LB–transform and denote

$$Lg(s) = \frac{1}{s}\, G\left(\frac{1}{s}\right),$$

  we obtain with $t = \frac{1}{s}$ that $G(t)$ satisfies

$$G(4t) = 2G(2t) + 1 - \frac{1}{1 + 4t} - \frac{4t}{(1 + 4t)^2} = 2G(2t) + 1 + O\left(\frac{1}{t}\right). \tag{5.2}$$

The general solution of (5.2) is

$$G(t) = -1 + tK(\log_2 t) + O\left(\frac{1}{t}\right),$$

where $K(t)$ is a 1–periodic function. Remember that Lemma 3.2 and equation (5.2) imply that $K(\log_2 t)$ is analytic for $t \in \mathbb{C} \setminus \{0\}$. Therefore, we obtain

$$\begin{aligned} F(t) &= \frac{1}{1-t}\, G\left(\frac{t}{1-t}\right) \\ &= -\frac{1}{1-t} + \frac{1}{(1-t)^2}\, K\left(\log_2 \frac{1}{1-t}\right) + O(1), \quad \text{for } t \to 1_-. \end{aligned} \tag{5.3}$$

Theorem 2.9 gives

$$f_n = n\bar{K}(\log_2 n) + O(1) \quad \text{for } n \to \infty,$$

where $\bar{K}(t)$ is a 1–periodic function. Kirschenhofer and Prodinger [KiP2] proved with other (rather complicated) methods

$$f_n = \frac{n}{\log 2}(1 + K_1(\log_2 n)) - 1 - K_2(\log_2 n) + O\left(\frac{1}{n}\right), \tag{5.4}$$

where $K_1(t)$ and $K_2(t)$ are 1–periodic functions with mean value zero.

- Generalized digital trees (cf. [FR])
  The recurrence can be written in the form

$$f_n = 1 + \sum_{k=0}^{n-m} \frac{1}{2^{n-m}} \binom{n-m}{k} (f_k + f_{n-m-k}) \qquad n \geq m \tag{5.5}$$

with initial conditions $f_0 = 0$, $f_1 = f_2 = \cdots = f_m = 1$. The exponential generating function satisfies the difference–differential equation

$$\frac{d^m}{dz^m} f(4z) = e^{4z} + 2e^{2z} f(2z), \quad m \in \mathbb{N}. \tag{5.6}$$

Since Theorem 4.1 has not been stated in that form the proof of Theorem 4.1, Case 5 has to be adopted. With the LB–transform it follows that

$$\frac{s^m}{4^{m+1}}\, \tilde{f}\left(\frac{s}{4}\right) - \frac{1}{4} \sum_{l=0}^{m-1} \left(\frac{s}{4}\right)^{m-1-l} f_l = \tilde{f}\left(\frac{s}{2} - 1\right) + \frac{1}{s-4}. \tag{5.7}$$

If we put $f(z) = e^z g(z)$ we get $\tilde{f}(s) = \tilde{g}(s-1)$ and $\frac{1}{s}\, F\left(\frac{1}{s}\right) := \tilde{f}(s) = \tilde{g}(s-1) =: \frac{1}{s-1}\, G\left(\frac{1}{s-1}\right)$ which implies with $\frac{1}{s} = t$

$$F(t) = \sum_{n=0}^{\infty} f_n t^n = \frac{1}{1-t}\, G\left(\frac{t}{1-t}\right). \tag{5.8}$$

Equation (5.7) implies with $s = 4\left(1 + \frac{1}{2t}\right)$ that $G(t)$ satisfies the functional equation

$$\left(1 + \frac{1}{2t}\right)^m G(2t) = 2G(t) + 1 + \frac{1}{2t} \sum_{l=0}^{m-1} \left(1 + \frac{1}{2t}\right)^{m-1-l} f_l. \tag{5.9}$$

Because of [DTTV, 3.11] it follows that

$$G(t) = tK(\log_2 t)\left(1 + O\left(\frac{1}{t}\right)\right), \quad \text{for } t \to \infty, \tag{5.10}$$

where $K(t)$ is a 1–periodic function. Remember that Lemma 3.2 and equation (5.9) imply that $K(\log_2 t)$ is analytic for $t \in \mathbb{C} \setminus \{0\}$. Therefore, we obtain

$$\begin{aligned} F(t) &= \frac{1}{1-t}\, G\left(\frac{t}{1-t}\right) \\ &= \frac{1}{(1-t)^2}\, K\left(\log_2 \frac{1}{1-t}\right)(1 + O(1-t)), \quad \text{for } t \to 1_-. \end{aligned}$$

Theorem 2.9 gives

$$f_n = n\tilde{K}(\log_2 n) + O(1). \tag{5.11}$$

Flajolet and Richmond proved by other methods that

$$f_n = n[q_0 + \hat{K}(\log_2 n)] + O\left(n^{\frac{1}{2}}\right), \tag{5.12}$$

where q_0 is a constant which can be calculated and $\hat{K}(t)$ is a 1–periodic function with mean value zero and known Fourier expansion.

- Mimimum order statistic of the Cantor distribution (cf. [KnP])
  The recurrence can be written in the form

$$f_n = \frac{1}{2^n - 2\phi}\left(\phi \sum_{k=1}^{n-1} \binom{n}{k} f_k + (1-\phi)\right), \quad n \geq 1,\ f_0 = 0,\ 0 < \phi \leq \frac{1}{2}.$$

The exponential generating function $f(x) = \sum_{n=1}^{\infty} \frac{f_n}{n!} x^n$ is a solution of the functional equation

$$f(4x) = \phi(1 + e^{2x}) f(2x) + (1-\phi)(e^{2x} - 1),\ f(0) = 0.$$

That means in Theorem 4.1 we have to take $a_0 = a_1 = \phi, b_0 = b_1 = 0, c_0 = \phi - 1, c_1 = 1 - \phi, \alpha = 2, \beta = 0, \gamma = 2$ and apply Case 1 ($\delta = \frac{\alpha}{2} = 1$) and receive

$$f_n = n^{\log_2 \phi} K(\log_2 n)\left(1 + O\left(\frac{1}{n}\right)\right), \tag{5.13}$$

while in [KnP]

$$f_n = n^{\log_2 \phi}[c(\phi) + K_1(\log_2 n)]\left[1 + O\left(\frac{1}{n}\right)\right], \tag{5.14}$$

with

$$c(\phi) = \frac{(1-\phi)(1-2\phi)}{\phi \log 2}\,\Gamma(-\log_2 \phi)\,\zeta(-\log_2 \phi)$$

was shown. $K(t)$ and $K_1(t)$ are 1–periodic functions. The mean value of $K_1(t)$ is zero. $\Gamma(x)$ is the Gamma function and $\zeta(x)$ is Riemann's Zeta function. Comparison shows that (5.14) is a little better because the constant $c(\phi)$ is explicitly known.

- Moments of the Cantor–Fibonacci distribution (cf. [P2])
  The recurrence has the form

$$f_n(a^2 - a\phi^n - \phi^{2n}) = \sum_{k=0}^{n-1} \binom{n}{k} (1-\phi)^{n-k}\phi^{2k} f_k \quad n \geq 1,$$

with $f_0 = 1$, $a = \frac{1+\sqrt{5}}{2}$ and $0 < \phi < \frac{1}{2}$. Thus the exponential generating function satisfies the functional equation

$$f(q^2x) = \frac{1}{a} f(qx) + \frac{1}{a^2} e^{(q^2-q)x} f(x) \quad \text{with } q = \frac{1}{\phi}. \tag{5.15}$$

That means applying Theorem 4.1, we have to take $a_0 = 0, a_1 = \frac{1}{a}, b_0 = 0, b_1 = \frac{1}{a^2}, c_0 = c_1 = 0, \alpha = 0, \beta = q^2 - q, \gamma = 0$. Case 2 ($\delta = \frac{q}{q+1} = \frac{1}{1+\phi}$) is fulfilled and it can be seen that

$$f_n = \frac{1}{(1+\phi)^n} n^{\log_\phi a} K(-\log_\phi n)\left(1 + O\left(\frac{1}{n}\right)\right) \tag{5.16}$$

holds with a 1–periodic function $K(t)$ whereas Prodinger [P2] showed by other well adopted methods

$$f_n = \frac{1}{(1+\phi)^n} n^{\log_\phi a} K_1(-\log_\phi n)\left(1 + O\left(\frac{1}{n}\right)\right), \tag{5.17}$$

where $K_1(t)$ is a 1–periodic function with known mean value.

Chapter 6

Conclusion

The methode of using the Laplace–Borel transformation, adopting well known results about q–difference equations (cf. Adams [Ad2]) and applying the transfer lemma of Flajolet and Odlyzko [FO] is a very general method to calcultate the asymptotic properties of Taylor coefficients defined by the unique analytic soltution of functional equations. The examples have shown that this method can be easily adopted to a large class of commensurable difference and functional–differential equations. But there are still open questions e.g. there has not been treated yet the case of a system of functional equations such as

$$\begin{pmatrix} f_1(qx) \\ f_2(qx) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 e^{\alpha x} & b_0 + b_1 e^{\beta x} \\ c_0 + c_1 e^{\gamma x} & d_0 + d_1 e^{\delta x} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} + \begin{pmatrix} m_0 + m_1 e^{\mu x} \\ n_0 + n_1 e^{\nu x} \end{pmatrix}.$$

Probably this case can by solved by a similar technique of using the LB–transform, applying the results of Trjitzinsky [Tr] and constructing a gerneralization of the transfer lemmas of Flajolet and Odlyzko [FO]. In the case of multiple incommensurable transformations, i.e.

$$f(qx) = (a_0 + a_1 e^{\alpha x})\, f(px) + \left(b_0 + b_1 e^{\beta x}\right) f(x) + (c_0 + c_1 e^{\gamma x})$$

with $q \neq p^k$, $k \in \mathbb{N}$, the Laplace transformation still works but the methods of Adams cannot be used. This is an interesting and open problem for further investigations.

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