DISSERTATION

Spectral Properties of a λ-Rational Sturm-Liouville Problem

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Introduction

In this thesis we consider the Sturm-Liouville problem

$$y'' - p\,y + \left(\lambda + \frac{q}{u-\lambda}\right) y = 0$$

on the interval $[0,1]$ with the boundary conditions

$$y(0) = 0\,,\ y(1) = 0.$$

Here p, $q \in L^1(0,1)$, $u \in L^\infty(0,1)$ and q is assumed to be non-negative. This Sturm-Liouville problem is a non-classical eigenvalue problem since the eigenvalue parameter λ appears not only linearly but rationally. In the differential equation there appears a singularity which depends on the eigenvalue parameter and which is called a 'floating singularity'. Because of this singularity the spectral theory of this problem is more involved. If $p \equiv 0$, this differential equation was studied by E. Bogomolova in 1985 [B] under the assumption that the functions q and u are analytic. She proved a decomposition of $L^2(0,1)$-functions in eigenfunctions and generalized eigenfunctions of the differential equation. Later H. Langer, R. Mennicken and M. Möller investigated the given problem and showed that it is equivalent to the λ-linear eigenvalue problem of the block operator matrix

$$\widetilde{A} = \begin{pmatrix} -\dfrac{d^2}{dx^2} + p & \sqrt{q} \\ \sqrt{q} & u \end{pmatrix}$$

in the Hilbert space $\widetilde{\mathcal{H}} = L^2(0,1) \oplus L^2(0,1)$. The eigenvalues of the λ-rational and the λ-linear problem coincide, and the left-hand side of the differential equation is the inverse of the Schur complement of $\widetilde{A}$, which appears in the resolvent of $\widetilde{A}$.

Block operator matrices have been studied by many authors and they appear in many applications, e.g. astrophysics, fluid dynamics, polymerisation chemistry and magnetohydrodynamics. One aim is to get spectral information about the matrix from the spectral data of the entries. We mention the papers [AdL] and [ALMS], where for example the essential spectrum of such a block operator matrix is considered.

There are two main results in this thesis: The first is a comparison of the eigenvalues of $\widetilde{A}$ which are embedded in the essential spectrum or which are to the right of it with the eigenvalues of the problem

$$y'' - p\,y + \lambda y = 0 \tag{1}$$

with the same boundary conditions. The latter differential equation arises from the original equation by setting $q \equiv 0$. When q increases the eigenvalues move to the right or can disappear, if they are embedded in the essential spectrum. In the latter case it is shown that such an eigenvalue 'moves' onto an 'unphysical sheet' of the Riemann surface of the resolvent of $\widetilde{A}$ and becomes a pole of the resolvent. Such a pole on the unphysical sheet is called a resonance pole.

The second main result is concerned with the continuous spectrum. We show under more restrictive assumptions on the functions p, q and u that the operator $\widetilde{A}$ has no singular continuous spectrum. Moreover, a Fourier transform onto the space L^2_σ with the 'spectral measure' σ is given. This Fourier transform and its inverse are written as integral transformations.

In the first chapter the operator pencil which corresponds to the differential equation

$$L(\lambda)\,y := y'' - p\,y + \left(\lambda + \frac{q}{u-\lambda}\right) y$$

is introduced. The block operator matrix $\widetilde{A}$ is constructed and its essential spectrum is determined, which is roughly speaking equal to the set $u([0,1])$. In Section 1.2 the existence of the wave operators of the pair $\widetilde{A}$ and

$$\widetilde{A}_0 = \begin{pmatrix} -\dfrac{d^2}{dx^2} + p & 0 \\ 0 & u \end{pmatrix},$$

which corresponds to the differential equation (1), is proved. With this result the absolutely continuous spectrum of the operator $\widetilde{A}$ can be determined. In Section 1.3 the spectral measure σ mentioned above is constructed by means of the Titchmarsh-Weyl coefficient, which was already defined in [LMM] as for ordinary Sturm-Liouville problems. Furthermore, the Fourier transform is defined on some subspace, which is done in a similar way as in [ALM]. A fundamental system of solutions of the differential equation, which takes into account the singularity, is constructed in Section 1.4. This fundamental system, which will be used in later chapters, is constructed under more restrictive conditions on the given functions, which we will assume to be fulfilled in the following chapters.

In Chapter 2 the eigenvalues are investigated. In particular the eigenvalues embedded in the essential spectrum, which is now an interval $[\alpha, \beta]$, are interesting. The main result in this chapter is the estimate mentioned above for the eigenvalues $> \alpha$ from below by the eigenvalues of the problem (1). This estimate is obtained in Theorem 2.3 by means of a new version of Sturm's comparison theorem which can be applied to differential equations with a singularity and which is proved in Section 2.1. As a corollary we get an upper bound for the number of embedded eigenvalues, and it is shown that if q is large enough, there does not exist any eigenvalue in $(\alpha, \beta]$. But α can be an eigenvalue. Moreover, the behaviour of the eigenvalues under a perturbation of q is studied. This behaviour is different for those eigenvalues $> \alpha$ or those $\leq \alpha$. It is very likely that an embedded eigenvalue disappears because the corresponding eigenfunction has to vanish at the singularity of the differential equation if $q > 0$.

In Chapter 3 it is shown that the operator $\widetilde{A}$ has no singular continuous spectrum, and the density of the absolutely continuous component of the spectral measure σ is calculated by means of the fundamental system of solutions, which was constructed in Section 1.4. From the absence of the singular continuous spectrum and the existence of the wave operators it follows that the Fourier transform is an isomorphism from the whole space $\widetilde{\mathcal{H}}$ onto L^2_σ. Under this isomorphism the operator $\widetilde{A}$ becomes the operator of multiplication by the independent variable. In Section 3.2 this Fourier transform and its inverse are written as integral transformations. For this purpose we introduce generalized eigenvectors for the continuous spectrum. They are not in the space $\widetilde{\mathcal{H}}$ since a Dirac delta distribution appears. If this integral transformation is applied to vectors the second component of which vanishes, one gets the spectral decomposition which was given in [B]. For these vectors the formulas are much simpler since there is no Dirac delta distribution involved. Finally, another isomorphism is constructed into a space of analytic functions defined by L. de Branges (cf. [dB]). This isomorphism can also be written as an integral transformation.

In Chapter 4 the resonances of $\widetilde{A}$ are studied. Resonances appear in many physical problems, but often there does not exist a rigorous proof for their existence. Under the assumption that the functions p, q and u are real analytic, the existence of resonances of $\widetilde{A}$, i. e. of poles on the unphysical sheet of the Riemann surface of the resolvent of $\widetilde{A}$, is shown. This sheet is the analytic continuation from the upper half plane $\mathbb{C}^+$ across the essential spectrum. Moreover, the location of this resonance pole is studied for small q. Finally, we prove the Fermi golden rule, a heuristic formula for the asymptotic behaviour of the imaginary part of a resonance pole, which has only been proved for special cases.

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Chapter 1

Preliminaries

In the first chapter the differential equation is introduced and a corresponding operator pencil is defined. It is shown that the corresponding eigenvalue problem is equivalent to the eigenvalue problem for a block operator matrix in some larger Hilbert space. The interplay between these two problems will be important in the sequel. Then the Titchmarsh-Weyl coefficient is introduced as for ordinary Sturm-Liouville problems. From this the spectral measure σ is obtained and an isomorphism onto the L^2_σ. Finally, a fundamental system of solutions of the differential equation is constructed.

1.1 The differential equation and its Titchmarsh-Weyl coefficient

We consider the differential equation

$$y'' - p\,y + \left(\lambda + \frac{q}{u-\lambda}\right) y = 0 \tag{1.1}$$

on the interval $[0,1]$ with the boundary conditions

$$y(0) = 0\,,\ y(1) = 0. \tag{1.2}$$

Here p, q and u are real functions on $[0,1]$ which satisfy the following conditions:

$$p,\ q \in L^1(0,1),\ \ u \in L^\infty(0,1),\ \ q(x) \geq 0 \ \text{ for } \ x \in [0,1]. \tag{1.3}$$

These assumptions will be sharpened in Section 1.4. The complex number λ in equation (1.1) is the eigenvalue parameter, which appears nonlinearly. Observe that the differential equation has a so-called 'floating singularity', i. e. a singularity which depends on the eigenvalue parameter.

For $\lambda \in \rho(u) := \{\mu \in \mathbb{C} \colon \operatorname*{ess\,inf}_{x\in[0,1]} |u(x) - \mu| > 0\}$ we define the operator

$$L(\lambda)\, y := y'' - p\, y + \Big(\lambda + \frac{q}{u-\lambda}\Big)\, y \tag{1.4}$$

on the Hilbert space $L^2 := L^2(0,1)$ with domain

$$\mathcal{D}(L(\lambda)) := \{y \in L^2 \colon\ y, y' \in \mathrm{AC},\ L(\lambda)\, y \in L^2,\ y(0)=y(1)=0\},$$

where AC denotes the set of absolutely continuous functions on the interval $[0,1]$. Recall that $\lambda_0 \in \rho(u)$ is called an *eigenvalue* of L, if there exists a $y_0 \in \mathcal{D}(L(\lambda_0))$, $y_0 \neq 0$, such that $L(\lambda_0) y_0 = 0$; then y_0 is a corresponding *eigenfunction*.

We introduce the non-negative function $v(x) := \sqrt{q(x)}$ and the self-adjoint operator

$$A\, y := -y'' + p\, y \tag{1.5}$$

with domain

$$\mathcal{D}(A) := \{y \in L^2 \colon\ y, y' \in \mathrm{AC},\ -y'' + p\, y \in L^2,\ y(0)=y(1)=0\}.$$

In the sequel u and v denote also the operators of multiplication by these functions in L^2 with their maximal domains $\mathcal{D}(u)$ and $\mathcal{D}(v)$. Let $\widetilde{\mathcal{H}}=\mathcal{H}_1 \oplus \mathcal{H}_2$, $\mathcal{H}_1=\mathcal{H}_2=L^2$, and define $\widetilde{A}$ as the closure of the matrix

$$\begin{pmatrix} A & v \\ v & u \end{pmatrix} \tag{1.6}$$

in the space $\widetilde{\mathcal{H}}$ with domain $\mathcal{D}(A) \oplus \mathcal{D}(v)$. It follows from [ALMS] that the operator $\widetilde{A}$ is self-adjoint with domain

$$\begin{aligned} \mathcal{D}(\widetilde{A}) &= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \widetilde{\mathcal{H}} \colon\ y_1 + \overline{(A-\lambda)^{-1} v}\, y_2 \in \mathcal{D}(A) \ \text{ for some non-real } \lambda \right\} \\ &= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \widetilde{\mathcal{H}} \colon\ y_1, y_1' \in \mathrm{AC},\ A\, y_1 + v\, y_2 \in L^2,\ y_1(0)=y_1(1)=0 \right\}, \end{aligned} \tag{1.7}$$

where $\overline{(A-\lambda)^{-1} v}$ denotes the closure of the operator $(A-\lambda)^{-1} v$, and that the essential spectrum of the operator $\widetilde{A}$ is equal to

$$\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(u) = \mathbb{C} \setminus \rho(u).$$

This relation can also be derived from Proposition 1.1 below. Observe that the essential spectrum is the set of those λ's where $\dfrac{1}{u-\lambda}$ is essentially unbounded. If u is continuous, then $\sigma_{\mathrm{ess}}(\widetilde{A}) = u([0,1])$.

Now we can see that the eigenvalues of L and $\widetilde{A}$ in $\rho(u)$ coincide. Let λ_0 be an eigenvalue of $\widetilde{A}$, then we have the following equations:

$$\begin{aligned} -y_1'' + p\,y_1 + v\,y_2 - \lambda_0 y_1 &= 0, \\ v\,y_1 + u\,y_2 - \lambda_0 y_2 &= 0. \end{aligned}$$

From the second equation we get $y_2 = -\frac{v}{u-\lambda_0} y_1$. Inserting this expression for y_2 in the first equation yields the differential equation (1.1). Because of (1.7) the function y_1 is in the domain of $L(\lambda_0)$, so λ_0 is an eigenvalue of L. The proof of the converse is similar: If y_0 is an eigenfunction of $L(\lambda_0)$, then

$$\begin{pmatrix} y_0 \\ -\dfrac{v}{u-\lambda_0}\,y_0 \end{pmatrix}$$

is an eigenvector of the operator $\widetilde{A}$. It is easy to verify that for $\lambda \in \rho(\widetilde{A})$ the resolvent of $\widetilde{A}$ is of the form

$$(\widetilde{A}-\lambda)^{-1} = \begin{pmatrix} -L(\lambda)^{-1} & L(\lambda)^{-1} v (u-\lambda)^{-1} \\ (u-\lambda)^{-1} v L(\lambda)^{-1} & (u-\lambda)^{-1} - (u-\lambda)^{-1} v L(\lambda)^{-1} v (u-\lambda)^{-1} \end{pmatrix}. \tag{1.8}$$

In order to define a Titchmarsh–Weyl coefficient for the problem (1.1), (1.2) we introduce for $\lambda \in \rho(u)$ the functions $\varphi(x;\lambda)$, $\psi(x;\lambda)$ as the solutions of the differential equation (1.1) with the initial conditions

$$\begin{aligned} \varphi(0;\lambda) &= 1, \qquad \varphi'(0;\lambda) = 0, \\ \psi(0;\lambda) &= 0, \qquad \psi'(0;\lambda) = 1. \end{aligned} \tag{1.9}$$

If $x \in [0,1]$ is fixed, these functions depend holomorphically on the parameter λ. Let $\chi(x;\lambda) = \varphi(x;\lambda) + m(\lambda)\psi(x;\lambda)$ be the linear combination of $\varphi(x;\lambda)$ and $\psi(x;\lambda)$ which satisfies the boundary condition at the right end point: $\chi(1;\lambda) = 0$. The function

$$m(\lambda) = -\frac{\varphi(1;\lambda)}{\psi(1;\lambda)}$$

is called the *Titchmarsh-Weyl coefficient* (cf. [LMM]) of the problem (1.1), (1.2). As we will see in Chapter 3 it contains the whole spectral information.

If $\lambda \in \rho(u)$ is an eigenvalue of L, then $\psi(\cdot;\lambda)$ is a corresponding eigenfunction and the Titchmarsh–Weyl coefficient m has a pole at λ. The Wronskian of ψ and χ is

$$\begin{aligned} W(\psi,\chi) &= \psi\chi' - \psi'\chi = \psi(\varphi' + m\psi') - \psi'(\varphi + m\psi) = \\ &= \psi\varphi' - \psi'\varphi \equiv -1. \end{aligned}$$

It is constant since there is no term with y' in the differential equation (1.1). The operator $L(\lambda)^{-1}$ can be written as an integral operator

$$(L(\lambda)^{-1}y)(x) = \int_0^1 G(x,\xi;\lambda)y(\xi)d\xi \tag{1.10}$$

with the Green's function

$$G(x,\xi;\lambda) = -\begin{cases} \psi(x;\lambda)\,\chi(\xi;\lambda), & x \le \xi, \\ \chi(x;\lambda)\,\psi(\xi;\lambda), & x \ge \xi. \end{cases} \tag{1.11}$$

Let us introduce the operator

$$\widetilde{A}_0 := \begin{pmatrix} A & 0 \\ 0 & u \end{pmatrix},$$

so $\widetilde{A}$ can be considered as a perturbation of $\widetilde{A}_0$. The next proposition proves that this perturbation is trace class for the resolvents, which will be used in the next section.

Proposition 1.1 *For non-real λ the operators*

$$L(\lambda)^{-1} \quad \textit{and} \quad (\widetilde{A}-\lambda)^{-1} - (\widetilde{A}_0-\lambda)^{-1}$$

are trace class operators.

Proof: First, we consider $L(\lambda)^{-1}$. The kernel $G(x,\xi;\lambda)$ of the operator $L(\lambda)^{-1}$ has the properties that $G(0,\xi;\lambda) = 0$ and that $\dfrac{\partial G(x,\xi;\lambda)}{\partial x}$ is bounded on $[0,1]\times[0,1]$. The operator $L(\lambda)^{-1}$ can be written as a product $L(\lambda)^{-1} = JK$ of the operator

$$(Jf)(x) := \int_0^x f(\xi)d\xi$$

and the integral operator K with kernel $\dfrac{\partial G(x,\xi)}{\partial x}$. Both operators are integral operators with bounded kernels. Hence they are Hilbert-Schmidt operators, which implies that $L(\lambda)^{-1}$ is a trace class operator. Setting $q \equiv 0$ we get that $(A-\lambda)^{-1}$ is also a trace class operator.

The operator $(\widetilde{A}-\lambda)^{-1} - (\widetilde{A}_0-\lambda)^{-1}$ can be written as

$$\begin{pmatrix} -JK - (A-\lambda)^{-1} & JKv(u-\lambda)^{-1} \\ (u-\lambda)^{-1}vJK & -(u-\lambda)^{-1}vJKv(u-\lambda)^{-1} \end{pmatrix}.$$

The operators Kv and vJ are Hilbert-Schmidt operators since $v \in L^2$. So all the entries of $(\widetilde{A}-\lambda)^{-1} - (\widetilde{A}_0-\lambda)^{-1}$ are trace class operators. ■

1.2 The wave operators

In this section we introduce wave operators, the main concept in mathematical scattering theory. Let H be an arbitrary self-adjoint operator in a Hilbert space $\mathcal{H}$ and $E(\,\cdot\,)$ its spectral family, which is defined for all Borel sets in $\mathbb{R}$. By $\mathcal{H}_{\rm ac}$ ($\mathcal{H}_{\rm sc}$) we denote the set of elements x for which $(E(\cdot)\,x, x)$ is absolutely continuous (singular continuous) with respect to the Lebesgue measure. The sets $\mathcal{H}_{\rm ac}$ and $\mathcal{H}_{\rm sc}$ are closed linear subspaces, which are H-invariant. They are called the spaces of *absolute continuity* and *singular continuity* with respect to H. If $\mathcal{H}_{\rm p}$ denotes the closed linear span of the eigenvectors of H, we have:

$$\mathcal{H} = \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm sc} \oplus \mathcal{H}_{\rm p}.$$

The *absolutely continuous spectrum* $\sigma_{\rm ac}(H)$ and the *singular continuous spectrum* $\sigma_{\rm sc}(H)$ are defined by

$$\sigma_{\rm ac}(H) := \sigma(H|_{\mathcal{H}_{\rm ac}}), \qquad \sigma_{\rm sc}(H) := \sigma(H|_{\mathcal{H}_{\rm sc}}).$$

Now let H_0 be another self-adjoint operator in $\mathcal{H}$ and P_0 the orthogonal projection onto the space of absolute continuity with respect to H_0. The wave operators $W_+(H, H_0)$ and $W_-(H, H_0)$ are defined by

$$W_\pm(H, H_0) := s\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} P_0$$

if the limits exist, where s-lim denotes the strong limit. If a wave operator exists, it is an isometry and has the following intertwining property:

$$H\, W_\pm(H, H_0) = W_\pm(H, H_0)\, H_0. \tag{1.12}$$

A wave operator is called *complete* if it maps $\mathcal{H}_{\rm ac}^{(0)}$ onto $\mathcal{H}_{\rm ac}$, where $\mathcal{H}_{\rm ac}^{(0)}$ and $\mathcal{H}_{\rm ac}$ are the spaces of absolute continuity of H_0 and H respectively. In this case, due to relation (1.12), the absolutely continuous parts of the operators H and H_0, i.e. the operators $H|_{\mathcal{H}_{\rm ac}}$ and $H_0|_{\mathcal{H}_{\rm ac}^{(0)}}$, are unitarily equivalent.

The following theorem due to Kato and Rosenblum (cf. [K], Theorem 4.12) gives a sufficient condition for the existence and completeness of the wave operators.

Theorem 1.2 *Let H and H_0 be some self-adjoint operators in a Hilbert space $\mathcal{H}$. If*

$$(H-\lambda)^{-1} - (H_0-\lambda)^{-1}$$

is a trace class operator for some $\lambda \in \rho(H) \cap \rho(H_0)$, then the wave operators $W_\pm(H, H_0)$ exist and are complete.

Using Proposition 1.1 we get the following corollary since the operator A has no absolutely continuous part.

Corollary 1.3 *The wave operators $W_\pm(\widetilde{A}, \widetilde{A}_0)$ exist. Moreover, the absolutely continuous parts of the operator $\widetilde{A}$ and of the operator of multiplication by u are unitarily equivalent.*

1.3 An isomorphism

In this section we use some formal calculations with δ_0, the Dirac delta distribution at zero. The resulting relations can also be verified without it. Recall that δ_0', the first derivative of δ_0, is defined by the relation $\int\limits_0^1 y\,\delta_0' dx = -y'(0)$ if the function y is continuously differentiable at 0. With $\boldsymbol{d} := \begin{pmatrix} -\delta_0' \\ 0 \end{pmatrix}$ we define

$$\mathbf{r}_\lambda(x) := \Big((\widetilde{A} - \lambda)^{-1}\boldsymbol{d}\Big)(x) = \left(\begin{array}{c} \chi(x;\lambda) \\ -\dfrac{v(x)}{u(x)-\lambda}\chi(x;\lambda) \end{array}\right). \tag{1.13}$$

The second equality follows from the relations

$$\left((\widetilde{A} - \lambda)^{-1}\begin{pmatrix} -\delta_0' \\ 0 \end{pmatrix}\right)(x) = \left(\begin{array}{c} \left(L(\lambda)^{-1}\delta_0'\right)(x) \\ -\dfrac{v(x)}{u(x)-\lambda}\left(L(\lambda)^{-1}\delta_0'\right)(x) \end{array}\right)$$

and

$$\left(L(\lambda)^{-1}\delta_0'\right)(x) = \int\limits_0^1 G(x,\xi;\lambda)\delta_0'(\xi)d\xi = \chi(x;\lambda)\psi'(0;\lambda) = \chi(x;\lambda).$$

Lemma 1.4 *For $\lambda, z \in \mathbb{C}\backslash\mathbb{R}$ the following relations hold:*

$$m(\lambda) = \Big((\widetilde{A} - \lambda)^{-1}\boldsymbol{d}, \boldsymbol{d}\Big), \tag{1.14}$$

$$(\mathbf{r}_\lambda, \mathbf{r}_z) = \frac{m(\lambda) - \overline{m(z)}}{\lambda - \overline{z}}, \tag{1.15}$$

$$(\widetilde{A} - \lambda)^{-1}\mathbf{r}_z = \frac{\mathbf{r}_\lambda - \mathbf{r}_z}{\lambda - z}. \tag{1.16}$$

The relation (1.14) means that m is the so-called $\boldsymbol{d}$*–resolvent* of $\widetilde{A}$ with the generalized element $\boldsymbol{d}$.

Proof: For $\lambda, z \in \mathbb{C}\backslash\mathbb{R}$ we have

$$\begin{aligned}
\Big((\widetilde{A}-\lambda)^{-1}\boldsymbol{d},\boldsymbol{d}\Big) &= \left(\begin{pmatrix}\chi(\,\cdot\,;\lambda)\\ -(u-\lambda)^{-1}v\chi(\,\cdot\,;\lambda)\end{pmatrix},\begin{pmatrix}-\delta_0'\\ 0\end{pmatrix}\right)\\
&= -\int_0^1 \chi(x;\lambda)\delta_0'(x)dx = \chi'(0;\lambda) = m(\lambda).
\end{aligned}$$

Further,

$$\begin{aligned}
(\mathbf{r}_\lambda,\mathbf{r}_z) &= \left(\begin{pmatrix}\chi(\,\cdot\,;\lambda)\\ -(u-\lambda)^{-1}v\chi(\,\cdot\,;\lambda)\end{pmatrix},\begin{pmatrix}\chi(\,\cdot\,;z)\\ -(u-z)^{-1}v\chi(\,\cdot\,;z)\end{pmatrix}\right)\\
&= \Big(\chi(\,\cdot\,;\lambda),\chi(\,\cdot\,;z)\Big) + \Big(q(u-\lambda)^{-1}(u-\overline{z})^{-1}\chi(\,\cdot\,;\lambda),\chi(\,\cdot\,;z)\Big)\\
&= (\lambda-\overline{z})^{-1}\int_0^1 \left(\lambda-\overline{z}+\frac{q(x)}{u(x)-\lambda}-\frac{q(x)}{u(x)-\overline{z}}\right)\chi(x;\lambda)\overline{\chi(x;z)}dx\\
&= (\lambda-\overline{z})^{-1}\int_0^1 \Big[\Big(-\chi''(x;\lambda)+p(x)\chi(x;\lambda)\Big)\overline{\chi(x;z)}\\
&\qquad +\chi(x;\lambda)\overline{\Big(\chi''(x;z)-p(x)\chi(x;z)\Big)}\Big]dx\\
&= (\lambda-\overline{z})^{-1}\Big(-\chi'(x;\lambda)\overline{\chi(x;z)}+\chi(x;\lambda)\overline{\chi'(x;z)}\Big)\Big|_0^1\\
&= (\lambda-\overline{z})^{-1}\Big(\chi'(0;\lambda)\overline{\chi(0;z)}-\chi(0;\lambda)\overline{\chi'(0;z)}\Big) \;=\; \frac{m(\lambda)-\overline{m(z)}}{\lambda-\overline{z}}.
\end{aligned}$$

The relation (1.16) is a direct consequence of the resolvent identity. ■

For the next proposition, where the Titchmarsh-Weyl coefficient is examined in more detail, we need the following definition:

A function f is called a *Nevanlinna function* if it is analytic on $\mathbb{C}\setminus\mathbb{R}$, $f(\overline{z})=\overline{f(z)}$ and $\operatorname{Im} f(z)\geq 0$ for $\operatorname{Im} z\geq 0$.

Proposition 1.5 *The Titchmarsh-Weyl coefficient m is a Nevanlinna function and it admits the following representation:*

$$m(z) = a+\int_{-\infty}^{+\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\sigma(t), \tag{1.17}$$

where $a = \operatorname{Re} m(i) = -\operatorname{Re} \dfrac{\varphi(1;i)}{\psi(1;i)}$, *and* σ *is a positive measure which satisfies* $\int\limits_{-\infty}^{\infty} \dfrac{d\sigma(t)}{1+t^2} < \infty$.

Proof: Relation (1.15) implies for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\frac{\operatorname{Im} m(\lambda)}{\operatorname{Im} \lambda} = \|\mathbf{r}_\lambda\|^2 > 0,$$

hence m is a Nevanlinna function and therefore it admits a representation

$$m(z) = a + bz + \int\limits_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t)$$

with a, σ as above and $b \geq 0$ (cf. [AG], Theorem 69.2). The number b can be calculated as follows:

$$b = \lim_{\eta \nearrow \infty} \frac{\operatorname{Im} m(i\eta)}{\eta} = \lim_{\eta \nearrow \infty} \|\mathbf{r}_{i\eta}\|^2.$$

Since, by (1.16),

$$\mathbf{r}_\lambda = (\lambda - z)(\widetilde{A} - \lambda)^{-1}\mathbf{r}_z + \mathbf{r}_z$$

and $(\lambda - z)(\widetilde{A} - \lambda)^{-1} \to -I$ in the strong sense if $\lambda = i\eta$, $\eta \nearrow \infty$, it follows that $\mathbf{r}_{i\eta} \to 0$ if $\eta \nearrow \infty$, and hence $b = 0$. ■

The absence of the z-linear term in the representation (1.17) is due to the fact that $\widetilde{A}$ is an operator and not a relation (i. e. a multi-valued operator).

In the following the space L^2_σ with the *spectral measure* σ in the representation (1.17) and the self-adjoint operator of multiplication by the independent variable in this space will play an important role. We also introduce the space $\widetilde{\mathcal{H}}_0 :=$ c. l. s. $\{\mathbf{r}_z\colon z \in \mathbb{C} \setminus \mathbb{R}\} \subseteq \widetilde{\mathcal{H}}$. Due to relation (1.16) it reduces the operator $\widetilde{A}$. We want to establish an isometry from $\widetilde{\mathcal{H}}_0$ onto L^2_σ. So let us define the operator $\mathcal{F}$ on span$\{\mathbf{r}_z\colon z \in \mathbb{C} \setminus \mathbb{R}\}$ by

$$(\mathcal{F}\mathbf{r}_z)(t) = \frac{1}{t-z}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.18}$$

This is an isometry since

$$(\mathbf{r}_\lambda, \mathbf{r}_z) = \frac{m(\lambda) - \overline{m(z)}}{\lambda - \overline{z}} = \int\limits_{-\infty}^{\infty} \frac{1}{(t-\lambda)(t-\overline{z})} d\sigma(t) = \left(\frac{1}{\cdot - \lambda}, \frac{1}{\cdot - \overline{z}} \right)_{L^2_\sigma}.$$

Therefore the operator $\mathcal{F}$ can be extended to $\widetilde{\mathcal{H}}_0$ by continuity, which we will also denote by $\mathcal{F}$.

Theorem 1.6 *The mapping $\mathcal{F} : \widetilde{\mathcal{H}}_0 \longmapsto L^2_\sigma$, defined by* (1.18) *is an isometry from $\widetilde{\mathcal{H}}_0$ onto L^2_σ. Under this isometry the operator $\widetilde{A}|_{\widetilde{\mathcal{H}}_0}$ in $\widetilde{\mathcal{H}}_0$ is unitarily equivalent to the operator of multiplication by the independent variable in L^2_σ.*

Proof: To show that the operator $\mathcal{F}$ is surjective it is sufficient to show that the linear span of the functions $\dfrac{1}{\cdot - z}$, $z \in \mathbb{C}\backslash\mathbb{R}$, is dense in L^2_σ. Assume that there exists an $f \in L^2_\sigma$ which is orthogonal on $\dfrac{1}{\cdot - z}$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. Then

$$F(z) := \int_{-\infty}^{\infty} \frac{f(t)}{t-z} d\sigma(t) \equiv 0.$$

From the Stieltjes inversion formula (cf. [AG]) it follows that $f(t)\sigma(t)$ is the zero measure. So f is zero in L^2_σ.

The equivalence of the operator $\widetilde{A}|_{\widetilde{\mathcal{H}}_0}$ to the operator of multiplication by the independent variable in L^2_σ follows from the relation

$$\begin{aligned} \mathcal{F}\left(\left(\widetilde{A}-\lambda\right)^{-1}\mathbf{r}_z\right)(t) &= \mathcal{F}\left(\frac{\mathbf{r}_\lambda - \mathbf{r}_z}{\lambda - z}\right)(t) = \frac{1}{\lambda - z}\left(\frac{1}{t-\lambda} - \frac{1}{t-z}\right) \\ &= \frac{1}{t-\lambda}\cdot\frac{1}{t-z} = \frac{1}{t-\lambda}(\mathcal{F}\mathbf{r}_z)(t). \end{aligned}$$

■

Under stronger assumptions about p, q and u this result will be sharpened (see Theorem 3.5), where we will get an isometry which is defined on $\widetilde{\mathcal{H}}$ instead of $\widetilde{\mathcal{H}}_0$.

1.4 A fundamental system of solutions

In order to find an explicit expression for the spectral density in Chapter 3, a special fundamental system of solutions of the differential equation (1.1) is needed. This fundamental system, which will be derived in the present section, is also useful for the discussion of the eigenvalues in Chapter 2. In the sequel we suppose that the following conditions on the real functions p, q and u are fulfilled:

(a) $p, q \in L^\infty(0,1)$;

(b) u is an increasing, Lipschitz-continuous function;

(c) there exists a $d > 0$ such that (1.19)
$|u(x_2) - u(x_1)| \geq d|x_2 - x_1|$ for $x_1, x_2 \in [0,1]$;

(d) $q(x) \geq q_0 > 0$.

Sufficient for these conditions (a) – (d) is for example

$$p, q \in \mathrm{C}[0,1],\ u \in \mathrm{C}^1[0,1] \quad \text{and} \quad q(x) > 0,\ u'(x) > 0 \text{ for } x \in [0,1].$$

If the functions p, q and u are analytic then the behaviour of the solutions of the differential equation near the singularity is well-known. The aim of this section is to get similar results when the functions are not analytic.

Set $[\alpha,\beta] := [u(0), u(1)]$, which coincides with the essential spectrum of $\widetilde{A}$, and

$$r(x;z) := z - p(x) + \frac{q(x)}{u(x) - z}, \tag{1.20}$$

which is the coefficient of y in the differential equation (1.1). Further, set

$$\Omega := \{z \,|\, z = \lambda + i\varepsilon,\ \lambda \in [\alpha,\beta],\ |\varepsilon| \le 1\}$$

and

$$\Omega^+ := \{z \,|\, z = \lambda + i\varepsilon,\ \lambda \in [\alpha,\beta],\ 0 < \varepsilon \le 1\}.$$

If $\lambda \in [\alpha,\beta]$, define

$$x_\lambda := u^{-1}(\lambda), \tag{1.21}$$

where u^{-1} denotes the inverse function of u.

Theorem 1.7 *Suppose that the assumptions* (a) – (d) *are satisfied. If* $z \in \Omega$, *then for* $\lambda = \operatorname{Re} z$ *the initial-value problem*

$$\begin{aligned} &y''(x) + r(x;z)\,y(x) = 0 \\ &y(x_\lambda) = 0\,,\ y'(x_\lambda) = 1 \end{aligned} \tag{1.22}$$

has a unique solution $y = g(\,\cdot\,;z)$. *This solution* $g(\,\cdot\,;z)$ *and its derivative* $g'(\,\cdot\,;z)$ *depend continuously on the parameter* z (*with respect to the norm of* $\mathrm{C}[0,1]$).

For non-real z *the function* $g(\,\cdot\,;z)$ *has no zeros in* $[0,1]$ *except* x_λ. *For* $\lambda \in [\alpha,\beta]$ *the function* $g(\,\cdot\,;\lambda)$ *is real and there exists a* $\delta > 0$, *which is independent of* λ, *such that* $g(\,\cdot\,;\lambda)$ *has no zeros in* $[0,1] \cap [x_\lambda - \delta, x_\lambda + \delta]$ *except* x_λ.

Proof: The problem (1.22) is equivalent to the integral equation

$$y(x) = x - x_\lambda - \int_{x_\lambda}^{x} (x-t) r(t;z) y(t) dt.$$

With $y(x) = (x - x_\lambda)w(x; z)$ we get the following equation for w:

$$w(x; z) = 1 - \int_{x_\lambda}^{x} \frac{x - t}{x - x_\lambda} r(t; z)(t - x_\lambda)w(t; z)dt. \tag{1.23}$$

The kernel

$$K(x, t; z) := \begin{cases} \dfrac{x - t}{x - x_\lambda} r(t; z)(t - x_\lambda) & \text{if } t \text{ lies between } x \text{ and } x_\lambda, \\ 0 & \text{else} \end{cases} \tag{1.24}$$

of the integral operator in equation (1.23) is uniformly bounded for $z \in \Omega$:

$$\begin{aligned} |K(x, t; z)| &\leq \max_{z \in \Omega} |z| + |p(t)| + \frac{q(t)}{|u(t) - \lambda|}|t - x_\lambda| \\ &\leq \max\left(\sqrt{\alpha^2 + 1}, \sqrt{\beta^2 + 1}\right) + \operatorname*{ess\,sup}_{\xi \in [0,1]} \left(|p(\xi)| + \frac{q(\xi)}{d}\right) =: M. \end{aligned}$$

If $(x_n, z_n) \to (x_0, z_0)$ in $[0, 1] \times \Omega$, then $K(x_n, t; z_n) \to K(x_0, t; z_0)$ for almost all $t \in [0, 1]$. With the dominated convergence theorem and the uniform estimate for the kernel we obtain that for each function $f(x; z)$ from the space $\mathrm{C}([0, 1] \times \Omega)$ the function

$$(\widehat{K} f)(x; z) := \int_{x_\lambda}^{x} K(x, t; z)f(t; z)dt \tag{1.25}$$

is continuous on $[0, 1] \times \Omega$ and

$$|(\widehat{K} f)(x; z)| \leq M\|f\|_\infty |x - x_\lambda|. \tag{1.26}$$

So $\widehat{K}$ is a bounded operator on $\mathrm{C}([0, 1] \times \Omega)$ with $\|\widehat{K}\| \leq M$. By induction we get

$$|(\widehat{K}^n f)(x; z)| \leq \frac{M^n \|f\|_\infty |x - x_\lambda|^n}{n!} \tag{1.27}$$

and

$$\|\widehat{K}^n\| \leq \frac{M^n}{n!}.$$

It follows that the operator $I + \widehat{K}$ is invertible in $\mathrm{C}([0, 1] \times \Omega)$ and its inverse admits a representation as a Neumann series which converges with respect to the operator norm:

$$(I + \widehat{K})^{-1} = \sum_{n=0}^{\infty} (-1)^n \widehat{K}^n.$$

Moreover, the following estimates are valid:

$$\left|\Big((I+\widehat{K})^{-1}f\Big)(x;z)\right| \le e^{M|x-x_\lambda|}\|f\|_\infty, \qquad \|(I+\widehat{K})^{-1}\| \le e^{M}.$$

If we apply $(I+\widehat{K})^{-1}$ to $w_0(x;z) := 1$, we get the unique solution of (1.23)

$$w(x;z) := \Big((I+\widehat{K})^{-1}w_0\Big)(x;z), \tag{1.28}$$

which is a function in $\mathrm{C}([0,1]\times\Omega)$ and satisfies the uniform estimate

$$|w(x;z)| \le e^{M|x-x_\lambda|} \tag{1.29}$$

for $z\in\Omega$. The function $g(x;z) := (x-x_\lambda)w(x;z)$ is the unique solution of the initial-value problem (1.22).

From

$$g'(x;z) = 1 - \int_{x_\lambda}^{x} r(t;z)(t-x_\lambda)w(t;z)dt$$

it follows that also $g'(x;z)\in\mathrm{C}([0,1]\times\Omega)$.

Since $K(x,t;\lambda)$ is real for $\lambda\in[\alpha,\beta]$, also the solution $g(x;\lambda)$ is real.

Assume that $g(\widehat{x};z) = 0$ for some non-real z and $\widehat{x} < x_\lambda$. Then z is an eigenvalue of $\widetilde{A}$ restricted to $L^2(\widehat{x},x_\lambda)\oplus L^2(\widehat{x},x_\lambda)$ with Dirichlet boundary conditions at $\widehat{x}$ and x_λ. But this is a self-adjoint operator, a contradiction.

By (1.23) and (1.29) we get for each $\lambda\in[\alpha,\beta]$ and $x > x_\lambda$

$$\begin{aligned}|w(x;\lambda)| &\ge 1 - \int_{x_\lambda}^{x} |K(x,t;\lambda)||w(t;\lambda)|dt \ \ge\ 1 - \int_{x_\lambda}^{x} Me^{M(t-x_\lambda)}dt \\ &= 2 - e^{M(x-x_\lambda)}.\end{aligned}$$

Therefore $w(x;\lambda)\ne 0$ for $x\in\Big[x_\lambda, x_\lambda+\dfrac{\ln 2}{M}\Big)\cap\Big[0,1\Big]$. In the same way $w(x;\lambda)\ne 0$ for $x\in\Big(x_\lambda-\dfrac{\ln 2}{M}, x_\lambda+\dfrac{\ln 2}{M}\Big)\cap\Big[0,1\Big]$. So we can choose e.g. $\delta := \dfrac{\ln 2}{2M}$. ∎

Remark 1: The existence and uniqueness of a solution remains valid for an arbitrary function r such that $(x-x_\lambda)r(x)$ belongs to L^1. This can be proved by means of a fixed point theorem.

Remark 2: For $z\in\Omega^+$ another estimate instead of (1.27) can be proved. With

$$\widetilde{M}_\varepsilon = \max\left(\sqrt{\alpha^2+1},\sqrt{\beta^2+1}\right) + \operatorname*{ess\,sup}_{x\in[0,1]}\left(|p(x)| + \frac{q(x)}{\varepsilon}\right), \quad \varepsilon = \operatorname{Im} z,$$

we have

$$\left|\left(\widehat{K}^n f\right)(x;z)\right| \le \frac{\widetilde{M}_\varepsilon^n |x-x_\lambda|^{2n}}{2^n n!}\|f\|_\infty. \tag{1.30}$$

Since the non-decreasing function x_λ satisfies a Lipschitz condition with constant $\frac{1}{d}$, we have the following lemma.

Lemma 1.8 *The function x_λ admits the representation*

$$x_\lambda = \int_\alpha^\lambda \rho(\mu)d\mu, \quad \lambda \in [\alpha,\beta] \tag{1.31}$$

with $\rho \in L^\infty(\alpha,\beta)$, $\rho \ge 0$, $\|\rho\|_\infty \le \frac{1}{d}$.

Define the function

$$\gamma(\lambda) := q(x_\lambda)\rho(\lambda) \tag{1.32}$$

which is in $L^\infty(\alpha,\beta)$. The following theorem now guarantees the existence of a second, linearly independent solution of the differential equation.

Theorem 1.9 *Suppose that the assumptions* (a) – (d) *from the beginning of this section are satisfied. If $z \in \Omega^+$, there exists a solution $h(\,\cdot\,;z)$ of the differential equation* (1.1) *which is linearly independent of $g(\,\cdot\,;z)$. If $x \in [0,1]$ is fixed, the values $h(x;z)$, $h'(x;z)$ depend continuously on z for $z \in \Omega^+$. Moreover, for almost every $\lambda \in [\alpha,\beta]$ there exists a function $\Theta(x;\lambda)$ such that the following relation holds:*

$$\lim_{\varepsilon\downarrow 0} h(x;\lambda+i\varepsilon) = \begin{cases} g(x;\lambda)\Theta(x;\lambda) & \text{if } x < x_\lambda, \\ g(x;\lambda)\Big(\Theta(x;\lambda) + i\pi\gamma(\lambda)\Big) & \text{if } x > x_\lambda. \end{cases} \tag{1.33}$$

The function $g(\,\cdot\,;\lambda)\Theta(\,\cdot\,;\lambda)$ is real and continuous on $[0,1]$*; it is a solution of* (1.1) *for $x \ne x_\lambda$ and has the asymptotic behaviour*

$$g(x;\lambda)\Theta(x;\lambda) = -1 + \mathrm{O}\Big((x-x_\lambda)\log|x-x_\lambda|\Big) \tag{1.34}$$

for $x \to x_\lambda$.

Proof: In order to find a solution $h(\,\cdot\,;z)$ of (1.1) which is linearly independent of $g(x;z)$, we write $h(\,\cdot\,;z) = g(\,\cdot\,;z)\tau(\,\cdot\,;z)$ for $z \in \Omega^+$. The Wronskian of g and h

$$W(g,h) = gh' - g'h = g^2\tau'$$

is constant in x since there is no term with y' in the differential equation. So we can set $W(g,h) \equiv 1$, which yields

$$\tau'(\,\cdot\,;z) = \frac{1}{g(\,\cdot\,;z)^2}.$$

With δ from Theorem 1.7 the function $w(x;\lambda) = \dfrac{g(x;\lambda)}{x - x_\lambda}$, has no zero in $[x_\lambda - \delta, x_\lambda + \delta]$. First, fix $\lambda_0 \in [\alpha, \beta]$ and set $x_1 := x_{\lambda_0} - \dfrac{\delta}{2}$, $x_2 := x_{\lambda_0} + \dfrac{\delta}{2}$. For each $\lambda \in \Big[u\Big(x_{\lambda_0} - \dfrac{\delta}{4}\Big), u\Big(x_{\lambda_0} + \dfrac{\delta}{4}\Big)\Big] =: [\lambda_1, \lambda_2]$ the function $w(x;\lambda)$ has no zero in the interval $[x_1, x_2]$ and $x_\lambda \in (x_1, x_2)$. For the following we choose $z = \lambda + i\varepsilon$ with $\lambda \in [\lambda_1, \lambda_2]$ and $0 < \varepsilon \leq 1$.

The solution h, which fulfills the initial conditions $h(x_1;z) = 0$, $h'(x_1;z) = \dfrac{1}{g(x_1;z)}$, can be written as follows for $x \in [x_1, x_\lambda)$:

$$\begin{aligned} h(x;z) &= g(x;z)\tau(x;z) \;=\; g(x;z)\int_{x_1}^{x} \frac{1}{g(t;z)^2}dt \\ &= g(x;z)\Big[\int_{x_1}^{x} \Big(\frac{1}{g(t;z)^2} - \frac{1}{(t-x_\lambda)^2}\Big)dt - \frac{1}{x - x_\lambda} + \frac{1}{x_1 - x_\lambda}\Big]. \end{aligned}$$

Since the integrand in the last line has no singularity at $t = x_\lambda$, this is a solution for $x \in [x_1, x_2]$. For fixed x it depends continuously on the parameter z.

In order to investigate the limit for $\varepsilon \to 0$, we rewrite the integral

$$\begin{aligned} \int_{x_1}^{x} \Big(\frac{1}{g(t;z)^2} - \frac{1}{(t-x_\lambda)^2}\Big)dt &= \int_{x_1}^{x} \frac{1 - w(t;z)^2}{(t-x_\lambda)^2 w(t;z)^2}dt \\ &= \int_{x_1}^{x} \frac{(1 - w(t;z))[(1 - w(t;z))(1 + 2w(t;z)) + 2w(t;z)^2]}{(t-x_\lambda)^2 w(t;z)^2}dt \\ &= \int_{x_1}^{x} \frac{(1 - w(t;z))^2}{(t-x_\lambda)^2 w(t;z)^2}\Big(1 + 2w(t;z)\Big)dt + \int_{x_1}^{x} \frac{2(1 - w(t;z))}{(t-x_\lambda)^2}dt. \end{aligned}$$

The first integral has a bounded integrand since by (1.23), (1.26) and (1.29) it holds

$$\left|\frac{1 - w(t;z)}{t - x_\lambda}\right| \leq \frac{M|t - x_\lambda|e^{M|t-x_\lambda|}}{|t - x_\lambda|} \leq Me^M.$$

So the integral is continuous in z also for $z \in \Omega$ and it is real for real z. The second integral can be rewritten using (1.23) and integration by parts:

$$\begin{aligned}
\int_{x_1}^{x} \frac{2(1-w(t;z))}{(t-x_\lambda)^2}dt &= \int_{x_1}^{x} \frac{2}{(t-x_\lambda)^3} \int_{x_\lambda}^{t} (t-s)r(s;z)(s-x_\lambda)w(s;z)ds\,dt \\
&= -\frac{1}{(t-x_\lambda)^2} \int_{x_\lambda}^{t} (t-s)r(s;z)(s-x_\lambda)w(s;z)ds\Big|_{t=x_1}^{x} \\
&\quad + \int_{x_1}^{x} \frac{1}{(t-x_\lambda)^2} \int_{x_\lambda}^{t} r(s;z)(s-x_\lambda)w(s;z)ds\,dt \\
&= -\frac{1}{(t-x_\lambda)^2} \int_{x_\lambda}^{t} (t-s)r(s;z)(s-x_\lambda)w(s;z)ds\Big|_{t=x_1}^{x} \\
&\quad - \frac{1}{t-x_\lambda} \int_{x_\lambda}^{t} r(s;z)(s-x_\lambda)w(s;z)ds\Big|_{t=x_1}^{x} \\
&\quad + \int_{x_1}^{x} r(t;z)(w(t;z)-1)dt + \int_{x_1}^{x} (z-p(t))dt \\
&\quad - \int_{0}^{x_1} \frac{q(t)}{u(t)-z}dt + \int_{0}^{x} \frac{q(t)}{u(t)-z}dt.
\end{aligned}$$

All integrals but the last one are bounded also for $\varepsilon \to 0$. We can write τ as

$$\tau(x;z) = -\frac{1}{x-x_\lambda} + \tau_1(x;z) + \tau_2(x;z) \tag{1.35}$$

where

$$\tau_2(x;z) := \int_{0}^{x} \frac{q(t)}{u(t)-z}dt \tag{1.36}$$

and $\tau_1(x;z)$ is continuous in z also for $0 \le \varepsilon \le 1$ and it is real for real z. Since the differential equation (1.1) has no singularity for $x \in [0,1] \setminus [x_1, x_2]$, the solution $h(x;z)$ can be continued to the whole interval $[0,1]$ as a function of x. Also τ and τ_1 can be defined there, but they have a pole where g vanishes.

Now we can also consider a different λ_0, say $\widehat{\lambda}_0$ and get a solution $\widehat{h}$ for $z = \lambda + i\varepsilon$, $\lambda \in [\widehat{\lambda}_1, \widehat{\lambda}_2]$. Since $W(g,h) = W(g,\widehat{h}) \equiv 1$, the solutions h and $\widehat{h}$ can differ only by a multiple of g for $\lambda \in [\lambda_1, \lambda_2] \cap [\widehat{\lambda}_1, \widehat{\lambda}_2]$. But we can choose $\widehat{h}$ so that $h = \widehat{h}$ for such a λ. So we have defined a solution $h(x;z)$ for $z \in \Omega^+$ which depends continuously on z there.

To examine the solution $h(\,\cdot\,;\lambda+i\varepsilon)$ for $\varepsilon\to 0$ we have only to consider τ_2. With $\gamma(\lambda)$ defined in (1.32) we can write

$$\tau_2(x;z)=\int_0^x \frac{q(t)}{u(t)-z}dt=\int_\alpha^{u(x)}\frac{q(x_\mu)\rho(\mu)}{\mu-z}d\mu=\int_\alpha^{u(x)}\frac{\gamma(\mu)}{\mu-z}d\mu.$$

Since $\gamma\in L^\infty(\alpha,\beta)$, it follows (cf. [RR], Theorem 5.30) that for a fixed x

$$\lim_{\varepsilon\downarrow 0}\tau_2(x;\lambda+i\varepsilon)=\text{p.v.}\int_\alpha^{u(x)}\frac{\gamma(\mu)}{\mu-\lambda}d\mu+i\pi\gamma(\lambda) \tag{1.37}$$

for almost every $\lambda\in[\alpha,u(x))$, where p.v. denotes the principal value at $\mu=\lambda$. For $\lambda\in(u(x),\beta]$ we have simply the integral and no imaginary part. Setting

$$\Theta(x;\lambda):=\lim_{\varepsilon\downarrow 0}\operatorname{Re}\tau(x;\lambda+i\varepsilon) \tag{1.38}$$

we get the desired relation (1.33).

For $x<x_\lambda$ the function $g(x;\lambda)\Theta(x;\lambda)$ is a solution of the differential equation. If the limit (1.37) exists then also the limits

$$\lim_{\varepsilon\downarrow 0}h(x;\lambda+i\varepsilon),\quad \lim_{\varepsilon\downarrow 0}h'(x;\lambda+i\varepsilon)$$

exist since $W(g,h)\equiv 1$. But the differential equation has no singularity for $x>x_\lambda$, so $\lim_{\varepsilon\downarrow 0}h(x;\lambda+i\varepsilon)$ is a solution there and also $g(x;\lambda)\Theta(x;\lambda)$.

The asymptotic behaviour of $g(x;\lambda)\Theta(x;\lambda)$ follows from (1.35) and the estimate

$$\left|\int_0^x\frac{q(t)}{u(t)-\lambda}dt\right|\le\frac{\sup q}{d}\int_0^x\frac{dt}{|t-x_\lambda|}=\frac{\sup q}{d}\Big(-\log|x-x_\lambda|+\log x_\lambda\Big)$$

for $x<x_\lambda$. For $x>x_\lambda$ one has to consider the integral from 1 to x. ∎

Note that $\lambda\in[\alpha,\beta]$ is an eigenvalue if and only if $g(0;\lambda)=g(1;\lambda)=0$, because $g\Theta$ cannot be an eigenfunction since its first derivative has a logarithmic singularity at x_λ.

If $q\in \mathrm{C}[0,1]$ and $u\in\mathrm{C}^1[0,1]$, then we get the following asymptotic formula for $x\to x_\lambda$:

$$g(x;\lambda)\Theta(x;\lambda)=-1+\frac{q(x_\lambda)}{u'(x_\lambda)}(x-x_\lambda)\log|x-x_\lambda|+\mathrm{O}(x-x_\lambda). \tag{1.39}$$

Chapter 2

Discussion of the Eigenvalues

In this chapter we investigate the eigenvalues of the operator $\widetilde{A}$, in particular those eigenvalues which are embedded in the essential spectrum. The main result is a comparison of the eigenvalues of $\widetilde{A}$ and $\widetilde{A}_0$. This yields also an upper bound for the number of embedded eigenvalues. Another interesting fact is that the eigenvalues $\le \alpha$ and those $> \alpha$ behave in a different manner under perturbation. If q increases, the eigenvalues $\le \alpha$ move to the left, those $> \alpha$ move to the right or if they are embedded they can disappear.

2.1 Auxiliary results

For the discussion of the eigenvalues we use some information about the number of zeros of the solutions of the differential equation. An important tool is Sturm's comparison theorem. Because of the floating singularity we need it in a form, where the coefficients may by unbounded.

Theorem 2.1 *Suppose that q_1 and q_2 are measurable functions on the interval (x_1, x_2) such that $q_1(x) > q_2(x)$ for almost every $x \in (x_1, x_2)$. Let f_1 and f_2 be non-trivial continuous solutions of the equations*

$$f_1'' + q_1 f_1 = 0,$$
$$f_2'' + q_2 f_2 = 0$$

on the interval (x_1, x_2), and assume that $q_1 f_1,\ q_2 f_2 \in L^1(x_1, x_2)$. Then the relation

$$f_2(x_1) = f_2(x_2) = 0$$

implies that there exists a number $\xi \in (x_1, x_2)$ such that $f_1(\xi) = 0$.

Proof: Suppose that $f_1(x) \neq 0$ for $x \in (x_1, x_2)$. We may assume that x_1 and x_2 are two successive zeros of f_2, so we suppose that $f_1(x) > 0$ and $f_2(x) > 0$ for $x \in (x_1, x_2)$. From

$$f_1 f_2'' - f_2 f_1'' + (q_2 - q_1) f_1 f_2 = 0$$

it follows

$$\begin{aligned} &\int_{x_1}^{x_2} \Big(q_1(x) - q_2(x)\Big) f_1(x) f_2(x) dx \qquad (2.1) \\ &= \int_{x_1}^{x_2} \Big(f_1(x) f_2''(x) - f_2(x) f_1''(x)\Big) dx \\ &= f_1(x) f_2'(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f_1'(x) f_2'(x) dx - f_2(x) f_1'(x)\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} f_1'(x) f_2'(x) dx \\ &= f_1(x_2) f_2'(x_2) - f_1(x_1) f_2'(x_1) \ \leq \ 0 \end{aligned}$$

since $f_1(x_1) \geq 0$, $f_1(x_2) \geq 0$ and $f_2'(x_1) \geq 0$, $f_2'(x_2) \leq 0$. But on the other hand the integral in (2.1) is positive, a contradiction. ■

Now we derive a correspondence between the embedded eigenvalues of L and $\widetilde{A}$. The next lemma proves that there is one more restrictive condition for an embedded eigenvalue, namely the eigenfunction must have a zero at the singularity. This is the main point that the comparison theorem can be applied. But on the other hand it implies that embedded eigenvalues are very unlikely.

Lemma 2.2 *Assume that the assumptions* (a) – (d) *are satisfied. Then the number* $\lambda \in [\alpha, \beta]$ *is an eigenvalue of* $\widetilde{A}$ *if and only if there exists a continuously differentiable non-trivial solution* y *of the differential equation* (1.1) *with the boundary conditions* (1.2). *In that case the solution has a zero at* x_λ *but does not vanish identically on any interval. Moreover, the eigenvalue is algebraically simple.*

Proof: If λ is an eigenvalue of $\widetilde{A}$, there exist functions y_1, $y_2 \in L^2$ with y_1, $y_1' \in \mathrm{AC}$ such that

$$\begin{aligned} -y_1'' + p\, y_1 + v\, y_2 - \lambda\, y_1 &= 0, \qquad (2.2) \\ v\, y_1 + u\, y_2 - \lambda\, y_2 &= 0. \qquad (2.3) \end{aligned}$$

Equation (2.3) implies

$$y_2(x) = -\frac{v(x)}{u(x) - \lambda} y_1(x)$$

for $x \neq x_\lambda$. As $y_2 \in L^2$, i. e.

$$\int_0^1 \frac{v(x)^2}{|u(x)-\lambda|^2} |y_1(x)|^2 dx < \infty,$$

it follows that $y_1(x_\lambda) = 0$ since otherwise in a neighbourhood of x_λ we would have

$$\frac{v(x)^2}{|u(x)-\lambda|^2} |y_1(x)|^2 \geq \frac{c}{|x-x_\lambda|^2}$$

with $c > 0$. Because g and $g\Theta$ build a fundamental system of solutions and $g(x_\lambda; \lambda) = 0$, the function y_1 has to be a multiple of g. Therefore y_1 is continuously differentiable and it is a solution of (1.1), (1.2).

Conversely, if $y \in C^1[0,1]$ is a solution of (1.1), (1.2), it has to be a multiple of g and therefore it must have a zero at x_λ. The functions $y_1 := y, y_2 := -\dfrac{v}{u-\lambda} y$ satisfy the equations (2.2), (2.3) with y_1, $y_1' \in$ AC, $y_2 \in L^2$ and $-y_1'' + p\, y_1 + v\, y_2 \in L^2$. Thus λ is an eigenvalue of $\widetilde{A}$.

That the solution cannot vanish identically on any interval and that the eigenvalue is algebraically simple follows from the fact that the solution is a multiple of g and that $g'(x_\lambda) = 1$. ■

2.2 Eigenvalues in the interval (α, ∞)

In this section we derive estimates for the eigenvalues in the interval (α, ∞). As a corollary we get conditions for $\widetilde{A}$ having no embedded eigenvalues in the interval $(\alpha, \beta]$. These considerations cannot be extended to the point α as will be seen in the next section. First, we consider the case $p \equiv 0$. To avoid the distinction, whether λ is an embedded eigenvalue or not, we set

$$x_\lambda := 1 \quad \text{for } \lambda > \beta.$$

Theorem 2.3 *If the conditions* (a) – (d) *are satisfied and* $p \equiv 0$*, then the point spectrum of* $\widetilde{A}$ *in the interval* (α, ∞) *consists of a sequence of simple eigenvalues* λ_n *:* $\lambda_1 < \lambda_2 < \ldots$ *with the only accumulation point infinity. Moreover,*

$$\lambda_n > \frac{n^2\pi^2}{x_{\lambda_n}^2} + \frac{\inf\limits_{x\in[0,1]} q(x)}{\lambda_n - \alpha}\ . \tag{2.4}$$

Proof: Let $\lambda < \widehat{\lambda}$ be two eigenvalues in (α, ∞) and $y, \widehat{y}$ be the corresponding eigenfunctions. For $x \in (0, x_\lambda)$ the following inequality holds:

$$\lambda + \frac{q(x)}{u(x)-\lambda} < \widehat{\lambda} + \frac{q(x)}{u(x)-\widehat{\lambda}}\ .$$

Assume that y has k zeros in $x \in (0, x_\lambda)$. Since also 0 and x_λ are zeros of y and $x_\lambda < x_{\widehat{\lambda}}$, according to the comparison theorem (Theorem 2.1) the function $\widehat{y}$ has at least $k+1$ zeros in the interval $(0, x_{\widehat{\lambda}})$. The comparison theorem can be applied because

$$\Big(\lambda + \frac{q(x)}{u(x) - \lambda}\Big) y(x)$$

is bounded at $x = x_\lambda$.

It follows that the eigenvalues in (α, ∞) cannot have a finite accumulation point. So we can denote these eigenvalues by λ_n: $\lambda_1 < \lambda_2 < \ldots$ and the corresponding eigenfunctions by y_n. By induction we get that y_n has at least $n-1$ zeros in the interval $(0, x_{\lambda_n})$.

To prove the inequality (2.4) assume that

$$\lambda_n \leq \frac{n^2\pi^2}{x_{\lambda_n}^2} + \frac{\inf\limits_{x\in[0,1]} q(x)}{\lambda_n - \alpha} .$$

Then it follows for all $x \in (0, x_{\lambda_n})$

$$\lambda_n + \frac{q(x)}{u(x) - \lambda_n} < \lambda_n - \frac{\inf\limits_{x\in[0,1]} q(x)}{\lambda_n - \alpha} \leq \frac{n^2\pi^2}{x_{\lambda_n}^2} .$$

Since y_n has at least $n-1$ zeros in the interval $(0, x_{\lambda_n})$, the comparison theorem implies that a solution $\tilde{y}$ of the equation

$$\tilde{y}'' + \frac{n^2\pi^2}{x_{\lambda_n}^2}\tilde{y} = 0, \quad \tilde{y}(0) = 0$$

has at least n zeros in the interval $(0, x_{\lambda_n})$. But $\tilde{y} = \sin \dfrac{n\pi x}{x_{\lambda_n}}$ has only $n-1$ zeros in $(0, x_{\lambda_n})$. Thus the inequality (2.4) is proved.

As $\widetilde{A}$ is semi-bounded from below but not bounded and the essential spectrum is bounded, it is clear that $+\infty$ is an accumulation point of the eigenvalues. ∎

As a corollary we get an estimate for the eigenvalues from below.

Corollary 2.4 *For the eigenvalues* $\lambda_n \in (\alpha, \beta]$ *it holds*

$$\lambda_n > n^2\pi^2 + \frac{\inf\limits_{x\in[0,1]} q(x)}{\beta - \alpha}, \tag{2.5}$$

for the eigenvalues $\lambda_n \in (\beta, \infty)$ it holds

$$\lambda_n > \frac{n^2\pi^2 + \alpha + \sqrt{(n^2\pi^2 - \alpha)^2 + 4 \inf_{x\in[0,1]} q(x)}}{2} . \tag{2.6}$$

In any case there is

$$\lambda_n > n^2\pi^2. \tag{2.7}$$

If $\beta < \pi^2$, the inequality (2.7) can be proved also by an extremal principle for eigenvalues as it is carried out in [AdL] because the spectra of A and u are disjoint and the eigenvalues of A are $n^2\pi^2$.

Now we can estimate the number of embedded eigenvalues of $\widetilde{A}$ in $(\alpha, \beta]$.

Corollary 2.5 *Let n be the smallest non-negative integer such that*

$$\beta \leq (n+1)^2\pi^2 + \frac{\inf_{x\in[0,1]} q(x)}{\beta - \alpha} . \tag{2.8}$$

Then $\widetilde{A}$ has at most n eigenvalues in $(\alpha, \beta]$.

This result admits the following interpretation: The eigenvalues of the problem (1.1), (1.2) with $q(x) \equiv 0$ are just the eigenvalues of A, namely $k^2\pi^2$. If n of these numbers are $< \beta$, then at most n eigenvalues of the problem with $q(x) > 0$ are in the interval $(\alpha, \beta]$. This is an improvement of the remark after Proposition 4 in [AdLM], where this assertion was proved for $n = 0$. And if the infimum of q is sufficiently large, $\widetilde{A}$ has no eigenvalue in the interval $(\alpha, \beta]$.

Theorem 2.3 implies also that for $n^2\pi^2 \in (\alpha, \beta)$ the inequality $\lambda_n \geq \widetilde{\lambda}_n$ holds, where $\widetilde{\lambda}_n$ is the unique solution of the equation

$$\lambda \cdot \left(u^{-1}(\lambda)\right)^2 = n^2\pi^2.$$

This solution $\widetilde{\lambda}_n$ is greater than $n^2\pi^2$ and independent of q. So λ_n cannot move continuously towards $n^2\pi^2$ if q tends to 0.

The corollaries hold true also if u is decreasing, the inequality (2.4) in Theorem 2.3 has to be modified slightly. In that case the zeros in the interval $(x_\lambda, 1)$ have to be considered instead of those in the interval $(0, x_\lambda)$.

Next we examine whether the inequality (2.4) in Theorem 2.3 is sharp.

Theorem 2.6 *Let u satisfy the conditions* (b) *and* (c). *If $\lambda \in (\alpha, \beta]$ and*

$$\lambda > \frac{\pi^2}{x_\lambda^2}, \tag{2.9}$$

then there exists a function $q \in \mathrm{C}[0,1]$, $q(x) > 0$, such that λ is an eigenvalue of $\widetilde{A}$.

Proof: Choose a number x' with $x' < x_\lambda$ such that

$$\lambda > \frac{\pi^2}{x'^2}.$$

If $\tilde{q}(x) > 0$ is chosen such that

$$\lambda + \frac{\tilde{q}(x)}{u(x) - \lambda} > \frac{\pi^2}{x'^2}$$

on the interval $[0, x']$, then the solution y of the differential equation (1.1) with $q = \tilde{q}$ and $y(x_\lambda) = 0$, $y'(x_\lambda) = 1$ has a zero in the interval $(0, x')$ because of the comparison theorem (comparison with $\sin \frac{\pi x}{x'}$).

Next we set $q(x) = t \cdot \tilde{q}(x)$, $t \geq 1$. There exists a $t_0 > 1$ such that

$$\lambda + \frac{t_0 \tilde{q}(x)}{u(x) - \lambda} < 0$$

on $[0, x_\lambda)$ since the denominator is negative there. The solution of the differential equation with this q has no zero in the interval $[0, x_\lambda)$. Between 1 and t_0 there exists a t such that the corresponding solution has a zero at 0.

Finally we change the function q in the interval $(x_\lambda, 1]$ such that 1 is also a zero of the solution. This can be done in a similar way as before. Since the term $\frac{q(x)}{u(x)-\lambda}$ is positive there, the solution will have a zero in $(x_\lambda, 1]$ if q is large enough. ■

As a corollary we get: If $\beta > \pi^2$, then there exist a number $\lambda \in (\alpha, \beta)$ and a function $q \in \mathrm{C}[0,1]$, $q(x) > 0$, such that λ is an eigenvalue of $\widetilde{A}$.

Theorem 2.7 *Let n be a positive integer. Then there exists a number λ in (α, ∞) such that equation* (1.1) *has a solution which has zeros at $x = 0$ and $x = x_\lambda$, and exactly $n - 1$ zeros in the interval $(0, x_\lambda)$.*

Proof: Define q and u on $(-\infty, 0)$ such that q is constant, u is a linear function there, $q(x) > 0$, $u'(x) > 0$ for $x \in (-\infty, 0)$, and u is continuous at 0. So $\lambda + \frac{q(x)}{u(x)-\lambda}$ tends to λ for $x \to -\infty$. For $\lambda \in (\alpha, \infty)$ let y_λ be the continuously differentiable solution of equation (1.1) on $(-\infty, 1]$ with $y_\lambda(x_\lambda) = 0$ and $y'_\lambda(x_\lambda) = 1$; for $\lambda \in (\alpha, \beta]$ the function y_λ is an extension of $g(\,\cdot\,;\lambda)$. We consider the zero $a_\lambda < x_\lambda$ of y_λ such that there are exactly $n-1$ zeros in the interval (a_λ, x_λ). It depends continuously on λ. If y_λ does not have n zeros in $(-\infty, x_\lambda)$, we set $a_\lambda = -\infty$. If $\lambda < \widehat{\lambda}$ and $a_\lambda \geq a_{\widehat{\lambda}}$, then the function y_λ would have as many zeros in (a_λ, x_λ) as $y_{\widehat{\lambda}}$ in $(a_{\widehat{\lambda}}, x_{\widehat{\lambda}})$ in contradiction to the comparison theorem. Hence a_λ is increasing in λ. For $\lambda = \alpha$ we have $a_\lambda < 0$ and for $\lambda \to \infty$ the number a_λ tends to 1. So there is a λ with $a_\lambda = 0$. ■

We call λ a *pseudo-eigenvalue* if there exists a solution of (1.1) which satisfies the boundary condition at $x = 0$ and vanishes at $x = x_\lambda$, but the boundary condition at $x = 1$ need not be fulfilled. Whether such a point λ is really an eigenvalue or not, depends on the function q. Theorem 2.3 and Corollaries 2.4, 2.5 remain valid if the λ_n are the pseudo-eigenvalues in the interval (α, ∞) since in the proofs only the interval $[0, x_\lambda]$ was considered. With increasing q these pseudo-eigenvalues also increase as is shown in the following theorem. For some q they may be eigenvalues, for $\lambda > \beta$ they are always eigenvalues. Pseudo-eigenvalues have the property that in contrast to eigenvalues they do not disappear, if q is changed slightly. This follows from the last theorem since the pseudo-eigenvalues can be identified by the number of zeros in the interval $(0, x_\lambda)$. We denote by λ_n^+ the pseudo-eigenvalue $> \alpha$ which has $n-1$ zeros in the interval $(0, x_{\lambda_n^+})$.

Theorem 2.8 *If $q(x) < \widehat{q}(x)$ almost everywhere, then for the corresponding pseudo-eigenvalues the inequality $\lambda_n^+ < \widehat{\lambda}_n^+$ holds.*

Proof: Assume that $\lambda_n^+ \geq \widehat{\lambda}_n^+$, then we have

$$\lambda_n^+ + \frac{q(x)}{u(x) - \lambda_n^+} > \widehat{\lambda}_n^+ + \frac{\widehat{q}(x)}{u(x) - \widehat{\lambda}_n^+}$$

for $x \in (0, x_{\widehat{\lambda}_n^+})$. But this implies that the solution of the differential equation corresponding to q and λ_n^+ has more zeros in this interval than the solution which corresponds to $\widehat{q}$ and $\widehat{\lambda}_n^+$, a contradiction. ■

Now we consider the case of an arbitrary $p \in L^\infty(0,1)$. Here we can estimate the eigenvalues by the eigenvalues μ_n: $\mu_1 < \mu_2 < \ldots$ of the operator A.

Theorem 2.9 *If the conditions* (a) – (d) *are satisfied, then the point spectrum of $\widetilde{A}$ in the interval* (α, ∞) *consists of a sequence of simple eigenvalues*

$\lambda_n\colon \lambda_1<\lambda_2<\ldots$ *with the only accumulation point infinity. Moreover,*

$$\lambda_n > \mu_n + \frac{\inf\limits_{x\in[0,1]} q(x)}{\lambda_n - \alpha} \ . \tag{2.10}$$

Proof: As in Theorem 2.3 we get that y_n, the eigenfunction corresponding to λ_n, has at least $n-1$ zeros in the interval $(0, x_{\lambda_n})$. If the inequality (2.10) does not hold, it follows

$$\lambda_n - p(x) + \frac{q(x)}{u(x)-\lambda_n} < \lambda_n - p(x) - \frac{\inf\limits_{x\in[0,1]} q(x)}{\lambda_n - \alpha} \leq \mu_n - p(x).$$

The comparison theorem yields that $\widetilde{y}_n$, an eigenfunction of A corresponding to the eigenvalue μ_n, has at least n zeros in the interval $(0,1)$. But $\widetilde{y}_n$ must have exactly $n-1$ zeros, which follows from similar considerations as in the proof of Theorem 2.7. ∎

2.3 The eigenvalue α

Now we will show that the point α plays a special role. Namely, α can be an eigenvalue for arbitrarily given p and u if q is chosen appropriately, even if $\beta < \mu_1$, where μ_1 is the least eigenvalue of $\widetilde{A}$ as in the last section.

Theorem 2.10 *If p and u satisfy the conditions* (a) – (c)*, there exists a positive, bounded function q such that α is an eigenvalue of $\widetilde{A}$. The function q can be chosen to have an arbitrarily large infimum.*

Proof: First we consider the case $u(x) = x$ and $p \equiv 0$. Then equation (1.1) with $\lambda = \alpha = 0$ becomes

$$y'' + \frac{q}{x} y = 0. \tag{2.11}$$

If we take q constant, one calculates that

$$y(x) = \sqrt{x} J_1(2\sqrt{qx}) \tag{2.12}$$

is a solution of (2.11), where J_1 is the Bessel function of order 1, i. e. it is a solution of the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2-1) J_1(x) = 0.$$

The boundary condition at $x=0$ is fulfilled. In order to satisfy the other boundary condition, q has to be chosen such that $J_1(2\sqrt{q}) = 0$, which means $q = \frac{c^2}{4}$, where c is a zero of J_1. Note that q can be taken arbitrarily large.

Next we consider arbitrary functions u and p. There exists a bounded function u_1 with positive infimum such that

$$u(x) - \alpha = x\, u_1(x).$$

Hence

$$\alpha - p(x) + \frac{q(x)}{u(x) - \alpha} = \frac{q(x) + \alpha x u_1(x) - x u_1(x) p(x)}{x u_1(x)} .$$

We set

$$\frac{q(x) + \alpha x u_1(x) - x u_1(x) p(x)}{u_1(x)} = \frac{c^2}{4},$$

where c is a zero of J_1. For q we get

$$q(x) = u_1(x) \left(\frac{c^2}{4} - \alpha x + x p(x) \right),$$

which has an arbitrarily large infimum if we take c appropriately. With this choice we get again the equation $y'' + \dfrac{c^2}{4x} y = 0$, where (2.12) is a solution. Thus α is an eigenvalue of $\widetilde{A}$. ■

2.4 Eigenvalues in the interval $(-\infty, \alpha]$

In this section we discuss the spectrum in the interval $(-\infty, \alpha]$. First we show that the eigenvalues cannot accumulate at α from below.

Theorem 2.11 *If the assumptions* (a) – (d) *are satisfied, then the operator* $\widetilde{A}$ *has an at most finite number of eigenvalues in the interval* $(-\infty, \alpha)$.

Proof: The function $\lambda - p(x) + \frac{q(x)}{u(x) - \lambda}$ is increasing in λ on the interval $(-\infty, \alpha]$ for fixed $x \in (0, 1)$. Let $\lambda < \widehat{\lambda} < \alpha$ be two eigenvalues and y, $\widehat{y}$ the corresponding eigenfunctions. As in Theorem 2.3 it follows by the comparison theorem that $\widehat{y}$ has at least as many zeros in $[0, 1]$ as y has.

The function $g(\,\cdot\,; \alpha)$ has a finite number of zeros in the interval $(0, 1)$. By the comparison theorem it follows that all eigenfunctions with eigenvalue $\lambda < \alpha$ have at most as many zeros as $g(\,\cdot\,; \alpha)$. But this implies that there is only a finite number of eigenvalues $< \alpha$. ■

It has to be mentioned that there can be an accumulation of eigenvalues at α from below if u is not monotonous. This case is considered in [MSS].

As in Theorem 2.8 it can be proved that the eigenvalues $\leq \alpha$ move to the left if q increases, because the denominator $u(x) - \lambda$ is positive in this case.

Next we show that there are eigenvalues in the interval $(-\infty, \alpha)$ if q is sufficiently large.

Theorem 2.12 *For given p and u, there exists a number C, $C \geq 0$, such that the relation $\inf\limits_{x\in[0,1]} q(x) > C$ implies that $\widetilde{A}$ has at least one eigenvalue in $(-\infty, \alpha)$.*

Proof: With $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{D}(\widetilde{A})$, $\|\boldsymbol{y}\| = 1$, we have

$$\begin{aligned}(\widetilde{A}\boldsymbol{y}, \boldsymbol{y}) &= (Ay_1, y_1) + (vy_2, y_1) + (vy_1, y_2) + (uy_2, y_2) \\ &= \int_0^1 |y_1'|^2 dx + \int_0^1 p|y_1|^2 dx + \int_0^1 vy_2\overline{y_1} dx + \int_0^1 vy_1\overline{y_2} dx + \int_0^1 u|y_2|^2 dx.\end{aligned}$$

Setting

$$y_1(x) := \sin \pi x, \quad y_2(x) := -\sin \pi x,$$

we get

$$\begin{aligned}(\widetilde{A}\boldsymbol{y}, \boldsymbol{y}) &= \pi^2 \int_0^1 \cos^2 \pi x \, dx + \int_0^1 \Big(p(x) + u(x) - 2v(x)\Big) \sin^2 \pi x \, dx \\ &\leq \frac{\pi^2}{2} + \frac{1}{2} \sup_{x\in[0,1]} \Big(p(x) + u(x) - 2v(x)\Big) \\ &\leq \frac{1}{2}\Big(\pi^2 + \sup_{x\in[0,1]} p(x) + \beta\Big) - \inf_{x\in[0,1]} v(x) \; .\end{aligned}$$

If

$$\inf_{x\in[0,1]} v(x) > \frac{1}{2}\Big(\pi^2 + \sup_{x\in[0,1]} p(x) + \beta\Big) - \alpha,$$

then $(\widetilde{A}\boldsymbol{y}, \boldsymbol{y}) < \alpha$ and therefore $\min \sigma(\widetilde{A}) < \alpha$. ∎

The next theorem gives a condition for the non-existence of eigenvalues in the interval $(-\infty, \alpha]$. A part of the proof is taken from Proposition 2 in [AdLM]. Denote by $(f(x))_+$ the positive part of $f(x)$:

$$(f(x))_+ := \begin{cases} f(x) & \text{for } f(x) \geq 0, \\ 0 & \text{for } f(x) < 0. \end{cases}$$

Theorem 2.13 *If the conditions* (a) – (d) *are satisfied and*

$$\int_0^1 \left(\alpha - p(x) + \frac{q(x)}{u(x) - \alpha}\right)_+ x\,(1 - x)\, dx < 1, \tag{2.13}$$

then $\widetilde{A}$ has no eigenvalue in the interval $(-\infty, \alpha]$.

Proof: (i) First we consider $\lambda < \alpha$. Let A_0 be the operator defined by

$$A_0 y := -y''$$

with domain

$$\mathcal{D}(A_0) := \{y \in L^2 :\ y, y' \in \mathrm{AC},\ -y'' \in L^2,\ y(0)=y(1)=0\}.$$

The operator A_0^{-1} is a positive trace class operator, which can be written as integral operator with the kernel

$$K_0(x,\xi) := \begin{cases} x\,(1-\xi)\,, & 0 \le x \le \xi \le 1, \\ (1-x)\,\xi\,, & 0 \le \xi \le x \le 1. \end{cases} \tag{2.14}$$

Further let $R(\lambda)$ be the operator of multiplication by the function $r(\,\cdot\,;\lambda)$ in L^2, where $r(x;\lambda)$ is defined in (1.20). Then λ, $\lambda < \alpha$, is an eigenvalue of $\widetilde{A}$ if and only if 1 is an eigenvalue of the self-adjoint trace class operator

$$T(\lambda) := A_0^{-\frac{1}{2}} R(\lambda) A_0^{-\frac{1}{2}}.$$

In an analogous way let $R_+(\lambda)$ be the operator of multiplication by the function $r_+(\,\cdot\,;\lambda) := (r(\,\cdot\,;\lambda))_+$ and

$$T_+(\lambda) := A_0^{-\frac{1}{2}} R_+(\lambda) A_0^{-\frac{1}{2}}, \tag{2.15}$$

which is a positive trace class operator. Since $r(x;\lambda) \le r_+(x;\lambda)$, we have

$$T(\lambda) \le T_+(\lambda) \quad (-\infty < \lambda < \alpha).$$

We can estimate the norm of $T_+(\lambda)$ as follows (see e. g. [GGK], Corollary VIII.6.2 and Theorem VII.2.3):

$$\begin{aligned} \|T_+(\lambda)\| &\le \operatorname{tr}(T_+(\lambda)) = \operatorname{tr}(A_0^{-1} R_+(\lambda)) \\ &= \int_0^1 x(1-x) r_+(x;\lambda)dx \le \int_0^1 x(1-x) r_+(x;\alpha)dx =: c < 1. \end{aligned}$$

Therefore we have $\sigma(T_+(\lambda)) \subseteq [0,c]$, and $\sigma(T(\lambda)) \subseteq (-\infty, c]$. It follows that 1 is not an eigenvalue of $T(\lambda)$ and the assertion is proved for $\lambda \in (-\infty, \alpha)$.

(ii) It remains to consider the case $\lambda = \alpha$. Instead of $T(\lambda)$ we consider the operator

$$V(\lambda) := R(\lambda) A_0^{-1}$$

for $\lambda \leq \alpha$. The operator $R(\alpha)$ is an unbounded operator with domain

$$\begin{aligned} \mathcal{D}(R(\alpha)) &= \left\{ y \in L^2 \middle| \, y(x) r(x;\alpha) \in L^2 \right\} \\ &= \left\{ y \in L^2 \middle| \, \frac{y(x)}{x} \in L^2 \right\}. \end{aligned}$$

Since $\mathcal{D}(A_0) \subseteq \mathcal{D}(R(\alpha))$, $V(\alpha) = R(\alpha) A_0^{-1}$ is an everywhere defined closed operator and therefore bounded. We will show that $V(\lambda)$ converges to $V(\alpha)$ in norm for $\lambda \nearrow \alpha$. Take $y \in L^2$. Since $A_0^{-1} y \in \mathcal{D}(A_0)$, we can write

$$(A_0^{-1} y)(x) = x\, y_1(x) \quad \text{with} \quad y_1 \in \mathrm{C}[0,1].$$

Observe that $\left|\frac{K_0(x,\xi)}{x}\right| \leq 1$, where K_0 is the kernel (2.14) of the operator A_0^{-1}. So we can estimate

$$|y_1(x)| \leq \int_0^1 \left| \frac{K_0(x,\xi)}{x} y(\xi) \right| d\xi \leq \|y\|.$$

Thus

$$\begin{aligned} \|V(\lambda)\, y - V(\alpha)\, y\|^2 &= \|R(\lambda) A_0^{-1}\, y - R(\alpha) A_0^{-1}\, y\|^2 \\ &= \|R(\lambda)(x\, y_1(x)) - R(\alpha)(x\, y_1(x))\|^2 \\ &= \int_0^1 \left| \Big(r(x;\lambda) - r(x;\alpha) \Big) x\, y_1(x) \right|^2 dx \\ &\leq \int_0^1 |r(x;\lambda)\, x - r(x;\alpha)\, x|^2 dx \cdot \|y\|^2 . \end{aligned}$$

If $\lambda \nearrow \alpha$, the function $r(x;\lambda)x$ converges pointwise and monotonously towards $r(x;\alpha)x$. By the dominated convergence theorem we have

$$\int_0^1 |r(x;\lambda)\, x - r(x;\alpha)\, x|^2 dx \to 0 \quad \text{for} \quad \lambda \nearrow \alpha$$

and therefore $V(\lambda) \to V(\alpha)$ in norm. By part (i) it follows that

$$\sigma(V(\lambda)) \setminus \{0\} = \sigma(T(\lambda)) \setminus \{0\} \subseteq (-\infty, c],$$

for $\lambda < \alpha$. But the convergence of $V(\lambda)$ towards $V(\alpha)$ implies that 1 is not an eigenvalue of $V(\alpha)$. Therefore α is not an eigenvalue of $\widetilde{A}$. ∎

By a simple estimation we can get a more convenient condition than (2.13).

Corollary 2.14 *If the conditions* (a) – (d) *are satisfied and*

$$\frac{1}{3}\sup_{x\in[0,1]}\Big(\alpha-p(x)\Big)_+ + \frac{1}{d}\sup_{x\in[0,1]} q(x) < 2,$$

then $\widetilde{A}$ *has no eigenvalue in the interval* $(-\infty,\alpha]$.

Proof: Because of assumption (c) we have

$$\frac{q(x)\,x}{u(x)-\alpha} \le \frac{\sup\limits_{t\in[0,1]} q(t)}{d}.$$

Now we can prove that the condition of Theorem 2.13 is satisfied:

$$\begin{aligned}
&\int_0^1 \Big(\alpha-p(x)+\frac{q(x)}{u(x)-\alpha}\Big)_+ x\,(1-x)\,dx \\
&\le \int_0^1 \sup_{t\in[0,1]}\Big(\alpha-p(t)\Big)_+ x\,(1-x)\,dx + \int_0^1 \frac{q(x)\,x}{u(x)-\alpha}(1-x)\,dx \\
&\le \frac{\sup\limits_{t\in[0,1]}\Big(\alpha-p(t)\Big)_+}{6} + \frac{\sup\limits_{t\in[0,1]} q(t)}{d}\int_0^1 (1-x)\,dx \\
&= \frac{\sup\limits_{t\in[0,1]}\Big(\alpha-p(t)\Big)_+}{6} + \frac{1}{2}\cdot\frac{\sup\limits_{t\in[0,1]} q(t)}{d} \;<\; 1.
\end{aligned}$$

■

If the norm of the operator $A_0^{-1}R_+(\lambda)$ is estimated by the Hilbert-Schmidt norm instead of the trace, the condition in the last corollary can be weakened.

Theorem 2.15 *If the conditions* (a) – (d) *are satisfied,* $\alpha \ge p(x)$ *and*

$$\frac{1}{3}\sup_{x\in[0,1]}\Big(\alpha-p(x)\Big) + \frac{1}{d}\sup_{x\in[0,1]} q(x) < 3,$$

then $\widetilde{A}$ *has no eigenvalue in the interval* $(-\infty,\alpha]$.

Proof: The operator $A_0^{-1}R_+(\lambda)$, $\lambda<\alpha$, is an integral operator with the kernel

$$K_1(x,\xi) := \begin{cases} x\,(1-\xi)\,r(\xi;\lambda)\,, & 0\le x\le\xi\le 1,\\ (1-x)\,\xi\,r(\xi;\lambda)\,, & 0\le\xi\le x\le 1.\end{cases}$$

This is a Hilbert-Schmidt operator. Hence we have the inequality

$$\|A_0^{-1}R_+(\lambda)\| \leq \|A_0^{-1}R_+(\lambda)\|_2,$$

with $\|\cdot\|_2$ denoting the Hilbert-Schmidt norm. The Hilbert-Schmidt norm of an integral operator can be easily computed:

$$\begin{aligned}
\|A_0^{-1}R_+(\lambda)\|_2^2 &= \int_0^1\int_0^1 |K_1(x,\xi)|^2 dx d\xi \\
&= \int_0^1\int_0^\xi (1-\xi)^2 x^2 \left(\lambda - p(\xi) + \frac{q(\xi)}{u(\xi)-\lambda}\right)^2 dx d\xi \\
&\quad + \int_0^1\int_\xi^1 (1-x)^2 \xi^2 \left(\lambda - p(\xi) + \frac{q(\xi)}{u(\xi)-\lambda}\right)^2 dx d\xi \\
&= \frac{1}{3}\int_0^1 \xi^3 (1-\xi)^2 \left(\lambda - p(\xi) + \frac{q(\xi)}{u(\xi)-\lambda}\right)^2 d\xi \\
&\quad + \frac{1}{3}\int_0^1 (1-\xi)^3 \xi^2 \left(\lambda - p(\xi) + \frac{q(\xi)}{u(\xi)-\lambda}\right)^2 d\xi \\
&\leq \frac{1}{3}\int_0^1 \left(\sup_{t\in[0,1]}\Big(\alpha - p(t)\Big)\xi + \frac{1}{d}\sup_{t\in[0,1]} q(t)\right)^2 \Big[\xi(1-\xi)^2 + (1-\xi)^3\Big] d\xi \\
&\text{with } c_1 := \sup_{t\in[0,1]}\Big(\alpha - p(t)\Big),\ c_2 := \frac{1}{d}\sup_{t\in[0,1]} q(t) \\
&= \frac{1}{3}\int_0^1 (1-\xi)^2 (c_1\xi + c_2)^2 d\xi \ = \ \frac{1}{9}\left(\frac{c_1^2}{10} + \frac{c_1 c_2}{2} + c_2^2\right) \\
&\leq \frac{1}{9}\left(\frac{c_1}{3} + c_2\right)^2 \ < \ 1.
\end{aligned}$$

Hence $\|A_0^{-1}R_+(\lambda)\| \leq c < 1$, which implies $\sigma(T_+(\lambda)) \subseteq [0,c]$, with $T_+(\lambda)$ defined in (2.15). The rest of the proof is the same as in Theorem 2.13. ■

It follows that there cannot be any eigenvalue $\leq \alpha$ if for example $\alpha < 9$, $p \equiv 0$ and q is chosen sufficiently small. If $\alpha > \pi^2$, there has to be such an eigenvalue, namely the perturbation of the smallest eigenvalue π^2 of the operator A. From the fact that 9 is quite near to $\pi^2 \approx 9,87$, we see that the condition in the last theorem is not too strong.

Chapter 3

The Spectral Density

In this chapter we will first compute the spectral density, i. e. the density of the measure σ, by means of the fundamental system of solutions. Then it is shown that the operator $\mathcal{F}$ is an isomorphism from the whole space $\widetilde{\mathcal{H}}$ onto L^2_σ. This isomorphism $\mathcal{F}$ is written as an integral transformation with the help of generalized eigenvectors for the continuous spectrum. Also another isomorphism into a space of analytic functions is constructed.

3.1 The spectral density of the Titchmarsh-Weyl coefficient

In the following, the solution $h(\,\cdot\,;z)$ from Theorem 1.9 is written in the form $h(\,\cdot\,;z) = g(\,\cdot\,;z)\tau(\,\cdot\,;z)$ and we express the Titchmarsh-Weyl coefficient of the problem (1.1), (1.2) by the functions g and τ. It is easy to see that for non-real z the functions $\psi(x;z)$ and $\varphi(x;z)$ from (1.9) can be represented as follows:

$$\begin{aligned}
\psi(x;z) &= g(0;z)g(x;z)(\tau(x;z)-\tau(0;z)),\\
\varphi(x;z) &= \frac{g(x;z)}{g(0;z)} - g'(0;z)g(x;z)(\tau(x;z)-\tau(0;z))\\
&= \frac{g(x;z)}{g(0;z)} - \frac{g'(0;z)}{g(0;z)}\psi(x;z).
\end{aligned}$$

Hence the Titchmarsh-Weyl coefficient becomes

$$\begin{aligned}
m(z) &= -\frac{\varphi(1;z)}{\psi(1;z)}\\
&= \frac{g'(0;z)}{g(0;z)} - \frac{1}{g(0;z)^2(\tau(1;z)-\tau(0;z))}\,.
\end{aligned}$$

Now it is not difficult to compute the density of the absolutely continuous component of the spectral measure σ, the measure in the representation (1.17) of the Titchmarsh-Weyl coefficient.

Theorem 3.1 *If the assumptions* (a) – (d) *are satisfied, then the measure* σ *in* (1.17) *has no singular continuous component (with respect to the Lebesgue measure). Its absolutely continuous component* σ_{ac} *has the density*

$$\sigma'_{\mathrm{ac}}(\lambda) = \frac{\gamma(\lambda)}{g(0;\lambda)^2\Big((\Theta(1;\lambda)-\Theta(0;\lambda))^2+\pi^2\gamma(\lambda)^2\Big)} \tag{3.1}$$

almost everywhere; it vanishes on the set $\{\lambda : \, g(1;\lambda)=0\}$.

Proof: The Stieltjes inversion formula implies

$$\sigma(\Gamma) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_\Gamma \operatorname{Im} m(\lambda+i\varepsilon)d\lambda \tag{3.2}$$

for each interval Γ with endpoints having σ–measure 0. It follows from Corollary 2.5, that there is only a finite number of zeros of $g(0;\lambda)$, which are exactly the pseudo-eigenvalues mentioned after Theorem 2.7. Let $\Gamma\subset(\alpha,\beta)$ be a closed interval with $g(0;\lambda)\neq 0$ for $\lambda\in\Gamma$. We show that σ is absolutely continuous on Γ.

Since u is Lipschitz continuous and $q\geq q_0>0$, there exists a positive constant c such that $\gamma(\lambda)\geq c$. Choose $\eta>0$ small enough such that for all $\varepsilon\in(0,\eta)$ and $\lambda\in\Gamma$ the following estimates hold:

$$\begin{aligned}\Big|\operatorname{Im}\Big(\tau_1(1;\lambda+i\varepsilon)-\tau_1(0;\lambda+i\varepsilon)\Big)\Big| &\leq \frac{\pi c}{8},\\ \operatorname{Arctan}\frac{\beta-\lambda}{\varepsilon}+\operatorname{Arctan}\frac{\lambda-\alpha}{\varepsilon} &\geq \frac{\pi}{4},\end{aligned}$$

where τ_1 is defined in (1.35). Then we obtain

$$\begin{aligned}&\Big|\operatorname{Im}\Big(\tau(1;\lambda+i\varepsilon)-\tau(0;\lambda+i\varepsilon)\Big)\Big|\\ &\geq \operatorname{Im}\Big(\tau_2(1;\lambda+i\varepsilon)-\tau_2(0;\lambda+i\varepsilon)\Big)-\Big|\operatorname{Im}\Big(\tau_1(1;\lambda+i\varepsilon)-\tau_1(0;\lambda+i\varepsilon)\Big)\Big|\\ &\geq \varepsilon\int_\alpha^\beta\frac{\gamma(\mu)}{(\mu-\lambda)^2+\varepsilon^2}d\mu-\frac{\pi c}{8}\\ &\geq c\Big(\operatorname{Arctan}\frac{\beta-\lambda}{\varepsilon}+\operatorname{Arctan}\frac{\lambda-\alpha}{\varepsilon}\Big)-\frac{\pi c}{8}\;\geq\;\frac{\pi c}{8}.\end{aligned}$$

Therefore $g(0;\lambda+i\varepsilon)^{-1}$, $g'(0;\lambda+i\varepsilon)$ and $(\tau(1;\lambda+i\varepsilon)-\tau(0;\lambda+i\varepsilon))^{-1}$ are bounded, which implies that also $m(\lambda+i\varepsilon)$ is bounded for $\lambda\in\Gamma$ and $\varepsilon\in$

$(0, \eta)$. By the dominated convergence theorem it follows that σ is absolutely continuous on Γ, and the density of σ is the pointwise limit of $\operatorname{Im} m(\lambda + i\varepsilon)$:

$$\begin{aligned}
\sigma'_{\mathrm{ac}}(\lambda) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} m(\lambda + i\varepsilon) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} \frac{-1}{g(0; \lambda + i\varepsilon)^2 (\tau(1; \lambda + i\varepsilon) - \tau(0; \lambda + i\varepsilon))} \\
&= \frac{1}{g(0;\lambda)^2 \pi} \operatorname{Im} \frac{-1}{\Theta(1;\lambda) - \Theta(0;\lambda) + i\pi\gamma(\lambda)} \\
&= \frac{\gamma(\lambda)}{g(0;\lambda)^2 \Big(\big(\Theta(1;\lambda) - \Theta(0;\lambda)\big)^2 + \pi^2 \gamma(\lambda)^2 \Big)} .
\end{aligned}$$

The density σ'_{ac} can vanish only if $\Theta(1; \lambda)$ or $\Theta(0; \lambda)$ is infinite. Since the term $g(0; \lambda)\Theta(0; \lambda)$ is always finite, σ'_{ac} vanishes only if $\Theta(1; \lambda)$ is infinite, which is equivalent to $g(1; \lambda) = 0$. ■

For the rest of the paper we assume that the following condition is fulfilled:

$$\text{(e)} \quad g(1; \lambda) \neq 0 \ \text{ a.e. on } \ [\alpha, \beta] \tag{3.3}$$

This implies that $\sigma'_{\mathrm{ac}}(\lambda) > 0$ almost everywhere. With $(f(x))_+$ denoting the positive part of $f(x)$ we have the following

Proposition 3.2 *Each of the two following conditions is sufficient for* (e)*:*

(i) *u is real analytic on* $[0, 1]$.

(ii) *The inequality*

$$\frac{1}{3}\Big(\beta - \inf_{x \in [0,1]} p(x)\Big)_+ + \frac{1}{d} \sup_{x \in [0,1]} q(x) < 2$$

holds.

Proof: (i) The equality $g(1; \lambda) = 0$ is equivalent to $\widehat{\chi}(u^{-1}(\lambda); \lambda) = 0$, where $\widehat{\chi}$ is a solution of the differential equation fulfilling the initial conditions $\widehat{\chi}(1; \lambda) = 0$ and $\widehat{\chi}'(1; \lambda) = 1$. But $\widehat{\chi}(u^{-1}(\lambda); \lambda)$ is real analytic in λ and has only a discrete set of zeros.

(ii) A generalization of Theorem 5.1 in [H] to differential equations with a singularity implies that if

$$\int_{x_\lambda}^{1} (x - x_\lambda)(1 - x)(r(x; \lambda))_+ dx \ < \ 1 - x_\lambda,$$

then $g(x;\lambda)$ has no zero in the interval $(x_\lambda, 1]$. By the assumption we can estimate

$$\begin{aligned}
&\int_{x_\lambda}^{1} \frac{(x-x_\lambda)(1-x)}{1-x_\lambda}(r(x;\lambda))_+ dx \\
&\quad\leq \int_{x_\lambda}^{1} \frac{(x-x_\lambda)(1-x)}{1-x_\lambda}\left((\beta-\inf p)_+ + \frac{\sup q}{d}\cdot\frac{1}{x-x_\lambda}\right) dx \\
&\quad= (\beta-\inf p)_+ \cdot \frac{(1-x_\lambda)^2}{6} + \frac{\sup q}{d}\cdot\frac{1-x_\lambda}{2} \\
&\quad\leq \frac{1}{6}(\beta-\inf p)_+ + \frac{1}{2}\cdot\frac{\sup q}{d} < 1.
\end{aligned}$$

■

As a corollary to Theorem 3.1 we get a connection between the absolutely continuous part of the operator $\widetilde{A}$ and the absolutely continuous part of the measure σ.

Corollary 3.3 *If the conditions* (a) – (e) *are satisfied, then the absolutely continuous part of the operator* $\widetilde{A}$ *is unitarily equivalent to the operator of multiplication by the independent variable in the space* $L^2_{\sigma_{\mathrm{ac}}}$.

Proof: According to Corollary 1.3, the absolutely continuous part of $\widetilde{A}$ is unitarily equivalent to the absolutely continuous part of the operator of multiplication by u in the space $L^2(0,1)$. The operator of multiplication by u in the space $L^2(0,1)$ is unitarily equivalent to the operator of multiplication by the independent variable in the space $L^2(\alpha,\beta)$ by the isometry

$$f(x) \mapsto f(u^{-1}(t))\sqrt{\rho(t)}$$

from $L^2(0,1)$ onto $L^2(\alpha,\beta)$ since $\rho(t) \geq c > 0$. This shows also that the operator of multiplication by u in $L^2(0,1)$ has purely absolutely continuous spectrum.

Further, the operators of multiplication by the independent variable in the spaces $L^2(\alpha,\beta)$ and $L^2_{\sigma'_{\mathrm{ac}}}(\alpha,\beta)$ are unitarily equivalent by the isometry

$$f(t) \mapsto \frac{f(t)}{\sqrt{\sigma'_{\mathrm{ac}}(t)}}$$

from $L^2(\alpha,\beta)$ onto $L^2_{\sigma'_{\mathrm{ac}}}(\alpha,\beta)$ since $\sigma'_{\mathrm{ac}} > 0$ almost everywhere. ■

In the following, we consider the components $\mathcal{H}_1, \mathcal{H}_2$ in the decomposition $\widetilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ also as subspaces of $\widetilde{\mathcal{H}}$. Recall that $\widetilde{\mathcal{H}}_0 = \text{c.l.s.}\{\mathbf{r}_z \colon z \in \mathbb{C}\setminus\mathbb{R}\}$.

Further, $\widetilde{\mathcal{H}}_{\rm ac}$ ($\widetilde{\mathcal{H}}_{\rm p}$, respectively) denotes the subspace of absolute continuity (the subspace spanned by all eigenvectors, respectively) of the operator $\widetilde{A}$.

Proposition 3.4 *If the conditions* (a) – (e) *are satisfied, then the following inclusions hold:*

$$\mathcal{H}_1 \subset \widetilde{\mathcal{H}}_{\rm ac} \oplus \widetilde{\mathcal{H}}_{\rm p} \subset \widetilde{\mathcal{H}}_0. \tag{3.4}$$

Proof: In order to prove the second inclusion observe that according to Theorem 1.6 the operator $\widetilde{A}|_{\widetilde{\mathcal{H}}_0}$ is unitarily equivalent to the operator of multiplication by the independent variable in L^2_σ. Therefore $\widetilde{A}|_{\widetilde{\mathcal{H}}_0\cap\widetilde{\mathcal{H}}_{\rm ac}}$ is unitarily equivalent to the operator of multiplication by the independent variable in $L^2_{\sigma_{\rm ac}}$. On the other hand, also $\widetilde{A}|_{\widetilde{\mathcal{H}}_{\rm ac}}$ is unitarily equivalent to this multiplication operator in $L^2_{\sigma_{\rm ac}}$ by Corollary 3.3. Since the spectra of the operators are simple, it follows that $\widetilde{\mathcal{H}}_0 \cap \widetilde{\mathcal{H}}_{\rm ac} = \widetilde{\mathcal{H}}_{\rm ac}$, hence $\widetilde{\mathcal{H}}_{\rm ac} \subset \widetilde{\mathcal{H}}_0$. The operator of multiplication by the independent variable in L^2_σ has the same eigenvalues as $\widetilde{A}$, therefore these eigenvectors belong also to $\widetilde{\mathcal{H}}_0$, hence $\widetilde{\mathcal{H}}_{\rm p} \subset \widetilde{\mathcal{H}}_0$, and the second inclusion of (3.4) is proved.

In order to prove the first inclusion, we have to show that for $\boldsymbol{f} = \binom{f}{0}$, f in a dense subspace of $\mathcal{H}_1$, the measure $(\widetilde{E}(\cdot)\boldsymbol{f},\boldsymbol{f})$ has no singular continuous component with respect to the Lebesgue measure, where $\widetilde{E}$ is the spectral family of $\widetilde{A}$. Let $\Gamma \subset (\alpha,\beta)$ be a closed interval where $g(0;\lambda) \neq 0$ as in Theorem 3.1. This implies that there is no eigenvalue in Γ. Then the Stieltjes inversion formula yields

$$(E(\Gamma)\boldsymbol{f},\boldsymbol{f}) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_\Gamma \mathrm{Im}\left((\widetilde{A}-\lambda-i\varepsilon)^{-1}\boldsymbol{f},\boldsymbol{f}\right)d\lambda.$$

It suffices to show that for some $s > 1$

$$\int_\Gamma \left|\left((\widetilde{A}-\lambda-i\varepsilon)^{-1}\boldsymbol{f},\boldsymbol{f}\right)\right|^s d\lambda$$

is bounded for $\varepsilon \downarrow 0$ (cf. [RS], Theorem XIII.19). We express the resolvent by means of the Green's function (1.11):

$$\begin{aligned}\left((\widetilde{A}-z)^{-1}\boldsymbol{f},\boldsymbol{f}\right) &= \left(-L(z)^{-1}f,f\right) = -\int_0^1\int_0^1 G(x,\xi;z)f(\xi)d\xi\,\overline{f(x)}\,dx \\ &= \int_0^1\left(\varphi(x;z)\overline{f(x)}\int_0^x \psi(\xi;z)f(\xi)d\xi + \psi(x;z)\overline{f(x)}\int_x^1 \varphi(\xi;z)f(\xi)d\xi\right)dx \\ &\quad + m(z)\int_0^1 \psi(\xi;z)f(\xi)d\xi\cdot\int_0^1 \psi(x;z)\overline{f(x)}dx. \end{aligned} \tag{3.5}$$

According to Theorem 3.1 the function $m(z)$ is bounded for $z = \lambda + i\varepsilon$, $\lambda \in \Gamma$ and $\varepsilon \in (0, \eta)$ for some $\eta > 0$. Take $f \in \mathrm{C}[0,1]$ and consider the integral

$$\int_0^1 \psi(x; z) f(x) dx.$$

Since g and τ_1 (see Theorem 1.9) are bounded, we have only to study the integral

$$\begin{aligned}
\int_0^1 g(x;z)\tau_2(x;z)f(x)dx &= \int_0^1 g(x;z)f(x)\int_\alpha^{u(x)} \frac{\gamma(\mu)}{\mu - z} d\mu\, dx \\
&= \int_\alpha^\beta \frac{\gamma(\mu)}{\mu - z} \int_{x_\mu}^1 g(x;z)f(x)dx\, d\mu \\
&= \int_\alpha^\beta \frac{\gamma(\mu)}{\mu - z} \int_{x_\mu}^{x_\lambda} g(x;z)f(x)dx\, d\mu + \int_\alpha^\beta \frac{\gamma(\mu)}{\mu - z} d\mu \cdot \int_{x_\lambda}^1 g(x;z)f(x)dx
\end{aligned}$$

with $\lambda = \operatorname{Re} z$. The first integral is bounded since

$$\left| \int_{x_\mu}^{x_\lambda} g(x;z)f(x)dx \right| \le C|\lambda - \mu|.$$

It follows from [RR], Theorem 5.32, that for $s > 1$

$$\sup_{\varepsilon \in (0,1]} \int_\Gamma \left| \int_\alpha^\beta \frac{\gamma(\mu)}{\mu - \lambda - i\varepsilon} d\mu \right|^s d\lambda < \infty.$$

Therefore we have for $s > \frac{1}{2}$

$$\sup_{\varepsilon \in (0,1]} \int_\Gamma \left| m(\lambda + i\varepsilon) \int_0^1 \psi(\xi; \lambda + i\varepsilon) f(\xi) d\xi \cdot \int_0^1 \psi(x; \lambda + i\varepsilon)\overline{f(x)} dx \right|^s d\lambda < \infty.$$

In a similar way it is shown that the integral of the s-th power of the first summand in (3.5) is bounded. So the spectrum on Γ is purely absolutely continuous and the first inclusion of (3.4) is proved. ■

Under the stronger conditions about the functions p, q and u we get now an improvement of Theorem 1.6.

Theorem 3.5 *If the conditions* (a) – (e) (*see* (1.19), (3.3)) *are satisfied, then the mapping* $\mathcal{F}$ *is an isometry from* $\widetilde{\mathcal{H}}$ *onto* L^2_σ. *Under this isometry the operator* $\widetilde{A}$ *is unitarily equivalent to the operator of multiplication by the independent variable in* L^2_σ*; it has no singular continuous spectrum.*

Proof: We will show in fact that $\widetilde{\mathcal{H}}_0 = \widetilde{\mathcal{H}}$. The inclusions $\mathcal{H}_1 \subset \widetilde{\mathcal{H}}_{\rm ac} \oplus \widetilde{\mathcal{H}}_{\rm p} \subset \widetilde{\mathcal{H}}_0$ imply that $\widetilde{\mathcal{H}}_0^\perp \subset \mathcal{H}_2$. The space $\widetilde{\mathcal{H}}_0^\perp$ is $\widetilde{A}$-invariant, and since $q \neq 0$ there is no non-trivial $\widetilde{A}$-invariant subspace in $\mathcal{H}_2$. Hence, $\widetilde{\mathcal{H}}_0^\perp$ is trivial. Finally, the last statement follows from the fact that the measure σ has no singular continuous component. ∎

Sufficient conditions for (e) were given in Proposition (3.2).

3.2 The integral transformation

Assume that the conditions (a) – (e) in (1.19), (3.3) are fulfilled. In this section we shall find an explicit expression for the isomorphism $\mathcal{F}$ as an integral transformation. To this end we introduce the 'generalized eigenvectors' for the continuous spectrum. If $\lambda \in [\alpha, \beta]$, let $\widehat{\psi}(\,\cdot\,;\lambda)$ be the solution of the differential equation (1.1) for $x \neq x_\lambda$ which is continuous at $x = x_\lambda$ and fulfills the boundary conditions

$$\widehat{\psi}(0;\lambda) = 0\,, \qquad \widehat{\psi}'(0;\lambda) = 1\,, \qquad \widehat{\psi}(1;\lambda) = 0. \tag{3.6}$$

We can prove the existence and uniqueness of $\widehat{\psi}$ except for the case that $g(0;\lambda) \neq 0$ and $g(1;\lambda) = 0$, but the set of such λ's has Lebesgue measure 0. To prove the existence and uniqueness of $\widehat{\psi}$ we write $\widehat{\psi}$ as a linear combination of g and $g\Theta$:

$$\widehat{\psi}(x;\lambda) = \begin{cases} a_1 g(x;\lambda) + b\, g(x;\lambda)\Theta(x;\lambda) & \text{if } x < x_\lambda, \\ a_2 g(x;\lambda) + b\, g(x;\lambda)\Theta(x;\lambda) & \text{if } x > x_\lambda \end{cases}$$

with a_1, a_2, b depending on λ. The coefficient b at $g\Theta$ is the same in both intervals since $\widehat{\psi}$ shall be continuous at x_λ. If $g(0;\lambda) \neq 0$ and $g(1;\lambda) \neq 0$, we fix some $b \neq 0$ and then choose a_1 and a_2 to satisfy $\widehat{\psi}(0;\lambda) = 0$ and $\widehat{\psi}(1;\lambda) = 0$. Multiplying $\widehat{\psi}$ with some constant also $\widehat{\psi}'(0;\lambda) = 1$ can be fulfilled. If $g(0;\lambda) = 0$, we choose $b = 0$.

In general the function $\widehat{\psi}$ does not have a continuous derivative at $x = x_\lambda$. So let us define the following limit:

$$\Delta\widehat{\psi}'(x_\lambda;\lambda) := \lim_{\varepsilon\downarrow 0} \Big(\widehat{\psi}'(x_\lambda+\varepsilon;\lambda) - \widehat{\psi}'(x_\lambda-\varepsilon;\lambda)\Big). \tag{3.7}$$

This limit exists for almost every λ since we have

$$(g(x;\lambda)\Theta(x;\lambda))'\Big|_{x_\lambda-\varepsilon}^{x_\lambda+\varepsilon} = (g(x;\lambda)\Theta(x;\lambda))'\Big|_0^1 + \int_0^{x_\lambda-\varepsilon} r(t;\lambda)g(t;\lambda)\Theta(t;\lambda)dt + \int_{x_\lambda+\varepsilon}^1 r(t;\lambda)g(t;\lambda)\Theta(t;\lambda)dt$$

and the asymptotic formula (1.34) for $g\Theta$, and because the principal value in (1.37) exists almost everywhere.

If λ is an eigenvalue, then $\widehat{\psi}$ is a corresponding eigenfunction and we can set $\psi(\,\cdot\,;\lambda) := \widehat{\psi}(\,\cdot\,;\lambda)$ for such a λ. If λ is not an eigenvalue, then $\widehat{\psi}'$ is not continuous at $x = x_\lambda$, therefore it does not belong to the domain of A.

The function $\widehat{\psi}$ is used as the first component of the generalized eigenvector. Inserting it into the eigenvalue equation, by a formal calculation we find as a candidate for the generalized eigenvector

$$\mathbf{v}_\lambda := \begin{pmatrix} \widehat{\psi}(\,\cdot\,;\lambda) \\ -\dfrac{v}{u-\lambda}\widehat{\psi}(\,\cdot\,;\lambda) + \dfrac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{v(x_\lambda)}\delta_{x_\lambda} \end{pmatrix} \tag{3.8}$$

where δ_{x_λ} is the Dirac delta distribution at x_λ. Now the isomorphism $\mathcal{F}$ becomes $\mathcal{F}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\lambda) = \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \mathbf{v}_\lambda\right)$. This is made precise in the following

Theorem 3.6 *Suppose that the assumptions* (a) – (e) *are satisfied. Then the isomorphism $\mathcal{F}$ is given by the formula*

$$\begin{aligned} \mathcal{F}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\lambda) &= \int_0^1 \widehat{\psi}(x;\lambda)f_1(x)dx - \text{p.v.}\int_0^1 \frac{v(x)}{u(x)-\lambda}\widehat{\psi}(x;\lambda)f_2(x)dx \\ &\quad + \frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{v(x_\lambda)}f_2(x_\lambda), \end{aligned} \tag{3.9}$$

for σ-almost every $\lambda \in \sigma(\widetilde{A})$, $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \operatorname{span}\{\mathbf{r}_z \,|\, z \in \mathbb{C}\backslash\mathbb{R}\}$, where p.v. *denotes the principal value at $x = x_\lambda$.*

If $\lambda \in \sigma_{\mathrm{p}}(\widetilde{A})$ the last term in (3.9) vanishes, $\widehat{\psi}(x;\lambda) = \psi(x;\lambda)$, and we get

$$\mathcal{F}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\lambda) = \int_0^1 \psi(x;\lambda)f_1(x)dx - \int_0^1 \frac{v(x)}{u(x)-\lambda}\psi(x;\lambda)f_2(x)dx, \quad \lambda \in \sigma_{\mathrm{p}}(\widetilde{A}).$$

Proof: It is sufficient to prove that $\big(\mathcal{F}\mathbf{r}_z\big)(\lambda) = \dfrac{1}{\lambda - z}$. This relation is obtained for $\lambda \in [\alpha, \beta]$ and with $I_\varepsilon := [0,1]\backslash(x_\lambda - \varepsilon, x_\lambda + \varepsilon)$ as follows:

$$
\begin{aligned}
\big(\mathcal{F}\mathbf{r}_z\big)(\lambda) &= \mathcal{F}\begin{pmatrix} \chi(\,\cdot\,; z) \\ -\dfrac{v}{u-z}\,\chi(\,\cdot\,; z) \end{pmatrix}(\lambda) \\
&= \int_0^1 \widehat{\psi}(x;\lambda)\,\chi(x;z)\,dx + \lim_{\varepsilon\downarrow 0}\int_{I_\varepsilon} \frac{-v(x)}{u(x)-\lambda}\,\widehat{\psi}(x;\lambda)\,\frac{-v(x)}{u(x)-z}\,\chi(x;z)\,dx \\
&\quad - \frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{v(x_\lambda)}\,\frac{v(x_\lambda)}{u(x_\lambda)-z}\,\chi(x_\lambda;z) \\
&= \int_0^1 \widehat{\psi}(x;\lambda)\,\chi(x;z)\,dx - \frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{\lambda - z}\,\chi(x_\lambda;z) \\
&\quad + \lim_{\varepsilon\downarrow 0}\int_{I_\varepsilon} \frac{1}{\lambda - z}\left(\frac{q(x)}{u(x)-\lambda} - \frac{q(x)}{u(x)-z}\right)\widehat{\psi}(x;\lambda)\,\chi(x;z)\,dx \\
&= \int_0^1 \widehat{\psi}(x;\lambda)\,\chi(x;z)\,dx - \frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{\lambda - z}\,\chi(x_\lambda;z) \\
&\quad + \lim_{\varepsilon\downarrow 0}\int_{I_\varepsilon} \frac{1}{\lambda - z}\Big(\Big(-\widehat{\psi}''(x;\lambda) - \lambda\widehat{\psi}(x;\lambda) + p(x)\widehat{\psi}(x;\lambda)\Big)\,\chi(x;z) \\
&\qquad\qquad + \Big(\chi''(x;z) + z\chi(x;z) - p(x)\chi(x;z)\Big)\,\widehat{\psi}(x;\lambda)\Big)\,dx \\
&= \frac{1}{\lambda - z}\,\lim_{\varepsilon\downarrow 0}\int_{I_\varepsilon}\Big(\widehat{\psi}(x;\lambda)\,\chi''(x;z) - \widehat{\psi}''(x;\lambda)\,\chi(x;z)\Big)\,dx \\
&\quad - \frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{\lambda - z}\,\chi(x_\lambda;z) \\
&= \frac{1}{\lambda - z}\,\lim_{\varepsilon\downarrow 0}\Big(\widehat{\psi}(x;\lambda)\,\chi'(x;z) - \widehat{\psi}'(x;\lambda)\,\chi(x;z)\Big)\left(\Big|_0^{x_\lambda-\varepsilon} + \Big|_{x_\lambda+\varepsilon}^1\right) \\
&\quad - \lim_{\varepsilon\downarrow 0}\frac{\widehat{\psi}'(x_\lambda+\varepsilon;\lambda) - \widehat{\psi}'(x_\lambda-\varepsilon;\lambda)}{\lambda - z}\,\chi(x_\lambda;z) \\
&= \frac{1}{\lambda - z}\,\widehat{\psi}'(0;\lambda)\,\chi(0;z) \;=\; \frac{1}{\lambda - z}.
\end{aligned}
$$

If $\lambda \notin [\alpha, \beta]$, the calculation is the same but no special consideration concerning the point x_λ is needed. ∎

Theorem 3.7 *Suppose that the conditions* (a) – (e) *are satisfied. Then for* $x \in [0,1]$ *the components of the inverse mapping* $\mathcal{F}^{-1}$ *are given by the formulas*

$$\left(\mathcal{F}^{-1}F\right)_1(x) = \int\limits_{\sigma(\widetilde{A})} \widehat{\psi}(x;\lambda)F(\lambda)d\sigma(\lambda), \tag{3.10}$$

$$\begin{aligned}\left(\mathcal{F}^{-1}F\right)_2(x) &= \text{p.v.}\int\limits_{\sigma(\widetilde{A})} \frac{-v(x)}{u(x)-\lambda}\widehat{\psi}(x;\lambda)F(\lambda)d\sigma(\lambda) \\ &\quad + \frac{\Delta\widehat{\psi}'(x;u(x))}{v(x)}\frac{\sigma'_{\rm ac}(u(x))}{\rho(u(x))}F(u(x)),\end{aligned} \tag{3.11}$$

where ρ *is defined in Lemma* 1.8.

Proof: With $\mathcal{F}\boldsymbol{f} = F$, $\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $\mathcal{F}\boldsymbol{g} = G$, $\boldsymbol{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $\boldsymbol{f},\boldsymbol{g} \in \operatorname{span}\{\mathbf{r}_z \,|\, z \in \mathbb{C}\backslash\mathbb{R}\}$, we find

$$\begin{aligned}(\boldsymbol{f},\boldsymbol{g}) &= (F,G)_{L^2_\sigma} = \int\limits_{\sigma(\widetilde{A})} F(\lambda)\overline{G(\lambda)}\,d\sigma(\lambda) \\ &= \int\limits_{\sigma(\widetilde{A})} F(\lambda)\left[\int\limits_0^1 \widehat{\psi}(x;\lambda)\,\overline{g_1(x)}\,dx - \text{p.v.}\int\limits_0^1 \frac{v(x)}{u(x)-\lambda}\,\widehat{\psi}(x;\lambda)\,\overline{g_2(x)}\,dx \right. \\ &\qquad \left. + \frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{v(x_\lambda)}\,\overline{g_2(x_\lambda)}\right] d\sigma(\lambda) \\ &= \int\limits_0^1 \int\limits_{\sigma(\widetilde{A})} F(\lambda)\,\widehat{\psi}(x;\lambda)\,d\sigma(\lambda)\,\overline{g_1(x)}\,dx \\ &\qquad + \int\limits_0^1 \text{p.v.}\int\limits_{\sigma(\widetilde{A})} F(\lambda)\,\frac{-v(x)}{u(x)-\lambda}\,\widehat{\psi}(x;\lambda)\,d\sigma(\lambda)\,\overline{g_2(x)}\,dx \\ &\qquad + \int\limits_{\sigma(\widetilde{A})} F(\lambda)\,\frac{\Delta\widehat{\psi}'(x_\lambda;\lambda)}{v(x_\lambda)}\overline{g_2(x_\lambda)}\,d\sigma(\lambda).\end{aligned}$$

The last integral equals

$$\begin{aligned}&\int\limits_\alpha^\beta F(\lambda)\,\frac{\Delta\widehat{\psi}'(u^{-1}(\lambda);\lambda)}{v(u^{-1}(\lambda))}\,\overline{g_2(u^{-1}(\lambda))}\,\sigma'_{\rm ac}(\lambda)\,d\lambda \\ &\qquad = \int\limits_0^1 F(u(x))\,\frac{\Delta\widehat{\psi}'(x;u(x))}{v(x)}\,\frac{\sigma'_{\rm ac}(u(x))}{\rho(u(x))}\,\overline{g_2(x)}\,dx,\end{aligned}$$

and we get finally

$$\int_0^1 \left(f_1(x) - \int_{\sigma(\widetilde{A})} F(\lambda)\,\widehat{\psi}(x;\lambda)\,d\sigma(\lambda) \right) \overline{g_1(x)} dx$$

$$+ \int_0^1 \left(f_2(x) - \text{p.v.} \int_{\sigma(\widetilde{A})} F(\lambda)\,\frac{-v(x)}{u(x)-\lambda}\,\widehat{\psi}(x;\lambda)\,d\sigma(\lambda) \right.$$

$$\left. -F(u(x))\,\frac{\Delta\widehat{\psi}'(x;u(x))}{v(x)}\,\frac{\sigma'_{\text{ac}}(u(x))}{\rho(u(x))} \right) \overline{g_2(x)}\,dx = 0.$$

Since the functions g_1 and g_2 are arbitrary, the claim follows. ■

If the integral over $\sigma(\widetilde{A})$ is split into the integrals over the eigenvalues and the continuous spectrum, with $\sigma_j = \sigma(\{\lambda_j\})$ we get the following formulas:

$$\begin{aligned}
\left(\mathcal{F}^{-1}F\right)_1(x) &= \sum_{\lambda_j \in \sigma_{\text{p}}(\widetilde{A})} \psi(x;\lambda_j)\,F(\lambda_j)\,\sigma_j + \int_\alpha^\beta \widehat{\psi}(x;\lambda)\,F(\lambda)\,\sigma'_{\text{ac}}(\lambda)\,d\lambda, \\
\left(\mathcal{F}^{-1}F\right)_2(x) &= \sum_{\lambda_j \in \sigma_{\text{p}}(\widetilde{A})} \frac{-v(x)}{u(x)-\lambda_j}\,\psi(x;\lambda_j)\,F(\lambda_j)\,\sigma_j \\
&\quad + \text{p.v.}\int_\alpha^\beta \frac{-v(x)}{u(x)-\lambda}\,\widehat{\psi}(x;\lambda)\,F(\lambda)\,\sigma'_{\text{ac}}(\lambda)\,d\lambda \\
&\quad + \frac{\Delta\widehat{\psi}'(x;u(x))}{v(x)}\,\frac{\sigma'_{\text{ac}}(u(x))}{\rho(u(x))}\,F(u(x)).
\end{aligned}$$

If the formulas of Theorems 3.6 and 3.7 are applied to vectors of the form $\begin{pmatrix} f \\ 0 \end{pmatrix} \in \widetilde{\mathcal{H}}$, one gets the spectral decomposition which was given in [B]. But this is not unique since the first components of the eigenvectors corresponding to eigenvalues $> \beta$ build a Riesz basis in $\mathcal{H}_1$ if the spectrum of A lies right of $[\alpha, \beta]$ (cf. [AdL]). So for the function f there exists an expansion into a series of these eigenfunctions without using the generalized eigenfunctions.

The space L^2_σ can also be considered as a reproducing kernel space $\mathcal{L}(m)$ of functions which are holomorphic in the upper and lower half planes (cf. [dB]). For $f \in L^2_\sigma$ the corresponding function $\Psi f \in \mathcal{L}(m)$ is

$$(\Psi f)(z) = \int_{-\infty}^{\infty} \frac{f(t)}{t-z} d\sigma(t), \quad z \in \mathbb{C} \backslash \mathbb{R}, \tag{3.12}$$

with the inner product being defined by

$$(\Psi f, \Psi g)_{\mathcal{L}(m)} := (f, g)_{L^2_\sigma}.$$

Indeed, the relation

$$\Psi\left(\frac{1}{\cdot - \overline{w}}\right)(z) = \int_{-\infty}^{\infty} \frac{1}{(t-z)(t-\overline{w})} d\sigma(t) = \frac{m(z) - \overline{m(w)}}{z - \overline{w}}$$

implies for $F = \Psi f$

$$\Big(F(z), \frac{m(z) - \overline{m(w)}}{z - \overline{w}}\Big)_{\mathcal{L}(m)} = \Big(f(\,\cdot\,), \frac{1}{\cdot - \overline{w}}\Big)_{L^2_\sigma} = \int_{-\infty}^{\infty} \frac{f(t)}{t-w} d\sigma(t) = F(w).$$

This means that $\dfrac{m(z) - \overline{m(w)}}{z - \overline{w}}$ is the reproducing kernel of $\mathcal{L}(m)$. Denote the image in $\mathcal{L}(m)$ of the operator of multiplication by the independent variable in L^2_σ by T. Then the resolvent of T is given by the relation

$$\Big((T-\lambda)^{-1}F\Big)(z) = \Psi\left(\frac{f(\,\cdot\,)}{\cdot - \lambda}\right)(z) = \int_{-\infty}^{\infty} \frac{f(t)}{(t-\lambda)(t-z)} d\sigma(t) = \frac{F(z) - F(\lambda)}{z - \lambda}.$$

An explicit expression for the corresponding isomorphism $\Phi := \Psi\mathcal{F}$ between $\widetilde{\mathcal{H}}$ and $\mathcal{L}(m)$ follows from the explicit form of the image of the element $\mathbf{r}_\lambda$:

$$\Phi\mathbf{r}_\lambda = \int_{-\infty}^{\infty} \frac{1}{t-z} \cdot \frac{1}{t-\lambda}\, d\sigma(t) = \left(\frac{1}{\cdot - \lambda}, \frac{1}{\cdot - \overline{z}}\right)_{L^2_\sigma} = (\mathbf{r}_\lambda, \mathbf{r}_{\overline{z}}).$$

Since the linear span of the elements $\mathbf{r}_\lambda$ is dense in $\widetilde{\mathcal{H}}$, we get for $\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \widetilde{\mathcal{H}}$ the relation $(\Phi\boldsymbol{f})(z) = (\boldsymbol{f}, \mathbf{r}_{\overline{z}})$. Written explicitly we get the following

Theorem 3.8 *Under the conditions* (a) – (e) *the isometry* Φ *from* $\widetilde{\mathcal{H}}$ *onto* $\mathcal{L}(m)$ *is given by*

$$(\Phi\boldsymbol{f})(z) = \int_0^1 \chi(x;z) f_1(x)dx - \int_0^1 \frac{v(x)}{u(x) - z}\chi(x;z) f_2(x)dx \tag{3.13}$$

for $\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \widetilde{\mathcal{H}}$.

Chapter 4

Resonances

In this chapter we examine what happens with embedded eigenvalues of $\widetilde{A}_0$ under a perturbation. In Chapter 2 we have seen that in general they will disappear. But in fact there is some kind of 'memory' of this eigenvalue, namely the spectral density has a bump in the neighbourhood of the original eigenvalue. And this bump comes from a non-real pole of the resolvent on an unphysical sheet. So the eigenvalue does not really disappear, it moves onto the unphysical sheet. This is a phenomenon often observed in physics, e. g. the autoionizing states of the helium atom. To make these things more precise we give the following definition (cf. [RS]).

Let H and H_0 be self-adjoint operators in the Hilbert space $\mathcal{H}$. Suppose that there is a dense set $\mathcal{D} \subseteq \mathcal{H}$ such that for $y \in \mathcal{D}$ the resolvents $((H_0 - z)^{-1}y, y)$ and $((H - z)^{-1}y, y)$ admit a meromorphic continuation in z from the upper half plane $\mathbb{C}^+$ to a part of the lower half plane $\mathbb{C}^-$ across the real axis. The point $z_0 \in \mathbb{C}^-$ is called a *resonance pole* if $((H_0 - z)^{-1}y, y)$ is analytic at $z = z_0$ and there exists a $y_0 \in \mathcal{D}$ such that $((H - z)^{-1}y_0, y_0)$ has a pole at $z = z_0$.

This second sheet of the Riemann surface of the resolvent of the operator H is called the *unphysical sheet*, and half of the modulus of the imaginary part of a resonance pole is called the *width* of the resonance.

4.1 Analytic continuation of the resolvent

Let $\mathcal{U}$ be an open, simply connected neighbourhood of the interval $[0, 1]$ in the complex plane where the functions p, q and u are analytic. Further, set $\mathcal{U}^+ := \mathcal{U} \cap \mathbb{C}^+$, $\mathcal{U}^- := \mathcal{U} \cap \mathbb{C}^-$ and $\mathcal{U}_0^- := \mathcal{U}^- \cup [0, 1]$.

The following lemma about analytic continuation of the solutions of the differential equation can be proved by well-known results about differential equations with analytic coefficients since the only singularity is at $x = u^{-1}(z)$.

Lemma 4.1 *For fixed $z \in \mathbb{C}^+$ the functions $\varphi(\,\cdot\,;z)$ and $\psi(\,\cdot\,;z)$ admit an analytic continuation to $\mathcal{U}^-$ which is a solution of the differential equation* (1.1) *there.*

The next lemma shows that the solutions of the differential equation have an analytic continuation in the eigenvalue parameter z into the unphysical sheet.

Lemma 4.2 *For fixed $x_0 \in (0,1]$ the functions $\varphi(x_0;\,\cdot\,)$ and $\psi(x_0;\,\cdot\,)$ admit an analytic continuation to $u(\mathcal{U}^-)$ across the interval $(\alpha, u(x_0))$.*

Proof: We consider only the function φ since for ψ the proof is the same. According to the previous lemma the function $\varphi(x;z)$ admits an analytic continuation in x to $\mathcal{U}^-$ if $z \in \mathbb{C}^+$. Let $\mathcal{C}$ be a curve in $\mathcal{U}^-$ from 0 to x_0 and $\mathcal{V} \subseteq \mathcal{U}$ a neighbourhood of $\mathcal{C}$. Then we can restrict the differential equation and the function $\varphi(x;z)$ to $x \in \mathcal{V}$. Since the differential equation has no singularity in $\mathcal{V}$ for $z \in u(\mathcal{U} \setminus \mathcal{V})$, we can continue the function $\varphi(x_0;z)$ to such z and it depends analytically on z there. As $\mathcal{C}$ and $\mathcal{V}$ are arbitrary, we can reach every $z \in u(\mathcal{U}^-)$ across the interval $(\alpha, u(x_0))$. ■

Lemma 4.3 *Let $f(x;z)$ be a function which fulfills the following conditions:*

(i) *For fixed $z \in \mathbb{C}^+$ the function $f(x;z)$ is analytic in x on $\mathcal{U}^-$ and continuous on $\mathcal{U}_0^-$.*

(ii) *For fixed $x \in \mathcal{U}_0^-$ the function $f(x;z)$ is analytic in z on $u(\mathcal{U} \setminus \{x\})$, where $u(x)$ may be a branch point.*

Then for $x_0 \in (0,1]$ the integral

$$\int_0^{x_0} f(x;z)dx$$

admits an analytic continuation in z to $u(\mathcal{U}^-)$ across the interval $(\alpha, u(x_0))$.

Proof: For $z \in \mathbb{C}^+$ we have

$$\int_0^{x_0} f(x;z)dx = \int_{\mathcal{C}} f(\zeta;z)d\zeta$$

because of (i), where $\mathcal{C}$ is a curve in $\mathcal{U}^-$ from 0 to x_0. The integrand of the integral at the right hand side is analytic in z on $u(\mathcal{U} \setminus \mathcal{C})$. So the integral is analytic there. Since the curve $\mathcal{C}$ is arbitrary in $\mathcal{U}^-$, we can reach every $z \in u(\mathcal{U}^-)$ across the interval $(\alpha, u(x_0))$. ■

Remark: In the last lemma the number x_0 can be complex; then instead of the integral over the interval $[0, x_0]$ the integral over a curve from 0 to x_0 has to be considered.

Theorem 4.4 *For* $\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, f_1 *and* f_2 *analytic, the* $\boldsymbol{f}$*-resolvent of the operator* $\widetilde{A}$

$$\Big((\widetilde{A}-z)^{-1}\boldsymbol{f}, \boldsymbol{f}\Big)$$

admits a meromorphic continuation into the unphysical sheet. Poles can only be at poles of the Titchmarsh-Weyl coefficient $m(z)$*. For every pole of* $m(z)$ *there exists an* $\boldsymbol{f}$ *such that the* $\boldsymbol{f}$*-resolvent has a pole there.*

Proof: The $\boldsymbol{f}$-resolvent of $\widetilde{A}$ can be calculated as follows:

$$\begin{aligned}\Big((\widetilde{A}-z)^{-1}\boldsymbol{f}, \boldsymbol{f}\Big) &= -\Big(L(z)^{-1}f_1, f_1\Big) + \Big(L(z)^{-1}\frac{v}{u-z}f_2, f_1\Big) \\ &+ \Big(\frac{v}{u-z}L(z)^{-1}f_1, f_2\Big) + \Big(\frac{1}{u-z}f_2, f_2\Big) - \Big(\frac{v}{u-z}L(z)^{-1}\frac{v}{u-z}f_2, f_2\Big).\end{aligned} \tag{4.1}$$

With $f_i^*(x) := \overline{f_i(\overline{x})}$ the first term is equal to

$$\begin{aligned}\Big(-L(z)^{-1}f_1, f_1\Big) &= -\int_0^1\int_0^1 G(x,\xi;z)f_1(\xi)d\xi\, f_1^*(x)\,dx \\ &= \int_0^1 \Big(\varphi(x;z)f_1^*(x)\int_0^x \psi(\xi;z)f_1(\xi)d\xi + \psi(x;z)f_1^*(x)\int_x^1 \varphi(\xi;z)f_1(\xi)d\xi\Big)dx \\ &\quad + m(z)\int_0^1 \psi(\xi;z)f_1(\xi)d\xi \cdot \int_0^1 \psi(x;z)f_1^*(x)dx.\end{aligned}$$

Since f_1 and f_1^* are analytic, the functions

$$\psi(x;z)f_1(x) \quad \text{and} \quad \psi(x;z)f_1^*(x)$$

fulfill the conditions of Lemma 4.3. So

$$\int_0^1 \psi(\xi;z)f_1(\xi)d\xi \quad \text{and} \quad \int_0^1 \psi(x;z)f_1^*(x)dx$$

admit analytic continuation. Also the function

$$\varphi(x;z)f_1^*(x)\int_0^x \psi(\xi;z)f_1(\xi)d\xi$$

fulfills these conditions. So the function $(L(z)^{-1}f_1, f_1)$ admits a meromorphic continuation, where poles can only come from poles of $m(z)$.

To get the other terms in (4.1) the functions f_1 has to be replaced by $\frac{v}{u-z}f_2$ at some places. But e. g. the function

$$\frac{v(x)}{u(x)-z}f_2(x)$$

also fulfills the conditions of Lemma 4.3. So the $\boldsymbol{f}$-resolvent can be continued meromorphically, where poles can only come from poles of $m(z)$. If $m(z)$ has a pole at $z = z_0$, we can find a function f_1 such that

$$\int_0^1 \psi(\xi;z)f_1(\xi)d\xi \neq 0, \quad \int_0^1 \psi(x;z)f_1^*(x)dx \neq 0.$$

With $\boldsymbol{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ the $\boldsymbol{f}$-resolvent has a pole at $z = z_0$. ■

This theorem implies that the operator $\widetilde{A}$, considered as a perturbation of $\widetilde{A}_0$, has a resonance pole at a pole of $m(z)$ since for $\mathcal{D}$ in the definition we can take the subspace of analytic functions. Further, observe that the poles of $m(z)$ are exactly the zeros of $\psi(1;z)$.

4.2 The width of the resonance

In this section we consider small perturbations of the operator $\widetilde{A}_0$ which depend linearly on a parameter. So we introduce the operator

$$\widetilde{A}_\vartheta := \begin{pmatrix} A & \vartheta v \\ \vartheta v & u \end{pmatrix} \tag{4.2}$$

where ϑ is a real parameter. We want to examine the asymptotic behaviour of the eigenvalues and resonance poles of $\widetilde{A}_\vartheta$ for $\vartheta \to 0$. The eigenvalue problem for $\widetilde{A}_\vartheta$ is equivalent to

$$y'' - p\,y + \left(z + \frac{\kappa q}{u-z}\right) y = 0 \tag{4.3}$$

$$y(0) = y(1) = 0 \tag{4.4}$$

with $\kappa = \vartheta^2$. For investigating resonance poles the differential equation has to be considered on a curve $\mathcal{C}$ which is in $\mathbb{C}^-$.

Let $\varphi(x; z, \kappa)$ and $\psi(x; z, \kappa)$ be solutions of (4.3) which fulfill the initial conditions

$$\varphi(0; z, \kappa) = 1, \quad \varphi'(0; z, \kappa) = 0, \tag{4.5}$$

$$\psi(0; z, \kappa) = 0, \quad \psi'(0; z, \kappa) = 1. \tag{4.6}$$

Observe that $\varphi(x; z, 1) = \varphi(x; z)$ and $\psi(x; z, 1) = \psi(x; z)$. The next theorem guarantees the existence of resonance poles, and the asymptotic behaviour in dependence of the parameter ϑ is examined.

Theorem 4.5 *Assume that p, q and u are analytic functions. Let λ_0 be an eigenvalue of $\widetilde{A}_0$, $\lambda_0 \neq \alpha$, $\lambda_0 \neq \beta$, and $\begin{pmatrix} y_0 \\ 0 \end{pmatrix}$ a normalized eigenvector. Then for small ϑ the operator $\widetilde{A}_\vartheta$ has an eigenvalue or resonance pole $\lambda(\vartheta)$ in a neighbourhood of λ_0. It depends analytically on the parameter ϑ and admits the Taylor expansion*

$$\lambda(\vartheta) = \sum_{j=0}^{\infty} a_{2j} \vartheta^{2j} \tag{4.7}$$

with $a_0 = \lambda_0$. If $\lambda_0 \in \mathbb{R} \setminus [\alpha, \beta]$, the coefficient a_2 is given by

$$a_2 = -\int_0^1 \frac{q(t)}{u(t) - \lambda_0} (y_0(t))^2 dt. \tag{4.8}$$

If $\lambda_0 \in (\alpha, \beta)$, we have

$$a_2 = -\text{p.v.}\int_0^1 \frac{q(t)}{u(t) - \lambda_0} |y_0(t)|^2 dt - i\pi \frac{q(x_0)}{u'(x_0)} |y_0(x_0)|^2, \tag{4.9}$$

where $x_0 = u^{-1}(\lambda_0)$ and p.v. *denotes the principal value at x_0.*

Proof: In Theorem 4.4 we have seen that the $\boldsymbol{f}$-resolvent of the operator $\widetilde{A}_\vartheta$ admits a meromorphic continuation into the unphysical sheet and that the resonance poles are the values $\lambda(\vartheta)$ such that $\psi(1; \lambda(\vartheta), \vartheta^2) = 0$ due to the equivalence of the eigenvalue problem for the operator $\widetilde{A}_\vartheta$ to the problem (4.3), (4.4) with $\kappa = \vartheta^2$. So we consider the function $\widehat{\lambda}(\kappa) := \lambda(\kappa^{\frac{1}{2}})$, which satisfies the relation $\psi(1; \widehat{\lambda}(\kappa), \kappa) = 0$. The function ψ depends analytically on κ since κ appears linearly in the differential equation. So for the existence and analyticity of $\widehat{\lambda}(\kappa)$ it is sufficient to show that $\psi_z(1; z, \kappa)\big|_{z=\lambda_0,\kappa=0} \neq 0$ where ψ_z denotes the partial derivative of ψ with respect to z.

The function $\widehat{\lambda}(\kappa)$ has the expansion $\widehat{\lambda}(\kappa) = \sum_{j=0}^{\infty} a_{2j}\kappa^j$ and the coefficient a_2 can be calculated as follows

$$a_2 = \frac{d\widehat{\lambda}(\kappa)}{d\kappa}\Big|_{\kappa=0} = -\frac{\psi_\kappa(1; z, \kappa)}{\psi_z(1; z, \kappa)}\Big|_{z=\lambda_0,\kappa=0}$$

where ψ_κ denotes the partial derivatives of ψ with respect to κ. The function ψ is a solution of the equation

$$\psi''(x; z, \kappa) + \Big(z - p(x) + \frac{\kappa q(x)}{u(x) - z}\Big)\psi(x; z, \kappa) = 0,$$

which fulfills the initial conditions (4.6). So for ψ_κ we get the differential equation

$$\psi''_\kappa(x; z, \kappa) + \Big(z - p(x) + \frac{\kappa q(x)}{u(x) - z}\Big)\psi_\kappa(x; z, \kappa) + \frac{q(x)}{u(x) - z}\psi(x; z, \kappa) = 0.$$

For $\kappa = 0$ we get

$$\psi''_\kappa(x; z, 0) + (z - p(x))\psi_\kappa(x; z, 0) = -\frac{q(x)}{u(x) - z}\psi(x; z, 0), \tag{4.10}$$

where $\psi(x; z, 0)$ is a solution of

$$\psi''(x; z, 0) + (z - p(x))\psi(x; z, 0) = 0.$$

The solution of (4.10) which fulfills the initial conditions (4.6) is

$$\begin{aligned}\psi_\kappa(x; z, 0) \;&=\; \Big(1 - \int\limits_{\mathcal{C}_{0,x}} \frac{q(t)}{u(t) - z}\,\varphi(t; z, 0)\psi(t; z, 0)dt\Big)\psi(x; z, 0)\\ &\quad + \int\limits_{\mathcal{C}_{0,x}} \frac{q(t)}{u(t) - z}(\psi(t; z, 0))^2 dt \cdot \varphi(x; z, 0),\end{aligned}$$

where $\mathcal{C}_{0,x}$ is some curve in $\mathcal{U}^-$ from 0 to x. If we set $x = 1$ and $z = \lambda_0$, we get

$$\psi_\kappa(1; \lambda_0, 0) = \int\limits_{\mathcal{C}_{0,1}} \frac{q(t)}{u(t) - \lambda_0}(\psi(t; \lambda_0, 0))^2 dt \cdot \varphi(1; \lambda_0, 0)$$

since $\psi(1;\lambda_0,0)=0$ as λ_0 is an eigenvalue of A. For ψ_z we get with $\kappa=0$ the equation

$$\psi_z''(x;z,0)+(z-p(x))\psi_z(x;z,0)+\psi(x;z,0)=0,$$

which has the solution

$$\begin{aligned}\psi_\kappa(x;z,0) &= \left(1-\int\limits_{\mathcal{C}_{0,x}}\varphi(t;z,0)\psi(t;z,0)dt\right)\psi(x;z,0)\\ &\quad+\int\limits_{\mathcal{C}_{0,x}}(\psi(t;z,0))^2dt\cdot\varphi(x;z,0).\end{aligned}$$

Setting $x=1$, $z=\lambda_0$ yields

$$\psi_z(1;\lambda_0,0)=\int\limits_{\mathcal{C}_{0,1}}(\psi(t;\lambda_0,0))^2dt\cdot\varphi(1;\lambda_0,0).$$

In particular this expression does not vanish. So the function $\widehat{\lambda}(\kappa)$ exists and the coefficient a_2 is

$$\begin{aligned}a_2 &= -\frac{\int\limits_{\mathcal{C}_{0,1}}\frac{q(t)}{u(t)-\lambda_0}(\psi(t;\lambda_0,0))^2dt}{\int\limits_{\mathcal{C}_{0,1}}(\psi(t;\lambda_0,0))^2dt}\\ &= -\int\limits_{\mathcal{C}_{0,1}}\frac{q(t)}{u(t)-\lambda_0}|y_0(t)|^2dt.\end{aligned}$$

For $\lambda\notin[\alpha,\beta]$ the integral is equal to the integral on the real line. For $\lambda\in(\alpha,\beta)$ we take for $\mathcal{C}_{0,1}$ the curve which consists of lines from 0 over $-i\varepsilon$ and $1-i\varepsilon$ to 1. Then taking the limit $\varepsilon\to 0$ we get the desired formula (cf. equation (1.37)). ■

For the case $\lambda\in\mathbb{R}\setminus[\alpha,\beta]$ the formula for a_2 can be rewritten by the mean value theorem for integrals since y_0 is normalized:

$$a_2=-\frac{q(\xi)}{u(\xi)-\lambda_0}$$

for some $\xi\in[0,1]$. This is positive for $\lambda_0>\beta$ and negative for $\lambda_0<\alpha$. So the eigenvalues move away from the essential spectrum at least for small ϑ. But this follows also from Theorem 2.8.

If $\lambda_0 \in (\alpha, \beta)$, $p \equiv 0$ and $\lambda(\vartheta)$ is an eigenvalue for some $\vartheta > 0$, it follows from the considerations after Corollary 2.5 that it has to be greater than some lower bound $\widetilde{\lambda}_0 > \lambda_0$ independent of ϑ. But as $\lambda(\vartheta)$ is analytic, we get that $\lambda(\vartheta)$ cannot be an eigenvalue for ϑ very small. So the eigenvalue λ_0 has to disappear in any case when the perturbation is switched on. Only for some greater ϑ there can be an embedded eigenvalue.

Now we will prove the Fermi golden rule for our problem, which allows to calculate the imaginary part of the coefficient a_2 – i. e. twice the width of the resonance in linear approximation – without solving the perturbed problem. Only the spectral family of the unperturbed operator is used. The Fermi golden rule is a general heuristic formula which has not been proved yet for many physical problems. But in our case a proof is not difficult.

Theorem 4.6 *Let λ_0 be an eigenvalue of $\widetilde{A}_0$, $\lambda_0 \in (\alpha, \beta)$, and $\widetilde{y} = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$ a normalized eigenvector. Further, let $\widetilde{E}_0(\,\cdot\,)$ be the spectral family of the operator $\widetilde{A}_0$, $\widehat{E}_0(\lambda) := \widetilde{E}_0((-\infty, \lambda] \setminus \{\lambda_0\})$ and $V := \widetilde{A} - \widetilde{A}_0$. Then*

$$-\mathrm{Im}\, a_2 = \pi \frac{d}{d\lambda} \Big(V \widehat{E}_0(\lambda) V \widetilde{y}_0, \widetilde{y}_0 \Big)\Big|_{\lambda=\lambda_0}. \tag{4.11}$$

Proof: The operator function $\widehat{E}_0(\lambda)$ is diagonal, and in the first component it is constant in a neighbourhood of λ_0. So it is sufficient to consider the spectral family of the operator u, which is equal to the operator of multiplication by the characteristic function $\chi_{(0,u^{-1}(\lambda)]}$. With $V = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$ we calculate

$$\begin{aligned} &\left(\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \chi_{(0,u^{-1}(\lambda)]} \end{pmatrix} \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}, \begin{pmatrix} y_0 \\ 0 \end{pmatrix} \right) \\ &= \ (v \chi_{(0,u^{-1}(\lambda)]} v y_0, y_0) = \int_0^{u^{-1}(\lambda)} (v(x))^2 |y_0(x)|^2 dx. \end{aligned}$$

Differentiating with respect to λ and setting $\lambda = \lambda_0$, $x_0 = u^{-1}(\lambda_0)$ yields

$$\frac{(v(x_0))^2 |y_0(x_0)|^2}{u'(x_0)}.$$

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