DISSERTATION

Sturm-Liouville operators with singular potentials

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I seem to have been only like a boy
playing on the seashore and diverting myself
in now and then finding a smoother pebble
or a prettier shell than ordinary,
whilst the great ocean of truth
lay all undiscovered before me.

Isaac Newton

Contents

Abstract

This dissertation deals with Sturm-Liouville differential equations of the form

$$-y''(x) + (V_\infty(x) - \lambda)y(x) = f(x), \quad x \in [a,b], a < 0 < b, \tag{0.1}$$

where $\lambda \in \mathbb{C}$ and the function $f \in \mathcal{L}^p([a,b])$ with $1 \le p \le \infty$. Boundary conditions in the boundary points a and b are fixed, for instance the Dirichlet conditions $y(a) = y(b) = 0$. In the standard theory, (see [N], [AG], [W], or [EE]), the potential V_∞ is required to be locally integrable on the whole open interval (a,b). In this dissertation, V_∞ is admitted to be singular in the inner point 0 of the interval $[a,b]$. The potential V_∞ is required to be locally integrable on the half open intervals $[a,0)$ and $(0,b]$ but not necessarily on the whole of $[a,b]$.

The whole dissertation is divided into three parts. The first part, Section 2, demonstrates a way to interpret the singular Sturm-Liouville problem in the context of the Hilbert space $\mathcal{L}^2([a,b])$. The idea is to consider the equation (0.1) first on the intervals $[a,0)$ and $(0,b]$ and then match solutions by interface conditions in 0. The second part, Section 3, provides a strong approximation result. The singular potential V_∞ can be approximated by regular potentials $V_j \in \mathcal{L}^1([a,b])$, $j \in \mathbb{N}$. To these potentials V_j correspond differential operators T_j, $j \in \mathbb{N}$. Under certain assumptions, it is possible to compute the limit of these operators if $j \to \infty$. It is a differential operator for the singular potential V_∞ with particular interface conditions in 0. These depend on the sequence of regular potentials V_j, $j \in \mathbb{N}$. The third part, Section 4, proves for a particular class of interface conditions in 0 that the spectrum $\sigma(T_\infty)$ of the corresponding differential operator T_∞ consists of isolated eigenvalues. These eigenvalues obey an asymptotic formula of the kind $\lambda_n \sim \frac{\pi^2 n^2}{(b-a)^2}$ if $n \to \infty$. An asymptotic formula for the eigenfunctions shows that the eigenfunctions and generalized eigenfunctions of T_∞ can be chosen to form a Bari basis of the space $\mathcal{L}^2([a,b])$ under certain conditions on V_∞.

Sturm-Liouville Operatoren mit singulärem Potential

Zusammenfassung

Diese Dissertation untersucht Sturm-Liouville-Differentialgleichungen der Form (0.1) wobei $\lambda \in \mathbb{C}$ und die Funktion $f \in \mathcal{L}^p([a,b])$ mit $1 \leq p \leq \infty$. An den Intervallenden a und b werden fixe Randbedingungen angenommen, etwa Dirichlet-Randbedingungen $y(a) = y(b) = 0$. In der üblichen Theorie (siehe [N], [AG], [W], oder [EE]) wird das Potential V_∞ auf dem gesamten offenen Intervall (a,b) als lokal summierbar vorausgesetzt. In dieser Dissertation darf V_∞ im inneren Punkt 0 des Intervalls $[a,b]$ singulär sein. Verlangt wird, daß V_∞ auf den halboffenen Intervallen $[a,0)$ und $(0,b]$ lokal summierbar ist, aber nicht notwendigerweise auf ganz $[a,b]$.

Die Dissertation gliedert sich in drei Teile. Der erste, Abschitt 2, zeigt eine Möglichkeit, eine singuläre Sturm-Liouville-Aufgabe im Hilbertraum $\mathcal{L}^2([a,b])$ zu interpretieren. Grundgedanke ist, die Gleichung (0.1) erst auf den Teilintervallen $[a,0)$ und $(0,b]$ zu betrachten und dann die Lösungen durch Übergangsbedingungen in 0 zu verbinden. Der zweite Teil, Abschnitt 3, liefert ein starkes Approximationsresultat. Das singuläre Potential V_∞ kann durch reguläre Potentiale $V_j \in \mathcal{L}^1([a,b])$, $j \in \mathbb{N}$, angenähert werden. Diesen Potentialen entsprechen Differentialoperatoren T_j, $j \in \mathbb{N}$. Unter bestimmten Voraussetzungen ist es möglich, den Grenzwert dieser Operatoren für $j \to \infty$ zu berechnen. Es ist dies ein Differentialoperator zum singulären Potential V_∞ mit bestimmten Übergangsbedingungen in 0. Diese hängen von der Folge der regulären Potentiale V_j, $j \in \mathbb{N}$, ab. Der dritte Teil, Abschnitt 4, beweist für eine gewisse Klasse von Übergangsbedingungen in 0, daß das Spektrum $\sigma(T_\infty)$ aus isolierten Eigenwerten besteht. Diese Eigenwerte weisen für $n \to \infty$ eine Asymptotik der Gestalt $\lambda_n \sim \frac{\pi^2 n^2}{(b-a)^2}$ auf. Eine asymptotische Formel für die Eigenfunktionen zeigt, daß unter gewissen Voraussetzungen über V_∞ eine Bari-Basis in $\mathcal{L}^2([a,b])$ aus Eigenfunktionen und verallgemeinerten Eigenfunktionen von T_∞ gebildet werden kann.

1 Introduction

Particular Sturm-Liouville equations with a singularity in the interior of the considered interval were studied as early as in [L]. The Schrödinger equation of the one-dimensional hydrogen atom was treated there. This equation has roughly the form

$$-y''(x) - \frac{y(x)}{|x|} = \lambda y(x), \quad x \in \mathbb{R}. \tag{1.1}$$

In [Bo], the very similar differential equation

$$-y''(x) - \frac{y(x)}{x} = \lambda y(x), \quad x \in [a,b], a < 0 < b, \tag{1.2}$$

with boundary conditions $y(a) = y(b) = 0$ was considered. In both cases, the usual approach was to find explicit solutions on the subintervals left and right of 0 in terms of Whittaker functions. These solutions then were matched in 0 by interface conditions. Regularly, there was some discussion as to which functions should be admitted as solutions because solutions with a logarithmic singularity at 0 in the derivative occur, see [EGZ] or [Ne].

Even when such solutions are admitted, it was not clear which interface conditions should be imposed in 0. In the case of (1.1), the Dirichlet conditions $y(0-) = y(0+) = 0$ were favoured. In the case of (1.2), interface conditions of the form $y(0-) = y(0+)$, $\lim_{x\to 0}(y'(x) - y'(-x)) = \gamma y(0)$ with $\gamma = 0$ (see [M1]) or $\gamma = -i\pi$ (see [Bo]) were proposed. This was sometimes justified by a limit procedure, where the singular potential was approximated by regular potentials; see [G] and [Bo]. However, in [Gu] it was pointed out that different ways to approximate the singular potential may yield different sets of interface conditions in 0. As to the author's opinion on this subject see Remark 3.63. Other ideas to justify interface conditions can be found in [Ku], where distributions are used, and in [FLM]. There, boundary conditions for (1.1) are formulated first with a linear relation in $\mathbb{C}^2$ as parameter, which must be chosen. All choices are equally acceptable from the mathematical point of view. Then it is shown that a particular stochastic interpretation of (1.1) corresponds to Dirichlet boundary conditions in 0.

The operator studied in [Bo] was not selfadjoint in spite of the opposite claim. This claim was based on a bilinear inner product on $\mathcal{L}^2([a,b])$. In fact, the operator is J-selfadjoint in the sense of [EE]. Anyway, the question was raised in [Bo], but remained open wether it is possible to find a basis of $\mathcal{L}^2([a,b])$ of eigenfunctions and generalized eigenfunctions. This question was solved in [BDL]. An asymptotic formula for the eigenvalues was proved there. This and dissipativity helped to show that the eigenfunctions and generalized eigenfunctions can be chosen to form a Bari-basis of $\mathcal{L}^2([a,b])$.

This dissertation speaks about three topics:

- How to cope with singularities of the potential in the interior of the interval?
- Which operators emerge when the potential is approximated by regular ones?
- Is there a basis of eigenfunctions and generalized eigenfunctions?

Although some additional assumptions must be made, several comparable theorems to be found in the literature are superseded by the theorems presented here (such theorems can be found in [Gu], [BDL], [G], and [Kl]).

Section 2 treats real valued singular potentials V_∞ in the Hilbert space case. This case is the nearest to the standard theory as it is presented in [N], [AG], [W], [CL], or [EE]. To define a differential operator T_∞ in $\mathcal{L}^2([a,b])$ from the singular potential V_∞, minimal and maximal differential operators first are considered in the subspaces $\mathcal{L}^2([a,0])$ and $\mathcal{L}^2([0,b])$. From these, a symmetric minimal operator S is constructed for the Sturm-Liouville equation (0.1) in the whole interval. Its adjoint S^* is the corresponding maximal operator. The interesting case is that the equation (0.1) is in the limit-circle case at 0 on both intervals. Then the operator S has deficiency index $(2,2)$. Differential operators for the equation (0.1) are characterized by interface conditions in 0. These operators extend S and are restricitons of S^*. Those interface conditions which imply selfadjoint or maximal dissipative differential operators are characterized. For similar ideas see [EZ], where only selfadjoint interface conditions are considered, and [BDL], where the particular equation (1.2) is studied.
The results obtained in Section 3 form a synthesis and strengthening of results obtained in [Gu] and [Kl]. Some restrictive technical assumptions previously imposed turn out to be superfluous, or at least their meaning becomes understandable. The method seems to be new, is quite elementary, and works fine for many complex valued singular potentials. Its core mechanism is the use of a family of integral series which is invoked to control solutions of the differential equation near the singularity. Results are first obtained on a compact interval $[a,b]$, $a<0<b$, and then similar statements on $\mathbb{R}$ are proved. To get a first impression of this chapter, read Corollary 3.58 and the examples following.
Section 4 takes up a vision expressed in [Bo] and supplies a useful basisness result for complex valued singular potentials on a compact interval $[a,b]$, $a<0<b$, and a particular class of interface and boundary conditions. For (1.2), a comparable result can be found in [BDL]. The basisness result in Section 4 is proved from an asymptotic formula for the eigenvalues of the form $\lambda_n \sim \frac{\pi^2 n^2}{(b-a)^2}$, $n\to\infty$, which also shows that the resolvent set of the operators in consideration cannot be empty. It turns out that under suitable conditions, the eigenfunctions are asymptotically equal to $\sin\frac{n\pi(\cdot - a)}{b-a}$ if $n\to\infty$ uniformly on the interval $[a,b]$. The whole section heavily relies upon the tools developed in Section 3 to access the singularity.

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Finally, I do not want to miss to thank my parents, too, who enabled and encouraged me to follow my various studies.

1.2 Notation

Let $\mathbb{N}$ be the set of nonnegative integers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. In the context of a complex vector space, complex numbers are also used as the corresponding scalar multiplication operators on these spaces. The upper complex halfplane is $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$, the lower one is $\mathbb{C}^- = \{z \in \mathbb{C} : \Im z < 0\}$. Closures in a topological space are expressed by overlining, for instance $\overline{\mathbb{C}^+} = \{z \in \mathbb{C} : \Im z \geq 0\}$ is the closed upper complex halfplane. Nevertheless, denote the wellknown compactification of $\mathbb{C}$ on the set $\mathbb{C} \cup \{\infty\}$ by $\overline{\mathbb{C}}$. The symbol $\overline{\mathbb{C}}$ will also be used to denote the underlying set $\mathbb{C} \cup \{\infty\}$. A local base of $\infty \in \overline{\mathbb{C}}$ can be chosen to consist of the open sets $\{z \in \mathbb{C} : |z| > k\} \cup \{\infty\}$, $k \in \mathbb{N}$. Subsets of $\overline{\mathbb{C}}$ like $\mathbb{N} \cup \{\infty\}$ are equipped with the trace topology. Limits in the compact space $\overline{\mathbb{C}}$ are denoted by the symbol $\overset{\overline{\mathbb{C}}}{\lim}$. Similarly, $\overset{[0,\infty)}{\sup}$ denotes a supremum in the usual linear ordering on the set $[0, \infty)$ of nonnegative real numbers.

Let the function ι be the identity on $\mathbb{R}$, this means $\iota(x) = x$ for all $x \in \mathbb{R}$. The constant function 1 is denoted by 1. For $x \in \mathbb{R}$ define the interval

$$I_x = \begin{cases} [x, 0) \text{ if } x < 0 \\ \quad \emptyset \quad \text{ if } x = 0 \\ (0, x] \text{ if } x > 0 \end{cases} .$$

For $0 < r \leq \infty$ let $\ell^r(\mathbb{Z})$ be the space of complex valued sequences $c = (c_k)_{k\in\mathbb{Z}}$ with indices in $\mathbb{Z}$ such that the norm

$$\|c\|_{\ell^r(\mathbb{Z})} = \sqrt[r]{\sum_{k\in\mathbb{Z}} |c_k|^r} < \infty$$

if $r \neq \infty$, and

$$\|c\|_{\ell^\infty(\mathbb{Z})} = \sup_{k\in\mathbb{Z}} |c_k| < \infty.$$

Note here that $\|\cdot\|_{\ell^r(\mathbb{Z})}$ does not fulfill the triangle inequality if $r < 1$. Nevertheless, it is called "norm" here. Analogously define the space $\ell^r(\mathbb{N})$ and $\|\cdot\|_{\ell^r(\mathbb{N})}$.

For $0 < r \leq \infty$ and a measurable set $M \subseteq \mathbb{R}$ let $\mathcal{L}^r(M)$ be the Lebesgue space with exponent r on M, that is the space of measurable functions f such that

$$\|f\|_{\mathcal{L}^r(M)} = \sqrt[r]{\int_M |f(x)|^r dx} < \infty$$

if $r < \infty$, and

$$\|f\|_{\mathcal{L}^\infty(M)} = \sup_{x\in M} |f(x)| < \infty,$$

with the usual identification of functions which coincide almost everywhere. The supremum sup means the essential supremum in this context. If f is a function such that its domain $\mathcal{D}(f) \supset M$, the expression $f \in \mathcal{L}^r(M)$ should express that the restriction of f to M is in this space. Analogously, the norm $\|f\|_{\mathcal{L}^r(M)}$ then means the respective norm of the restriction of f to M. As for $\ell^r(\mathbb{Z})$, $\|\cdot\|_{\mathcal{L}^r(M)}$ is called "norm" even if $r < 1$.

For $0 < r \leq \infty$ and a measurable set $M \subseteq \mathbb{R}$ let $\mathcal{L}^r_{\text{loc}}(M)$ be the space of functions on M which are in $\mathcal{L}^r(M')$ for all compact subsets $M' \subseteq M$. Also define the set

$$\mathcal{M}^r(M) = \{u \in \mathcal{L}^r(M) : u(\xi) \geq u(x) \geq 0 \text{ if } x \in M \text{ and } \xi \in I_x \cap M\}$$

of nonnegative real valued functions in $\mathcal{L}^r(M)$ which are increasing when the function argument tends to zero.
For an interval $I \subseteq \mathbb{R}$ let $\mathcal{AC}(I)$ be the set of all absolutely continuous functions y on I. For a finite union $M \subseteq \mathbb{R}$ of intervals let $\mathcal{AC}_{\text{loc}}(M)$ be the set of all functions on M which are in $\mathcal{AC}(I)$ for all compact subintervals $I \subseteq M$ (compare [EE, III.10.1]). For $y \in \mathcal{AC}_{\text{loc}}(M)$, the derivative y' is defined in the Radon-Nikodym sense almost everywhere in the interior of M. Set

$$\mathcal{AC}^2(I) = \{y \in \mathcal{AC}(I) : y' \in \mathcal{AC}(I)\}, \quad \mathcal{AC}^2_{\text{loc}}(M) = \{y \in \mathcal{AC}_{\text{loc}}(M) : y' \in \mathcal{AC}_{\text{loc}}(M)\}.$$

For the functions $y \in \mathcal{AC}^2_{\text{loc}}(M)$, the second derivative y'' is defined almost everywhere in the interior of M. Identify a function defined only on a subset of I with its extension which is zero on the rest of I. Particularly, if $y \in AC^2_{\text{loc}}(I \setminus \{0\})$, then y' and y'', when they are undefined in 0, are treated as if they were 0 there. This convention considerably simplifies the notation when differential expressions with singular potentials are studied. No distributions are considered, like it is done in [Ku].
For a number $r \in \mathbb{R} \cup \{\infty\}$ let $\hat{r} \in \mathbb{R} \cup \{\infty\}$ fulfill $\frac{1}{r} + \frac{1}{\hat{r}} = 1$. This number can be computed as $\hat{r} = \frac{r}{r-1}$ with the understanding that $\frac{1}{1-1} = \infty$ and $\frac{\infty}{\infty-1} = 1$. Sometimes it will be convenient to use the formal rules $\infty \pm 1 = \infty$, $x^0 = \sqrt[\infty]{x} = 1$, $\frac{x}{\infty} = 0$, and $\frac{\infty}{x} = \infty$ for $x \in \mathbb{R} \setminus \{0\}$.
Let $\mathcal{X}$ be a Banach space. Define the the closed unit ball $\mathcal{B}(\mathcal{X}) = \{v \in \mathcal{X} : \|v\|_{\mathcal{X}} \leq 1\}$.
Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Then a linear relation T from $\mathcal{X}$ to $\mathcal{Y}$ is a linear subspace of the product space $\mathcal{X} \times \mathcal{Y}$. Particularly, a linear operator T from $\mathcal{X}$ to $\mathcal{Y}$ is canonically identified with the linear relation $\{(u, Tu) : u \in \mathcal{D}(T)\}$. Many notions extend naturally from linear operators to linear relations, see [Br]. In the following, the attribute "linear" will be omitted when speaking about operators and relations. The domain $\mathcal{D}(T)$ of a relation T is its projection to the first component, thus $\mathcal{D}(T) \subseteq \mathcal{X}$; the range $\mathcal{R}(T)$ of T is the projection of T to the second component, thus $\mathcal{R}(T) \subseteq \mathcal{Y}$. When $\mathcal{X} = \mathcal{Y}$, T is said to be a relation in $\mathcal{X}$.
Assume that T is a relation in $\mathcal{X}$. The resolvent set $\rho(T)$ is the set of all numbers $\lambda \in \mathbb{C}$ such that $(T - \lambda)^{-1}$ is a bounded operator on $\mathcal{X}$. The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$. The operator norm of a bounded linear operator T from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\|T\|_{\mathcal{X}\to\mathcal{Y}}$. If $\mathcal{X} = \mathcal{Y}$, the abbreviation $\|T\|_{\mathcal{X}}$ is used. When there is no possibility of confusion, the norm may simply be denoted by $\|T\|$.
Let $\mathcal{X}$ be a metric space, $v \in \mathcal{X}$, and $\mathcal{U} \subseteq \mathcal{X}$. Then $\operatorname{dist}(v, \mathcal{U})$ is the infimum of all distances of elements $u \in \mathcal{U}$ from v.
Assume that $\mathcal{H}$ is a Hilbert space. Then its inner product is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. When there is no possibility of confusion, the inner product may simply be denoted by $\langle \cdot, \cdot \rangle$. This sesquilinear form is linear in its first argument and conjugate linear in its second. For a subspace $\mathcal{U} \subset \mathcal{H}$ the orthogonal complement is $\mathcal{U}^{\perp}$. If $u, v \in \mathcal{H}$, then the orthogonality relation $\langle u, v \rangle_{\mathcal{H}} = 0$ is written as $u \perp v$.
Let $I \subseteq \mathbb{R}$ be a possibly unbounded interval such that 0 is an inner point of I. Set $I_- = I \cap (-\infty, 0)$ and $I_+ = I \cap (0, \infty)$. Assume $V_\infty \in \mathcal{L}^1_{\text{loc}}(I \setminus \{0\})$ and define the differential expression l_∞ by

$$l_\infty[y] = -y'' + V_\infty y$$

for all $y \in \mathcal{AC}^2_{\text{loc}}(I \setminus \{0\})$. Equivalently, l_∞ can be defined first on the spaces $\mathcal{AC}^2_{\text{loc}}(I_\pm)$ and then be extended to their direct sum. Compare [EZ] and [BDL].

Finally, choose a number $p \in [1, \infty]$. The canonical identification

$$\mathcal{L}^p(I) = \mathcal{L}^p(I \setminus \{0\}) = \mathcal{L}^p(I_-) \dot{+} \mathcal{L}^p(I_+) \tag{1.3}$$

will always be used.

1.3 Basic results

The following result is [W, Theorem 2.1].

Result 1.1 *Assume $m \in \mathbb{N}$, that A is a family of $m \times m$ matrices on a real interval I with entries in $\mathcal{L}^1_{\mathrm{loc}}(I)$, and $f \in \mathcal{L}^1_{\mathrm{loc}}(I)$. Consider the differential equation*

$$y'(x) = A(x)y(x) + f(x), \quad x \in I. \tag{1.4}$$

Then the following assertions hold:

1. *For every number $x_0 \in I$ and vector $y_0 \in \mathbb{C}^m$, there is a unique solution y of (1.4) satisfying the initial condition $y(x_0) = y_0$.*
2. *If A has the form $A = A_1 + \lambda A_2$ where A_1 and A_2 are families of $m \times m$-matrices on I with entries in $\mathcal{L}^1_{\mathrm{loc}}(I)$ and $\lambda \in \mathbb{C}$ is a parameter, then the solution $y(z, x)$ is an entire function of z for every $x \in I$.*
3. *If $A_j \to A$ and $f_j \to f$ in $\mathcal{L}^1_{\mathrm{loc}}(I)$ and $y_{j,0} \to y_0$ in $\mathbb{C}^m$ if $j \to \infty$, then the corresponding solutions y_j converge to y in $\mathcal{L}^\infty_{\mathrm{loc}}(I)$.*

This result will frequently be used in the following adapted version.

Corollary 1.2 *Assume that $I' \subset \mathbb{R}$ is a compact interval, that $V_j \in \mathcal{L}^1(I')$, $f_j \in \mathcal{L}^1(I')$, $x_{j,0} \in I'$, and $\tilde{y}_{j,0} \in \mathbb{C}^2$ for $j \in \mathbb{N} \cup \{\infty\}$. Consider the differential equations*

$$-y''_{j,\lambda} + (V_j - \lambda)y_{j,\lambda} = f_j, \quad x \in I'. \tag{1.5}$$

Then for all $j \in \mathbb{N} \cup \{\infty\}$ and $\lambda \in \mathbb{C}$ there is a unique solution $y_{j,\lambda}$ of (1.5) satisfying the initial condition

$$\begin{pmatrix} y_{j,\lambda}(x_{j,0}) \\ y'_{j,\lambda}(x_{j,0}) \end{pmatrix} = \tilde{y}_{j,0}.$$

For all numbers $j \in \mathbb{N} \cup \{\infty\}$ and $x \in I'$, the functions mapping $\lambda \in \mathbb{C}$ to $y_{j,\lambda}(x)$ are entire. If the convergence relations $V_j \to V_\infty$ and $f_j \to f_\infty$ in $\mathcal{L}^1(I')$, $\tilde{y}_{0,j} \to \tilde{y}_{\infty,0}$ in $\mathbb{C}^2$, and $x_{j,0} \to x_{\infty,0}$ in I' hold if $j \to \infty$, then $y_{j,\lambda} \to y_{\infty,\lambda}$ and $y'_{j,\lambda} \to y'_{\infty,\lambda}$ in $\mathcal{L}^\infty(I')$.

Proof. Sketched. The assertion is obtained from Result 1.1 when the second order linear differential equations are written as linear first order systems

$$\begin{pmatrix} y'_{j,\lambda}(x) \\ y''_{j,\lambda}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V_j(x) - \lambda & 0 \end{pmatrix} \begin{pmatrix} y_{j,\lambda}(x) \\ y'_{j,\lambda}(x) \end{pmatrix} - \begin{pmatrix} 0 \\ f(x) \end{pmatrix}, \quad x \in I',$$

of differential equations for all $j \in \mathbb{N} \cup \{\infty\}$. To see that the solutions depend continuously also on the numbers $x_{j,0}$, note that the shift operators form a continuous group of operators in $\mathcal{L}^1(I')$. Apply a shift by $x_{\infty,0} - x_{j,0}$ to the functions V_j and f_j and its inverse to the solution $y_{j,\lambda}$. This reduces the original problem to a problem with fixed initial point. □

1.4 Table of notation

The following table gives the page references of most symbols introduced. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}^{\pm}$, $\overline{\mathbb{C}}$, I_x, $I_{\pm}$, 1, ι, ℓ^r, $\mathcal{L}^r$, $\mathcal{L}^r_{\mathrm{loc}}$, $\mathcal{AC}$, $\mathcal{M}^r$, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\perp$, $\mathcal{D}(\cdot)$, and $\mathcal{R}(\cdot)$ have been defined in Section 1.2.

$[\cdot,\cdot]_{\mathcal{H}^2}$	11	$[\cdot,\cdot]_x$	14	$\lceil\cdot,\cdot\rfloor_x$	21
$\mathtt{b}$	16	$\mathtt{b}_{\pm}$	15	$\mathtt{b}_{j,\lambda,x}$	31
$B_{j,\lambda,x}$	31	B	68	$F_{j,\lambda,x}$	23
$g_{j,\lambda,\infty,x}$	29	$g_{\infty,\lambda,\infty,x}$	29	gap	35
$G_{j,\lambda,x}$	49	$H_{j,\lambda,n,x}$	23	$H_{j,\lambda,\infty,x}$	23
I	8	J	17	J_0	15
$K_{j,\lambda}$	33	Λ	19	$l_j[\cdot]$	19
$l_\infty[\cdot]$	8	L_λ	58	M_λ	68
μ	19, 58	$\hat{\mu}$	67	ν	46
p	9, 19	q	46, 58	$S_{\pm}$	15
S	16	$S^*_{\pm}$	14	S^*	16
T_∞	68	$v_{\pm}$	68	$x_{\mathtt{b}}$	61
X	19	$X_{j,\lambda,\pm x}$	62	$\Xi_{j,\lambda,x,\xi}$	58
$z_{\pm}$	37, 68				

2 Singular differential operators in the Hilbert space

Throughout this section assume that the potential $V_\infty \in \mathcal{L}^1_{\mathrm{loc}}(I \setminus \{0\})$ is real valued.

2.1 Extensions of a symmetric operator

Definition 2.1 Assume that $\mathcal{H}$ is a Hilbert space and T a relation in $\mathcal{H}$. Then the adjoint of T is the relation

$$T^* = \{(u,v) \in \mathcal{H} \times \mathcal{H} : \langle f, u\rangle = \langle y, v\rangle \text{ for all } (y,f) \in T\}.$$

Definition 2.2 Assume that $\mathcal{H}$ is a Hilbert space. Then the relation D in $\mathcal{H}$ is dissipatve if

$$\Im \langle f, y\rangle \geq 0, \quad (y,f) \in D.$$

It is maximal dissipative if it does not admit a dissipative proper extension. Similarly, the relation S in $\mathcal{H}$ is symmetric if $S \subseteq S^*$. This means that

$$\Im \langle f, y\rangle = 0, \quad (y,f) \in S.$$

It is maximal symmetric if it does not admit a symmetric proper extension. The relation A in $\mathcal{H}$ is selfadjoint if $A = A^*$.

Definition 2.3 Assume that $\mathcal{X}$ is a Banach space and T a relation in $\mathcal{X}$. A canonical extension of T is a relation $T' \supseteq T$ in $\mathcal{X}$. Throughout this thesis, only canonical extensions are considered. So the word "canonical" is omitted when speaking about extensions. In the case that an extension T' of T is an operator, it is called an operator extension of T. If $T' \neq T$, T' is called a proper extension of T. The relation T is called a restriction of a relation T', if $T \subseteq T'$. If $T \neq T'$, it is a proper restriction. If $T \subseteq T'$ and n is the dimension of the factor space T'/T, then T' is called an n-dimensional extension of T and T is called an n-dimensional restriction of T'.

Definition 2.4 Assume that $\mathcal{H}$ is a Hilbert space and that S is a symmetric relation in $\mathcal{H}$. An extension T of S is called a regular extension of S if $T \subseteq S^*$.

Assume that $\mathcal{H}$ is a Hilbert space. Consider the space $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$ with the indefinite inner product $[\cdot,\cdot]_{\mathcal{H}^2}$ given by

$$[(y_1,f_1),(y_2,f_2)]_{\mathcal{H}^2} = \frac{\langle f_1, y_2\rangle - \langle y_1, f_2\rangle}{i}, \quad (y_1,f_1),(y_2,f_2) \in \mathcal{H}^2.$$

As to indefinite inner product spaces see [B].

Definition 2.5 A subspace $\mathcal{U}$ of $\mathcal{H}^2$ is called $[\cdot,\cdot]_{\mathcal{H}^2}$-positive, $[\cdot,\cdot]_{\mathcal{H}^2}$-negative, or $[\cdot,\cdot]_{\mathcal{H}^2}$-neutral if $[u,u]_{\mathcal{H}^2} \geq 0$, $[u,u]_{\mathcal{H}^2} \leq 0$, or $[u,u]_{\mathcal{H}^2} = 0$, respectively, for all $u \in \mathcal{U}$. It is called maximal $[\cdot,\cdot]_{\mathcal{H}^2}$-positive, if there is no $[\cdot,\cdot]_{\mathcal{H}^2}$-positive space $\mathcal{U}' \subseteq \mathcal{H}^2$ such that $\mathcal{U}$ is properly contained in $\mathcal{U}'$. The definitions for maximal $[\cdot,\cdot]_{\mathcal{H}^2}$-negative and maximal $[\cdot,\cdot]_{\mathcal{H}^2}$-neutral are analogous.
Assume that $\mathcal{U}$ is a subspace of $\mathcal{H}^2$. Then the $[\cdot,\cdot]_{\mathcal{H}^2}$-orthogonal companion of $\mathcal{U}$ is the space

$$\{v \in \mathcal{H}^2 : [v,u]_{\mathcal{H}^2} = 0 \text{ for all } u \in \mathcal{U}\}.$$

Easy computation shows that a relation is (maximal) dissipative if and only if it is a (maximal) $[\cdot,\cdot]_{\mathcal{H}^2}$-positive subspace of $\mathcal{H}\times\mathcal{H}$, and it is (maximal) symmetric if and only if it is a (maximal) $[\cdot,\cdot]_{\mathcal{H}^2}$-neutral subspace. The adjoint of an operator is its $[\cdot,\cdot]_{\mathcal{H}^2}$-orthogonal companion. For longer proofs of the following statement see [GG, Theorem III.1.3] or [Kz, Lemma I.3.7].

Lemma 2.6 *Assume that $\mathcal{H}$ is a Hilbert space, S a symmetric relation in $\mathcal{H}$, and D a dissipative extension of S. Then D is a regular extension of S.*

Proof. Since D is $[\cdot,\cdot]_{\mathcal{H}^2}$-positive semidefinite and $S\subseteq D$ is $[\cdot,\cdot]_{\mathcal{H}^2}$-neutral, the Schwarz inequality yields (compare [B], Lemma I.4.4)

$$\left|[(y_1,f_1),(y_2,f_2)]_{\mathcal{H}^2}\right| \le \sqrt{[(y_1,f_1),(y_1,f_1)]_{\mathcal{H}^2}}\sqrt{[(y_2,f_2),(y_2,f_2)]_{\mathcal{H}^2}} = 0,$$

for all pairs $(y_1,f_1)\in S$ and $(y_2,f_2)\in D$. So D is $[\cdot,\cdot]_{\mathcal{H}^2}$-orthogonal to S, which means $D\subseteq S^*$. □

Now the following standard assertion (see [GG, p. 150]) can be proved easily without the use of deficiency indices.

Lemma 2.7 *Assume that $\mathcal{H}$ is a Hilbert space and A a relation in $\mathcal{H}$. Then A is selfadjoint if and only if both A and $-A$ are maximal dissipative.*

Proof. If A is selfadjoint, it is dissipative. Moreover, Lemma 2.6 implies that it has no dissipative proper extensions. Hence, A is maximal dissipative. The same argument applies to $-A$.
Conversely, if $\pm A$ are maximal dissipative, A is symmetric, which means $A\subseteq A^*$. Moreover, the space $A\subseteq\mathcal{H}^2$ is maximal $[\cdot,\cdot]_{\mathcal{H}^2}$-positive and maximal $[\cdot,\cdot]_{\mathcal{H}^2}$-negative. So by [B, Lemma I.6.3], its orthogonal companion A^* is $[\cdot,\cdot]_{\mathcal{H}^2}$-negative and $[\cdot,\cdot]_{\mathcal{H}^2}$-positive. Hence, the maximality properties of $\mp A$ prove $A^*\subseteq A$. □

Lemma 2.8 *Assume that $\mathcal{H}$ is a Hilbert space and D a dissipative relation in $\mathcal{H}$ with dense domain $\mathcal{D}(D)$. Then D is an operator.*

Proof. Assume $(0,u)\in D$ for some $u\in\mathcal{H}$. Then for all vectors $(y,f)\in D$ and numbers $\lambda\in\mathbb{C}$, the dissipativity of D implies

$$0\le \Im\langle\lambda u+f,y\rangle = \Im\langle\lambda u,y\rangle + \Im\langle f,y\rangle .$$

For large numbers λ, the second term can be neglected, which proves

$$0\le\Im\langle\lambda u,y\rangle,\quad \lambda\in\mathbb{C}.$$

This is only possible if $u\perp y$. Since this must hold for all $y\in\mathcal{D}(D)$, $u=0$, and thus D is an operator. □

Note here that a dissipative operator T may have dissipative extensions which are proper relations, but may admit no dissipative operator extensions. The following Corollary 2.9 shows that the notion "maximal dissipative" can be handled without the use of relations when speaking about densely defined operators.

Corollary 2.9 *Assume that $\mathcal{H}$ is a Hilbert space and D a dissipative operator in $\mathcal{H}$. Then D is maximal dissipative if and only if it is densely defined and has no dissipative proper operator extension.*

Proof. If D is densely defined, then all dissipative extensions are so, too, and thus are operators by Lemma 2.8. So D is maximal dissipative if and only if it has no dissipative proper operator extension.
If D is not densely defined, then the relation

$$D' = D \dot{+} \{(0,u) : u \perp \mathcal{D}(D)\}$$

is a dissipative proper extension of D. □

Definition 2.10 (see [GG, p. 159]) Assume that $\mathcal{X}$ is a Banach space and T a relation in $\mathcal{X}$. Then T is solvable if its resolvent set $\rho(T)$ is not empty.

The proof of the following standard assertion is based on the proof of [GG, Theorem III.1.2] for the operator case. Also compare the definitions and comments in [K, p. 279].

Lemma 2.11 *Assume that $\mathcal{H}$ is a Hilbert space and D a maximal dissipative relation in $\mathcal{H}$. Then D is solvable.*

Proof. Choose a number $\lambda \in \mathbb{C}^-$. Dissipativity of D implies that the numerical range

$$\Theta(D) = \{\langle f, y\rangle : (y,f) \in D, \|y\| = 1\}$$

is contained in the closed upper halfplane $\overline{\mathbb{C}^+}$. Since a pair $(y,f) \in D$ with $\|y\| = 1$ fulfills

$$\|f - \lambda y\| \geq \langle f - \lambda y, y\rangle = \langle f, y\rangle - \lambda \geq \operatorname{dist}(\lambda, \Theta(D)),$$

the resolvent $(D-\lambda)^{-1}$ is a bounded operator on the space $\mathcal{R}(D-\lambda) \subseteq \mathcal{H}$. The closure of a dissipative relation is dissipative again, thus the maximal dissipative relation D is closed and so is the range $\mathcal{R}(D-\lambda)$.
Assume that $(D-\lambda)^{-1}$ is not defined everywhere on $\mathcal{H}$. So there is a nonzero vector $v \in (\mathcal{R}(D-\lambda))^\perp$. Then the proper extension $D' = D \dot{+} \operatorname{span}\{(v, \overline{\lambda} v)\}$ of D is dissipative again because every pair $(y,f) \in D$ fulfills $\langle f - \lambda y, v\rangle = 0$ and hence the imaginary part of the expression

$$\begin{aligned} \left\langle \overline{\lambda} v + f, v + y \right\rangle &= \langle f, y\rangle + \langle f, v\rangle + \left\langle \overline{\lambda} v, y\right\rangle + \left\langle \overline{\lambda} v, v\right\rangle \\ &= \langle f, y\rangle + \langle f, v\rangle + \overline{\langle \lambda y, v\rangle} + \overline{\lambda}\langle v, v\rangle \\ &= \langle f, y\rangle + \langle f, v\rangle + \overline{\langle f, v\rangle} + \overline{\lambda}\langle v, v\rangle = \langle f, y\rangle + 2\Re\langle f, v\rangle + \overline{\lambda}\langle v, v\rangle \end{aligned}$$

is positive. This contradicts the assumption that D is maximal dissipative. □

As to the converse of Lemma 2.11 note that solvable extensions of symmetric operators need not even be regular as the construction in the following example shows.

Example 2.12 Assume that $\mathcal{H}$ is a Hilbert space and S a densely defined closed symmetric operator in $\mathcal{H}$ with deficiency index $(1,1)$. Choose a number $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and a vector $g \in \mathcal{H} \setminus \mathcal{R}(S-\lambda)$. Then choose a vector $u \in \mathcal{D}(S)$ such that $\langle g, u\rangle \neq 0$. Finally choose a vector $y \in \mathcal{H} \setminus \mathcal{D}(S)$ such that

$$\langle g, u\rangle \neq \left\langle y, (S-\overline{\lambda})u\right\rangle. \tag{2.1}$$

This is possible because the set $\mathcal{H} \setminus \mathcal{D}(S)$ is dense in $\mathcal{H}$. Consider the extension $T \supset S$ defined by

$$\begin{aligned} \mathcal{D}(T) &= \mathcal{D}(S)\dot{+}\operatorname{span}\{y\}, \\ Tu &= Su, \quad u \in \mathcal{D}(S), \\ Ty &= g + \lambda y. \end{aligned}$$

Then (2.1) implies

$$\langle Ty, u\rangle = \langle g + \lambda y, u\rangle = \langle g, u\rangle + \left\langle y, \overline{\lambda} u\right\rangle \neq \left\langle y, (S-\overline{\lambda})u\right\rangle + \left\langle y, \overline{\lambda} u\right\rangle = \langle y, Su\rangle.$$

Thus T is not contained in S^*. For $f \in \mathcal{H}$ define the functional χ such that $f = f_0 + \chi[f]g$ with $f_0 \in \mathcal{R}(S-\lambda)$. Then χ is continuous because $\mathcal{R}(S-\lambda)$ is closed. The equality

$$(T-\lambda)^{-1}f = (S-\lambda)^{-1}(f - \chi[f]g) + \chi[f]y, \quad f \in \mathcal{H},$$

shows that T is solvable. Of course by Lemma 2.6, neither T nor $-T$ is dissipative.

2.2 The differential expression on the subintervals

The differential expression l_∞ on each of the intervals $I_\pm$ induces a minimal operator $S_\pm$ and a maximal operator, which is the adjoint $S_\pm^*$ of $S_\pm$, compare [EE, Theorem III.10.7]. For simplicity assume that l_∞ is in the limit point case at the nonzero end points of $I_\pm$. To find similar results if l_∞ is regular at the nonzero end points of $I_\pm$, suitable selfadjoint boundary conditions in these end points must be imposed. These could be of the form $y(a) = y(b) = 0$, for instance, if $I = [a,b]$. The maximal operator $S_\pm^*$ on $I_\pm$ is given by

$$\begin{aligned} \mathcal{D}(S_\pm^*) &= \left\{ y \in \mathcal{L}^2(I_\pm) \cap \mathcal{AC}^2_{\mathrm{loc}}(I_\pm) : -y'' + V_\infty y \in \mathcal{L}^2(I_\pm) \right\}, \\ S_\pm^* y &= V_\infty y, \quad y \in \mathcal{D}(S_\pm^*). \end{aligned}$$

For $x, x_1, x_2 \in I_\pm$ define the sesquilinear forms $[\cdot,\cdot]_x$ and $[\cdot,\cdot]_{x_1}^{x_2}$ by

$$[u,v]_x = u(x)\overline{v'(x)} - u'(x)\overline{v(x)}, \quad [u,v]_{x_1}^{x_2} = [u,v]_{x_2} - [u,v]_{x_1}, \quad u, v \in \mathcal{D}(S_\pm^*).$$

Then Green's Formula

$$[u,v]_{x_1}^{x_2} = \int_{x_1}^{x_2} \left(l_\infty[u](x)\overline{v(x)} - u(x)\overline{l_\infty[v](x)} \right) dx \tag{2.2}$$

holds for $u, v \in \mathcal{D}(S_\pm^*)$ and $x_1, x_2 \in I_\pm$, see [EE, III.10.1] or [W, Theorem 2.2]. Since $S_\pm^* \subseteq \mathcal{L}^2(I_\pm)^2$, (2.2) implies that the limits $[u,v]_{\inf I_\pm} = \lim_{x \to \inf I_\pm}[u,v]_x$, $[u,v]_{\sup I_\pm} = \lim_{x \to \sup I_\pm}[u,v]_x$ exist and are finite for $u, v \in \mathcal{D}(S_\pm^*)$. Moreover,

$$[y_1, y_2]_{\inf I_\pm}^{\sup I_\pm} = \left\langle S_\pm^* y_1, y_2 \right\rangle - \left\langle y_1, S_\pm^* y_2 \right\rangle, \quad y_1, y_2 \in \mathcal{D}(S_\pm^*). \tag{2.3}$$

Since $S_\pm$ is the adjoint of $S_\pm^*$, the domain of the minimal operator $S_\pm$ can be described as

$$\mathcal{D}(S_\pm) = \left\{ y \in \mathcal{D}(S_\pm^*) : [y,u]_{0\pm} = 0 \text{ for all } u \in \mathcal{D}(S_\pm^*) \right\}. \tag{2.4}$$

By (2.3) and (2.4), the operator $S_\pm$ is symmetric. If the differential expression l_∞ is in the limit point case at $0\pm$ in $I_\pm$, then $S_\pm = S_\pm^*$ is a selfadjoint operator. Else $S_\pm$ is a symmetric operator with deficiency index $(1,1)$, and $S_\pm^*$ is a two dimensional extension of $S_\pm$. For the rest of this section assume this latter case for $I_\pm$. Then the factor space $\mathcal{D}(S_\pm^*)/\mathcal{D}(S_\pm)$ is two-dimensional. It is spanned by the equivalence classes of two functions $v_\pm$ and $w_\pm$ in $\mathcal{D}(S_\pm^*)$:

$$\mathcal{D}(S_\pm^*) = \mathcal{D}(S_\pm) \dot{+} \operatorname{span}\{v_\pm, w_\pm\}. \tag{2.5}$$

The von Neumann formula (see [GG, III, Theorem 1.1])

$$\mathcal{D}(S_\pm^*) = \mathcal{D}(S) \dot{+} \ker(S_\pm^* - \lambda) \dot{+} \ker(S_\pm^* - \overline{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.6}$$

implies that the functions $v_\pm$ and $w_\pm$ can be found from the solutions of the equations $l_\infty[y] = \lambda y$ and $l_\infty[y] = \overline{\lambda} y$ on $I_\pm$. Since V_∞ was assumed to be real valued, the conjugate $\overline{y}$ of a solution y of the equation $S_\pm^* y = \lambda y$ solves $S_\pm^* \overline{y} = \overline{\lambda}\overline{y}$. Moreover, by (2.6), the functions y and $\overline{y}$ are linearly independent modulo $\mathcal{D}(S_\pm)$. Now assume $y_\pm \in \ker(S_\pm^* - \lambda) \setminus \{0\}$ and define the real valued functions $v_\pm = y_\pm + \overline{y}_\pm$ and $w_\pm = -i(y_\pm - \overline{y}_\pm)$. Without loss of generality, it is assumed that $[v_\pm, w_\pm]_{0\pm} = -1$. This can always be achieved by multiplication of y with a suitable complex number. Note that $[v_\pm, w_\pm]_{0\pm} = 0$ is impossible because the sets $\{v_\pm, w_\pm\}$ are linearly independent modulo $\mathcal{D}(S_\pm)$. The antisymmetric definition of the sesquilinear form $[\cdot,\cdot]_x$, $x \in I_\pm$, then yields

$$[w_\pm, v_\pm]_{0\pm} = -[v_\pm, w_\pm]_{0\pm} = 1, \quad [v_\pm, v_\pm]_{0\pm} = [w_\pm, w_\pm]_{0\pm} = 0. \tag{2.7}$$

From (2.4), it follows

$$\mathcal{D}(S_\pm) = \{ y \in \mathcal{D}(S_\pm^*) : [y, v_\pm]_{0\pm} = [y, w_\pm]_{0\pm} = 0 \}. \tag{2.8}$$

Define the boundary operators $\mathtt{b}_\pm : \mathcal{D}(S_\pm^*) \to \mathbb{C}^2$ by

$$\mathtt{b}_\pm y = \begin{pmatrix} \mp [y, v_\pm]_{0\pm} \\ \mp [y, w_\pm]_{0\pm} \end{pmatrix}, \quad y \in \mathcal{D}(S_\pm^*). \tag{2.9}$$

Lemma 2.13 *Assume that J_0 is the 2×2-matrix*

$$J_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \ker \mathtt{b}_\pm &= \mathcal{D}(S_\pm), \\ \mathcal{R}(\mathtt{b}_\pm) &= \mathbb{C}^2, \\ \frac{\left\langle S_\pm^* y_1, y_2 \right\rangle - \left\langle y_1, S_\pm^* y_2 \right\rangle}{i} &= \mp (\mathtt{b}_\pm y_2)^* J_0 \mathtt{b}_\pm y_1, \quad y_1, y_2 \in \mathcal{D}(S_\pm^*). \end{aligned}$$

Proof. The equality (2.8) implies the first assertion. The second one follows from the properties (2.7) of $v_\pm$ and $w_\pm$ and (2.9) since

$$\mathtt{b}_\pm v_\pm = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}, \quad \mathtt{b}_\pm w_\pm = \begin{pmatrix} \mp 1 \\ 0 \end{pmatrix}. \tag{2.10}$$

Assume $y_1, y_2 \in \mathcal{D}(S_\pm^*)$. These vectors by (2.5) can be decomposed as

$$y_k = u_k + \alpha_k v_\pm + \beta_k w_\pm, \quad k \in \{1,2\},$$

where $u_1, u_2 \in \mathcal{D}(S_\pm)$. Then the relations $u_k \in \ker \mathtt{b}_\pm$ and (2.10) yield

$$\mathtt{b}_\pm y_k = \begin{pmatrix} \mp\beta_k \\ \pm\alpha_k \end{pmatrix}, \quad k \in \{1,2\},$$

and from (2.9), it follows

$$\begin{aligned} [y_1, y_2]_{0\pm} &= \alpha_1\overline{\alpha_2}[v_\pm, v_\pm] + \beta_1\overline{\alpha_2}[w_\pm, v_\pm] + \alpha_1\overline{\beta_2}[v_\pm, w_\pm] + \beta_1\overline{\beta_2}[w_\pm, w_\pm] \\ &= \begin{pmatrix} -\overline{\beta_2} & \overline{\alpha_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\beta_1 \\ \alpha_1 \end{pmatrix} = i(\mathtt{b}_\pm y_2)^* J_0 \mathtt{b}_\pm y_1. \end{aligned}$$

Now (2.3) yields

$$\frac{\left\langle S_\pm^* y_1, y_2 \right\rangle - \left\langle y_1, S_\pm^* y_2 \right\rangle}{i} = \frac{[y_1, y_2]_{\inf I_\pm}^{\sup I_\pm}}{i} = \mp\frac{[y_1, y_2]_{0\pm}}{i} = \mp(\mathtt{b}_\pm v)^* J_0 \mathtt{b}_\pm u, \quad u, v \in \mathcal{D}(S_\pm^*),$$

since the terms $[\cdot,\cdot]_{\inf I}$ and $[\cdot,\cdot]_{\sup I}$ vanish as a consequence of the limit point case in the end points of I. □

2.3 Interface conditions in 0

Define the closed symmetric operator $S = S_- \oplus S_+$ in $\mathcal{L}^2(I)$. Its adjoint is $S^* = S_-^* \oplus S_+^*$. This section starts the study of the regular extensions of S. Other extensions than regular ones are not contained in the orthogonal sum of the maximal operators. So they are not considered to represent the differential expression l_∞. For example, an extension constructed from S with the help of the method used in Example 2.12 typically is such an extension. The vector y used there might be a discontinuous function and its image chosen rather arbitrarily.

The cases when the differential expression l_∞ is in the limit point case at 0 in one of the intervals $I_\pm$ are not interesting in the present context of a differential expression on a direct sum space. This was pointed out in [EZ]. For instance assume that l_∞ is in the limit point case at 0 in I_-. Then S_- is selfadjoint in $\mathcal{L}^2(I_-)$ and all regular extensions T of S are of the form $S_- \oplus T_+$, where T_+ is a regular extension of S_+. Hence the study of regular extensions of S reduces to that of regular extensions of S_+ in $\mathcal{L}^2(I_+)$. In the case where l_∞ is in the limit point case at 0 in both intervals $I_\pm$, S itself is selfadjoint.

The assumption is made that l_∞ is regular or in the limit circle case at $0\pm$ in $I_\pm$. So the operators $S_\pm$ are symmetric with deficiency index $(1,1)$ each and S has deficiency index $(2,2)$. Define the boundary mapping $\mathtt{b} : \mathcal{D}(S^*) \to \mathbb{C}^4$ by

$$\mathtt{b}y = \begin{pmatrix} \mathtt{b}_- y_- \\ \mathtt{b}_+ y_+ \end{pmatrix} = \begin{pmatrix} [y, v_-]_{0-} \\ [y, w_-]_{0-} \\ -[y, v_+]_{0+} \\ -[y, w_+]_{0+} \end{pmatrix}, \quad y = y_- + y_+, y_\pm \in \mathcal{D}(S_\pm^*).$$

Lemma 2.13 yields

$$\mathcal{R}(\mathtt{b}) = \mathbb{C}^4, \qquad \ker \mathtt{b} = \mathcal{D}(S). \tag{2.11}$$

So the boundary mapping $\mathtt{b}$ has the important property that every regular extension T of S can be determined by a complex 4-column-matrix B such that

$$\mathcal{D}(T) = \ker(B\mathtt{b}). \tag{2.12}$$

2.4 Maximal dissipative and selfadjoint interface conditions

Let J_0 be the 2×2-matrix defined in Lemma 2.13 and J be the 4×4-matrix

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}.$$

Lemma 2.14 *Assume $y_1, y_2 \in \mathcal{D}(S^*)$. Then it holds*

$$\frac{\langle S^* y_1, y_2\rangle - \langle y_1, S^* y_2\rangle}{i} = (\mathtt{b}y_2)^* J \mathtt{b} y_1.$$

Proof. Lemma 2.13 yields for $y_k = y_{k,-} + y_{k,+}$, $y_{k,\pm} \in \mathcal{D}(S^*_\pm)$, $k \in \{1,2\}$,

$$\begin{aligned} \frac{\langle S^* y_1, y_2\rangle - \langle y_1, S^* y_2\rangle}{i} &= \frac{\left\langle S^*_- y_{1,-}, y_{2,-}\right\rangle - \left\langle y_{1,-}, S^*_- y_{2,-}\right\rangle}{i} \\ &\quad + \frac{\left\langle S^*_+ y_{1,+}, y_{2,+}\right\rangle - \left\langle y_{1,+}, S^*_+ y_{2,+}\right\rangle}{i} \\ &= (\mathtt{b}_- y_{2,-})^* J_0 \mathtt{b}_- y_{1,-} - (\mathtt{b}_+ y_{2,+})^* J_0 \mathtt{b}_+ y_{1,+} = (\mathtt{b} y_2)^* J \mathtt{b} y_1. \end{aligned}$$

□

Definition 2.15 A subspace $\mathcal{U}$ of $\mathbb{C}^4$ is called J-positive, J-negative, or J-neutral if and only if $u^* J u \geq 0$, $u^* J u \leq 0$, or $u^* J u = 0$, respectively, for all $u \in \mathcal{U}$. A J-positive subspace $\mathcal{U} \subseteq \mathbb{C}^4$ is called maximal J-positive, if all J-positive subspaces $\mathcal{U}'$ of $\mathbb{C}^4$ with $\mathcal{U} \subseteq \mathcal{U}'$ equal $\mathcal{U}$. The definitions of maximal J-negative and maximal J-neutral are analogous.

Corollary 2.16 *Assume that T is a regular extension of S and $\mathcal{U} = \{\mathtt{b}y : y \in \mathcal{D}(T)\}$. Then T is (maximal) dissipative if and only if $\mathcal{U}$ is (maximal) J-positive, $-T$ is (maximal) dissipative if and only if $\mathcal{U}$ is (maximal) J-negative, and T is (maximal) symmetric if $\mathcal{U}$ is (maximal) J-neutral.*

Proof. Lemma 2.14 implies

$$0 \leq 2\Im \langle Ty, y\rangle = \frac{\langle Ty, y\rangle - \langle y, Ty\rangle}{i} = \frac{\langle S^* y, y\rangle - \langle y, S^* y\rangle}{i} = (\mathtt{b}y)^* J \mathtt{b} y, \quad y \in \mathcal{D}(T). \tag{2.13}$$

So T is dissipative if and only if $\mathcal{U}$ is J-positive. Next assume that T' is a dissipative proper extension of T. Then it is a regular extension of S since $T'^* \subset T^* \subseteq S^*$ and thus also $\{\mathtt{b}y : y \in \mathcal{D}(T')\} \supset \mathcal{U}$ is J-positive. Hence $\mathcal{U}$ is not maximal J-positive. Conversely, if $\mathcal{U}$ is not maximal J-positive, there is a J-positive subspace $\mathcal{U}' \supset \mathcal{U}$ of $\mathbb{C}^4$. As a consequence of (2.11), this subspace is the image of a subspace $T' \supseteq S$ of S^* under the mapping $\mathtt{b}$. Then T' is a dissipative proper extension of T. The proof for the assertion on $-T$ follows the same pattern, only the symbol "$\leq$" in (2.13) must be replaced by "$\geq$". Symmetric extensions are treated analogously using the equality symbol "$=$" in place of the inequality symbol in (2.13). □

Corollary 2.17 *Assume that T is an extension of S such that T or $-T$ is maximal dissipative. Then the domain $\mathcal{D}(T)$ can be given by (2.12) with a 2×4-matrix B which has rank 2.*

Proof. Lemma 2.6 implies that T is a regular extension of S and therefore its domain can be given by (2.12) with some 4-column matrix B. Since the matrix J has the eigenvalues ± 1, each of multiplicity 2, the maximal J-positive and maximal J-negative subspaces of $\mathbb{C}^4$ are of dimension 2, see [B, Lemma IX.1.2]. This and Corollary 2.16 yield that the subspace $\mathcal{U} = \{\mathtt{b}y : y \in \mathcal{D}(T)\} = \ker B$ of $\mathbb{C}^4$ is two-dimensional. This implies that the matrix B has rank 2, and without loss of generality, B is a 2×4-matrix. □

Theorem 2.18 *Assume that T is a regular extension of S and that its domain is given by (2.12) with a 2×4-matrix B which has rank 2. Assume that B is decomposed as*

$$B = \begin{pmatrix} B_- & B_+ \end{pmatrix},$$

where $B_\pm$ are 2×2-matrices. Then T or $-T$ is maximal dissipative if and only if

$$B_- J_0 B_-{}^* \leq B_+ J_0 B_+{}^* \quad \text{or} \quad B_- J_0 B_-{}^* \geq B_+ J_0 B_+{}^*,$$

respectively. The operator T is selfadjoint if and only if

$$B_- J_0 B_-{}^* = B_+ J_0 B_+{}^*.$$

Proof. It suffices to prove the criterion for T to be maximal dissipative. The corresponding criterion for $-T$ is proved analogously. The assertion about selfadjointness then follows from these criteria using Lemma 2.7.
The space $\mathcal{U} = \ker B$ in Corollary 2.16 is maximal J-positive if and only if its J-orthogonal companion

$$\{v \in \mathbb{C}^4 : \langle Ju, v\rangle = 0 \text{ for all } u \in \mathcal{U}\} = \mathcal{R}(JB^*)$$

is J-negative, compare [B, Theorem V.4.4]. Since $J = J^* = J^{-1}$, the space $\mathcal{R}(JB^*)$ J-negative if and only if $\mathcal{R}(B^*)$ is. This condition is equivalent to $B_- J_0 B_-{}^* \leq B_+ J_0 B_+{}^*$. □

3 Approximation by regular potentials

For the potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$, assume

$$V_j \in \mathcal{L}^1_{\text{loc}}(I), \quad j \in \mathbb{N}, \qquad V_\infty \in \mathcal{L}^1_{\text{loc}}(I \setminus \{0\}), \tag{3.1}$$

$$\int_{I'} |V_\infty(x) - V_j(x)|\, dx \;\to\; 0, \quad j \to \infty, \text{ for all compact sets } I' \subset I \setminus \{0\}, \tag{3.2}$$

$$\int_x^0 |\xi V_j(\xi)| d\xi \;\to\; 0 \tag{3.3}$$

uniformly with respect to $j \in \mathbb{N} \cup \{\infty\}$ if $x \to 0$. The conditions (3.1) and (3.3) contain a restriction for the potential V_∞ to which the following theory is applicable. The condition (3.2) only expresses that the regular potentials V_j, $j \in \mathbb{N}$, converge to V_∞ in the sense of $\mathcal{L}^1_{\text{loc}}(I \setminus \{0\})$. For $j \in \mathbb{N} \cup \{\infty\}$, the differential expression l_j is given by

$$l_j[y] = -y'' + V_j y$$

for all $y \in \mathcal{AC}^2_{\text{loc}}(I \setminus \{0\})$. Equivalently, the differential expressions l_j can first be defined for $y \in \mathcal{AC}^2_{\text{loc}}(I_\pm)$. Then l_j can be extended to all $y \in \mathcal{AC}^2_{\text{loc}}(I \setminus \{0\})$ using the identification (1.3). The Sturm-Liouville equations corresponding to l_j are

$$-y'' + (V_j - \lambda)y \;=\; 0, \tag{3.4}$$

$$-y'' + (V_j - \lambda)y \;=\; f, \tag{3.5}$$

where $f \in \mathcal{L}^p(I)$ and $\lambda \in \Lambda \subset \mathbb{C}$. The set Λ is some sufficiently large bounded subset of $\mathbb{C}$ with $0 \in \Lambda$. For the purposes of this section, $\Lambda = \{z \in \mathbb{C} : |z| \le 1\}$ suffices. The conditions (3.1), (3.2), and (3.3) are understood as implicit premises in all statements of this section. The same is true for the properties of the set Λ and the constant $p \in [1, \infty]$ which defines the considered space of functions $\mathcal{L}^p(I)$, compare (1.3).

3.1 Some integral inequalities and the function μ

Lemma 3.1 *Assume $j \in \mathbb{N} \cup \{\infty\}$ and $x \in I$. Then it holds*

$$\int_x^0 \int_x^\xi |V_j(\varsigma)| d\varsigma d\xi = \left| \int_x^0 |\xi V_j(\xi)| d\xi \right| \to 0$$

uniformly with respect to $j \in \mathbb{N} \cup \{\infty\}$ if $x \to 0$.

Proof. Reverse the order of integration in

$$\int_x^0 \int_x^\xi |V_j(\varsigma)| d\varsigma d\xi = \int_x^0 \int_\varsigma^0 |V_j(\varsigma)| d\xi d\varsigma = -\int_x^0 \varsigma |V_j(\varsigma)| d\varsigma.$$

Then the assumption (3.3) yields the convergence asserted. □

Define

$$\mu(x) = \sup_{j \in \mathbb{N} \cup \{\infty\}} \int_x^0 \int_x^\xi |V_j(\varsigma)| d\varsigma d\xi \tag{3.6}$$

for all $x \in I$. Lemma 3.1 implies $\mu(x) \to 0$ if $x \to 0$. Let $X \subseteq I$ be a symmetric open interval around 0 with $\sup\limits_{\lambda \in \Lambda, x \in X} (\mu(x) + x^2 |\lambda|) < \frac{1}{2}$.

Lemma 3.2 *Assume $r \in [1,\infty]$, $x \in \mathbb{R} \setminus \{0\}$, $u \in \mathcal{L}^r(I_x)$, and $\xi \in I_x \cup \{0\}$. Then it holds*

$$\left| \int_x^\xi u(\varsigma) d\varsigma \right| \leq \sqrt[\hat{r}]{|x|} \, \|u\|_{\mathcal{L}^r(I_x)} \, .$$

Proof. To simplify the notation, assume $x < 0$. If $1 < r < \infty$, Hölder's inequality yields

$$\left| \int_x^\xi |u(\varsigma)| d\varsigma \right| \leq \left| \int_x^0 |u(\varsigma)| d\varsigma \right| \leq \sqrt[\hat{r}]{\int_x^0 1 d\xi} \sqrt[r]{\int_x^0 |u(\varsigma)|^r d\varsigma} = \sqrt[\hat{r}]{|x|} \, \|u\|_{\mathcal{L}^r(I_x)} \, .$$

If $r \in \{1, \infty\}$, the assertion of the lemma is trivial. □

Corollary 3.3 *Assume $r \in [1,\infty]$, $x \in \mathbb{R} \setminus \{0\}$, $u \in \mathcal{L}^r(I_x)$, and $\xi \in I_x \cup \{0\}$. Then it holds*

$$\left| \int_x^\xi \sqrt[r]{\varsigma} u(\varsigma) d\varsigma \right| \leq |x| \, \|u\|_{\mathcal{L}^r(I_x)} \, .$$

Proof. Lemma 3.2 yields

$$\left| \int_x^\xi \sqrt[r]{\varsigma} |u(\varsigma)| d\varsigma \right| \leq \left| \sqrt[r]{x} \int_x^0 |u(\varsigma)| d\varsigma \right| \leq \sqrt[r]{|x|} \sqrt[\hat{r}]{|x|} \, \|u\|_{\mathcal{L}^r(I_x)} = |x| \, \|u\|_{\mathcal{L}^r(I_x)} \, .$$

□

Corollary 3.4 *Assume $r, s \in (0,\infty]$, $r \geq s$, $x \in \mathbb{R} \setminus \{0\}$, and $u \in \mathcal{L}^r(I_x)$. Then it holds*

$$\|u\|_{\mathcal{L}^s(I_x)} \leq |x|^{s^{-1} - r^{-1}} \|u\|_{\mathcal{L}^r(I_x)}$$

Proof. Without loss of generality assume $s < \infty$. Consider the function $u^s \in \mathcal{L}^{\frac{r}{s}}$. Application of Lemma 3.2 yields

$$\begin{aligned} \|u\|_{\mathcal{L}^s(I_x)} &= \sqrt[s]{\|u^s\|_{\mathcal{L}^1(I_x)}} \leq \sqrt[s]{\sqrt[\widehat{r/s}]{|x|} \, \|u^s\|_{\mathcal{L}^{\frac{r}{s}}(I_x)}} \\ &= |x|^{s^{-1}\left(1 - \frac{s}{r}\right)} \|u\|_{\mathcal{L}^r(I_x)} = |x|^{s^{-1} - r^{-1}} \|u\|_{\mathcal{L}^r(I_x)} \, . \end{aligned}$$

□

Lemma 3.5 *Assume $r \in (0,\infty]$, $x \in \mathbb{R} \setminus \{0\}$, and $u \in \mathcal{M}^r(I_x)$. Then it holds*

$$|\sqrt[r]{x} u(x)| \leq \|u\|_{\mathcal{L}^r(I_x)} \, .$$

Proof. If $r = \infty$, the statement is trivial. If $0 < r < \infty$, monotonicity of u implies

$$|x u^r(x)| = \left| \int_x^0 u^r(x) d\xi \right| \leq \left| \int_x^0 u^r(\xi) d\xi \right| = \|u\|^r_{\mathcal{L}^r(I_x)} \, .$$

□

3.2 Basic properties of the solutions of (3.5)

Lemma 3.6 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f \in \mathcal{L}^p(I_x)$, and that the function y is a solution of (3.5) on I_x. Then*

$$\|y\|_{\mathcal{L}^\infty(I_x)} \leq \frac{|y(x)| + |xy'(x)| + |x\sqrt[\hat{p}]{x}|\,\|f\|_{\mathcal{L}^p(I_x)}}{1-\mu(x)-x^2|\lambda|} \leq 2|y(x)| + 2|xy'(x)| + 2|x\sqrt[\hat{p}]{x}|\,\|f\|_{\mathcal{L}^p(I_x)}.$$

Proof. In the following, all functions $C_n(\ldots)$ which are applied for error estimation fulfill $|C_n(\ldots)| \leq 1$. The differential equation (3.5) and the Lemmas 3.1 and 3.2 for $\xi \in I_x$ imply

$$\begin{aligned} y(\xi) &= y(x) + (\xi - x)y'(x) + \int_x^\xi \int_x^\varsigma (V_j(\tau) - \lambda)y(\tau) - f(\tau)d\tau d\varsigma \\ &= y(x) + (\xi - x)y'(x) + C_1(x,\xi,f)x\sqrt[\hat{p}]{x}\,\|f\|_{\mathcal{L}^p(I_x)} \\ &\quad + C_2(j,\lambda,x,\xi,y)(\mu(x) + x^2|\lambda|)\sup_{\varsigma\in I_x\setminus I_\xi}|y(\varsigma)|. \end{aligned}$$

For the supremum $\sup_{\varsigma\in I_x\setminus I_\xi}|y(\varsigma)|$ this yields the equation

$$\begin{aligned} \sup_{\varsigma\in I_x\setminus I_\xi}|y(\varsigma)| &= C_3(x,\xi,y)y(x) + C_4(x,\xi,y)xy'(x) + C_5(x,\xi,f)x\sqrt[\hat{p}]{x}\,\|f\|_{\mathcal{L}^p(I_x)} \\ &\quad + C_6(j,\lambda,x,\xi,y)(\mu(x) + x^2|\lambda|)\sup_{\varsigma\in I_x\setminus I_\xi}|y(\varsigma)|. \end{aligned}$$

It can be solved when $\mu(x) + x^2|\lambda| < 1$ and then gives

$$\begin{aligned} \sup_{\varsigma\in I_x\setminus I_\xi}|y(\varsigma)| &= \frac{C_3(x,\xi,y)y(x) + C_4(x,\xi,y)xy'(x) + C_5(x,\xi,f)x\sqrt[\hat{p}]{x}\,\|f\|_{\mathcal{L}^p(I_x)}}{1 - C_6(j,\lambda,x,\xi,y)(\mu(x) + x^2|\lambda|)} \\ &\leq \frac{|y(x)| + |xy'(x)| + |x\sqrt[\hat{p}]{x}|\,\|f\|_{\mathcal{L}^p(I_x)}}{1-\mu(x)-x^2|\lambda|}. \end{aligned}$$

This implies the assertion because $x \in X$ and the last expression on the right side is independent of $\xi \in I_x$. □

Corollary 3.7 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in I \setminus \{0\}$, and $f \in \mathcal{L}^p(I_x)$. Then all solutions y of (3.4) and (3.5) are bounded on I_x.*

Proof. Choose $\xi \in X \cap I_x$. First consider a solution y on $I_x \setminus I_\xi$. Since $V_j \in \mathcal{L}^1(I_x \setminus I_\xi)$, both y and y' are bounded there. On the interval I_ξ, Lemma 3.6 proves that y is bounded. □

If $p = 2$ and V_j is real valued, this implies that the differential expression l_j on $I_\pm$ is regular or in limit circle case at $0\pm$. If $p = 2$ and V_j is complex valued, this implies that the differential expression l_j on $I_\pm$ is regular or in case II or III at $0\pm$. For the definition of case II and III see [EE, p. 159].
Assume $x \in I \setminus \{0\}$. Then define the bilinear form $\lceil\cdot,\cdot\rfloor_x$ by

$$\lceil y_1, y_2\rfloor_x = y_1(x)y_2'(x) - y_1'(x)y_2(x),$$

for all $y_1, y_2 \in \mathcal{AC}^2_{\mathrm{loc}}(I \setminus \{0\})$. Then Green's formula

$$\lceil y_1, y_2 \rfloor_{x_2} - \lceil y_1, y_2 \rfloor_{x_1} = \int_{x_1}^{x_2} l_\infty[y_1](x) y_2(x) - y_1(x) l_\infty[y_2](x) dx \tag{3.7}$$

holds if both $x_1, x_2 \in I_-$, or $x_1, x_2 \in I_+$. See, for instance, [EE, III, (10.6)], but note that it is more convenient in the present context to use a bilinear form $\lceil \cdot, \cdot \rfloor_x$ instead of the sesquilinear form $[\cdot, \cdot]_x$ introduced in (2.2). The reason is that the potentials dealt with are complex valued now.

Corollary 3.8 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f_1, f_2 \in \mathcal{L}^p(I_x)$, and that the functions y_1 and y_2 are solutions of (3.5) on I_x for $f = f_1$ and $f = f_2$, respectively. Then the limits*

$$\lim_{\xi \to 0\pm} \lceil y_1, y_2 \rfloor_\xi$$

exist and are finite.

Proof. Lemma 3.6 implies $y_1, y_2 \in \mathcal{L}^\infty(I_x)$. Moreover, $l_j[y_1] = f_1, l_j[y_2] = f_2 \in \mathcal{L}^p(I_x) \subseteq \mathcal{L}^1(I_x)$. Hence, Green's formula (3.7) yields the assertion. □

Corollary 3.9 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in I \setminus \{0\}$, $f \in \mathcal{L}^p(I_x)$, and that the function y is a solution of (3.5) on I_x. Then there exists a function $u \in \mathcal{M}^1(I_x)$ with*

$$\|u\|_{\mathcal{L}^1(I_x)} \leq (\mu(x) + x^2|\lambda|) \|y\|_{\mathcal{L}^\infty(I_x)} + |x \sqrt[\hat{p}]{x}| \|f\|_{\mathcal{L}^p(I_x)}$$

such that $|y'(\xi) - y'(x)| \leq u(\xi)$ for $\xi \in I_x$. Particularly, $y' \in \mathcal{L}^1(I_x)$.

Proof. By Corollary 3.7, $\|y\|_{\mathcal{L}^\infty(I_x)}$ is finite. For $\xi \in I_x$, it holds

$$y'(\xi) - y'(x) = \int_x^\xi y''(\varsigma) d\varsigma = \int_x^\xi (V_j(\varsigma) - \lambda) y(\varsigma) - f(\varsigma) d\varsigma.$$

Let the function u for $\xi \in I_x$ be given by

$$u(\xi) = \left| \int_x^\xi |V_j(\varsigma)| + |\lambda| d\varsigma \right| \|y\|_{\mathcal{L}^\infty(I_x)} + \sqrt[\hat{p}]{|x|} \|f\|_{\mathcal{L}^p(I_x)}.$$

Then $|y'(\xi) - y'(x)| \leq u(\xi)$ for all $\xi \in I_x$ by Lemma 3.2. Lemma 3.1 and the relation (3.6) imply the assertions on u. □

Corollary 3.10 *Assume $\lambda \in \Lambda$, $x \in I \setminus \{0\}$, $f \in \mathcal{L}^p(I_x)$, and that the function y is a solution of (3.5) on $I_x \cup I_{-x}$. Then the limits*

$$y(0\pm) = \lim_{x \to 0\pm} y(x)$$

exist and are finite.

For a result similar to Lemma 3.6, see [EE, Theorem III.10.17], where estimates for the solutions in the case of a nonsingular potential are given. For a result comparable to Corollary 3.7 see [W, Theorem 6.4].

3.3 The operators $H_{j,\lambda,n,x}$ and their properties

For $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in I \setminus \{0\}$, and $n \in \mathbb{N}$ define the integral operators $F_{j,\lambda,x}$ and $H_{j,\lambda,n,x}$ on $\mathcal{L}^\infty(I_x)$ by

$$(F_{j,\lambda,x}y)(\xi) = \int_0^\xi \int_x^\varsigma (V_j(\tau) - \lambda)y(\tau)d\tau d\varsigma, \quad \xi \in I_x, y \in \mathcal{L}^\infty(I_x), \tag{3.8}$$

$$H_{j,\lambda,n,x} = \sum_{m=0}^{n-1} F_{j,\lambda,x}^m. \tag{3.9}$$

Similar operators are well known from the Picard-Lindelöf procedure, also compare Corollary 3.17 below. The definition of $H_{j,\lambda,n,x}$ implies the recursion formulas

$$H_{j,\lambda,0,x} = 0, \tag{3.10}$$

$$H_{j,\lambda,n+1,x} = 1 + F_{j,\lambda,x}H_{j,\lambda,n,x}, \quad n \in \mathbb{N}. \tag{3.11}$$

Lemma 3.11 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in I \setminus \{0\}$. Then it holds*

$$\|F_{j,\lambda,x}\|_{\mathcal{L}^\infty(I_x)} \le \mu(x) + x^2|\lambda|.$$

Proof. For $y \in \mathcal{L}^\infty(I_x)$, Lemma 3.1 and (3.6) imply

$$|(F_{j,\lambda,x}y)(\xi)| \le \int_x^0 \int_x^\varsigma (|V_j(\tau)| + |\lambda|)|y(\tau)|d\tau d\varsigma \le (\mu(x) + x^2|\lambda|) \|y\|_{\mathcal{L}^\infty(I_x)}.$$

□

Corollary 3.12 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in X \setminus \{0\}$. Then $F_{j,\lambda,x}^n \to 0$ and $H_{j,\lambda,n,x}$ converges if $n \to \infty$. These convergence relations are uniform with respect to $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in X \setminus \{0\}$.*

For $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in X \setminus \{0\}$ define $H_{j,\lambda,\infty,x}$ as

$$H_{j,\lambda,\infty,x} = \lim_{n\to\infty} H_{j,\lambda,n,x} = \sum_{m=0}^{\infty} F_{j,\lambda,x}^m. \tag{3.12}$$

Lemma 3.13 *Assume $\lambda \in \Lambda$, $n \in \mathbb{N} \cup \{\infty\}$, and $x \in X \setminus \{0\}$. Then*

$$F_{j,\lambda,x} \to F_{\infty,\lambda,x}, \quad H_{j,\lambda,n,x} \to H_{\infty,\lambda,n,x}$$

if $j \to \infty$, uniformly with respect to all other variables.

Proof. For $\xi \in I_x$, it holds

$$\begin{aligned}\|F_{\infty,\lambda,x} - F_{j,\lambda,x}\|_{\mathcal{L}^\infty(I_x)} &\le \int_x^0 \int_x^\varsigma |V_\infty(\tau) - V_j(\tau)|d\tau d\varsigma \\ &= \int_x^\xi \int_x^\varsigma |V_\infty(\tau) - V_j(\tau)|d\tau d\varsigma + \int_\xi^0 \int_x^\xi |V_\infty(\tau) - V_j(\tau)|d\tau d\varsigma \\ &\quad + \int_\xi^0 \int_\xi^\varsigma |V_\infty(\tau) - V_j(\tau)|d\tau d\varsigma.\end{aligned}$$

For $\epsilon > 0$ choose the number $\xi \in I_x$ such that the absolute value of the last integral on the right side is less than ϵ for all $j \in \mathbb{N}$. This is possible because of Lemma 3.1. The first two integrals on the right side then tend to zero if $j \to \infty$ in consequence of (3.2). Consequently for $n \in \mathbb{N}$, $F^n_{j,\lambda,x} \to F^n_{\infty,\lambda,x}$ if $j \to \infty$, and this convergence is uniform with respect to $\lambda \in \Lambda$ and $x \in X \setminus \{0\}$. The convergence assertion on $H_{j,\lambda,n,x}$, $n \in \mathbb{N} \cup \{\infty\}$ now follows because these operators are defined as sums with an absolutely convergent upper bound by Lemma 3.11. □

Lemma 3.14 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda_1, \lambda_2 \in \Lambda$, $n \in \mathbb{N} \cup \{\infty\}$, $x \in X \setminus \{0\}$, and $y \in \mathcal{L}^\infty(I_x)$. Then it holds*

$$\begin{aligned} |(H_{j,\lambda_1,n,x}y - H_{j,\lambda_2,n,x}y)'(\xi)| &\leq 4|x(\lambda_1 - \lambda_2)|\, \|y\|_{\mathcal{L}^\infty(I_x)}, \\ |(H_{j,\lambda_1,n,x}y - H_{j,\lambda_2,n,x}y)(\xi)| &\leq 4|\xi x(\lambda_1 - \lambda_2)|\, \|y\|_{\mathcal{L}^\infty(I_x)}, \end{aligned}$$

and if $j \to \infty$, the differences $(H_{j,\lambda_1,n,x}y - H_{j,\lambda_2,n,x}y)'(\xi)$ and $(H_{j,\lambda_1,n,x}y - H_{j,\lambda_2,n,x}y)(\xi)$ converge uniformly with respect to $n \in \mathbb{N} \cup \{\infty\}$, $\xi \in I_x$, and $y \in \mathcal{B}(\mathcal{L}^\infty(I_x))$.

Proof. Without loss of generality assume $|\lambda_1| \leq |\lambda_2|$. By interlaced induction on $m \in \mathbb{N} \setminus \{0\}$, first the following two pairs of hypotheses are proved. Firstly,

$$|(F^m_{j,\lambda_1,x}y - F^m_{j,\lambda_2,x}y)'(\xi)| \leq \left|(\lambda_1 - \lambda_2)x(\mu(x) + x^2|\lambda_2|)^{m-1}\right| m\, \|y\|_{\mathcal{L}^\infty(I_x)},$$

and if $j \to \infty$, the expression $(F^m_{j,\lambda_1,x}y - F^m_{j,\lambda_2,x}y)'$ converges in $\mathcal{L}^\infty(I_x)$ uniformly with respect to $y \in \mathcal{B}(\mathcal{L}^\infty(I_x))$. Secondly,

$$|(F^m_{j,\lambda_1,x}y - F^m_{j,\lambda_2,x}y)(\xi)| \leq \left|(\lambda_1 - \lambda_2)x(\mu(x) + x^2|\lambda_2|)^{m-1}\xi\right| m\, \|y\|_{\mathcal{L}^\infty(I_x)},$$

and if $j \to \infty$, the quotient $\frac{(F^m_{j,\lambda_1,x}y - F^m_{j,\lambda_2,x}y)(\xi)}{\xi}$ converges uniformly with respect to $\xi \in I_x$ and $y \in \mathcal{B}(\mathcal{L}^\infty(I_x))$. It holds

$$\begin{aligned} (F_{j,\lambda_1,x}y - F_{j,\lambda_2,x}y)'(\xi) &= \int_x^\xi (\lambda_2 - \lambda_1)y(\varsigma)d\varsigma, \\ |(F_{j,\lambda_1,x}y - F_{j,\lambda_2,x}y)'(\xi)| &\leq |x(\lambda_1 - \lambda_2)|\, \|y\|_{\mathcal{L}^\infty(I_x)}, \end{aligned}$$

which proves the first pair of hypotheses for $m = 1$. If the first pair of hypotheses holds for $m \in \mathbb{N}$, the second pair follows by integration. Now assume that the second pair of hypotheses holds for $m \in \mathbb{N}$. For $\xi \in I_x$ compute

$$\begin{aligned} (F^{m+1}_{j,\lambda_1,x}y - F^{m+1}_{j,\lambda_2,x}y)'(\xi) &= (F_{j,\lambda_1,x}(F^m_{j,\lambda_1,x} - F^m_{j,\lambda_2,x})y + (F_{j,\lambda_1,x} - F_{j,\lambda_2,x})F^m_{j,\lambda_2,x}y)'(\xi) \\ &= \int_x^\xi (V_j(\varsigma) - \lambda_1)\varsigma \frac{(F^m_{j,\lambda_1,x}y - F^m_{j,\lambda_2,x}y)(\varsigma)}{\varsigma} d\varsigma + (\lambda_2 - \lambda_1)\int_x^\xi (F^m_{j,\lambda_2,x}y)(\varsigma)d\varsigma. \end{aligned}$$

If $j \to \infty$, the second pair of hypotheses, (3.2), and (3.3) imply that the first integral converges and Lemma 3.13 yields that the second integral converges. Both convergence relations are uniform with respect to $\xi \in I_x$. Moreover, the second pair of hypotheses implies that for all $j \in \mathbb{N} \cup \{\infty\}$, it holds

$$\begin{aligned} &|(F^{m+1}_{j,\lambda_1,x}y - F^{m+1}_{j,\lambda_2,x}y)'(\xi)| \\ &\leq \left(\left|(\lambda_1 - \lambda_2)mx(\mu(x) + x^2|\lambda_2|)^{m-1} \int_x^\xi |V_j(\varsigma) - \lambda_1|\, |\varsigma| d\varsigma\right| \right. \\ &\quad \left. + \left|(\lambda_1 - \lambda_2)\int_x^\xi (\mu(x) + x^2|\lambda_2|)^m d\varsigma\right| \right) \|y\|_{\mathcal{L}^\infty(I_x)} \\ &\leq |(\lambda_1 - \lambda_2)(m+1)x(\mu(x) + x^2|\lambda_2|)^m|\, \|y\|_{\mathcal{L}^\infty(I_x)} \end{aligned}$$

for $m \in \mathbb{N} \setminus \{0\}$ using (3.3). For $H_{j,\lambda_1,n,x}$ and $H_{j,\lambda_1,n,x}$, this and $x \in X$ yield

$$\begin{aligned}|(H_{j,\lambda_1,n,x}y - H_{j,\lambda_2,n,x}y)'(\xi)| &\le \sum_{m=0}^{n-1} \left|(F^m_{j,\lambda_1,x}y - F^m_{j,\lambda_2,x}y)'(\xi)\right| \\ &\le |(\lambda_1-\lambda_2)x|\, \|y\|_{\mathcal{L}^\infty(I_x)} \sum_{m=1}^{n} m(\mu(x)+x^2|\lambda_2|)^{m-1} \\ &= \frac{|(\lambda_1-\lambda_2)x|}{(1-\mu(x)-x^2|\lambda_2|)^2} \|y\|_{\mathcal{L}^\infty(I_x)} \le 4|(\lambda_1-\lambda_2)x|\, \|y\|_{\mathcal{L}^\infty(I_x)}\,.\end{aligned}$$

The second assertion of the lemma follows by integration. The convergence of the differences follow because the expressions are sums of terms which converge if $j \to \infty$ and which admit an absolutely convergent upper bound. □

Lemma 3.15 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $n \in \mathbb{N}$, $x \in I \setminus \{0\}$, $f \in \mathcal{L}^p(I_x)$, and that the function y is a solution of (3.5) on I_x. Let the function u be given by the formula*

$$u(\xi) = \int_0^\xi \int_x^\varsigma f(\tau)d\tau d\varsigma, \quad \xi \in I_x. \tag{3.13}$$

Then for all $\xi \in I_x$ it holds

$$y(\xi) = y(0)(H_{j,\lambda,n,x}1)(\xi) + y'(x)(H_{j,\lambda,n,x}\iota)(\xi) - (H_{j,\lambda,n,x}u)(\xi) + (F^n_{j,\lambda,x}y)(\xi).$$

Proof. If y fulfills (3.5), then Corollary 3.9 implies $y' \in \mathcal{L}^1(I_x)$. For $\xi \in I_x$, it follows

$$\begin{aligned} y(\xi) &= y(0) + \int_0^\xi y'(\varsigma)d\varsigma = y(0) + \xi y'(x) + \int_0^\xi \int_x^\varsigma y''(\tau)d\tau d\varsigma \\ &= y(0) + \xi y'(\xi) + \int_0^\xi \int_x^\varsigma (V_j(\tau)-\lambda)y(\tau) - f(\tau)d\tau d\varsigma \\ &= y(0) + \xi y'(x) - u(\xi) + (F_{j,\lambda,x}y)(\xi). \end{aligned} \tag{3.14}$$

Now the proof of the Lemma is by induction on n. For $n=0$, the assertion is implied by (3.10). For arbitrary $n \in \mathbb{N}$, (3.14), the induction hypothesis, and (3.11) imply

$$\begin{aligned} y(\xi) &= y(0) + \xi y'(x) - u(\xi) \\ &\quad + y(0)(F_{j,\lambda,x}H_{j,\lambda,n,x}1)(\xi) + y'(x)(F_{j,\lambda,x}H_{j,\lambda,n,x}\iota)(\xi) - (F_{j,\lambda,x}H_{j,\lambda,n,x}u)(\xi) \\ &\quad + (F_{j,\lambda,x}F^n_{j,\lambda,x}y)(\xi) \\ &= y(0)(H_{j,\lambda,n+1,x}1)(\xi) + y'(x)(H_{j,\lambda,n+1,x}\iota)(\xi) - (H_{j,\lambda,n+1,x}u)(\xi) + (F^{n+1}_{j,\lambda,x}y)(\xi). \end{aligned}$$

□

Corollary 3.16 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f \in \mathcal{L}^p(I_x)$, and that the function y is a solution of (3.5) on I_x. Let the function u be given by the double integral formula (3.13). Then for $\xi \in I_x$, it holds*

$$y(\xi) = y(0)(H_{j,\lambda,\infty,x}1)(\xi) + y'(x)(H_{j,\lambda,\infty,x}\iota)(\xi) - (H_{j,\lambda,\infty,x}u)(\xi).$$

Proof. If $n \to \infty$ in the assertion of Lemma 3.15, then $F^n_{j,\lambda,x} \to 0$, and $H_{j,\lambda,n,x} \to H_{j,\lambda,\infty,x}$ by Corollary 3.12. □

Corollary 3.17 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in X \setminus \{0\}$. Then the functions $H_{j,\lambda,\infty,x}1$ and $H_{j,\lambda,\infty,x}\iota$ are solutions of (3.4) with the boundary conditions*

$$\begin{array}{c} (H_{j,\lambda,\infty,x}1)(0)=1,\ (H_{j,\lambda,\infty,x}1)'(x)=0, \\ (H_{j,\lambda,\infty,x}\iota)(0)=0,\ (H_{j,\lambda,\infty,x}\iota)'(x)=1. \end{array}$$

Assume additionally $f \in \mathcal{L}^p(I_x)$ and that the function u is given by the double integral formula (3.13). Then the function $-H_{j,\lambda,\infty,x}u$ is a solution of (3.5) with the boundary conditions

$$(-H_{j,\lambda,\infty,x}u)(0) = 0, \quad (-H_{j,\lambda,\infty,x}u)'(x) = 0.$$

Remark 3.18 The function $-H_{j,\lambda,\infty,x}u$ which is defined in Corollary 3.17 solves the boundary value problem stated there for all $\lambda \in \Lambda$. Hence, if $x \in X \setminus \{0\}$, Λ is contained in the resolvent set of the differential operator on $\mathcal{L}^p(I_x)$ which corresponds to this boundary value problem.

Corollary 3.19 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, and $\xi \in I_x$. Then on I_ξ it holds*

$$\begin{aligned} H_{j,\lambda,\infty,\xi}1 &= H_{j,\lambda,\infty,x}1 - \frac{(H_{j,\lambda,\infty,x}1)'(\xi)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)} H_{j,\lambda,\infty,x}\iota, \\ H_{j,\lambda,\infty,\xi}\iota &= \frac{1}{(H_{j,\lambda,\infty,x}\iota)'(\xi)} H_{j,\lambda,\infty,x}\iota. \end{aligned}$$

Proof. Corollaries 3.16 and 3.17 imply

$$H_{j,\lambda,\infty,x}\iota = (H_{j,\lambda,\infty,x}\iota)(0)H_{j,\lambda,\infty,\xi}1 + (H_{j,\lambda,\infty,x}\iota)'(\xi)H_{j,\lambda,\infty,\xi}\iota = (H_{j,\lambda,\infty,x}\iota)'(\xi)H_{j,\lambda,\infty,\xi}\iota.$$

This proves the second assertion. Corollaries 3.16, 3.17, and finally the second assertion of the lemma imply

$$\begin{aligned} H_{j,\lambda,\infty,x}1 &= (H_{j,\lambda,\infty,x}1)(0)H_{j,\lambda,\infty,\xi}1 + (H_{j,\lambda,\infty,x}1)'(\xi)H_{j,\lambda,\infty,\xi}\iota \\ &= H_{j,\lambda,\infty,\xi}1 + (H_{j,\lambda,\infty,x}1)'(\xi)H_{j,\lambda,\infty,\xi}\iota = H_{j,\lambda,\infty,\xi}1 + \frac{(H_{j,\lambda,\infty,x}1)'(\xi)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)} H_{j,\lambda,\infty,x}\iota. \end{aligned}$$

This proves the first assertion. □

Also the following two-sided variant of Corollary 3.16 will be used. It provides a very useful decomposition of solutions y of (3.5) near $x = 0$.

Corollary 3.20 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \cap I_+$, $f \in \mathcal{L}^p([-x,x])$, and that the function y is a solution of (3.5) on $[-x,x] \setminus \{0\}$. Then for $\xi \in [-x,x] \setminus \{0\}$, it holds*

$$y(\xi) = \left\{ \begin{array}{ll} y(0-)(H_{j,\lambda,\infty,-x}1)(\xi) + y'(-x)(H_{j,\lambda,\infty,-x}\iota)(\xi) - (H_{j,\lambda,\infty,-x}u)(\xi) & \text{if } \xi \in I_{-x} \\ y(0+)(H_{j,\lambda,\infty,x}1)(\xi) + y'(x)(H_{j,\lambda,\infty,x}\iota)(\xi) - (H_{j,\lambda,\infty,x}u)(\xi) & \text{if } \xi \in I_x \end{array} \right. ,$$

where the function u for $\xi \in [-x,x]$ is given by

$$u(\xi) = \left\{ \begin{array}{ll} \int_0^\xi \int_{-x}^\varsigma f(\tau) d\tau d\varsigma & \text{if } \xi \in I_{-x} \\ \int_0^\xi \int_x^\varsigma f(\tau) d\tau d\varsigma & \text{if } \xi \in I_x \end{array} \right. . \tag{3.15}$$

Lemma 3.21 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $n \in \mathbb{N}$, $x \in I \setminus \{0\}$, and $y \in \mathcal{AC}(I_x)$ with $y(0) = 0$ and $y' \in \mathcal{L}^\infty(I_x)$. Then for $\xi \in I_x$, it holds*

$$\begin{aligned} |(F^n_{j,\lambda,x}y)'(\xi)| &\leq (\mu(x) + x^2|\lambda|)^n \, \|y'\|_{\mathcal{L}^\infty(I_x)}, \\ |(F^n_{j,\lambda,x}y)(\xi)| &\leq |\xi|(\mu(x) + x^2|\lambda|)^n \, \|y'\|_{\mathcal{L}^\infty(I_x)}. \end{aligned}$$

Proof. The proof is by interlaced induction for both assertions. For $n = 0$, the first assertion is $|y'(\xi)| \leq \|y'\|_{\mathcal{L}^\infty(I_x)}$ for $\xi \in I_x$. This is true.
If the first assertion holds for $n \in \mathbb{N}$, the second one for n directly follows by integration.
If the second assertion holds for $n \in \mathbb{N}$, it follows

$$\begin{aligned} |(F^{n+1}_{j,\lambda,x}y)'(\xi)| &= \left| \int_x^\xi (V_j(\varsigma) - \lambda)(F^n_{j,\lambda,x}y)(\varsigma)d\varsigma \right| \\ &\leq (\mu(x) + x^2|\lambda|)^n \, \|y'\|_{\mathcal{L}^\infty(I_x)} \int_x^\xi (|V_j(\varsigma)| + |\lambda|)\varsigma d\varsigma \leq (\mu(x) + x^2|\lambda|)^{n+1} \, \|y'\|_{\mathcal{L}^\infty(I_x)} \end{aligned}$$

by (3.6) and Lemma 3.1. □

Lemma 3.22 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $n \in (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$, $x \in X \setminus \{0\}$, and $f \in \mathcal{L}^p(I_x)$. Let the function u be given by the double integral formula (3.13). Then for all $\xi \in I_x$, it holds*

$$\begin{aligned} |(H_{j,\lambda,n,x}1)(\xi) - 1| &\leq 2(\mu(x) + x^2|\lambda|), \\ (H_{j,\lambda,n,x}1)' &\in \mathcal{L}^1(I_x), \\ \|(H_{j,\lambda,n,x}1)'\|_{\mathcal{L}^1(I_x)} &\leq 2(\mu(x) + x^2|\lambda|), \\ |(H_{j,\lambda,n,x}\iota)(\xi) - \xi| &\leq 2|\xi|(\mu(x) + x^2|\lambda|), \\ |(H_{j,\lambda,n,x}\iota)'(\xi) - 1| &\leq 2(\mu(x) + x^2|\lambda|), \\ |(H_{j,\lambda,n,x}u)(\xi)| &\leq 2|\xi| \sqrt[\hat{p}]{|x|} \, \|f\|_{\mathcal{L}^p(I_x)}, \\ |(H_{j,\lambda,n,x}u)'(\xi)| &\leq 2 \sqrt[\hat{p}]{|x|} \, \|f\|_{\mathcal{L}^p(I_x)}. \end{aligned}$$

Moreover, the absolute value of the function $(H_{j,\lambda,n,x}1)'$ is bounded by a function $v \in \mathcal{M}^1(I_x)$ with $\|v\|_{\mathcal{L}^1(I_x)} \leq 2(\mu(x) + x^2|\lambda|)$.

Proof. Lemma 3.11 for $m \in \mathbb{N}$ implies $|(F^m_{j,\lambda,x}1)(\xi)| \leq (\mu(x) + x^2|\lambda|)^m$. Summation according to (3.9) or (3.12) and $x \in X$ yield

$$|(H_{j,\lambda,n,x}1)(\xi) - 1| \leq \frac{\mu(x) + x^2|\lambda|}{1 - \mu(x) - x^2|\lambda|} \leq 2(\mu(x) + x^2|\lambda|).$$

For the derivative, (3.11) yields

$$\begin{aligned} |(H_{j,\lambda,n,x}1)'(\xi)| &= \left| \int_x^\xi (V_j(\varsigma) - \lambda)(H_{j,\lambda,n-1,x}1)(\varsigma)d\varsigma \right| \\ &\leq (1 + 2(\mu(x) + x^2|\lambda|)) \left| \int_x^\xi |V_j(\varsigma)| + |\lambda| d\varsigma \right| \leq 2 \left| \int_x^\xi |V_j(\varsigma)| + |\lambda| d\varsigma \right|. \end{aligned}$$

Call the rightmost expression $v(\xi)$. Now Lemma 3.1 and (3.6) imply the assertions on v and $(H_{j,\lambda,n,x}1)'$.
Lemma 3.21 for $m \in \mathbb{N}$ implies $|(F^m_{j,\lambda,x}\iota)'(\xi)| \le (\mu(x) + x^2|\lambda|)^m$. Summation according to (3.9) or (3.12) and $x \in X$ yield

$$|(H_{j,\lambda,n,x}\iota)'(\xi) - 1| \le \frac{\mu(x) + x^2|\lambda|}{1 - \mu(x) - x^2|\lambda|} \le 2(\mu(x) + x^2|\lambda|),$$

The assertion on $H_{j,\lambda,n,x}\iota$ follows from this by integration.
Lemma 3.2 yields $\|u'\|_{\mathcal{L}^\infty(I_x)} \le \sqrt[\hat{p}]{|x|}\,\|f\|_{\mathcal{L}^p(I_x)}$. This and Lemma 3.21 for $m \in \mathbb{N}$ imply

$$|(F^m_{j,\lambda,x}u)'(\xi)| \le (\mu(x) + x^2|\lambda|)^m \sqrt[\hat{p}]{|x|}\,\|f\|_{\mathcal{L}^p(I_x)}.$$

Summation according to (3.9) or (3.12) and $x \in X$ yield

$$|(H_{j,\lambda,n,x}u)'(\xi)| \le \frac{\sqrt[\hat{p}]{|x|}\,\|f\|_{\mathcal{L}^p(I_x)}}{1 - \mu(x) - x^2|\lambda|} \le 2\sqrt[\hat{p}]{|x|}\,\|f\|_{\mathcal{L}^p(I_x)}.$$

The assertion on $H_{j,\lambda,n,x}u$ follows from this by integration. □

Corollary 3.23 *Assume $j \in \mathbb{N}\cup\{\infty\}$, $\lambda \in \Lambda$, $n \in \mathbb{N}\cup\{\infty\}$, $x \in X\setminus\{0\}$, and $f \in \mathcal{L}^p(I_x)$. Let the function u be given by the double integral formula (3.13). Then it holds*

$$\begin{aligned}(H_{j,\lambda,n,x}\iota)'(\xi) &\to (H_{j,\lambda,n,x}\iota)'(0),\\ (H_{j,\lambda,n,x}u)'(\xi) &\to (H_{j,\lambda,n,x}u)'(0)\end{aligned}$$

if $\xi \to 0$.

Proof. The case $n = 0$ is trivial by (3.10). If $n > 0$ use (3.11) and, if $n = \infty$, additionally (3.12). With Lemma 3.22 and (3.3) compute

$$\begin{aligned}|(H_{j,\lambda,n,x}\iota)'(0) - (H_{j,\lambda,n,x}\iota)'(\xi)| &= \left|\int_\xi^0 (V_j(\varsigma) - \lambda)(H_{j,\lambda,n-1,x}\iota)(\varsigma)d\varsigma\right|\\ &\le 2\int_\xi^0 (|V_j(\varsigma)| + |\lambda|)\varsigma d\varsigma \to 0.\end{aligned}$$

The assertion about $(H_{j,\lambda,n,x}u)'(\xi)$ is proved in the same fashion with ι replaced by u. □

Lemma 3.24 *Assume $j \in \mathbb{N}\cup\{\infty\}$, $\lambda \in \Lambda$, $x \in X\setminus\{0\}$, and $\xi \in I_x$. Then*

$$|\xi(H_{j,\lambda,\infty,x}1)'(\xi)| \le 2(\mu(x) + x^2|\lambda|).$$

Proof. By Lemma 3.22, $|(H_{j,\lambda,\infty,x}1)'(\xi)| \le u(\xi)$ for some function $u \in \mathcal{M}^1(I_x)$ with $\|u\|_{\mathcal{L}^1(I_x)} \le 2(\mu(x) + x^2|\lambda|)$. Then Lemma 3.5 yields

$$|\xi(H_{j,\lambda,\infty,x}1)'(\xi)| \le |\xi u(\xi)| \le \|u\|_{\mathcal{L}^1(I_x)} \le 2(\mu(x) + x^2|\lambda|).$$

□

3.4 Definition of $g_{j,\lambda,n,x}$ and interface conditions in 0

For $j \in \mathbb{N}$, $\lambda \in \Lambda$, $n \in \mathbb{N} \cup \{\infty\}$, and $x \in X \setminus \{0\}$ define

$$g_{j,\lambda,n,x} = (H_{j,\lambda,n,-x}1)'(0) - (H_{j,\lambda,n,x}1)'(0) \tag{3.16}$$

and for all numbers $\lambda \in \Lambda$, $n \in \mathbb{N} \cup \{\infty\}$, and $x \in X \setminus \{0\}$ where the limit exists in the compact space $\overline{\mathbb{C}}$ define

$$g_{\infty,\lambda,n,x} = \overset{\overline{\mathbb{C}}}{\lim_{j\to\infty}} g_{j,\lambda,n,x}. \tag{3.17}$$

Lemma 3.25 *Either $g_{\infty,\lambda,\infty,x}$ is defined for all $\lambda \in \Lambda$ and $x \in X \setminus \{0\}$, or it is undefined for all $\lambda \in \Lambda$ and $x \in X \setminus \{0\}$. Moreover in the first case, if $\lambda \in \Lambda$, $x \in X$, and $\xi \in I_x$, then it holds*

$$\begin{aligned} g_{\infty,\lambda,\infty,\xi} - g_{\infty,\lambda,\infty,x} &= \frac{(H_{j,\lambda,\infty,x}1)'(\xi)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)}(H_{j,\lambda,\infty,x}\iota)'(0) \\ &\quad - \frac{(H_{j,\lambda,\infty,-x}1)'(-\xi)}{(H_{j,\lambda,\infty,-x}\iota)'(-\xi)}(H_{j,\lambda,\infty,-x}\iota)'(0), \\ |g_{\infty,\lambda,\infty,x} - g_{\infty,0,\infty,x}| &\leq 8|x\lambda|. \end{aligned}$$

Particularly, if $g_{\infty,\lambda,\infty,x} = \infty$ for one choice of numbers $\lambda \in \Lambda$ and $x \in X \setminus \{0\}$, then this is so for all such choices.

Proof. Trivially, $g_{\infty,\lambda,\infty,0} = 0$ is always defined. Assume $x \in X \setminus \{0\}$ and $\xi \in I_x$. For $j \in \mathbb{N}$, the relation (3.16) and Corollary 3.19 imply

$$\begin{aligned} g_{j,\lambda,\infty,\xi} - g_{j,\lambda,\infty,x} &= (H_{j,\lambda,\infty,-\xi}1)'(0) - (H_{j,\lambda,\infty,-x}1)'(0) \\ &\quad -(H_{j,\lambda,\infty,\xi}1)'(0) + (H_{j,\lambda,\infty,x}1)'(0) \\ &= -\frac{(H_{j,\lambda,\infty,-x}1)'(-\xi)}{(H_{j,\lambda,\infty,-x}\iota)'(-\xi)}(H_{j,\lambda,\infty,-x}\iota)'(0) + \frac{(H_{j,\lambda,\infty,x}1)'(\xi)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)}(H_{j,\lambda,\infty,x}\iota)'(0). \end{aligned}$$

The terms on the right side converge if $j \to \infty$. Note that the terms in the denominators on the right side by Lemma 3.22 cannot tend to zero if $x \in X \setminus \{0\}$. So the definition (3.17) of $g_{\infty,\lambda,\infty,x}$ and $g_{\infty,\lambda,\infty,\xi}$ implies that one of these numbers is defined if and only if the other is too. Moreover in this case, the above equality also holds for $j = \infty$.
As to the dependence on λ, the proof is similar. The definition (3.16) for $j \in \mathbb{N}$ implies

$$g_{j,\lambda,\infty,x} - g_{j,0,\infty,x} = (H_{j,\lambda,n,-x}1)'(0) - (H_{j,0,n,-x}1)'(0) - (H_{j,\lambda,n,x}1)'(0) + (H_{j,0,n,x}1)'(0) \tag{3.18}$$

and Lemma 3.14 proves that the right side converges if $j \to \infty$. So $g_{\infty,\lambda,\infty,x}$ is defined if and only if $g_{\infty,0,\infty,x}$ is. The estimate for the expression $|g_{\infty,\lambda,\infty,x} - g_{\infty,0,\infty,x}|$ also follows from (3.18) and Lemma 3.14. □

Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$ and $f \in \mathcal{L}^p(I_{-x} \cup I_x)$. Now interface conditions in 0 are formulated for a solution y of (3.5) in dependence of $g_{j,\lambda,\infty,x}$. If $g_{j,\lambda,\infty,x} \in \mathbb{C}$, then consider the pair

$$\begin{aligned} y(0-) &= y(0+), & (3.19) \\ \lim_{\xi\to 0}\left((H_{j,\lambda,\infty,\xi}\iota)'(0)y'(\xi) - (H_{j,\lambda,\infty,-\xi}\iota)'(0)y'(-\xi) - g_{j,\lambda,\infty,\xi}y(0)\right) &= 0 & (3.20) \end{aligned}$$

of interface conditions. If $g_{j,\lambda,\infty,x} = \infty$, then consider the Dirichlet interface conditions

$$y(0-) = y(0+) = 0. \tag{3.21}$$

Lemma 3.26 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f \in \mathcal{L}^p(I_{-x} \cup I_x)$, and that the function y is a solution of (3.5) on $I_{-x} \cup I_x$. Further assume $g_{j,\lambda,\infty,x} \in \mathbb{C}$ and that $\tilde{g}$ and $\tilde{h}$ are functions on $I_{-x} \cup I_x$ such that*

$$\lim_{\xi \to 0} (g_{j,\lambda,\infty,\xi} - \tilde{g}(\xi)) = 0, \quad \limsup_{\xi \to 0} \left| \frac{(H_{j,\lambda,\infty,\xi}\iota)'(0) - \tilde{h}(\xi)}{\xi} \right| < \infty.$$

Then the condition

$$\lim_{\xi \to 0} (\tilde{h}(\xi) y'(\xi) - \tilde{h}(-\xi) y'(-\xi) - \tilde{g}(\xi) y(0)) = 0 \tag{3.22}$$

is equivalent to (3.20).

Proof. By Corollary 3.9, the absolute value of the derivative y' is bounded by a function in $\mathcal{M}^1(I_{-x} \cup I_x)$. Then Lemma 3.5 and the assumption on $\tilde{h}$ imply $((H_{j,\lambda,\infty,\xi}\iota)'(0) - \tilde{h}(\xi))y'(\xi) \to 0$ if $\xi \to 0$. This and the assumption on $\tilde{g}$ finally yield

$$\begin{aligned} &\lim_{\xi \to 0} (\tilde{h}(\xi) y'(\xi) - \tilde{h}(-\xi) y'(-\xi) - \tilde{g}(\xi) y(0)) \\ &\quad = \lim_{\xi \to 0} ((H_{j,\lambda,\infty,\xi}\iota)'(0) y'(\xi) - (H_{j,\lambda,\infty,-\xi}\iota)'(0) y'(-\xi) - g_{j,\lambda,\infty,\xi} y(0)). \end{aligned}$$

□

The following result ensures that that the considered boundary conditions do not depend on λ.

Corollary 3.27 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f \in \mathcal{L}^p(I_{-x} \cup I_x)$, and that the function y is a solution of (3.5) on $I_{-x} \cup I_x$. Further assume $g_{j,\lambda,\infty,x} \in \mathbb{C}$. Then y fulfills (3.20) if and only if it fulfills (3.20) with λ replaced by an arbitrary number in Λ.*

Proof. The asserted independence follows from Lemmas 3.14, 3.25, and 3.26. □

Corollary 3.28 *Assume $j \in \mathbb{N}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f \in \mathcal{L}^p(I_{-x} \cup I_x)$, and that the function y is a solution of (3.5) on $I_{-x} \cup I_x$. Then y fulfills (3.19) and (3.20) if and only if $y(0-) = y(0+)$ and $y'(0-) = y'(0+)$.*

Proof. The formulas (3.1), (3.11), (3.12), and Lemma 3.22 imply

$$|(H_{j,\lambda,\infty,\xi}1)'(0)| = \left| \int_\xi^0 (V_j(\varsigma) - \lambda)(H_{j,\lambda,\infty,\xi}1)(\varsigma) d\varsigma \right| \le 2 \|V_\infty - \lambda\|_{\mathcal{L}^1(I_x)} \to 0$$

and hence $g_{j,\lambda,\infty,\xi} \to 0$ if $\xi \to 0$. Additionally use Corollary 3.3 to prove

$$|(H_{j,\lambda,\infty,\xi}\iota)'(0) - 1| = \left| \int_\xi^0 (V_j(\varsigma) - \lambda)(H_{j,\lambda,\infty,\xi}\iota)(\varsigma) d\varsigma \right| \le 2 \int_\xi^0 (|V_j(\varsigma)| + |\lambda|)\varsigma d\varsigma \le 2|\xi|.$$

So the constant functions $\tilde{g} = 0$ and $\tilde{h} = 1$ fulfill the assumptions of Lemma 3.26. Since $j \in \mathbb{N}$, (3.1) implies that the differential expression l_j is regular in 0 and the limits

$y'(0\pm)$ exist by Corollary 1.2. Hence, the condition (3.22) can be written in the form $y'(0-) = y'(0+)$. □

Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in X \cap I_+$. Define the boundary operator $\mathtt{b}_{j,\lambda,x}$ for all y which solve (3.5) on $[-x, x] \setminus \{0\}$ for some $f \in \mathcal{L}^p([-x, x])$ by

$$\mathtt{b}_{j,\lambda,x} y = \lim_{\xi \to 0+} \begin{pmatrix} \frac{\lceil y, H_{j,\lambda,\infty,-x}\iota \rfloor_{-\xi}}{(H_{j,\lambda,\infty,-x}\iota)'(0)} \\ \lceil y, H_{j,\lambda,\infty,-x}1 \rfloor_{-\xi} \\ -\frac{\lceil y, H_{j,\lambda,\infty,x}\iota \rfloor_{\xi}}{(H_{j,\lambda,\infty,x}\iota)'(0)} \\ -\lceil y, H_{j,\lambda,\infty,x}1 \rfloor_{\xi} \end{pmatrix}. \tag{3.23}$$

Note that $\mathtt{b}_{j,\lambda,x}$ is well defined by Corollary 3.8.

Lemma 3.29 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \cap I_+$, $f \in \mathcal{L}^p([-x, x])$, that the function u is given by (3.15), and that the function y is a solution of (3.5) on $[-x, x] \setminus \{0\}$. Then it holds*

$$\mathtt{b}_{j,\lambda,x} y = \begin{pmatrix} y(0-) \\ -(H_{j,\lambda,\infty,-x}\iota)'(0) y'(-x) + (H_{j,\lambda,\infty,-x}u)'(0) \\ -y(0+) \\ (H_{j,\lambda,\infty,x}\iota)'(0) y'(x) - (H_{j,\lambda,\infty,x}u)'(0) \end{pmatrix}.$$

Proof. Use Lemma 3.22, Corollary 3.23, and Lemma 3.24 to compute the following expressions.

$$\begin{aligned}
&\lim_{\xi \to 0+} \lceil H_{j,\lambda,\infty,x}1, H_{j,\lambda,\infty,x}\iota \rfloor_{\xi} \\
&\quad = \lim_{\xi \to 0+} ((H_{j,\lambda,\infty,x}1)(\xi)(H_{j,\lambda,\infty,x}\iota)'(\xi) - (H_{j,\lambda,\infty,x}1)'(\xi)(H_{j,\lambda,\infty,x}\iota)(\xi)) = (H_{j,\lambda,\infty,x}\iota)'(0), \\
&\lim_{\xi \to 0+} \lceil H_{j,\lambda,\infty,x}1, H_{j,\lambda,\infty,x}u \rfloor_{\xi} \\
&\quad = \lim_{\xi \to 0+} ((H_{j,\lambda,\infty,x}1)(\xi)(H_{j,\lambda,\infty,x}u)'(\xi) - (H_{j,\lambda,\infty,x}1)'(\xi)(H_{j,\lambda,\infty,x}u)(\xi)) \\
&\quad = (H_{j,\lambda,\infty,x}u)'(0), \\
&\lim_{\xi \to 0+} \lceil H_{j,\lambda,\infty,x}\iota, H_{j,\lambda,\infty,x}u \rfloor_{\xi} \\
&\quad = \lim_{\xi \to 0+} ((H_{j,\lambda,\infty,x}\iota)(\xi)(H_{j,\lambda,\infty,x}u)'(\xi) - (H_{j,\lambda,\infty,x}\iota)'(\xi)(H_{j,\lambda,\infty,x}u)(\xi)) = 0.
\end{aligned}$$

Further note that only the sign of $\lceil \cdot, \cdot \rfloor_{\xi}$, $\xi \in I_x$, changes if the arguments are exchanged. These equalities and the decomposition of y according to Corollary 3.20 prove

$$\begin{aligned}
\lim_{\xi \to 0+} \lceil y, H_{j,\lambda,\infty,x}1 \rfloor &= -y(x)(H_{j,\lambda,\infty,x}\iota)'(0) + (H_{j,\lambda,\infty,x}u)'(\xi), \\
\lim_{\xi \to 0+} \lceil y, H_{j,\lambda,\infty,x}\iota \rfloor &= y(0+)(H_{j,\lambda,\infty,x}\iota)'(0).
\end{aligned}$$

The analogous formulas hold for $-x$ in place of x, when the limits $\xi \to 0+$ are replaced by the limits $\xi \to 0-$. Together, these formulas prove the assertion. □

Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X$, and that $g_{j,\lambda,\infty,x}$ is defined. Then define the 2×4-matrix $B_{j,\lambda,x}$ by

$$B_{j,\lambda,x} = \begin{cases} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & g_{j,\lambda,\infty,x} & 1 \end{pmatrix} & \text{if } g_{j,\lambda,\infty,x} \in \mathbb{C} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \text{if } g_{j,\lambda,\infty,x} = \infty \end{cases}.$$

Lemma 3.30 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \cap I_+$, $f \in \mathcal{L}^p([-x,x])$, and that the function y is a solution of (3.5) on $[-x,x] \setminus \{0\}$. If $g_{j,\lambda,\infty,x} \in \mathbb{C}$, then the interface condition*

$$B_{j,\lambda,x} \mathsf{b}_{j,\lambda,x} y = 0$$

is equivalent to the pair (3.19) and (3.20) of interface conditions; if $g_{j,\lambda,\infty,x} = \infty$, then it is equivalent to the interface conditions (3.21).

Proof. The assertion on the case $g_{j,\lambda,\infty,x} = \infty$ is obvious from Lemma 3.29. Now consider the case $g_{j,\lambda,\infty,x} \in \mathbb{C}$, which is always true if $j \in \mathbb{N}$. By Lemma 3.29, the interface condition (3.19), which means continuity, is equivalent to

$$\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \mathsf{b}_{j,\lambda,x} y = 0. \tag{3.24}$$

This corresponds to the first row of the matrix $B_{j,\lambda,x}$. As to the second row of this matrix and (3.20), it now suffices to consider the case that y is continuous. So $y(0)$ may be used instead of separate values $y(0\pm)$. Use the decomposition proved in Corollary 3.20 and with the help of Corollary 3.19 and Lemma 3.25 compute for $\xi \in I_x$

$$\begin{aligned}
&(H_{j,\lambda,\infty,\xi}\iota)'(0)y'(\xi) - (H_{j,\lambda,\infty,-\xi}\iota)'(0)y'(-\xi) - g_{j,\lambda,\infty,\xi}y(0) \\
= \;& \frac{(H_{j,\lambda,\infty,x}\iota)'(0)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)}(y(0)(H_{j,\lambda,\infty,x}1)'(\xi) + y'(x)(H_{j,\lambda,\infty,x}\iota)'(\xi) - (H_{j,\lambda,\infty,x}u)'(\xi)) \\
& - \frac{(H_{j,\lambda,\infty,-x}\iota)'(0)}{(H_{j,\lambda,\infty,-x}\iota)'(-\xi)}(y(0)(H_{j,\lambda,\infty,-x}1)'(-\xi) + y'(-x)(H_{j,\lambda,\infty,-x}\iota)'(-\xi) \\
& \quad -(H_{j,\lambda,\infty,-x}u)'(-\xi)) \\
& - \left(g_{j,\lambda,\infty,x} - \frac{(H_{j,\lambda,\infty,-x}1)'(-\xi)}{(H_{j,\lambda,\infty,-x}\iota)'(-\xi)}(H_{j,\lambda,\infty,-x}\iota)'(0) + \frac{(H_{j,\lambda,\infty,x}1)'(\xi)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)}(H_{j,\lambda,\infty,x}\iota)'(0) \right) y(0) \\
= \;& \frac{(H_{j,\lambda,\infty,x}\iota)'(0)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)}(y'(x)(H_{j,\lambda,\infty,x}\iota)'(\xi) - (H_{j,\lambda,\infty,x}u)'(\xi)) \\
& - \frac{(H_{j,\lambda,\infty,-x}\iota)'(0)}{(H_{j,\lambda,\infty,-x}\iota)'(-\xi)}(y'(-x)(H_{j,\lambda,\infty,-x}\iota)'(-\xi) - (H_{j,\lambda,\infty,-x}u)'(-\xi)) - g_{j,\lambda,\infty,x}y(0) \\
= \;& (H_{j,\lambda,\infty,x}\iota)'(0)y'(x) - \frac{(H_{j,\lambda,\infty,x}\iota)'(0)}{(H_{j,\lambda,\infty,x}\iota)'(\xi)}(H_{j,\lambda,\infty,x}u)'(\xi) \\
& -(H_{j,\lambda,\infty,-x}\iota)'(0)y'(-x) + \frac{(H_{j,\lambda,\infty,-x}\iota)'(0)}{(H_{j,\lambda,\infty,-x}\iota)'(-\xi)}(H_{j,\lambda,\infty,-x}u)'(-\xi) - g_{j,\lambda,\infty,x}y(0).
\end{aligned}$$

Now take the limit $\xi \to 0$ and note that then

$$(H_{j,\lambda,\infty,\pm x}\iota)'(\xi) \to (H_{j,\lambda,\infty,\pm x}\iota)'(0) \neq 0, \quad (H_{j,\lambda,\infty,\pm x}u)'(\xi) \to (H_{j,\lambda,\infty,\pm x}u)'(0)$$

by Corollary 3.23 and Lemma 3.22 to obtain

$$\begin{aligned}
&\lim_{\xi \to 0+} ((H_{j,\lambda,\infty,\xi}\iota)'(0)y'(\xi) - (H_{j,\lambda,\infty,-\xi}\iota)'(0)y'(-\xi) - g_{j,\lambda,\infty,\xi}y(0)) \\
= \;& (H_{j,\lambda,\infty,x}\iota)'(0)y'(x) - (H_{j,\lambda,\infty,x}u)'(0) - (H_{j,\lambda,\infty,-x}\iota)'(0)y'(-x) + (H_{j,\lambda,\infty,-x}u)'(0) \\
& -g_{j,\lambda,\infty,x}y(0) \\
= \;& \begin{pmatrix} 0 & 1 & g_{j,\lambda,\infty,x} & 1 \end{pmatrix} \mathsf{b}_{j,\lambda,x} y.
\end{aligned}$$

Here Lemma 3.29 has given the last equality. Hence, (3.20) is equivalent to

$$\begin{pmatrix} 0 & 1 & g_{j,\lambda,\infty,x} & 1 \end{pmatrix} \mathsf{b}_{j,\lambda,x} y = 0$$

if one of the equivalent conditions (3.19) or (3.24) holds. This corresponds to the second row of the matrix $B_{j,\lambda,x}$. □

3.5 The integral operator $K_{j,\lambda}$

The following lemma allows to compute a solution of (3.5) on a compact interval around 0.

Lemma 3.31 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $a, b \in I$ with $a < 0 < b$, and $f \in \mathcal{L}^1([a,b])$. Then the solution y of (3.5) on $[a,b] \setminus \{0\}$ with boundary conditions $y(a) = y'(a) = y(b) = y'(b) = 0$ is given by*

$$y(x) = (K_{j,\lambda} f)(x) = \int_a^b k_{j,\lambda}(x,\xi) f(\xi) d\xi,$$

where the kernel $k_{j,\lambda}(\cdot,\cdot)$ is uniformly bounded with respect to $x, \xi \in [a,b]$, $j \in \mathbb{N} \cup \{\infty\}$, and $\lambda \in \Lambda$.

Proof. Define the kernel $k_{j,\lambda}$ on $[a,b] \times [a,b]$ in the following way. Assume $\xi \in [a,b]$. If $x \in [a,b] \setminus I_\xi$, then set

$$k_{j,\lambda}(x,\xi) = 0. \tag{3.25}$$

On the interval I_ξ let $k_{j,\lambda}(\cdot,\xi)$ be the unique solution of the initial value problem

$$\begin{aligned} -\frac{\partial^2 k_{j,\lambda}}{\partial x^2}(x,\xi) + (V_j(x) - \lambda) k_{j,\lambda}(x,\xi) &= 0, \\ k_{j,\lambda}(\xi,\xi) &= 0, \\ \frac{\partial k_{j,\lambda}}{\partial x}(\xi\mp,\xi) &= \pm 1, \end{aligned}$$

where the lower sign holds if $\xi < 0$ and the upper one if $\xi > 0$. By (3.2), and Corollaries 1.2, and 3.7, this function is uniformly bounded as asserted. The integral kernel $k_{j,\lambda}$ is defined in analogy to that in [BDL]. The structure of this integral kernel (compare Figure 1) implies that the integral

$$\int_a^b k_{j,\lambda}(x,\xi) f(\xi) d\xi$$

equals

$$\int_a^x k_{j,\lambda}(x,\xi) f(\xi) d\xi \quad \text{or} \quad \int_x^b k_{j,\lambda}(x,\xi) f(\xi) d\xi$$

if $x < 0$ or $x > 0$, respectively. Assume $x \in [a,0)$, the case $x \in (0,b]$ can be treated analogously. Then

$$(K_{j,\lambda} f)'(x) = \int_a^x \frac{\partial k_{j,\lambda}}{\partial x}(x,\xi) f(\xi) d\xi + k_{j,\lambda}(x,x-) f(x).$$

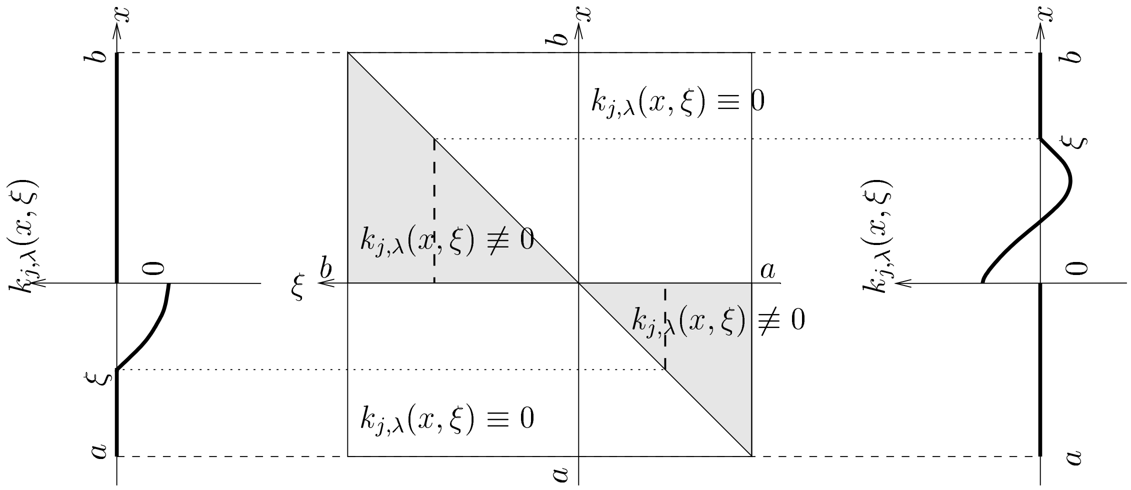

Figure 1: The kernel $k_{j,\lambda}(\cdot,\cdot)$, sketched. (In general, it is complex valued, of course.) Left if $\xi < 0$, right if $\xi > 0$; the picture in the middle shows, where $k_{x,\xi}$ is not identically zero. The kernel $k_{j,\lambda}$ is continuously differentiable on the set $\{(x,\xi) \in [a,b]^2 : x \neq 0, x \neq \xi\}$, and continuous on the set $\{(x,\xi) \in [a,b]^2 : x \neq 0\} \cup \{(0,0)\}$.

The last term vanishes because $k_{j,\lambda}$ is continuous in all points $(x,\xi) \in [a,b]^2$ with $x = \xi$ and vanishes there. Hence

$$(K_{j,\lambda}f)''(x) = \int_a^x \frac{\partial^2 k_{j,\lambda}}{\partial x^2}(x,\xi)f(\xi)d\xi + \frac{\partial k_{j,\lambda}}{\partial x}(x,x-)f(x).$$

Since $\frac{\partial k_{j,\lambda}}{\partial x}(x,\xi)$ continuously depends on the pair (x,ξ) if $\xi < x$ and since $\frac{\partial k_{j,\lambda}}{\partial x}(\xi+,\xi) = -1$, it holds $\frac{\partial k_{j,\lambda}}{\partial x}(x,x-) = -1$. That $K_{j,\lambda}f$ fulfills (3.5), now follows from

$$\begin{aligned} &-(K_{j,\lambda}f)''(x) + (V_j(x)-\lambda)(K_{j,\lambda}f)(x) \\ &= \int_{[a,b]\setminus\{0,x\}} \left(-\frac{\partial^2 k_{j,\lambda}}{\partial x^2}(x,\xi) + (V_j(x)-\lambda)k_{j,\lambda}(x,\xi)\right) f(\xi)d\xi + f(x) = f(x). \end{aligned}$$

Now prove the boundary conditions in a and b. It suffices to consider a, the proof for b is analogous. By Corollary 1.2, there is a number $x_0 \in [a,0)$ such that $|k_{j,\lambda}(x,\xi)| \leq 2$ and $|\frac{\partial k_{j,\lambda}}{\partial x}(x,\xi)| \leq 2$ if $x,\xi \in [a,x_0]$. This and (3.25) yield

$$|(K_{j,\lambda}f)(x)| = \left|\int_a^b k_{j,\lambda}(x,\xi)f(\xi)d\xi\right| = \left|\int_a^x k_{j,\lambda}(x,\xi)f(\xi)d\xi\right| \leq 2\,\|f\|_{[a,x]}$$

if $x \in [a,x_0]$. The right side of this equality tends to zero if $x \to a$ proving $(K_{j,\lambda}f)(a) = 0$. The analogous argument applies to the derivative to show $(K_{j,\lambda}f)'(a) = 0$. □

Lemma 3.32 *Assume $\lambda \in \Lambda$, $a,b \in I$ with $a < 0 < b$, $x \in X \cap I_+$, and that for all $j \in \mathbb{N} \cup \{\infty\}$, $K_{j,\lambda}$ is the integral operator introduced in Lemma 3.31. Then the mappings $\mathsf{b}_{j,\lambda,x}K_{j,\lambda}$, $j \in \mathbb{N} \cup \{\infty\}$, from $\mathcal{L}^1([a,b])$ to $\mathbf{C}^4$ are continuous and the convergence relation $\mathsf{b}_{j,\lambda,x}K_{j,\lambda} \to \mathsf{b}_{\infty,\lambda,x}K_{\infty,\lambda}$ holds if $j \to \infty$.*

Proof. From Corollary 1.2 and Lemma 3.31, it follows that the functionals mapping f to $(K_{j,\lambda}f)(\pm x)$ and $(K_{j,\lambda}f)'(\pm x)$ are continuous and that $(K_{j,\lambda}f)(\pm x) \to (K_{\infty,\lambda}f)(\pm x)$ and $(K_{j,\lambda}f)'(\pm x) \to (K_{\infty,\lambda}f)'(\pm x)$ if $j \to \infty$. Let the function u be given by the double integral formula (3.15). Then Corollary 3.20 implies

$$y(0\pm) = \frac{y(\pm x) - y'(\pm x)(H_{j,\lambda,\infty,\pm x}\iota)(\pm x) + (H_{j,\lambda,\infty,\pm x}u)(\pm x)}{(H_{j,\lambda,\infty,\pm x}1)(\pm x)}, \tag{3.26}$$

and the denominator of this expression is nonzero by Lemma 3.22. So also the numbers $(K_{j,\lambda}f)(0\pm)$ depend continuously on f and $(K_{j,\lambda}f)(0\pm) \to (K_{\infty,\lambda}f)(0\pm)$ if $j \to \infty$. The continuous dependence of $(H_{j,\lambda,\infty,\pm x}u)'(0\pm)$ on f follows from Lemma 3.22, the convergence of these numbers if $j \to \infty$ from Corollary 3.23. So Lemma 3.29 yields the assertion. □

Lemma 3.33 *Assume $\lambda \in \Lambda$, $a, b \in I$ with $a < 0 < b$, $x \in X \cap I_+$, and that for all $j \in \mathbb{N} \cup \{\infty\}$, $K_{j,\lambda}$ is the integral operator introduced in Lemma 3.31. Then there are four functions $f_1, f_2, f_3, f_4 \in \mathcal{L}^\infty([a,b])$ such that the vectors $\mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_1$, $\mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_2$, $\mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_3$, and $\mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_4$ form a basis of $\mathbb{C}^4$ for each sufficiently large $j \in \mathbb{N} \cup \{\infty\}$.*

Proof. Choose four functions $y_k \in \mathcal{AC}^2_{\mathrm{loc}}([a,b] \setminus \{0\})$, $k \in \{1,2,3,4\}$, which are linearly independent solutions of (3.4) with $j = \infty$ on $\left[\frac{a}{3}, \frac{b}{3}\right] \setminus \{0\}$ and which vanish on $\left[a, \frac{2a}{3}\right] \cup \left[\frac{2b}{3}, b\right]$. Then their images $\mathtt{b}_{\infty,\lambda,x}y_k$, $k \in \{1,2,3,4\}$, form a basis of $\mathbb{C}^4$. This is a consequence of Corollary 3.20 and Lemma 3.29. The functions $l_\infty[y_k] - \lambda y_k$, $k \in \{1,2,3,4\}$, vanish on $[a, \frac{2a}{3}) \cup (\frac{a}{3}, \frac{b}{3}) \cup (\frac{2b}{3}, b]$. This and (3.1) imply $l_\infty[y_k] - \lambda y_k \in \mathcal{L}^1([a,b])$, $k \in \{1,2,3,4\}$. Moreover, the equalities $K_{\infty,\lambda}(l_\infty[y_k] - \lambda y_k) = y_k$, $k \in \{1,2,3,4\}$, hold by Lemma 3.31. From Corollary 1.2 and Lemma 3.32, it follows that the mapping $\mathtt{b}_{\infty,\lambda,x}K_{\infty,\lambda}$ from $\mathcal{L}^1([a,b])$ to $\mathbb{C}^4$ is continuous. Since $\mathcal{L}^\infty([a,b])$ is dense in $\mathcal{L}^1([a,b])$, there are functions $f_k \in \mathcal{L}^\infty([a,b])$ near $l_\infty[y_k] - \lambda y_k$, $k \in \{1,2,3,4\}$, such that the vectors $\mathtt{b}_{\infty,\lambda,x}K_{\infty,\lambda}f_k$, $k \in \{1,2,3,4\}$, form a basis of $\mathbb{C}^4$. Lemma 3.32 implies that $\mathtt{b}_{j,\lambda,x}K_{j,\lambda} \to \mathtt{b}_{\infty,\lambda,x}K_{\infty,\lambda}$ if $j \to \infty$. Hence the vectors $\mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_k$, $k \in \{1,2,3,4\}$, form a basis of $\mathbb{C}^4$ for all sufficiently large numbers $j \in \mathbb{N} \cup \{\infty\}$. □

3.6 The convergence results

In the first place, convergence results will be proved as convergence in the generalized sense. This is convergence in the gap topology, which can be defined by the Hausdorff metric applied to the intersections of the respective operators with the unit sphere. Closely related to this metric and better suited to computation is the gap function, which is no metric itself, but defines the same topology, see [K, IV§2.1].

Definition 3.34 Let $\mathcal{X}$ be a Banach space. For a vector $v \in \mathcal{X}$ and a subset $\mathcal{V} \subseteq \mathcal{X}$ define the distance

$$\operatorname{dist}(u, \mathcal{V}) = \inf_{v \in \mathcal{V}} \|u - v\|_{\mathcal{X}}.$$

For two subspaces $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{X}$ define the gap

$$\operatorname{gap}(\mathcal{U}, \mathcal{V}) = \max\left(\sup_{u \in \mathcal{B}(\mathcal{U})} \operatorname{dist}(u, \mathcal{V}), \sup_{v \in \mathcal{B}(\mathcal{V})} \operatorname{dist}(v, \mathcal{U}) \right).$$

The topology defined by gap $(\cdot,\cdot)$ on the set of subspaces of $\mathcal{X}$ is called the gap topology. For two relations S and T from $\mathcal{X}$ to $\mathcal{Y}$ define gap (S,T) as their gap as subspaces of the product space $\mathcal{X}\times\mathcal{Y}$ with the norm given by

$$\|(u,v)\|_{\mathcal{X}\times\mathcal{Y}} = \max(\|u\|_{\mathcal{X}}, \|v\|_{\mathcal{Y}}), \quad (u,v)\in\mathcal{X}\times\mathcal{Y}.$$

A sequence of relations T_j, $j\in\mathbb{N}$, converges to a relation T_∞ in the generalized sense if gap $(T_j,T_\infty)\to 0$ if $j\to\infty$.

The definition of the gap given here is a slight simplification of that given in [K, IV§2], where also the basic properties of the gap topology are proved. Although only operators are considered there, most definitions and statements concerning the generalized convergence straightforwardly extend to linear relations. Note that the norm on $\mathcal{X}\times\mathcal{Y}$ may be defined differently without change of the gap topology, compare [K, p. 164, Footnote 2]. Also note that here the gap has been defined without the reqirement that the arguments are closed sets. This is technically advantageous.

Lemma 3.35 *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, that $\mathcal{U}_1$ and $\mathcal{U}_2$ are subspaces of $\mathcal{X}$, and that X_1 and X_2 are injective operators from $\mathcal{X}$ to $\mathcal{Y}$ such that*

$$C = \max\left(\sup_{x\in\mathcal{X}\setminus\{0\}}\frac{\|x\|_{\mathcal{X}}}{\|X_1x\|_{\mathcal{Y}}}, \sup_{x\in\mathcal{X}\setminus\{0\}}\frac{\|x\|_{\mathcal{X}}}{\|X_2x\|_{\mathcal{Y}}}\right) < \infty.$$

Then

$$\text{gap}\,(X_1\mathcal{U}_1, X_2\mathcal{U}_2) \le 2C\,\|X_1-X_2\|_{\mathcal{X}\to\mathcal{Y}} + C\min(\|X_1\|_{\mathcal{X}\to\mathcal{Y}}, \|X_2\|_{\mathcal{X}\to\mathcal{Y}})\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2).$$

Proof. Without loss of generality assume $\|X_1\|_{\mathcal{X}\to\mathcal{Y}}\le\|X_2\|_{\mathcal{X}\to\mathcal{Y}}$. Choose a vector $y_1\in\mathcal{B}(X_1\mathcal{U}_1)$. Thus there is a vector $x_1\in\mathcal{U}_1$ such that $y_1=X_1x_1$ and $\|x_1\|_{\mathcal{X}}\le C$. Choose $\epsilon>0$ arbitrarily. Then there is a vector $x_2\in\mathcal{U}_2$ such that $\|x_1-x_2\|_{\mathcal{X}}\le C(\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2)+\epsilon)$. Thus $\|x_2\|_{\mathcal{X}}\le C(1+\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2)+\epsilon)\le(2+\epsilon)C$. and for $y_2=X_2x_2$, it holds

$$\begin{aligned}\|y_1-y_2\|_{\mathcal{Y}} &= \|X_1x_1-X_2x_2\|_{\mathcal{Y}} \le \|X_1x_1-X_1x_2\|_{\mathcal{Y}} + \|X_1x_2-X_2x_2\|_{\mathcal{Y}}\\ &\le C\,\|X_1\|_{\mathcal{X}\to\mathcal{Y}}(\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2)+\epsilon) + (2+\epsilon)C\,\|X_1-X_2\|_{\mathcal{X}\to\mathcal{Y}}. \quad (3.27)\end{aligned}$$

Since the choice of $\epsilon>0$ was arbitrary, this proves

$$\sup_{y_1\in\mathcal{B}(X_1\mathcal{U}_1)}\text{dist}\,(y_1,X_2\mathcal{U}_2) \le 2C\,\|X_1-X_2\|_{\mathcal{X}\to\mathcal{Y}} + C\min(\|X_1\|_{\mathcal{X}\to\mathcal{Y}}, \|X_2\|_{\mathcal{X}\to\mathcal{Y}})\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2).$$

The complementary relation to estimate gap $(X_1\mathcal{U}_1,X_2\mathcal{U}_2)$ follows similarly, but now start from an arbitrary vector $y_2\in\mathcal{B}(X_2\mathcal{U}_2)$. Then there is a vector $x_2\in\mathcal{U}_2$ such that $y_2=X_2x_2$ and $\|x_2\|_{\mathcal{X}}\le C$. For each $\epsilon>0$ there is a vector $x_1\in\mathcal{U}_1$ such that $\|x_1-x_2\|_{\mathcal{X}}\le C(\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2)+\epsilon)$. Then $\|x_1\|_{\mathcal{X}}\le C(1+\text{gap}\,(\mathcal{U}_1,\mathcal{U}_2)+\epsilon)\le(2+\epsilon)C$. Now finish the proof using (3.27) as above. □

Corollary 3.36 *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, that T_1 and T_2 are closed operators from $\mathcal{X}$ to $\mathcal{Y}$, and that S_1 and S_2 are bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. Then*

$$\begin{aligned}\text{gap}\,(T_1+S_1,T_2+S_2) &\le (1+\max(\|S_1\|_{\mathcal{X}\to\mathcal{Y}}, \|S_2\|_{\mathcal{X}\to\mathcal{Y}}))(2\,\|S_1-S_2\|_{\mathcal{X}\to\mathcal{Y}}\\ &\quad +(1+\min(\|S_1\|_{\mathcal{X}\to\mathcal{Y}}, \|S_2\|_{\mathcal{X}\to\mathcal{Y}}))\text{gap}\,(T_1,T_2)).\end{aligned}$$

Proof. Consider the operators X_1 and X_2 in $\mathcal{X} \times \mathcal{Y}$ mapping a pair (y, f) to $(y, f + S_k y)$, $k \in \{1, 2\}$. Their inverses map a pair (y, f) to $(y, f - S_k y)$, $k \in \{1, 2\}$. Evidently, $\|X_k\|_{\mathcal{X}\times\mathcal{Y}} \leq 1 + \|S_k\|_{\mathcal{X}\to\mathcal{Y}}$, $\left\|X_k^{-1}\right\|_{\mathcal{X}\times\mathcal{Y}} \leq 1 + \|S_k\|$ for $k \in \{1, 2\}$, and $\|X_1 - X_2\|_{\mathcal{X}\times\mathcal{Y}} = \|S_1 - S_2\|_{\mathcal{X}\to\mathcal{Y}}$. Since $T_k + S_k = \{X_k(y, f) : (y, f) \in T_k\}$ for $k \in \{1, 2\}$, Lemma 3.35 proves the assertion. □

Corollary 3.37 *Assume $\lambda \in \Lambda$, $x \in X \cap I_+$, and that $g_{\infty,\lambda,\infty,x}$ is defined. Then the kernel spaces fulfill the convergence relation*

$$\ker B_{j,\lambda,x} \to \ker B_{\infty,\lambda,x}$$

in the gap topology if $j \to \infty$.

Proof. If $g_{\infty,\lambda,\infty,x} \in \mathbb{C}$, then $\ker B_{j,\lambda,x} = \mathcal{R}(X_j)$ with the matrices

$$X_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ g_{j,\lambda,\infty,x} & -1 \end{pmatrix}, \quad j \in \mathbb{N} \cup \{\infty\}.$$

The mappings X_j from $\mathbb{C}^2$ to $\mathbb{C}^4$ converge to X_∞ if $j \to \infty$. From the first and second row of X_j, it can be seen that

$$C = \sup_{j\in\mathbb{N}\cup\{\infty\}} \sup_{x\in\mathcal{X}\setminus\{0\}} \frac{\|x\|_{\mathcal{X}}}{\|X_j x\|_{\mathcal{Y}}} \leq 1.$$

Hence, Lemma 3.35 can be applied to estimate the gap between the ranges of the operators X_∞ and X_j, $j \in \mathbb{N}$. This yields

$$\begin{aligned} \operatorname{gap}(\ker B_{\infty,\lambda,x}, \ker B_{j,\lambda,x}) \leq\ & 2C\, \|X_\infty - X_j\|_{\mathbb{C}^2\to\mathbb{C}^4} \\ & + C \min(\|X_1\|_{\mathbb{C}^2\to\mathbb{C}^4}, \|X_2\|_{\mathbb{C}^2\to\mathbb{C}^4}) \operatorname{gap}(\mathbb{C}^2, \mathbb{C}^2). \end{aligned}$$

Since the gap between a space and itself is always zero, the rightmost term vanishes and the relation $X_j \to X_\infty$ proves the assertion for the case $g_{\infty,\lambda,\infty,x} \in \mathbb{C}$.
The proof for the case $g_{\infty,\lambda,\infty,x} = \infty$ is similar, using the matrices

$$X_j = \begin{pmatrix} (g_{j,\lambda,\infty,x})^{-1} & (g_{j,\lambda,\infty,x})^{-1} \\ 1 & 0 \\ -(g_{j,\lambda,\infty,x})^{-1} & -(g_{j,\lambda,\infty,x})^{-1} \\ 0 & 1 \end{pmatrix}, \quad j \in \mathbb{N}, \quad X_\infty = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

In the case of a compact interval $I = [a, b]$ with $-\infty < a < 0 < b < +\infty$, (3.1) implies that the differential expressions l_j, $j \in \mathbb{N} \cup \{\infty\}$, are regular at a and b. So boundary conditions are introduced in these points. They are of the form

$$y'(a) = z_- y(a), \quad y'(b) = z_+ y(b), \tag{3.28}$$

where $z_\pm \in \overline{\mathbb{C}}$, always with the understanding that such a boundary condition is to be read as Dirichlet boundary condition $y(a) = 0$ or $y(b) = 0$ if $z_- = \infty$ or $z_+ = \infty$, respectively.

Theorem 3.38 *Assume that the potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$, on the compact interval $I = [a,b]$, $a < 0 < b$, fulfill (3.1), (3.2), and (3.3). Assume $z_\pm \in \overline{\mathbb{C}}$. Define the operators T_j, $j \in \mathbb{N}$, by*

$$\begin{aligned} \mathcal{D}(T_j) &= \{y \in \mathcal{AC}^2([a,b]) : -y'' + V_j y \in \mathcal{L}^p([a,b]), y'(a) = z_- y(a), y'(b) = z_+ y(b)\}, \\ T_j y &= -y'' + V_j y, \quad y \in \mathcal{D}(T_j). \end{aligned}$$

For those $x \in [a,b] \cap [-b,-a]$ where the series converge and, in the case of $\tilde{g}$, the limit exists in the compact space $\overline{\mathbb{C}}$, define

$$\begin{aligned} \tilde{g}(x) &= \overset{\overline{\mathbb{C}}}{\lim_{j\to\infty}} \left(\sum_{m=1}^{\infty} \left[\int_{\varsigma_m=-x}^{0} V_j(\varsigma_m) \int_{\xi_{m-1}=0}^{\varsigma_m} \left[\int_{\varsigma_{m-1}=-x}^{\xi_{m-1}} V_j(\varsigma_{m-1}) \int_{\xi_{m-2}=0}^{\varsigma_{m-1}} \left[\dots \right.\right.\right.\right. \\ &\qquad \dots \left[\int_{\varsigma_2=-x}^{\xi_2} V_j(\varsigma_2) \int_{\xi_1=0}^{\varsigma_2} \left[\int_{\varsigma_1=-x}^{\xi_1} V_j(\varsigma_1) d\varsigma_1 \right] d\xi_1 d\varsigma_2 \right] \dots \\ &\qquad \left.\left.\left. \dots \right] d\xi_{m-2} d\varsigma_{m-1} \right] d\xi_{m-1} d\varsigma_m \right] \\ &\qquad - \sum_{m=1}^{\infty} \left[\int_{\varsigma_m=x}^{0} V_j(\varsigma_m) \int_{\xi_{m-1}=0}^{\varsigma_m} \left[\int_{\varsigma_{m-1}=x}^{\xi_{m-1}} V_j(\varsigma_{m-1}) \int_{\xi_{m-2}=0}^{\varsigma_{m-1}} \left[\dots \right.\right.\right. \\ &\qquad \dots \left[\int_{\varsigma_2=x}^{\xi_2} V_j(\varsigma_2) \int_{\xi_1=0}^{\varsigma_2} \left[\int_{\varsigma_1=x}^{\xi_1} V_j(\varsigma_1) d\varsigma_1 \right] d\xi_1 d\varsigma_2 \right] \dots \\ &\qquad \left.\left.\left.\left. \dots \right] d\xi_{m-2} d\varsigma_{m-1} \right] d\xi_{m-1} d\varsigma_m \right] \right), \\ \tilde{h}(x) &= 1 + \sum_{m=1}^{\infty} \left[\int_{x}^{0} V_\infty(\varsigma_m) \int_{0}^{\varsigma_m} \left[\int_{\varsigma_{m-1}=x}^{\xi_{m-1}} V_\infty(\varsigma_{m-1}) \int_{\xi_{m-2}=0}^{\varsigma_{m-1}} \left[\dots \right.\right.\right. \\ &\qquad \dots \left[\int_{x}^{\xi_2} V_\infty(\varsigma_2) \int_{0}^{\varsigma_2} \left[\int_{x}^{\xi_1} V_\infty(\varsigma_1) \varsigma_1 d\varsigma_1 \right] d\xi_1 d\varsigma_2 \right] \dots \\ &\qquad \left.\left.\left. \dots \right] d\xi_{m-2} d\varsigma_{m-1} \right] d\xi_{m-1} d\varsigma_m \right], \end{aligned}$$

The function $\tilde{h}$ is always defined and finite in a neighbourhood of 0. Additionally define the set $\mathcal{D}_\infty$ as the set of all functions $\{y \in \mathcal{AC}^2_{\mathrm{loc}}([a,b] \setminus \{0\})$ such that $-y'' + V_\infty y \in \mathcal{L}^p([a,b])$, $y'(a) = z_- y(a)$, $y'(b) = z_+ y(b)$, and $y(0-) = y(0+)$. Then exactly one of the following three cases holds:

1. *The function $\tilde{g}$ is defined in a neighbourhood of 0 as a complex valued function. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \left\{ y \in \mathcal{D}_\infty : \lim_{x\to 0} (\tilde{h}(x) y'(x) - \tilde{h}(-x) y'(-x) - \tilde{g}(x) y(0)) = 0 \right\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

2. *The function $\tilde{g}$ is constantly equal to infinity in a deleted neighbourhood of 0. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \{ y \in \mathcal{D}_\infty : y(0-) = y(0+) = 0 \}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

3. *The function $\tilde{g}$ is undefined in a deleted neighbourhood of* 0. *Then the sequence of operators T_j, $j \in \mathbb{N}$, has two subsequences with different limits in the generalized sense.*

Proof. It holds $\tilde{g}(x) = g_{\infty,0,\infty,x}$ and $\tilde{h}(x) = (H_{\infty,0,\infty,x}\iota)'(0)$. So the function $\tilde{h}$ is defined on the neighbourhood X of 0 and Lemma 3.25 shows that the function $\tilde{g}$ is either complex valued, constantly infinite, or everywhere undefined on $X \setminus \{0\}$. Since X is a neighbourhood of 0 and every neighbourhood of 0 intersects $X \setminus \{0\}$, exactly one of the three alternative cases mentioned in the assertion holds true. It remains to prove the convergence assertions. First assume that $g_{\infty,\lambda,\infty,x} \in \overline{\mathbb{C}}$ for $x \in X \setminus \{0\}$.
From now on fix an arbitrary number $\lambda \in \Lambda$. Choose for $j = \infty$ two nonzero solutions $y_{\infty,\pm}$ of (3.4) which satisfy the boundary conditions (3.28) and vanish on $I_\mp$ such that $y_{\infty,\pm}(\pm x) = 1$ for some number $x \in X \cap I_+$. Fix this number x. Then choose for all $j \in \mathbb{N} \cup \{\infty\}$ two nonzero solutions $y_{j,\pm}$ of (3.4) which satisfy the boundary conditions (3.28) and vanish on $I_\mp$. For large $j \in \mathbb{N} \cup \{\infty\}$ impose the additional requirement $y_{j,\pm}(\pm x) = 1$. To show that this can be fulfilled consider the functions $y_{j,-}$, the argument for $y_{j,+}$ is analogous. Define solutions $\tilde{y}_{j,\pm}$, $j \in \mathbb{N}$, of (3.4) on $[a, 0)$ which fulfill $\tilde{y}_{j,-}(a) = y_{\infty,-}(a)$ and $\tilde{y}'_{j,-}(a) = y'_{\infty,-}(a)_-$. Now (3.2) and Corollary 1.2 show that $\tilde{y}_{j,-}(-x) \to y_{\infty,-}(-x)$. Since x has been chosen such that $y_{\infty,-}(-x) \neq 0$, the functions $y_{j,-}$ can be defined by division of $\tilde{y}_{j,-}$ by $\tilde{y}_{j,-}(-x)$. These functions fulfill the boundary conditions (3.28) since $y_{\infty,-}$ does.
and additional inhomogeneous boundary conditions in a and b which can be chosen to converge. Then apply Corollary 1.2. This also implies $y_{j,\pm} \to y_{\infty,\pm}$ uniformly if $j \to \infty$. Choose a number $\epsilon > 0$ and a number $j_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
y_{j,\pm}(\pm x) &= 1, && (3.29)\\
\|y_{\infty,\pm} - y_{j,\pm}\|_{\mathcal{L}^\infty[a,b]} &< \epsilon, && (3.30)\\
\|K_{\infty,\lambda} - K_{j,\lambda}\|_{\mathcal{L}^p([a,b]) \to \mathcal{L}^\infty([a,b])} &< \epsilon, && (3.31)\\
\|\mathsf{b}_{\infty,\lambda,x} y_{\infty,\pm} - \mathsf{b}_{j,\lambda,x} y_{j,\pm}\|_{\mathbb{C}^4} &< \epsilon, && (3.32)\\
\|\mathsf{b}_{\infty,\lambda,x} K_{\infty,\lambda} - \mathsf{b}_{j,\lambda,x} K_{j,\lambda}\|_{\mathcal{L}^p([a,b]) \to \mathbb{C}^4} &< \epsilon, && (3.33)\\
\operatorname{gap}(\ker B_{\infty,\lambda,x}, \ker B_{j,\lambda,x}) &< \epsilon, && (3.34)
\end{aligned}
$$

for all $j \geq j_0$. This is possible in consequence of Corollary 1.2, Lemmas 3.29, 3.31, 3.32, and Corollary 3.37. Lemma 3.30 and Corollary 3.28 yield that the domains of the operators T_j, $j \in \mathbb{N} \cup \{\infty\}$, are the sets $\mathcal{D}(T_j)$ of all functions $y \in \mathcal{AC}^2_{\text{loc}}([a, b] \setminus \{0\})$ which fulfill $l_j[y] \in \mathcal{L}^p([a, b])$, (3.28), and $B_{j,\lambda,x}\mathsf{b}_{j,\lambda,x} y = 0$. Choose $(y_\infty, f_\infty) \in \mathcal{B}(T_\infty - \lambda)$, this means

$$(T_\infty - \lambda) y_\infty = f_\infty, \quad \max\left(\|y_\infty\|_{\mathcal{L}^p([a,b])}, \|f_\infty\|_{\mathcal{L}^p([a,b])}\right) \leq 1.$$

Then

$$y_\infty = c_- y_{\infty,-} + c_+ y_{\infty,+} + K_{\infty,\lambda} f_\infty,$$

where the complex numbers $c_\pm$ can be computed as $c_\pm = y_\infty(\pm x) - (K_{\infty,\lambda} f_\infty)(\pm x)$ by the assumptions on $y_{\infty,\pm}$. From (3.29) and Corollary 1.2, it follows $|c_\pm| \leq C_1$ for some constant $C_1 \in \mathbb{R}$ which is independent of the chosen $j \geq j_0$, y_∞, and f_∞. To find a pair $(y_j, f_j) \in T_j - \lambda$ near (y_∞, f_∞) use the ansatz

$$f_j = f_\infty + f_\Delta, \quad y_j = c_- y_{j,-} + c_+ y_{j,+} + K_{j,\lambda} f_\infty + K_{j,\lambda} f_\Delta,$$

where the function f_Δ has to be chosen such that $B_{j,\lambda,x}\mathtt{b}_{j,\lambda,x}y_j = 0$.
The inequality (3.34) implies that there is a vector $v \in \ker B_{j,\lambda,x}$ such that

$$\|v - \mathtt{b}_{\infty,\lambda,x}y_\infty\| \le \epsilon \|\mathtt{b}_{\infty,\lambda,x}y_\infty\| \le C_2\epsilon,$$

where the bound $C_2 \in \mathbb{R}$ can be chosen independently of the chosen j, y_∞, and f_∞. Choose the function f_Δ such that $\mathtt{b}_{j,\lambda,x}y_j = v$. It follows

$$\begin{aligned}
\|\mathtt{b}_{j,\lambda,x}f_\Delta\| &= \|v - c_-\mathtt{b}_{j,\lambda,x}y_{j,-} - c_+\mathtt{b}_{j,\lambda,x}y_{j,+} - \mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_\infty\|_{\mathbb{C}^4} \\
&\le \|v - \mathtt{b}_{\infty,\lambda,x}y_\infty\| \\
&\quad + |c_-|\,\|\mathtt{b}_{\infty,\lambda,x}y_{\infty,-} - \mathtt{b}_{j,\lambda,x}y_{j,-}\| + |c_+|\,\|\mathtt{b}_{\infty,\lambda,x}y_{\infty,+} - \mathtt{b}_{j,\lambda,x}y_{j,+}\| \\
&\quad + \|\mathtt{b}_{\infty,\lambda,x}K_{\infty,\lambda}f_\infty - \mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_\infty\| \\
&\le C_2\epsilon + 2C_1\epsilon + \epsilon = C_3\epsilon,
\end{aligned}$$

using (3.32) and (3.33). Lemma 3.33 implies that for sufficiently large $j \in \mathbb{N} \cup \{\infty\}$, say $j \ge j_1 \ge j_0$, f_Δ may be found as a linear combination of four fixed functions $f_1, f_2, f_3, f_4 \in \mathcal{L}^\infty([a,b]) \subseteq \mathcal{L}^p([a,b])$. The coefficients of this linear combination converge if $j \to \infty$ as a consequence of Lemma 3.32; so particularly, they stay bounded. The bound can be computed as

$$C_4\epsilon = C_3\epsilon \max_{k\in\{1,2,3,4\}} \|f_k\|_{\mathcal{L}^p([a,b])} \sup_{j\ge j_1} \left\|A_j^{-1}\right\|,$$

where the 4×4-matrices A_j are given by their columns $\mathtt{b}_{j,\lambda,x}K_{j,\lambda}f_k$, $k \in \{1,2,3,4\}$. This proves $\|f_\Delta\|_{\mathcal{L}^p([a,b])} \le C_4\epsilon$, and together with (3.30) and (3.31) the convergence

$$\sup_{(y_\infty,f_\infty)\in\mathcal{B}(T_\infty-\lambda)} \operatorname{dist}\left((y_\infty, f_\infty), T_j - \lambda\right) \to 0, \quad j \to \infty.$$

The complementary relation

$$\sup_{(y_j,f_j)\in\mathcal{B}(T_j-\lambda)} \operatorname{dist}\left((y_j, f_j), T_\infty - \lambda\right) \to 0, \quad j \to \infty$$

follows in the same fashion if the symbols j and ∞ are exchanged in the above estimates. Together with Corollary 3.36, these relations yield the assertion $\operatorname{gap}(T_\infty, T_j) \to 0$ if $j \to \infty$.
The divergence assertion for the third case, when the function $\tilde{g}$ is undefined near 0, follows from the first and second case and the fact that the space $\overline{\mathbb{C}}$ is compact and thus the sequence of numbers $g_{j,0,\infty,x}$, $j \in \mathbb{N}$, has convergent subsequences with different limits. □

Remark 3.39 Since the interval $[a,b]$ is compact and thus solutions of (3.5) converge uniformly on this interval, the proof of Theorem 3.38 in fact shows the slight strengthening of the generalized convergence which is obtained if the function $\operatorname{dist}(\cdot,\cdot)$ in the definition of the gap is defined from the metric of the space $\mathcal{L}^\infty([a,b]) \times \mathcal{L}^p([a,b])$ in place of that of $\mathcal{L}^p([a,b]) \times \mathcal{L}^p([a,b])$. In the context of differential operators, this is not surprising.

Lemma 3.40 *If $r \in [p,\infty]$ then $\ell^p(\mathbb{Z}) \subseteq \ell^r(\mathbb{Z})$ and $\|c\|_{\ell^r(\mathbb{Z})} \le \|c\|_{\ell^p(\mathbb{Z})}$ for all $c \in \ell^p(\mathbb{Z})$.*

Proof. The assertion for $r = \infty$ is evident, so assume $r < \infty$. Then for a sequence $c \in \ell^p(\mathbb{Z})$, the Minkowski inequality in the space $\ell^{\frac{r}{p}}(\mathbb{Z})$ yields

$$\|c\|_{\ell^r(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |c_k|^r\right)^{\frac{1}{r}} = \left(\left(\sum_{k\in\mathbb{Z}} (|c_k|^p)^{\frac{r}{p}}\right)^{\frac{p}{r}}\right)^{\frac{1}{p}} \le \left(\sum_{k\in\mathbb{Z}} |c_k|^p\right)^{\frac{1}{p}} = \|c\|_{\ell^p(\mathbb{Z})}.$$

□

In [Kl], integral kernels are used which are similar to that in the following lemma.

Lemma 3.41 *Assume that the operator A is given by*

$$\begin{aligned} \mathcal{D}(A) &= \left\{y \in \mathcal{AC}^2_{\mathrm{loc}}(\mathbb{R}) \cap \mathcal{L}^p(\mathbb{R}) : y'' \in \mathcal{L}^p(\mathbb{R})\right\}, \\ Ay &= -y'', \quad y \in \mathcal{D}(A). \end{aligned}$$

Further assume $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $f \in \mathcal{L}^p(\mathbb{R})$, $r \in [p, \infty]$, and $x \in \mathbb{R}$. Then

$$((A-\lambda)^{-1}f)(x) = \frac{-1}{2i\sqrt{\lambda}\,\mathrm{sgn}\,\Im\lambda} \int_{\mathbb{R}} e^{i|\xi - x|\sqrt{\lambda}\,\mathrm{sgn}\,\Im\lambda} f(\xi)d\xi \tag{3.35}$$

and $(A-\lambda)^{-1}$ is continuous from $\mathcal{L}^p(\mathbb{R})$ to $\mathcal{L}^r(\mathbb{R})$. Additionally, the linear functional mapping f to $((A-\lambda)^{-1}f)'(x)$ is continuous.

Proof. For simplicity assume $\Im\lambda > 0$. The case $\Im\lambda < 0$ can be treated analogously. For $x \in \mathbb{R}$ let the function y be given by the integral expression on the right side of (3.35). Then

$$\begin{aligned} y(x) &= \frac{-1}{2i\sqrt{\lambda}} \int_{-\infty}^{x} e^{i(x-\xi)\sqrt{\lambda}} f(\xi)d\xi + \frac{1}{2i\sqrt{\lambda}} \int_{x}^{\infty} e^{i(\xi-x)\sqrt{\lambda}} f(\xi)d\xi, \\ y'(x) &= -\frac{1}{2} \int_{-\infty}^{x} e^{i(x-\xi)\sqrt{\lambda}} f(\xi)d\xi + \frac{1}{2} \int_{x}^{\infty} e^{i(\xi-x)\sqrt{\lambda}} f(\xi)d\xi, \qquad (3.36) \\ y''(x) &= -f(x) - \frac{i\sqrt{\lambda}}{2} \int_{-\infty}^{x} e^{i(x-\xi)\sqrt{\lambda}} f(\xi)d\xi - \frac{i\sqrt{\lambda}}{2} \int_{x}^{\infty} e^{i(\xi-x)\sqrt{\lambda}} f(\xi)d\xi = -\lambda y - f(x). \end{aligned}$$

To estimate $\|y\|_{\mathcal{L}^r(\mathbb{R})}$, write y as $y = \sum_{k\in\mathbb{Z}} y_k$ where y_k, $k \in \mathbb{Z}$, is given by

$$y_k(x) = \frac{-1}{2i\sqrt{\lambda}} \int_k^{k+1} e^{i|\xi - x|\sqrt{\lambda}} f(\xi)d\xi, \quad x \in \mathbb{R}.$$

Lemma 3.2 for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$ yields

$$|y_k(x)| \le \frac{e^{-\mathrm{dist}\,(x,[k,k+1])\Im\sqrt{\lambda}}}{2\sqrt{|\lambda|}} \int_k^{k+1} |f(\xi)|d\xi \le \frac{e^{-\mathrm{dist}\,(x,[k,k+1])\Im\sqrt{\lambda}}}{2\sqrt{|\lambda|}} \|f\|_{\mathcal{L}^p([k,k+1])}. \tag{3.37}$$

In the case $r < \infty$, this and the Lemmas 3.2 and 3.40 imply

$$\begin{aligned} \|y\|_{\mathcal{L}^r(\mathbb{R})} &= \sqrt[r]{\sum_{m\in\mathbb{Z}} \|y\|^r_{\mathcal{L}^r([m,m+1])}} \le \sqrt[r]{\sum_{m\in\mathbb{Z}} \|y\|^r_{\mathcal{L}^\infty([m,m+1])}} \\ &\le \sqrt[p]{\sum_{m\in\mathbb{Z}} \|y\|^p_{\mathcal{L}^\infty([m,m+1])}} \le \sqrt[p]{\sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \|y_k\|^p_{\mathcal{L}^\infty([m,m+1])}} \\ &\le \frac{1}{2\sqrt{|\lambda|}} \sqrt[p]{\sum_{k\in\mathbb{Z}} \|f\|^p_{\mathcal{L}^p([k,k+1])} \sum_{m\in\mathbb{Z}} e^{-p\Im\sqrt{\lambda}\min(|k-m|,|k+1-m|)}} \\ &\le \frac{\sqrt[p]{2}\,\|f\|_{\mathcal{L}^p(\mathbb{R})}}{2\sqrt{|\lambda|}\sqrt[p]{1 - e^{-p\Im\sqrt{\lambda}}}}. \end{aligned}$$

The assertion for the case $r = \infty$ follows directly from (3.37) via

$$|y(x)| \leq \sum_{k\in\mathbb{Z}} |y_k(x)| \leq \frac{1}{2\sqrt{|\lambda|}} \sum_{k\in\mathbb{Z}} e^{-\text{dist}\,(x,[k,k+1])\Im\sqrt{\lambda}} \|f\|_{\mathcal{L}^p(\mathbb{R})} \leq \frac{\|f\|_{\mathcal{L}^p(\mathbb{R})}}{\sqrt{|\lambda|}(1-e^{-\Im\sqrt{\lambda}})}$$

for all $x \in \mathbb{R}$. The assertion on the expression $((A-\lambda)^{-1}f)'(x)$ similarly follows from (3.36). □

Theorem 3.42 *Assume that the potentials V_j, $j \in \mathbb{N}\cup\{\infty\}$, on $I = \mathbb{R}$, fulfill (3.1), (3.2), (3.3), and that there is an interval $[a,b]$, $a < 0 < b$, such that*

$$\begin{aligned} V_j &\in \mathcal{L}^\infty(\mathbb{R}\setminus[a,b]), \quad j \in \mathbb{N}\cup\{\infty\}, \\ \|V_\infty - V_j\|_{\mathcal{L}^\infty(\mathbb{R}\setminus[a,b])} &\to 0, \quad j\to\infty. \end{aligned}$$

Define the operators T_j, $j \in \mathbb{N}$, by

$$\begin{aligned} \mathcal{D}(T_j) &= \{y \in \mathcal{AC}^2(\mathbb{R})\cap\mathcal{L}^p(\mathbb{R}) : -y'' + V_j y \in \mathcal{L}^p(\mathbb{R})\}, \\ T_j y &= -y'' + V_j y, \quad y \in \mathcal{D}(T_j). \end{aligned}$$

For those $x \in \mathbb{R}$ where the series converge and, in the case of $\tilde{g}$, the limit exists in the compact space $\overline{\mathbb{C}}$, define the functions $\tilde{g}(x)$ and $\tilde{h}(x)$ by the same formulas as in Theorem 3.38. Additionally define

$$\mathcal{D}_\infty = \{y \in \mathcal{AC}^2_{\text{loc}}(\mathbb{R}\setminus\{0\})\cap\mathcal{L}^p(\mathbb{R}) : -y'' + V_\infty y \in \mathcal{L}^p(\mathbb{R}), y(0-) = y(0+)\}.$$

Then exactly one of the following three cases holds:

1. *The function $\tilde{g}$ is defined in a neighbourhood of 0 as a complex valued function. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \left\{y \in \mathcal{D}_\infty : \lim_{x\to 0}(\tilde{h}(x)y'(x) - \tilde{h}(-x)y'(-x) - \tilde{g}(x)y(0)) = 0\right\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

2. *The function $\tilde{g}$ is constantly equal to infinity in a deleted neighbourhood of 0. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \{y \in \mathcal{D}_\infty : y(0-) = y(0+) = 0\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

3. *The function $\tilde{g}$ is undefined in a deleted neighbourhood of 0. Then the sequence of operators T_j, $j \in \mathbb{N}$, has two subsequences with different limits in the generalized sense.*

Proof. In its main part, the proof is nearly the same as that of Theorem 3.38. Therefore, only the differences are noted here. Particularly, only the case is treated when the function $\tilde{g}$ is defined in a neighbourhood of 0, since the divergence assertion then follows from the compactness of $\overline{\mathbb{C}}$.
Assume $V_j(x) = 0$ for all $j \in \mathbb{N} \cup \{\infty\}$ and $x \in \mathbb{R} \setminus [a, b]$. The original assertion then can be proved with Corollary 3.36 from the assumptions concerning the potentials on $\mathbb{R} \setminus [a, b]$. Fix $\lambda \in \Lambda$ such that $\Im\lambda > 0$. Then the functions mapping x to $e^{\pm i\sqrt{\lambda}x}$ are solutions of (3.5) on $\mathbb{R} \setminus [a, b]$. Choose the parameters $z_\pm = \pm i\sqrt{\lambda}$. Then choose the solutions $y_{j,\pm}$, $j \in \mathbb{N} \cup \{\infty\}$, on $[a, b]$ as in the proof of Theorem 3.38 and continue them onto $\mathbb{R}$ setting

$$\begin{aligned} y_{j,\pm}(x) &= y_{j,\pm}(a)e^{-i\sqrt{\lambda}(x-a)}, \quad x < a, \\ y_{j,\pm}(x) &= y_{j,\pm}(b)e^{i\sqrt{\lambda}(x-b)}, \quad x > b. \end{aligned}$$

These functions are contained in all spaces $\mathcal{L}^r(\mathbb{R})$, $r \in [p, \infty]$. Let $P_{[a,b]}$ be the canonical projection of $\mathcal{L}^p(\mathbb{R})$ onto $\mathcal{L}^p([a, b])$. Using the notation of Lemma 3.41, define the operators $\tilde{K}_{j,\lambda}$ for $j \in \mathbb{N} \cup \{\infty\}$ by

$$(\tilde{K}_{j,\lambda}f)(x) = \begin{cases} K_{j,\lambda}P_{[a,b]}f + y_f & \text{if } x \in [a, b] \\ ((A-\lambda)^{-1}(1 - P_{[a,b]})f)(x) & \text{if } x \in \mathbb{R} \setminus [a, b] \end{cases},$$

where the function y_f is that solution of (3.4) on $[a, b] \setminus \{0\}$ which fulfills boundary conditions in a and b such that $\tilde{K}_{j,\lambda}f \in \mathcal{AC}^2_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. Lemma 3.41 and Corollary 1.2 imply that y_f depends continuously on $f \in \mathcal{L}^p(\mathbb{R})$. Similar considerations as in the proof of Theorem 3.38 prove that the operator $\tilde{K}_{x,\lambda}$ has all properties required there from $K_{j,\lambda}$. Particularly, for every number $\epsilon > 0$ there is a number $j_0 \in \mathbb{N}$ such that the conditions (3.29), (3.30), (3.31), (3.32), (3.33), and (3.34) hold when $K_{\cdot,\lambda}$ is replaced by $\tilde{K}_{\cdot,\lambda}$ and every pair $(y_j, f_j) \in T_j - \lambda$, $j \in \mathbb{N} \cup \{\infty\}$, fulfills

$$y_j = c_- y_{j,-} + c_+ y_{j,+} + \tilde{K}_{j,\lambda} f_\infty,$$

with the functions $y_{j,\pm}$ on $\mathbb{R}$ defined above. The remainder of the proof then is easily performed rewriting the proof of Theorem 3.38 accordingly. Only note that the spaces $\mathcal{L}^r(\mathbb{R})$ have to be considered separately for all values r in place of $\mathcal{L}^\infty([a, b])$, which suffices in the case of a compact interval. □

A second approximation theorem on $\mathbb{R}$ is obtained from slightly changed assumptions on the potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$.

Theorem 3.43 *Use the notation of Theorem 3.42. Assume that the potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$, on $I = \mathbb{R}$, fulfill (3.1), (3.2), (3.3), and*

$$\lim_{|x| \to \infty} \sup_{j \in \mathbb{N} \cup \{\infty\}} |V_j(x)| = 0.$$

Then the conclusion of Theorem 3.42 holds true.

Proof. Assume that the function $\tilde{g}$ is defined in a neighbourhood of 0. The divergence assertion follows again from the compactness of $\overline{\mathbb{C}}$.
Choose a number $\epsilon > 0$ and numbers $a, b \in \mathbb{R}$, $a < 0 < b$, such that

$$\sup_{x \in \mathbb{R} \setminus [a,b]} \sup_{j \in \mathbb{N} \cup \{\infty\}} |V_j(x)| < \epsilon.$$

For $j \in \mathbb{N} \cup \{\infty\}$ define the potentials W_j by

$$W_j(x) = \begin{cases} V_j(x) & \text{if } x \in [a,b] \\ W_j(x) = 0 & \text{if } x \in \mathbb{R} \setminus [a,b] \end{cases}$$

and define operators $\dot{T}_j$ from them like the operators T_j are defined from the potentials V_j. Then $\dot{T}_j \to \dot{T}_\infty$ if $j \to \infty$ by Theorem 3.42. This and Corollary 3.36 yield gap $(T_\infty, T_j) < 5\epsilon$ if $j \in \mathbb{N}$ is sufficiently large. □

A nice application of Theorem 3.43 is the following generalisation of the result obtained in [Kl]. Note that pointwise convergence of the potentials is used there and a uniform bound of the form $c\frac{1+|x|^{\gamma-\epsilon}}{|x|^\gamma}$ with $0 < \gamma < 2$, $0 < \epsilon < \gamma$, and $c \in \mathbb{R}$. These assumptions imply (3.2) and (3.3) which are used here.

Corollary 3.44 *Use the notation and assumptions of Theorem 3.43. Additionally assume that for some number $x > 0$,*

$$\begin{aligned} C = \limsup_{j\to\infty} \frac{\int_{-x}^{x} |V_j(\xi)| d\xi}{\int_{-x}^{x} V_j(\xi) d\xi} &< \infty, \\ \int_{-x}^{x} V_j(\xi) d\xi &\to \infty, \quad j \to \infty, \end{aligned}$$

in $\overline{\mathbb{C}}$. Then the operators T_j converge in the generalized sense to the operator T_∞ given by

$$\begin{aligned} \mathcal{D}(T_\infty) &= \{y \in \mathcal{D}_\infty : y(0-) = y(0+) = 0\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

Proof. First note that the limit relations

$$\int_{-x}^{x} V_j(\xi) d\xi \to \infty, \quad \int_{-x}^{x} |V_j(\xi)| d\xi \to \infty, \quad j \to \infty,$$

in $\overline{\mathbb{C}}$ are independent of the choice of x as long as $x \neq 0$. As a consequence of this and the assumption (3.2), the number C is independent of x as long as $x \neq 0$. So without loss of generality assume $x \in X \setminus \{0\}$ and $\mu(x) < (5C)^{-1}$. For $j \in \mathbb{N}$, the relations (3.16), (3.11), and (3.12) imply

$$\begin{aligned} g_{j,0,\infty,x} &= \int_{-x}^{0} V_j(\xi)(H_{j,0,\infty,-x}1)(\xi) d\xi - \int_{x}^{0} V_j(\xi)(H_{j,0,\infty,x}1)(\xi) d\xi \\ &= \int_{-x}^{x} V_j(\xi) d\xi \\ &\quad + \int_{-x}^{0} V_j(\xi)(F_{j,0,\infty,-x}H_{j,0,\infty,-x}1)(\xi) d\xi - \int_{x}^{0} V_j(\xi)(F_{j,0,\infty,x}H_{j,0,\infty,x}1)(\xi) d\xi. \end{aligned}$$

Due to Lemmas 3.11 and 3.22 the two rightmost integrals can be estimated by

$$\left| \int_{\pm x}^{0} V_j(\xi)(F_{j,0,\infty,\pm x}H_{j,0,\infty,\pm x}1)(\xi) d\xi \right| \leq 2\mu(\pm x) \left| \int_{\pm x}^{0} |V_j(\xi)| d\xi \right|.$$

So the assumptions imply

$$\left| g_{j,\lambda,\infty,x} - \int_{-x}^{x} V_j(\xi) d\xi \right| \leq 2\mu(\pm x) \left| \int_{-x}^{x} |V_j(\xi)| d\xi \right| \leq 2C\mu(x) \left| \int_{-x}^{x} V_j(\xi) d\xi \right| \leq \frac{1}{2} \left| \int_{-x}^{x} V_j(\xi) d\xi \right|$$

for all sufficiently large numbers $j \in \mathbb{N}$. Hence, $\tilde{g}(x) = g_{\infty,0,\infty,x} = \infty$. Then Theorem 3.43 yields the assertion. □

As it is seen from the proofs of Theorems 3.42 and 3.43, the difference between the cases of a compact interval $I = [a, b]$ and an unbounded interval $I = \mathbb{R}$ is not essential as to the interface conditions in 0. In fact, Theorems 3.42 and 3.43 have been given to demonstrate this similarity. It is possible to get better results, for instance the assumption that V_∞ is bounded on $\mathbb{R} \setminus [a, b]$ for some interval $[a, b]$ is an artificial restriction. It should be possible to change all potentials by arbitrary functions in $\mathcal{L}^1(\mathbb{R})$ and still get similar results. Also it is not difficult to find analogous theorems on semibounded intervals like $[a, \infty)$ with $a < 0$. In the following, statements will only be formulated for the case $I = [a, b]$. Note that in contrast to the situation in this section, the results of Section 4 indeed depend on the compactness of the underlying interval.

Definition 3.45 A sequence of operators T_j converges to T_∞ in the norm resolvent sense (in the strong resolvent sense) if, firstly, $\rho(T_\infty) \neq \emptyset$, secondly, for one and hence every number $\lambda \in \rho(T_\infty)$ the resolvent $(T_j - \lambda)^{-1}$ is defined if $j \in \mathbb{N}$ is sufficiently large, and, thirdly, $(T_j - \lambda)^{-1} \to (T_\infty - \lambda)^{-1}$ in the operator norm topology (in the strong operator topology) if $j \to \infty$.

Compare the definition in [RS, VIII.7] for the case of selfadjoint operators in a Hilbert space. Norm resolvent sense is equivalent to the generalized convergence if the limit has nonempty resolvent set. The advantage of norm resolvent convergence is its intuitive simplicity. The following is the norm resolvent version of Theorem 3.38 also using Remark 3.39.

Corollary 3.46 *Assume the conditions and use the notation of Theorem 3.38. If the function $\tilde{g}$ is defined in a neighbourhood of 0, define the operator T_∞ by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \left\{ y \in \mathcal{D}_\infty : \lim_{x \to 0} \left(\tilde{h}(x) y'(x) - \tilde{h}(-x) y'(-x) - \tilde{g}(x) y(0) \right) = 0 \right\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty), \end{aligned}$$

if the function $\tilde{g}$ is defined in a neighbourhood of 0 as a complex valued function, or by

$$\begin{aligned} \mathcal{D}(T_\infty) &= \{ y \in \mathcal{D}_\infty : y(0-) = y(0+) = 0 \}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty), \end{aligned}$$

if the function $\tilde{g}$ is constantly equal to infinity in a deleted neighbourhood of 0.
Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in norm resolvent sense if and only if the function $\tilde{g}$ is defined in a neighbourhood of 0 and $\rho(T_\infty) \neq \emptyset$. The norm resolvent limit of the sequence is T_∞. Moreover, the stronger convergence relation

$$\lim_{j \to \infty} \sup_{f \in \mathcal{B}(\mathcal{L}^p([a,b]))} \left\| (T_j - \lambda)^{-1} f - (T_\infty - \lambda)^{-1} f \right\|_{\mathcal{L}^\infty([a,b])} = 0$$

holds for all numbers $\lambda \in \rho(T_\infty)$.

Corollary 3.47 *Assume the conditions and use the notation of Theorem 3.38. Additionally assume $\rho(T_\infty) \neq \emptyset$. Then the sequence of operators T_j converges in the norm resolvent sense if and only if it converges in the strong resolvent sense for $j \to \infty$.*

Faced with eigenvalue problems with singular potential, many authors require eigenfunctions to be continuous in the singularity. This psychologic automatism now has found a motivation:

Corollary 3.48 *Assume that T_∞ is a differential operator such that $T_\infty y = l_\infty[y]$ and the boundary conditions (3.28) hold for all $y \in \mathcal{D}(T_\infty)$. Next assume that the operator T_∞ is approximable in the generalized sense by a sequence of differential operators T_j defined as in Theorem 3.38 from regular potentials V_j, $j \in \mathbb{N}$. Further assume that for these potentials the relations (3.1), (3.2), and (3.3) hold. Then all functions $y \in \mathcal{D}(T_\infty)$ are continuous in the singularity.*

However, this is not true under more general conditions than (3.1), (3.2), and (3.3) used in this section. Particularly, if (3.3) is omitted, there may be functions in the domain of the limit operator T_∞ which are not continuous in 0. This should not be surprising because then the derivative of Dirac's δ-distribution can be approximated.

3.7 Computation of the interface conditions

Throughout this section assume (3.1), (3.2), and

$$\int_x^0 \left| \int_x^\xi |V_j(\varsigma)| d\varsigma \right|^q d\xi \to 0 \tag{3.38}$$

uniformly with respect to $j \in \mathbb{N} \cup \{\infty\}$ if $x \to 0$ for some fixed real number $q \geq 1$. The Lemmas 3.1 and 3.2 show that (3.38) implies (3.3) and that both assumptions are equivalent in the case $q = 1$. For $x \in I$ define

$$\nu(x) = \sup_{\lambda \in \Lambda} \sup_{j \in \mathbb{N} \cup \{\infty\}} \left| \int_x^0 \left| \int_x^\xi |V_j(\varsigma)| + |\lambda| d\varsigma \right|^q d\xi \right|. \tag{3.39}$$

Then (3.38) implies

$$\nu(x) \to 0, \quad x \to 0. \tag{3.40}$$

Lemma 3.49 *Assume $s \in (0, 1]$, $x \in \mathbb{R}$, $v \in \mathcal{M}^s(I_x)$, and that the function w is given by*

$$w(\xi) = \int_x^\xi v(\varsigma) d\varsigma, \quad \xi \in I_x. \tag{3.41}$$

Then for $r = \frac{s}{1-s}$, it holds

$$\begin{aligned} w &\in \mathcal{L}^r(I_x), \\ \|w\|_{\mathcal{L}^r(I_x)} &\leq \|v\|_{\mathcal{L}^s(I_x)}. \end{aligned}$$

Proof. Monotonicity implies $v \in \mathcal{L}^1_{\mathrm{loc}}(I_x)$, hence w is defined. For $s = 1$, the assertion is trivial. The relation $0 < s < 1$ implies $1 < r < \infty$. Hence $\mathcal{L}^r(I_x)$ and $\mathcal{L}^{\hat{r}}(I_x)$ are the dual spaces of each other by [DS, Theorem IV.8.1]. Let $u \in \mathcal{L}^{\hat{r}}(I_x)$ be arbitrary. Then interchange of the order of integration yields

$$\begin{aligned} \int_x^0 u(\xi) w(\xi) d\xi &= \int_x^0 u(\xi) \int_x^\xi v(\varsigma) d\varsigma d\xi \\ &= \int_x^0 \int_\varsigma^0 u(\xi) d\xi v(\varsigma) d\varsigma = \int_x^0 \int_\varsigma^0 u(\xi) d\xi v(\varsigma)^{1-s} v(\varsigma)^s d\varsigma. \end{aligned}$$

Lemma 3.2 gives

$$\int_0^{\varsigma} |u(\xi)|d\xi \le \sqrt[r]{\xi}\, \|u\|_{\mathcal{L}^{\hat r}(I_x)}, \quad \varsigma \in I_x.$$

The relation $\frac{s}{r} = 1 - s$ implies $v^{1-s} \in \mathcal{M}^r(I_x)$ and Lemma 3.5 yields

$$|\sqrt[r]{\varsigma} v(\varsigma)^{1-s}| \le \left\|v^{1-s}\right\|_{\mathcal{L}^r(I_x)} = \|v\|^{1-s}_{\mathcal{L}^s(I_x)}, \quad \varsigma \in I_x.$$

Thus,

$$\left|\int_x^0 u(\xi)w(\xi)d\xi\right| \le \|u\|_{\mathcal{L}^{\hat r}(I_x)} \|v\|^{1-s}_{\mathcal{L}^s(I_x)} \left|\int_x^0 |v(\xi)|^s d\xi\right| = \|u\|_{\mathcal{L}^{\hat r}(I_x)} \|v\|_{\mathcal{L}^s(I_x)}.$$

Hence $w \in \mathcal{L}^r(I_x)$ and

$$\|w\|_{\mathcal{L}^r(I_x)} = \sup_{u \in \mathcal{L}^{\hat r}(I_x)\setminus\{0\}} \frac{\left|\int_x^0 u(\xi)w(\xi)d\xi\right|}{\|u\|_{\mathcal{L}^{\hat r}(I_x)}} \le \|v\|_{\mathcal{L}^s(I_x)}.$$

□

Lemma 3.50 *Assume $r \in [1,\infty]$, $x \in \mathbb{R}$, and $u \in \mathcal{L}^r_{\mathrm{loc}}(I_x)$. Then there is a function $v \in \mathcal{M}^r(I_x)$ such that*

$$\|u\|_{\mathcal{L}^r(I_x)} = \|v\|_{\mathcal{L}^r(I_x)}, \tag{3.42}$$

$$\left|\int_0^{\xi} u(\varsigma)d\varsigma\right| \le \left|\int_0^{\xi} v(\varsigma)d\varsigma\right|, \quad \xi \in I_x. \tag{3.43}$$

Proof. For $c \in [0,\infty)$ define the function m by

$$m(c) = \text{measure}\,(\{\varsigma \in I_x : |u(\varsigma)| \le c\}),$$

where measure $(\cdot)$ means the Lebesgue measure on $\mathbb{R}$. The function m is monotonously increasing, continuous from the right side, and $\lim_{c\to\infty} m(c) = |x|$. For $\xi \in I_x$ define the function v by

$$v(\xi) = \overset{[0,\infty)}{\sup}\{c \in [0,\infty) : m(c) < |x-\xi|\}.$$

The supremum should be taken in the linear ordering $[0,\infty)$, where the supremum of the empty set is 0. Then for all $d \in [0,\infty)$, the relation

$$\begin{aligned}
&\text{measure}\,(\{\xi \in I_x : v(\xi) \le d\}) \\
&\quad = \text{measure}\,(\{\xi \in I_x : \overset{[0,\infty)}{\sup}\{c \in [0,\infty) : m(c) < |x-\xi|\} \le d\}) \\
&\quad = \text{measure}\,(\{\xi \in I_x : m(d) \ge |x-\xi|\}) \\
&\quad = \text{measure}\,(\{\xi \in I_x : m(d) \ge |\xi|\}) = m(d)
\end{aligned}$$

holds, which proves (3.42) and

$$\|v\|_{\mathcal{L}^1(I_x)} = \|u\|_{\mathcal{L}^1(I_x)}. \tag{3.44}$$

From (3.42) and the definition of v, it follows $v \in \mathcal{M}^r(I_x)$. Now assume $\xi \in I_x$. What has been proved so far, particularly (3.44), applied to u on the interval I_ξ in place of of I_x, implies that the function w defined by

$$w(\varsigma) = \overset{[0,\infty)}{\sup}\{c \in \mathbb{R} : \text{measure}\,(\{\tau \in I_\xi : |u(\tau)| \le c\}) < |\xi - \varsigma|\}$$

fulfills $\|w\|_{\mathcal{L}^1(I_\xi)} = \|u\|_{\mathcal{L}^1(I_\xi)}$. Since

$$\text{measure}\,(\{\tau \in I_\xi : |u(\tau)| \le c\}) + |x - \xi| \ge \text{measure}\,(\{\tau \in I_x : |u(\tau)| \le c\}),$$

the condition measure $(\{\tau \in I_\xi : |u(\tau)| \le c\}) < |\xi - \varsigma|$ implies measure $(\{\tau \in I_x : |u(\tau)| \le c\}) < |x - \varsigma|$ for $c \in [0, \infty)$. Hence $w(\varsigma) \le v(\varsigma)$ for all $\varsigma \in I_\xi$, which yields (3.43). □

Lemma 3.51 *Assume $x \in \mathbb{R}$ and that $v \in \mathcal{L}^1_{\text{loc}}(I_x)$ is a nonnegative real valued function such that the function w which is given by (3.41) lies in $\mathcal{L}^q(I_x)$. Assume $1 \le r \le \hat{q}$ and $u \in \mathcal{L}^r(I_x)$. Then $s = (q^{-1} + r^{-1} - 1)^{-1} \in [1, \infty]$ and the function h given by*

$$h(\xi) = \int_x^\xi v(\varsigma) \int_0^\varsigma u(\tau) d\tau d\varsigma, \quad \xi \in I_x,$$

fulfills

$$\begin{aligned} h &\in \mathcal{L}^s(I_x), \\ \|h\|_{\mathcal{L}^s(I_x)} &\le 2\,\|w\|_{\mathcal{L}^q(I_x)}\,\|u\|_{\mathcal{L}^r(I_x)}\,. \end{aligned}$$

Proof. Assume $u \in \mathcal{M}^r(I_x)$. Lemma 3.50 implies that this is no loss of generality. Change of the order of integration or, equivalently, integration by parts yields

$$h(\xi) = \underbrace{\int_0^\xi u(\varsigma) d\varsigma w(\xi)}_{h_1(\xi)} - \underbrace{\int_x^\xi u(\varsigma) w(\varsigma) d\varsigma}_{h_2(\xi)}, \quad \xi \in I_x.$$

Lemma 3.2 implies

$$\left|\int_0^\xi u(\varsigma) d\varsigma\right| \le \sqrt[\hat{r}]{|\xi|}\,\|u\|_{\mathcal{L}^r(I_x)}\,, \quad \xi \in I_x.$$

Lemma 3.5 applied to the absolute value of the function w yields

$$\sqrt[q]{|\xi|}|w(\xi)| \le \|w\|_{\mathcal{L}^q(I_x)}\,, \quad \xi \in I_x.$$

So using the relation $1 - \frac{q}{\hat{r}} = 1 - q + \frac{q}{r} = q\,(q^{-1} + r^{-1} - 1) = \frac{q}{s}$,

$$\begin{aligned} |h_1(\xi)| &= \left|\int_0^\xi u(\varsigma) d\varsigma\right| |w(\xi)|^{\frac{q}{\hat{r}}} |w(\xi)|^{1-\frac{q}{\hat{r}}} \\ &\le \|u\|_{\mathcal{L}^r(I_\xi)}\,\|w\|^{\frac{q}{\hat{r}}}_{\mathcal{L}^q(I_\xi)}\,|w(\xi)|^{1-\frac{q}{\hat{r}}} = \|u\|_{\mathcal{L}^r(I_\xi)}\,\|w\|^{1-\frac{q}{s}}_{\mathcal{L}^q(I_\xi)}\,|w(\xi)|^{\frac{q}{s}} \end{aligned}$$

for all $\xi \in I_x$, it follows

$$\|h_1\|_{\mathcal{L}^s(I_x)} \le \|u\|_{\mathcal{L}^r(I_\xi)}\,\|w\|^{1-\frac{q}{s}}_{\mathcal{L}^q(I_\xi)}\,\left\|w^{\frac{q}{s}}\right\|_{\mathcal{L}^s(I_x)} = \|u\|_{\mathcal{L}^r(I_x)}\,\|w\|^{1-\frac{q}{s}}_{\mathcal{L}^q(I_\xi)}\,\|w\|^{\frac{q}{s}}_{\mathcal{L}^q(I_x)}\,.$$

Thus $\|h_1\|_{\mathcal{L}^s(I_x)} \leq \|u\|_{\mathcal{L}^r(I_x)} \|w\|_{\mathcal{L}^q(I_x)}$. As to h_2, Hölder's inequality implies that the product $uw \in \mathcal{M}^{(q^{-1}+r^{-1})^{-1}}(I_x)$ and

$$\|uw\|_{\mathcal{L}^{(q^{-1}+r^{-1})^{-1}}(I_x)} \leq \|u\|_{\mathcal{L}^r(I_x)} \|w\|_{\mathcal{L}^q(I_x)}.$$

The assumption $r \leq \hat{q}$ yields $(q^{-1}+r^{-1})^{-1} \leq (q^{-1}+\hat{q}^{-1})^{-1} = 1$, hence Lemma 3.49 gives

$$\|h_2\|_{\mathcal{L}^s(I_x)} \leq \|u\|_{\mathcal{L}^r(I_x)} \|w\|_{\mathcal{L}^q(I_x)}.$$

□

For $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in I \setminus \{0\}$ define the integral operator $G_{j,\lambda,x}$ on $\mathcal{L}^1(I_x)$ by

$$(G_{j,\lambda,x}y)(\xi) = \int_x^\xi (V_j(\varsigma) - \lambda) \int_0^\varsigma y(\tau) d\tau d\varsigma$$

for $y \in \mathcal{L}^1(I_x)$. The operator $G_{j,\lambda,x}$ is defined such that (3.8) implies

$$(F^n_{j,\lambda,x}y)' = G^{n-1}_{j,\lambda,x}(F_{j,\lambda,x}y)' \tag{3.45}$$

for $n \in \mathbb{N} \setminus \{0\}$ and $y \in \mathcal{L}^\infty(I_x)$.

Corollary 3.52 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in I \setminus \{0\}$, $r \in [1, \infty]$, and*

$$s = \begin{cases} (q^{-1} + r^{-1} - 1)^{-1} & \text{if } r < \hat{q} \\ \infty & \text{if } r \geq \hat{q} \end{cases}.$$

Then $G_{j,\lambda,x}$ maps $\mathcal{L}^r(I_x)$ to $\mathcal{L}^s(I_x)$ and

$$\|G_{j,\lambda,x}\|_{\mathcal{L}^r(I_x)\to\mathcal{L}^s(I_x)} \leq \begin{cases} 2\nu(x) & \text{if } r \leq \hat{q} \\ 2x^{\hat{q}^{-1}-r^{-1}}\nu(x) & \text{if } r > \hat{q} \end{cases}.$$

Proof. If $r \leq \hat{q}$, then the assertion is a direct consequence of Lemma 3.51 and (3.38). If $r > \hat{q}$ and $y \in \mathcal{L}^r(I_x)$, then Corollary 3.4 implies $y \in \mathcal{L}^{\hat{q}}$ and $\|y\|_{\mathcal{L}^{\hat{q}}} \leq x^{\hat{q}^{-1}-r^{-1}} \|y\|_{\mathcal{L}^{\hat{r}}}$. This and the assertion for the case $r \leq \hat{q}$ prove the general assertion. □

Corollary 3.53 *Assume $\lambda \in \Lambda$, $n \in \mathbb{N} \cup \{\infty\}$, $x \in X \setminus \{0\}$, $r \in [1, \infty]$ and s as in Corollary 3.52. Then*

$$G_{j,\lambda,x} \to G_{\infty,\lambda,x}$$

in norm as operators from $\mathcal{L}^r(I_x)$ to $\mathcal{L}^s(I_x)$ if $j \to \infty$, uniformly with respect to all other variables.

Proof. For $y \in \mathcal{L}^r(I_x)$, $\varsigma \in I_x$, and $\xi \in I_\varsigma$, it holds

$$\begin{aligned}
|G_{\infty,\lambda,x}y - G_{j,\lambda,x}y|(\varsigma) &= \left|\int_x^\varsigma (V_\infty(\tau) - V_j(\tau)) \int_0^\tau y(\upsilon)d\upsilon d\tau\right| \\
&\leq \int_x^\xi |V_\infty(\tau) - V_j(\tau)| \int_0^\tau |y(\upsilon)d\upsilon| d\tau + \left|\int_\xi^\varsigma (V_\infty(\tau) - V_j(\tau)) \int_0^\tau y(\upsilon)d\upsilon d\tau\right| \\
&\leq \int_x^\xi |V_\infty(\tau) - V_j(\tau)| d\tau \, \|y\|_{\mathcal{L}^1(I_x)} + |G_{\infty,\lambda,\xi}y - G_{j,\lambda,\xi}y|(\varsigma) \\
&\leq \int_x^\xi |V_\infty(\tau) - V_j(\tau)| d\tau \, \|y\|_{\mathcal{L}^1(I_x)} + |G_{\infty,\lambda,\xi}y|(\varsigma) + |G_{j,\lambda,\xi}y|(\varsigma).
\end{aligned} \tag{3.46}$$

Now choose $\epsilon > 0$ arbitrarily. Fix the number $\xi \in I_\varsigma$ so small that $\|G_{\infty,\lambda,\xi}\|_{\mathcal{L}^r(I_\xi)\to\mathcal{L}^s(I_\xi)} < \epsilon$ and $\|G_{j,\lambda,\xi}\|_{\mathcal{L}^r(I_\xi)\to\mathcal{L}^s(I_\xi)} < \epsilon$. This is possible by Corollary 3.52 and (3.40). Moreover, the first integral term in (3.46) tends to 0 if $j \to \infty$. □

Lemma 3.54 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $n \in \mathbb{N} \setminus \{0\}$, $x \in I \setminus \{0\}$, and $y \in \mathcal{L}^\infty(I_x)$. Define*

$$q_n = \begin{cases} (1 - \frac{n}{\hat{q}})^{-1} & \text{if } n < \hat{q} \\ \infty & \text{if } n \geq \hat{q} \end{cases}.$$

Then it holds

$$(F^n_{j,\lambda,x} y)' \in \mathcal{L}^{q_n}(I_x), \tag{3.47}$$

$$\left\| (F^n_{j,\lambda,x} y)' \right\|_{\mathcal{L}^{q_n}(I_x)} \leq \begin{cases} (2\nu(x))^n \, \|y\|_{\mathcal{L}^\infty(I_x)} & \text{if } n \leq \hat{q} \\ \left(\sqrt[\hat{q}]{|x|}\right)^{n-\hat{q}} (2\nu(x))^n \, \|y\|_{\mathcal{L}^\infty(I_x)} & \text{if } n > \hat{q} \end{cases}. \tag{3.48}$$

Moreover, $(F^n_{j,\lambda,x} y)' \to (F^n_{\infty,\lambda,x} y)'$ in $\mathcal{L}^{q_n}(I_x)$, if $j \to \infty$.

Proof. The proof is by induction on n. For $n = 1$, the relation

$$|(F_{j,\lambda,x} y)'(\xi)| \leq \left| \int_x^\xi |V_j(\varsigma)| + |\lambda| d\varsigma \right| \|y\|_{\mathcal{L}^\infty(I_x)}, \quad \xi \in I_x,$$

and (3.39) imply (3.47) and (3.48). To prove the convergence assertion for $j \to \infty$, choose a number $\epsilon > 0$ and decompose the integral

$$\begin{aligned} \int_x^0 |(F_{\infty,\lambda,x} y - F_{j,\lambda,x} y)'(\varsigma)|^q \, d\varsigma &= \int_\xi^0 |(F_{\infty,\lambda,x} y - F_{j,\lambda,x} y)'(\varsigma)|^q \, d\varsigma \\ &\quad + \int_x^\xi |(F_{\infty,\lambda,x} y - F_{j,\lambda,x} y)'(\varsigma)|^q \, d\varsigma \end{aligned}$$

with a number $\xi \in I_x$ chosen such that the first integral on the right side is less than $\epsilon \|y\|_{\mathcal{L}^\infty(I_x)}$. This is possible because of (3.38). The second integral on the right side tends to 0 if $j \to \infty$ due to (3.2).
Next assume that the assertions have been proved for $n-1$. Use $r = q_{n-1}$ in the premises of Corollary 3.52. Then the inequality

$$r = q_{n-1} = \left(1 - \frac{n-1}{\hat{q}}\right)^{-1} < \left(1 - \frac{\hat{q}-1}{\hat{q}}\right)^{-1} = (1 - q^{-1})^{-1} = \hat{q}$$

holds if and only if $n < \hat{q}$. In this case, the corresponding number s equals

$$\left(\frac{1}{q} + \frac{1}{q_{n-1}} - 1\right)^{-1} = \left(\frac{1}{q} + 1 - \frac{n-1}{\hat{q}} - 1\right)^{-1} = \left(1 - \frac{1}{\hat{q}} - \frac{n-1}{\hat{q}}\right)^{-1} = \left(1 - \frac{n}{\hat{q}}\right)^{-1} = q_n.$$

Else, $s = \infty = q_n$, so $s = q_n$ in both cases. Now (3.45) and Corollary 3.52 yield (3.47) and (3.48) for n. The convergence assertion for the limit $j \to \infty$ follows from Corollary 3.53. □

The following is a strengthening of Lemma 3.26 for the case $q > 1$.

Lemma 3.55 *Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, $x \in X \setminus \{0\}$, $f \in \mathcal{L}^p(I_{-x} \cup I_x)$, and that the function y is a solution of (3.5) on $I_{-x} \cup I_x$. Further assume $g_{j,\lambda,\infty,x} \in \mathbb{C}$ and that $\tilde{g}$ and $\tilde{h}$ are functions on $I_{-x} \cup I_x$ such that*

$$\lim_{\xi \to 0} (g_{j,\lambda,\infty,\xi} - \tilde{g}(\xi)) = 0, \quad \limsup_{\xi \to 0} \left| \frac{(H_{j,\lambda,\infty,\xi} \iota)'(0) - \tilde{h}(\xi)}{\sqrt[q]{\xi}} \right| < \infty.$$

Then the condition (3.22) is equivalent to (3.20).

Proof. The proof is analogous to that of Lemma 3.26, but now the absolute value of the derivative y' is bounded by a function in $\mathcal{M}^q(I_{-x} \cup I_x)$. This follows from the condition (3.38) in the same way as Corollary 3.9 has been proved. □

Lemma 3.56 *The asymptotic relations*

$$\mu(x) = o\left(\sqrt[\hat{q}]{x}\right)$$

holds if $x \to 0$.

Proof. The assumption (4.2) means that the function w given by

$$w(x) = \begin{cases} \int_a^x |V_\infty(\xi)| d\xi \text{ if } x < 0 \\ \int_x^b |V_\infty(\xi)| d\xi \text{ if } x > 0 \end{cases} \tag{3.49}$$

is contained in $\mathcal{L}^q([a,b])$. So Lemma 3.2 helps to prove

$$\mu(x) \leq \left| \int_0^x w(\xi) d\xi \right| \leq \sqrt[\hat{q}]{|x|}\, \|w\|_{\mathcal{L}^q(I_x)} = o\left(\sqrt[\hat{q}]{|x|}\right).$$

□

Theorem 3.57 *Assume that the potentials* V_j, $j \in \mathbb{N} \cup \{\infty\}$, *on the compact interval* $I = [a,b]$, $a < 0 < b$, *fulfill (3.1), (3.2), and (3.38) for a number* $q > 1$. *Assume* $z_\pm \in \overline{\mathbb{C}}$. *Define the operators* T_j, $j \in \mathbb{N}$, *and the set* $\mathcal{D}_\infty$ *as in Theorem 3.38. For those* $x \in [a,b] \cap [-b,-a]$ *where the limit defining* $\tilde{g}(x)$ *exists in the compact space* $\overline{\mathbb{C}}$, *define*

$$\begin{aligned}
\tilde{g}_q(x) \;=\; & \overset{\overline{\mathbb{C}}}{\lim_{j\to\infty}} \Bigg(\sum_{1 \leq m < \hat{q}} \Bigg[\int_{\varsigma_m=-x}^{0} V_j(\varsigma_m) \int_{\xi_{m-1}=0}^{\varsigma_m} \Bigg[\int_{\varsigma_{m-1}=-x}^{\xi_{m-1}} V_j(\varsigma_{m-1}) \int_{\xi_{m-2}=0}^{\varsigma_{m-1}} \Bigg[\cdots \\
& \cdots \Bigg[\int_{\varsigma_2=-x}^{\xi_2} V_j(\varsigma_2) \int_{\xi_1=0}^{\varsigma_2} \Bigg[\int_{\varsigma_1=-x}^{\xi_1} V_j(\varsigma_1) d\varsigma_1 \Bigg] d\xi_1 d\varsigma_2 \Bigg] \cdots \\
& \cdots \Bigg] d\xi_{m-2} d\varsigma_{m-1} \Bigg] d\xi_{m-1} d\varsigma_m \Bigg] \\
& - \sum_{1 \leq m < \hat{q}} \Bigg[\int_{\varsigma_m=x}^{0} V_j(\varsigma_m) \int_{\xi_{m-1}=0}^{\varsigma_m} \Bigg[\int_{\varsigma_{m-1}=x}^{\xi_{m-1}} V_j(\varsigma_{m-1}) \int_{\xi_{m-2}=0}^{\varsigma_{m-1}} \Bigg[\cdots \\
& \cdots \Bigg[\int_{\varsigma_2=x}^{\xi_2} V_j(\varsigma_2) \int_{\xi_1=0}^{\varsigma_2} \Bigg[\int_{\varsigma_1=x}^{\xi_1} V_j(\varsigma_1) d\varsigma_1 \Bigg] d\xi_1 d\varsigma_2 \Bigg] \cdots \\
& \cdots \Bigg] d\xi_{m-2} d\varsigma_{m-1} \Bigg] d\xi_{m-1} d\varsigma_m \Bigg] \Bigg), \\
\tilde{h}_q(x) \;=\; & 1 + \sum_{1 \leq m < \hat{q}-1} \Bigg[\int_x^0 V_\infty(\varsigma_m) \int_0^{\varsigma_m} \Bigg[\int_{\varsigma_{m-1}=x}^{\xi_{m-1}} V_\infty(\varsigma_{m-1}) \int_{\xi_{m-2}=0}^{\varsigma_{m-1}} \Bigg[\cdots \\
& \cdots \Bigg[\int_x^{\xi_2} V_\infty(\varsigma_2) \int_0^{\varsigma_2} \Bigg[\int_x^{\xi_1} V_\infty(\varsigma_1) \varsigma_1 d\varsigma_1 \Bigg] d\xi_1 d\varsigma_2 \Bigg] \cdots \\
& \cdots \Bigg] d\xi_{m-2} d\varsigma_{m-1} \Bigg] d\xi_{m-1} d\varsigma_m \Bigg].
\end{aligned}$$

Then exactly one of the following three cases holds:

1. *The function $\tilde{g}_q$ is defined in a neighbourhood of* 0 *as a complex valued function. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \left\{ y \in \mathcal{D}_\infty : \lim_{x \to 0} \left(\tilde{h}_q(x) y'(x) - \tilde{h}_q(-x) y'(-x) - \tilde{g}_q(x) y(0) \right) = 0 \right\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

2. *The function $\tilde{g}_q$ is constantly equal to infinity in a deleted neighbourhood of* 0*. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \{ y \in \mathcal{D}_\infty : y(0-) = y(0+) = 0 \}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

3. *The function $\tilde{g}_q$ is undefined in a deleted neighbourhood of* 0*. Then the sequence of operators T_j, $j \in \mathbb{N}$, has two subsequences with different limits in the generalized sense.*

Proof. Set $n = \min\{m \in \mathbb{N} : m \geq \hat{q}\}$. Assume $j \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \Lambda$, and $x \in X$ with $|x| \leq 1$ and $\nu(x) < \frac{1}{2}$, Then Lemma 3.54 implies

$$\begin{aligned} (H_{j,\lambda,\infty,x} 1 - H_{j,\lambda,n,x} 1)' &= \sum_{m=n}^{\infty} (F^m_{j,\lambda,x} 1)', \\ \|(H_{j,\lambda,\infty,x} 1 - H_{j,\lambda,n,x} 1)'\| &\leq \sum_{m=n}^{\infty} (2\nu(x))^m = \frac{(2\nu(x))^n}{1 - 2\nu(x)}. \end{aligned}$$

So both series are absolutely convergent in $\mathcal{L}^\infty(I_x)$. From this and the convergence relation $(F^m_{j,\lambda,x} 1)' \to (F^m_{\infty,\lambda,x} 1)'$ in $\mathcal{L}^\infty(I_x)$ if $j \to \infty$ for all $m \geq n$, it follows

$$(H_{j,\lambda,\infty,x} 1 - H_{j,\lambda,n,x} 1)' \to (H_{\infty,\lambda,\infty,x} 1 - H_{\infty,\lambda,n,x} 1)'$$

in $\mathcal{L}^\infty(I_x)$ if $j \to \infty$. This and (3.16) yields the convergence of the expression

$$g_{j,\lambda,\infty,x} - g_{j,\lambda,n,x} = (H_{j,\lambda,\infty,-x} 1)'(0) - (H_{j,\lambda,n,-x} 1)'(0) - (H_{j,\lambda,\infty,-x} 1)'(0) + (H_{j,\lambda,n,x} 1)'(0)$$

if $j \to \infty$. Hence, similar to the ideas used in the proof of Lemma 3.25, $g_{\infty,\lambda,n,x}$ is defined if and only if $g_{\infty,\lambda,\infty,x}$ is. Moreover for $x \in X \cap I_+$, it holds

$$\begin{aligned} |g_{\infty,\lambda,\infty,x} - g_{\infty,\lambda,n,x}| &\leq \|(H_{j,\lambda,\infty,-x} 1)' - (H_{j,\lambda,n,-x} 1)'\|_{\mathcal{L}^\infty(I_x)} \\ &\quad + \|(H_{j,\lambda,\infty,x} 1)' - (H_{j,\lambda,n,x} 1)'\|_{\mathcal{L}^\infty(I_x)} \\ &\leq \frac{2^{n+1} \nu(x)^n}{1 - 2\nu(x)}. \end{aligned}$$

As to $\tilde{h}_q$ observe that (3.9) or (3.11) and the Lemmas 3.11, 3.22, and finally 3.56 imply

$$|(H_{\infty,\lambda,\infty,x} \iota)'(0) - (H_{\infty,\lambda,n-1,x} \iota)'(0)| = \left| (F^{n-1}_{\infty,\lambda,x} H_{\infty,\lambda,\infty,x} \iota)'(0) \right| \leq 2\mu(x)^{n-1} \leq C \left| \sqrt[\hat{q}]{x}^{\,n-1} \right|$$

for some constant C. The choice of n yields

$$\frac{n-1}{\hat{q}} \geq \frac{\hat{q}-1}{\hat{q}} = \frac{1}{q}$$

and thus $\left|\sqrt[\hat{q}]{x}^{n-1}\right| \leq |\sqrt[q]{x}|$ if $|x| \leq 1$. Since $\tilde{g}_q(x) = g_{\infty,0,n,x}$ and $\tilde{h}_q(x) = (H_{\infty,0,n-1,x}\iota)'(0)$, this and the convergence relation (3.40) prove

$$\lim_{x\to 0}(\tilde{g}_q(x) - g_{\infty,0,\infty,x}) = 0, \quad \limsup_{x\to 0}\left|\frac{\tilde{h}_q(x) - (H_{\infty,0,\infty,x}\iota)'(0)}{\sqrt[q]{x}}\right| < \infty.$$

Recall the functions $\tilde{g}$ and $\tilde{h}$ defined in Theorem 3.38. The equalities $\tilde{g}(x) = g_{\infty,0,\infty,x}$, and $\tilde{h}(x) = (H_{\infty,0,\infty,x}\iota)'(0)$ for $x \in X \setminus \{0\}$, Lemma 3.55 and Theorem 3.38 finally yield the assertion. □

In the case $q \geq 2$, Theorem 3.57 becomes the following more convenient statement.

Corollary 3.58 *Assume that the potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$, on the compact interval $I = [a,b]$, $a < 0 < b$, fulfill (3.1), (3.2), and (3.38) with $q \geq 2$. Assume $z_\pm \in \overline{\mathbb{C}}$. Define the operators T_j, $j \in \mathbb{N}$, and the set $\mathcal{D}_\infty$ as in Theorem 3.38. For those $x \in [a,b]$ where the limit exists in the compact space $\overline{\mathbb{C}}$ define*

$$\tilde{g}_2(x) = \overset{\overline{\mathbb{C}}}{\lim_{j\to\infty}} \int_{-x}^{x} V_j(\xi)d\xi,$$

Then exactly one of the following three cases holds:

1. *The function $\tilde{g}_2$ is defined in a neighbourhood of 0 as a complex valued function. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \left\{ y \in \mathcal{D}_\infty : y(0-) = y(0+), \lim_{x\to 0}\left(y'(x) - y'(-x) - \tilde{g}_2(x)y(0)\right) = 0 \right\}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

2. *The function $\tilde{g}_2$ is constantly equal to infinity in a deleted neighbourhood of 0. Then the sequence of operators T_j, $j \in \mathbb{N}$, converges in the generalized sense to the operator T_∞ given by*

$$\begin{aligned} \mathcal{D}(T_\infty) &= \{ y \in \mathcal{D}_\infty : y(0-) = y(0+) = 0 \}, \\ T_\infty y &= -y'' + V_\infty y, \quad y \in \mathcal{D}(T_\infty). \end{aligned}$$

3. *The function $\tilde{g}_2$ is undefined in a deleted neighbourhood of 0. Then the sequence of operators T_j, $j \in \mathbb{N}$, has two subsequences with different limits in the the generalized sense.*

3.8 Examples

The following problem was studied in [Bo], [Gu], and [BDL].

Example 3.59 Consider the differential operators T_j, $j \in \mathbb{N}$, in $\mathcal{L}^2([a,b])$, $a < 0 < b$, defined by

$$\begin{aligned} \mathcal{D}(T_j) &= \{y \in \mathcal{AC}^2([a,b]) : y'' \in \mathcal{L}^2([a,b]), y(a) = y(b) = 0\}, \\ (T_j y)(x) &= -y''(x) - \frac{y(x)}{x + \frac{i}{j}}, \quad y \in \mathcal{D}(T_j), x \in [a,b]. \end{aligned}$$

The corresponding potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$, are given by

$$V_j(x) = -\frac{1}{x - \frac{i}{j}}, \quad x \in [a,b].$$

Their limit in the sense of (3.2) is given by

$$V_\infty(x) = -\frac{1}{x}, \quad x \in [a,b] \setminus \{0\}.$$

The potentials V_j, $j \in \mathbb{N}$, are regular and V_∞ is singular only in 0, hence the condition (3.1) is fulfilled. Since for all $j \in \mathbb{N} \cup \{\infty\}$, $x \in [a,b] \setminus \{0\}$, and $\xi \in I_x$, it holds

$$\left| \int_x^\xi |V_j(\varsigma)| d\varsigma \right| \leq \left| \int_x^\xi \frac{1}{|\varsigma|} d\varsigma \right| = |\ln \xi - \ln x|,$$

and since the logarithm is square integrable near 0, (3.38) holds with $q = 2$.
The function $\tilde{g}_2$ is given by

$$\tilde{g}_2(x) = -\overline{\lim_{j\to\infty}}^{\mathbb{C}} \int_{-x}^x \frac{1}{\xi - \frac{i}{j}} d\xi = \overline{\lim_{j\to\infty}}^{\mathbb{C}} \left(\ln\left(-x - \frac{i}{j}\right) - \ln\left(x - \frac{i}{j}\right) \right) = -\pi i \qquad (3.50)$$

for all $x \in (0, \min(-a,b)]$ and changes only its sign if $x \in [\max(a,-b), 0)$. That $\tilde{g}_2$ is constant on these intervals is a consequence of the odd symmetry of the potential V_∞. In general, this is not true. Now Corollary 3.58 yields that the operators T_j converge in the generalized sense to the operator T_∞ which is given by

$$(T_\infty y)(x) = -y''(x) + V_\infty(x) y(x) = -y''(x) - \frac{y(x)}{x}, \quad x \in [a,b] \setminus \{0\}, \qquad (3.51)$$

for all $y \in \mathcal{D}(T_\infty)$. The domain $\mathcal{D}(T_\infty)$ consists of all functions $y \in \mathcal{AC}^2_{\text{loc}}([a,b] \setminus \{0\})$ such that the function on the right side of (3.51) is in $\mathcal{L}^2([a,b])$,

$$y(0-) = y(0+), \quad \lim_{x\to 0+} (y'(x) - y'(-x)) = -\pi i y(0),$$

and $y(a) = y(b) = 0$. The one-sided limit is a way to cope with the sign change of $\tilde{g}_2$ at 0. Since the operator $-T_\infty$ is dissipative, see [BDL], and thus T_∞ has nonempty resolvent set, $T_j \to T_\infty$ also in the norm resolvent sense. Note that the nonempty resolvent set can also be proved by Corollary 4.18.

The following problem was dealt with in [BDL] for the case $\gamma \in \mathbb{C}$.

Example 3.60 For a number $\gamma \in \mathbb{C}$ consider the differential operators $T_{\gamma,j}$, $j \in \mathbb{N}$, in $\mathcal{L}^2([a,b])$, $a < 0 < b$, defined by

$$\begin{aligned} \mathcal{D}(T_{\gamma,j}) &= \{y \in \mathcal{AC}^2([a,b]) : y'' \in \mathcal{L}^2([a,b]), y(a) = y(b) = 0\}, \\ (T_{\gamma,j}y)(x) &= -y''(x) - \frac{1}{2}\left(\frac{1+\gamma/i\pi}{x+\frac{i}{j}} + \frac{1-\gamma/i\pi}{x-\frac{i}{j}}\right) y(x), \quad y \in \mathcal{D}(T_{\gamma,j}), x \in [a,b]. \end{aligned}$$

Like in Example 3.59, the corresponding potentials $V_{\gamma,j}$, $j \in \mathbb{N} \cup \{\infty\}$, given by

$$V_{\gamma,j}(x) = -\frac{1}{2}\left(\frac{1+\gamma/i\pi}{x+\frac{i}{j}} + \frac{1-\gamma/i\pi}{x-\frac{i}{j}}\right), \quad x \in [a,b], \quad V_\infty(x) = -\frac{1}{x}, \quad x \in [a,b] \setminus \{0\},$$

fulfill (3.1), (3.2), and (3.38) with $q = 2$. Using the limit computed in (3.50), the function $\tilde{g}_{\gamma,2}$ is seen to be given by

$$\begin{aligned} \tilde{g}_{\gamma,2}(x) &= -\frac{1+\gamma/\pi i}{2} \lim_{j\to\infty} \int_{-x}^{x} \frac{1}{\xi + \frac{i}{j}} d\xi - \frac{1-\gamma/\pi i}{2} \lim_{j\to\infty} \int_{-x}^{x} \frac{1}{\xi - \frac{i}{j}} d\xi \\ &= \frac{(1+\gamma/\pi i)\pi i - (1-\gamma/\pi i)\pi i}{2} = \gamma \end{aligned}$$

for all $x \in (0, \min(-a,b)]$. Now Corollary 3.58 yields that the operators $T_{\gamma,j}$ converge in the generalized sense to the operator $T_{\gamma,\infty}$ which is given by

$$(T_{\gamma,\infty}y)(x) = -y''(x) - \frac{y(x)}{x}, \quad x \in [a,b] \setminus \{0\}, \tag{3.52}$$

for all $y \in \mathcal{D}(T_{\gamma,\infty})$. The domain $\mathcal{D}(T_{\gamma,\infty})$ consists of all functions $y \in \mathcal{AC}^2_{\text{loc}}([a,b] \setminus \{0\})$ such that the function on the right side of (3.52) is in $\mathcal{L}^2([a,b])$,

$$y(0-) = y(0+), \quad \lim_{x\to 0} (y'(x) - y'(-x)) = \gamma y(0),$$

and $y(a) = y(b) = 0$. Since one of the operators $\pm T_{\gamma,\infty}$ is dissipative, see [BDL], $T_{\gamma,\infty}$ has nonempty resolvent set and $T_{\gamma,j} \to T_{\gamma,\infty}$ also in the norm resolvent sense. As in Example 3.59, also Corollary 4.18 yields $\rho(T_\infty) \neq \emptyset$.
Next consider the "diagonal" sequence of operators T_{j,j^2}, $j \in \mathbb{N}$. From

$$\begin{aligned} \int_{-x}^{x} \frac{1+\frac{j}{\pi i}}{\xi + \frac{i}{j^2}} &= \left(1 + \frac{j}{\pi i}\right)\left(\ln\left(x + \frac{i}{j^2}\right) - \left(\ln\left(-x + \frac{i}{j^2}\right)\right)\right) \\ &= \left(1 + \frac{j}{\pi i}\right)\left(-\pi i + O(j^{-2})\right) \\ &= -j - \pi i + O(j^{-1}) \end{aligned}$$

and the complex conjugate of this equality, it follows

$$\tilde{g}_2(x) = \overset{\overline{\mathbb{C}}}{\lim_{j\to\infty}} \left(-\frac{-j-\pi i + O(j^{-1})}{2} - \frac{-j+\pi i+O(j^{-1})}{2}\right) = \infty$$

for all $x \in (0, \min(-a, b)]$. So $T_{j,j^2} \to T_{\infty,\infty}$ in the generalized sense if $j \to \infty$, where $T_{\infty,\infty}$ is given by

$$(T_{\infty,\infty}y)(x) = -y''(x) - \frac{y(x)}{x}, \quad x \in [a, b] \setminus \{0\}, \tag{3.53}$$

for all $y \in \mathcal{D}(T_{\infty,\infty})$. The domain $\mathcal{D}(T_{\infty,\infty})$ consists of all $y \in \mathcal{AC}^2_{\text{loc}}([a, b] \setminus \{0\})$ such that the function on the right side of (3.53) is in $\mathcal{L}^2([a, b])$ and

$$y(0-) = y(0+) = y(a) = y(b) = 0.$$

This operator is selfadjoint and hence $T_{j,j^2} \to T_{\infty,\infty}$ also in the norm resolvent sense if $j \to \infty$.

The following examples are formulated on the real line $\mathbb{R}$. Since the additional assumption of Theorem 3.43, is fulfilled here, this is possible. Treatment of the next problem can be found in [L] and [G].

Example 3.61 Consider the differential operators T_j, $j \in \mathbb{N}$, in $\mathcal{L}^2(\mathbb{R})$, defined by

$$\begin{aligned} \mathcal{D}(T_j) &= \{y \in \mathcal{L}^2(\mathbb{R}) \cap \mathcal{AC}^2(\mathbb{R}) : y'' \in \mathcal{L}^2(\mathbb{R})\}, \\ (T_j y)(x) &= -y''(x) - \frac{y(x)}{|x| + \frac{1}{j}}, \quad y \in \mathcal{D}(T_j), x \in \mathbb{R}. \end{aligned}$$

Like in Example 3.59, the corresponding potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$ given by

$$V_j(x) = -\frac{1}{|x| + \frac{1}{j}}, \quad x \in \mathbb{R}, \quad V_\infty(x) = -\frac{1}{|x|}, \quad x \in \mathbb{R} \setminus \{0\},$$

fulfill (3.1), (3.2), and (3.38) with $q = 2$. Evidently, $\tilde{g}_2(x) = \infty$ for all $x \in \mathbb{R} \setminus \{0\}$. Hence Corollary 3.58 yields that the operators T_j converge in the generalized sense to the operator T_∞ which is given by

$$(T_\infty y)(x) = -y''(x) + V_\infty(x) = -y''(x) - \frac{y(x)}{|x|}, \quad x \in \mathbb{R} \setminus \{0\}, \tag{3.54}$$

for all functions $y \in \mathcal{D}(T_\infty)$. The domain $\mathcal{D}(T_\infty)$ consists of all functions $y \in \mathcal{L}^2(\mathbb{R}) \cap \mathcal{AC}^2_{\text{loc}}(\mathbb{R} \setminus \{0\})$ such that the function on the right side of (3.54) is in $\mathcal{L}^2(\mathbb{R})$ and

$$y(0-) = y(0+) = 0.$$

A different way to prove the approximation result would be the application of Corollary 3.44. Since the operator T_∞ is selfadjoint and thus has nonempty resolvent set, $T_j \to T_\infty$ also in the norm resolvent sense.

Example 3.62 Consider operators T_j as in Example 3.61 but with potentials V_j, $j \in \mathbb{N} \cup \{\infty\}$, given by

$$V_j(x) = \begin{cases} -\frac{1}{|x|} \text{ if } |x| \geq \frac{1}{j} \\ j \ln j \text{ if } |x| < \frac{1}{j} \end{cases}, \quad j \in \mathbb{N}, x \in \mathbb{R}, \quad V_\infty(x) = -\frac{1}{|x|}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Then due to the even symmetry of the potentials,

$$\tilde{g}_2(x) = \overline{\mathbb{C}}\lim_{j\to\infty}\left(2\int_0^{j^{-1}} j\ln j\,d\xi - 2\int_{j^{-1}}^{x}\frac{1}{|\xi|}d\xi\right) = 2\,\overline{\mathbb{C}}\lim_{j\to\infty}(\ln j - \ln x - \ln j) = -2\ln x,$$

for all numbers $x \in (0,\infty)$. Consequently, the operators T_j converge in the generalized sense to the operator T_∞ which is given by

$$(T_\infty y)(x) = -y''(x) - \frac{y(x)}{|x|}, \quad x \in \mathbb{R}\setminus\{0\}, \tag{3.55}$$

for all $y \in \mathcal{D}(T_\infty)$. The domain $\mathcal{D}(T_\infty)$ consists of all $y \in \mathcal{L}^2(\mathbb{R}) \cap \mathcal{AC}^2_{\text{loc}}(\mathbb{R}\setminus\{0\})$ such that the function on the right side of (3.55) is in $\mathcal{L}^2(\mathbb{R})$ and

$$y(0-) = y(0+), \quad \lim_{x\to 0+}(y'(x) - y'(-x) + y(0)\ln x) = 0.$$

Remark 3.63 There was a discussion concerning the difference of the potentials given by $-x^{-1}$ and $-|x|^{-1}$ for $x \in \mathbb{R}\setminus\{0\}$, see [M1], [Ne], [M2], and [Ku]. The apparent difference was seen that $-|x|^{-1}$ implied Dirichlet interface conditions in 0 whereas $-x^{-1}$ implied boundary conditions of the form

$$y(0-) = y(0+), \quad \lim_{x\to 0}(y'(x) - y'(-x)) = 0, \tag{3.56}$$

which physically bears the possibility of a particle travelling through the origin. This conclusion was proposed using a distributional interpretation of both potentials in [Ku]. The distributional interpretation of $-|x|^{-1}$ would not define a selfadjoint operator in $\mathcal{L}^2(\mathbb{R})$, so Dirichlet interface conditions were suggested.

The Examples 3.61 and 3.62 offer a different view: The proper choice of the interface conditions is no mathematical matter, but depends on the physical context. As to the one-dimensional hydrogen atom, one would expect that due to quantum phenomena the true potential is some regular potential, but rather one of a single-signed form like the potentials treated in Example 3.61 than some potential with a funny shape like the positive peak near 0 as in Example 3.62. Corollary 3.58 then states that the solutions for such a "reasonable" choice of potential are similar to those of the potential $-|x|^{-1}$ with Dirichlet interface conditions in 0. (This holds for "not all too high energies" because of the restriction $\lambda \in \Lambda$.)

Analogous considerations apply to the potential given by $-x^{-1}$, $x \in \mathbb{R}\setminus\{0\}$. Now, for instance, the physical context could motivate the assumption that the original regular potential is an odd function. Since odd potentials V_j always yield $\tilde{g}_2(x) = 0$ for $x \in \mathbb{R}$, boundary conditions (3.56) may be a good choice to approximate the original problem.

4 Resolvent and basisness

Throughout this section, $I = [a, b]$, $a < 0 < b$, is a compact interval. For the potential V_∞ assume

$$V_\infty \in \mathcal{L}^1_{\text{loc}}([a,b] \setminus \{0\}), \tag{4.1}$$

$$\int_x^0 \left| \int_x^\xi |V_\infty(\varsigma)| d\varsigma \right|^q d\xi \to 0 \tag{4.2}$$

if $x \to 0$. The number $q \in [1, \infty)$ is fixed throughout the whole section like the number $p \in [1, \infty]$, which defines the considered space of functions $\mathcal{L}^p(I)$, compare (1.3). Since the potentials V_j of Section 3 are not needed here, it is convenient to set $V_j = 0$, $j \in \mathbb{N}$. The function μ then can be computed from V_∞ only:

$$\mu(x) = \int_x^0 \int_x^\xi |V_\infty(\varsigma)| d\varsigma d\xi.$$

When the whole spectrum and eigenfunctions of the differential operator T_∞ is examined in this section, it is no longer appropriate to restrict the eigenvalue parameter to a bounded set $\Lambda \subset \mathbb{C}$. So the following extension of notation becomes necessary. The operators $H_{j,\lambda,\infty,x}$, $j \in \{0, \infty\}$, are used for numbers $\lambda \in \mathbb{C}$ of arbitrarily large absolute value. The series (3.12) defining the operator $H_{\infty,\lambda,\infty,x}$ converges if $\mu(x) + x^2|\lambda| < 1$ that defining $H_{0,\lambda,\infty,x}$ converges if $x^2|\lambda| < 1$. Note that the results in Section 3, have been proved for λ and x in fixed sets Λ and X. When such a result is needed in the present context for numbers $\lambda \in \mathbb{C}$ and $x \in [a, b]$, first set Λ such that it includes all numbers λ which are mentioned in the result. For instance $\Lambda = \{0, \lambda\}$ is adequate in many cases. Then define X in dependence of this set Λ as before and such that $x \in X$. This is possible, if $\mu(\pm x) + x^2 \sup_{\lambda \in \Lambda} |\lambda| < \frac{1}{2}$ or under the weaker condition $x^2 \sup_{\lambda \in \Lambda} |\lambda| < \frac{1}{2}$ for $H_{0,\lambda,\infty,x}$. Thus, such a condition on x must and always will be assured.

4.1 Estimates concerning the fundamental matrices

Assume $j \in \{0, \infty\}$, $\lambda \in \mathbb{C}$, $x, \xi \in I_\pm$, and that y_1 and y_2 are solutions of (3.4) such that $y_1(x) = 1$, $y_1'(x) = 0$, $y_2(x) = 0$, and $y_2'(x) = 1$. Then define the 2×2 fundamental matrix $\Xi_{j,\lambda,x,\xi}$ to be

$$\Xi_{j,\lambda,x,\xi} = \begin{pmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{pmatrix}.$$

Since $V_0 = 0$ and thus (3.4) for $j = 0$ becomes $-y'' - \lambda y = 0$, it is easy to see that

$$\Xi_{0,\lambda,x,\xi} = \begin{pmatrix} \cos\left(\sqrt{\lambda}(\xi - x)\right) & \sqrt{\lambda}^{-1} \sin\left(\sqrt{\lambda}(\xi - x)\right) \\ -\sqrt{\lambda} \sin\left(\sqrt{\lambda}(\xi - x)\right) & \cos\left(\sqrt{\lambda}(\xi - x)\right) \end{pmatrix} \tag{4.3}$$

if $\lambda \in \mathbb{C} \setminus \{0\}$. Additionally, for $\lambda \in \mathbb{C}$ define the diagonal 2×2-matrix

$$L_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}.$$

Then

$$L_\lambda^{-1} \Xi_{0,\lambda,x,\xi} L_\lambda = \begin{pmatrix} \cos\left(\sqrt{\lambda}(\xi - x)\right) & \sin\left(\sqrt{\lambda}(\xi - x)\right) \\ -\sin\left(\sqrt{\lambda}(\xi - x)\right) & \cos\left(\sqrt{\lambda}(\xi - x)\right) \end{pmatrix}$$

if $\lambda \in \mathbb{C} \setminus \{0\}$, $x \in [a, b]$, and $\xi \in I_x$.

Lemma 4.1 *Assume that $\mathcal{X}$ is a Banach space, $n \in \mathbb{N}$, and that A_k, A_k^{-1}, and B_k, $k \in \{1, \dots, n\}$, are bounded operators on $\mathcal{X}$. Then*

$$\left\| \prod_{k=1}^{n} (A_k + B_k) - \prod_{k=1}^{n} A_k \right\| \leq \left(\exp \sum_{k=1}^{n} \left\| A_k^{-1} B_k \right\| - 1 \right) \prod_{k=1}^{n} \|A_k\| .$$

Proof. The proof is by induction on n. For $n = 0$, the product terms are equal to the identity operator on $\mathcal{X}$ and the sum expression is zero. So the hypothesis is fulfilled. For arbitrary $n \in \mathbb{N}$ and sequences of operators A_k, B_k, $k \in \{1, \dots, n+1\}$, the induction hypothesis for n implies

$$\begin{aligned}
&\left\| \prod_{k=1}^{n+1} (A_k + B_k) - \prod_{k=1}^{n+1} A_k \right\| \\
&\quad = \left\| (A_{n+1} + B_{n+1}) \prod_{k=1}^{n} (A_k + B_k) - (A_{n+1} + B_{n+1}) \prod_{k=1}^{n} A_k + B_{n+1} \prod_{k=1}^{n} A_k \right\| \\
&\quad \leq \|A_{n+1}\| \left(\left\| 1 + A_{n+1}^{-1} B_{n+1} \right\| \left\| \prod_{k=1}^{n} (A_k + B_k) - \prod_{k=1}^{n} A_k \right\| + \left\| A_{n+1}^{-1} B_{n+1} \right\| \left\| \prod_{k=1}^{n} A_k \right\| \right) \\
&\quad \leq \|A_{n+1}\| \left(\left(1 + \left\| A_{n+1}^{-1} B_{n+1} \right\| \right) \left(\exp \sum_{k=1}^{n} \left\| A_k^{-1} B_k \right\| - 1 \right) \prod_{k=1}^{n} \|A_k\| \right. \\
&\qquad \left. + \left\| A_{n+1}^{-1} B_{n+1} \right\| \prod_{k=1}^{n} \|A_k\| \right) \\
&\quad = \left(\left(1 + \left\| A_{n+1}^{-1} B_{n+1} \right\| \right) \left(\exp \sum_{k=1}^{n} \left\| A_k^{-1} B_k \right\| - 1 \right) + \left\| A_{n+1}^{-1} B_{n+1} \right\| \right) \prod_{k=1}^{n+1} \|A_k\| \\
&\quad = \left(\left(1 + \left\| A_{n+1}^{-1} B_{n+1} \right\| \right) \exp \sum_{k=1}^{n} \left\| A_k^{-1} B_k \right\| - 1 \right) \prod_{k=1}^{n+1} \|A_k\| \\
&\quad \leq \left(\exp \left\| A_{n+1}^{-1} B_{n+1} \right\| \exp \sum_{k=1}^{n} \left\| A_k^{-1} B_k \right\| - 1 \right) \prod_{k=1}^{n+1} \|A_k\| \\
&\quad = \left(\exp \sum_{k=1}^{n+1} \left\| A_k^{-1} B_k \right\| - 1 \right) \prod_{k=1}^{n+1} \|A_k\| ,
\end{aligned}$$

since $1 + c \leq e^c$ for all numbers $c \geq 0$. □

Lemma 4.2 *Assume $z \in \mathbb{C}$. Then the Euclidean norm*

$$\left\| \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix} \right\|_{\mathbb{C}^2} = \exp |\Im z| .$$

Proof. Define the matrix $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the matrix identity

$$\begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix} = \exp (z J_1)$$

is valid. Since the matrix J_1 is normal and has the eigenvalues $\pm i$,

$$\|\exp (z J_1)\|_{\mathbb{C}^2} = \max \left(|e^{iz}|, |e^{-iz}| \right) = e^{|\Im z|} .$$

□

Lemma 4.3 *Assume $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$. Then there is a number $\delta > 0$ such that for all $z \in \mathbb{C}$ with $|z| \leq \delta$ the Euclidean norm*

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \cos(\alpha z) & \alpha^{-1}\sin(\alpha z) \\ -\alpha \sin(\alpha z) & \cos(\alpha z) \end{pmatrix}^{-1} \begin{pmatrix} \cos(\beta z) & \beta^{-1}\sin(\beta z) \\ -\beta\sin(\beta z) & \cos(\beta z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| \leq \left| \frac{\alpha^2 - \beta^2}{\alpha} \right| |z| + \epsilon |z|. \tag{4.4}$$

Proof. The Taylor expansions of the cosine and sine functions yield that there is a number $\delta_0 > 0$ such that

$$|\cos z - 1| \leq |z|^2, \quad |\sin z - z| \leq |z|^3,$$

if $|z| \leq \delta_0$. Set $\gamma = \max(|\alpha|, |\alpha|^{-1}, |\beta|)$ and assume $\gamma|z| \leq \delta_0$. Then

$$|\cos(\alpha z) - 1| \leq |\alpha z|^2, \quad |\sin(\alpha z) - \alpha z| \leq |\alpha z|^3,$$

and similarly for $\sin(\beta z)$ and $\cos(\beta z)$. Use these relations to estimate the considered norm

$$\begin{aligned}
&\left\| \begin{pmatrix} \cos(\alpha z) & -\alpha^{-1}\sin(\alpha z) \\ \sin(\alpha z) & \alpha^{-1}\cos(\alpha z) \end{pmatrix} \begin{pmatrix} \cos(\beta z) & \alpha\beta^{-1}\sin(\beta z) \\ -\beta\sin(\beta z) & \alpha\cos(\beta z) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|_{\mathbb{C}^2} \\
&\leq \left\| \begin{pmatrix} 1 & -z \\ \alpha z & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha z \\ -\beta^2 z & \alpha \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|_{\mathbb{C}^2} \\
&\quad + (|\alpha^2| + |\beta^2|)|z|^2 + |\alpha^2\beta^2||z|^4 + (|\alpha^2\beta^2| + |\beta^4|)|z|^4 + |\alpha^2\beta^4||z|^6 \\
&\quad + (|\alpha^3| + |\alpha\beta^2|)|z^3| + |\alpha^3\beta^2||z|^5 + (|\alpha^3| + |\alpha\beta^2|)|z|^3 + |\alpha^3\beta^2||z|^5 \\
&\quad + (|\alpha^3| + |\alpha\beta^2|)|z|^3 + |\alpha^3\beta^2||z|^5 + (|\alpha\beta^2| + |\alpha^{-1}\beta^4|)|z|^3 + |\alpha\beta^4||z|^5 \\
&\quad + (|\alpha^4| + |\alpha^2\beta^2|)|z|^4 + |\alpha^4\beta^2||z|^6 + (|\alpha^2| + |\beta^2|)|z|^2 + |\alpha^2\beta^2||z|^4 \\
&\leq \left\| \begin{pmatrix} 1 + \beta^2 z^2 & 0 \\ (\alpha - \alpha^{-1}\beta^2)z & 1 + \alpha^2 z^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|_{\mathbb{C}^2} \\
&\quad + 4\gamma^2|z|^2 + 8\gamma^3|z|^3 + 6\gamma^4|z|^4 + 4\gamma^5|z|^5 + 2\gamma^6|z|^6 \\
&\leq \left| \frac{\alpha^2 - \beta^2}{\alpha} \right| |z| + 6\gamma^2|z|^2 + (7\gamma^3 + \gamma^5)|z|^3 + 6\gamma^4|z|^4 + 4\gamma^5|z|^5 + 2\gamma^6|z|^6.
\end{aligned}$$

From this, it follows that for every $\epsilon > 0$ there is a constant $\delta > 0$ with $\gamma\delta \leq \delta_0$ such that (4.4) holds. □

The following important lemma shows that off the singularity the effect of the potential tends to zero if $\lambda \to \infty$.

Lemma 4.4 *Assume $\lambda \in \mathbb{C} \setminus \{0\}$, $x \in [a, b] \setminus \{0\}$, and $\xi \in I_x$. Then*

$$\left\| L_\lambda^{-1} (\Xi_{\infty,\lambda,x,\xi} - \Xi_{0,\lambda,x,\xi}) L_\lambda \right\|_{\mathbb{C}^2} \leq \left(\exp \frac{\|V_\infty\|_{\mathcal{L}^1(I_x \setminus I_\xi)}}{\sqrt{|\lambda|}} - 1 \right) \exp \left| (\xi - x) \Im\sqrt{\lambda} \right|.$$

Proof. To simplify the notation, assume $x < 0$. The expression on the left side of the asserted inequality is continuous in V_∞ with respect to the norm of $\mathcal{L}^1([x, \xi])$ as a consequence of Corollary 1.2. The expression on the right side trivially is continuous with respect to that norm. Since the set of step functions which are continuous from the right side is dense in the set of continuous functions and thus in $\mathcal{L}^1([x,\xi])$, it suffices to prove the inequality for such functions V_∞. Moreover, it may be assumed that $V_\infty(\varsigma) \neq \lambda$ for all $\varsigma \in [x,\xi]$ because also the set of step functions which do not attain the value λ is dense in $\mathcal{L}^1([x,\xi])$.

Let V_∞ be a step function which is continuous from the right side and choose a number $\epsilon > 0$. Then there is a number $n \in \mathbb{N}$ and a finite sequence of numbers ς_k, $k \in \{1,\ldots,n+1\}$, such that

$$x = \varsigma_1 < \varsigma_2 < \ldots < \varsigma_n < \varsigma_{n+1} = \xi$$

and such that V_∞ is constant on the intervals $[\varsigma_k, \varsigma_{k+1})$, $k \in \{1,\ldots,n\}$. Moreover, the intervals $[\varsigma_k, \varsigma_{k+1})$, $k \in \{1,\ldots,n\}$, can be chosen so small that the estimate (4.4) holds for $\alpha = \sqrt{\lambda}$, $\beta = \sqrt{\lambda - V_\infty(\varsigma_k)}$ and $z = \varsigma_{k+1} - \varsigma_k$ for each $k \in \{1,\ldots,n\}$.

The properties of fundamental matrices yield

$$L_\lambda^{-1}\Xi_{j,\lambda,x,\xi}L_\lambda = \prod_{k=1}^{n}\left(L_\lambda^{-1}\Xi_{j,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda\right), \quad j \in \{0,\infty\}. \tag{4.5}$$

In order to apply Lemma 4.1, set

$$\begin{aligned} A_k &= L_\lambda^{-1}\Xi_{0,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda, \\ B_k &= L_\lambda^{-1}\Xi_{\infty,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda - L_\lambda^{-1}\Xi_{0,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda \end{aligned}$$

for $k \in \{1,\ldots,n\}$. Then Lemma 4.3 helps to bound the norm

$$\begin{aligned} \left\|A_k^{-1}B_k\right\|_{\mathbb{C}^2} &= \left\|\left(L_\lambda^{-1}\Xi_{0,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda\right)^{-1}\left(L_\lambda^{-1}\Xi_{\infty,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda - L_\lambda^{-1}\Xi_{0,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda\right)\right\|_{\mathbb{C}^2} \\ &= \left\|L_\lambda^{-1}\Xi_{0,\lambda,\varsigma_k,\varsigma_{k+1}}^{-1}\Xi_{\infty,\lambda,\varsigma_k,\varsigma_{k+1}}L_\lambda - 1\right\|_{\mathbb{C}^2} \\ &\leq \left(\left|\frac{V_\infty(\varsigma_k)}{\sqrt{\lambda}}\right| + \epsilon\right)(\varsigma_{k+1} - \varsigma_k). \end{aligned}$$

By Lemma 4.2, $\|A_k\|_{\mathbb{C}^2} = e^{|(\varsigma_{k+1}-\varsigma_k)\Im\sqrt{\lambda}|}$. Now (4.5) and Lemma 4.1 imply

$$\begin{aligned} &\left\|L_\lambda^{-1}\Xi_{\infty,\lambda,x,\xi}L_\lambda - L_\lambda^{-1}\Xi_{0,\lambda,x,\xi}L_\lambda\right\| \\ &\quad\leq \left(\exp\left(\sum_{k=1}^{n}\left(\frac{V_\infty(\varsigma_k)}{\sqrt{|\lambda|}} + \epsilon\right)(\varsigma_{k+1} - \varsigma_k)\right) - 1\right)\prod_{i=1}^{n}\exp\left|(\varsigma_{k+1} - \varsigma_k)\Im\sqrt{\lambda}\right| \\ &\quad= \left(\exp\left(\frac{\|V_\infty\|_{\mathcal{L}^1([\varsigma_k,\varsigma_{k+1}])}}{\sqrt{|\lambda|}} + \epsilon|\xi - x|\right) - 1\right)\exp\left|(\xi - x)\Im\sqrt{\lambda}\right|. \end{aligned}$$

Since this inequality holds for all $\epsilon > 0$, the assertion is proved. □

4.2 Estimates concerning the interface conditions

To describe interface conditions in 0, fix a positive number $x_{\mathtt{b}} \in [a,b] \cap [-b,-a]$ such that $\mu(\pm x_{\mathtt{b}}) < \frac{1}{2}$. This determines a boundary operator $\mathtt{b}_{j,0,x_{\mathtt{b}}}$.

Lemma 4.5 *Assume $j \in \{0,\infty\}$, $\lambda \in \mathbb{C}$, $x \in (0,x_{\mathtt{b}}]$, such that $\mu(\pm x) + x^2|\lambda| < 1$, and that the function y is a solution of (3.4) on $[-x,x]\setminus\{0\}$. Then*

$$\mathtt{b}_{j,0,x_{\mathtt{b}}} y = \begin{pmatrix} X_{j,\lambda,-x}\begin{pmatrix} y(-x) \\ y'(-x)\end{pmatrix} \\ -X_{j,\lambda,x}\begin{pmatrix} y(x) \\ y'(x)\end{pmatrix}\end{pmatrix},$$

where the 2×2-Matrices $X_{j,\lambda,\pm x}$ are given by

$$\begin{pmatrix} \frac{1}{(H_{j,\lambda,\infty,\pm x}1)(\pm x)} & -\frac{(H_{j,\lambda,\infty,\pm x}\iota)(\pm x)}{(H_{j,\lambda,\infty,\pm x}1)(\pm x)} \\ \frac{h_{j,\lambda,\pm x}}{(H_{j,\lambda,\infty,\pm x}1)(\pm x)} & -(H_{j,\lambda,\infty,\pm x}\iota)'(0) - \frac{(H_{j,\lambda,\infty,\pm x}\iota)(\pm x)h_{j,\lambda,\pm x}}{(H_{j,\lambda,\infty,\pm x}1)(\pm x)}\end{pmatrix}$$

with $h_{j,\lambda,\pm x} = \lim_{\xi\to 0\pm}\lceil H_{j,\lambda,\infty,\pm x}1, H_{j,0,\infty,\pm x_{\mathtt{b}}}1\rfloor_\xi$.

Proof. It suffices to consider the third and fourth component of the vector $\mathtt{b}_{j,0,x_{\mathtt{b}}}y$. The asserted equalities for the first and second one are proved in the same way when x is replaced by $-x$ and $x_{\mathtt{b}}$ by $-x_{\mathtt{b}}$.
The equality (3.26) and Corollary 3.16 imply

$$y = \frac{y(x) - y'(x)(H_{j,\lambda,\infty,x}\iota)(x)}{(H_{j,\lambda,\infty,x}1)(x)} H_{j,\lambda,\infty,x}1 + y'(x)H_{j,\lambda,\infty,x}\iota. \tag{4.6}$$

Use Lemma 3.22, Corollary 3.23, and Lemma 3.5 to compute the following expressions.

$$\begin{aligned}
&\lim_{\xi\to 0+}\lceil H_{j,\lambda,\infty,x}\iota, H_{j,0,\infty,x_{\mathtt{b}}}\iota\rfloor_\xi \\
&\quad = \lim_{\xi\to 0+}\left((H_{j,\lambda,\infty,x}\iota)(\xi)(H_{j,0,\infty,x_{\mathtt{b}}}\iota)'(\xi) - (H_{j,\lambda,\infty,x}\iota)'(\xi)(H_{j,0,\infty,x_{\mathtt{b}}}\iota)(\xi)\right) = 0,\\
&\lim_{\xi\to 0+}\lceil H_{j,\lambda,\infty,x}1, H_{j,0,\infty,x_{\mathtt{b}}}\iota\rfloor_\xi \\
&\quad = \lim_{\xi\to 0+}\left((H_{j,\lambda,\infty,x}1)(\xi)(H_{j,0,\infty,x_{\mathtt{b}}}\iota)'(\xi) - (H_{j,\lambda,\infty,x}1)'(\xi)(H_{j,0,\infty,x_{\mathtt{b}}}\iota)(\xi)\right)\\
&\quad = (H_{j,0,\infty,x_{\mathtt{b}}}\iota)'(0),\\
&\lim_{\xi\to 0+}\lceil H_{j,\lambda,\infty,x}\iota, H_{j,0,\infty,x_{\mathtt{b}}}1\rfloor_\xi = -(H_{j,\lambda,\infty,x}\iota)'(0).
\end{aligned} \tag{4.7}$$

Also compare the proof of Lemma 3.29. This, (4.6), and the definition (3.23) of the boundary operator yield the statement. □

The following statement is similar to Lemma 3.13.

Lemma 4.6 *Assume $\lambda \in \mathbb{C}$ and $x \in [a,b]\cap[-b,-a]\setminus\{0\}$ such that $\mu(\pm x) + x^2|\lambda| < \frac{1}{2}$. Then it holds*

$$\|H_{\infty,\lambda,\infty,x} - H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)} \le 4\mu(x).$$

Proof. From (3.11), (3.12), and Lemma 3.11, it follows

$$\begin{aligned}
\|H_{\infty,\lambda,\infty,x} - H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)} &= \|1 + F_{\infty,\lambda,x}H_{\infty,\lambda,\infty,x} - 1 - F_{0,\lambda,x}H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)}\\
&\le \|F_{\infty,\lambda,x} - F_{0,\lambda,x}\|_{\mathcal{L}^\infty(I_x)}\|H_{\infty,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)}\\
&\quad + \|F_{0,\lambda,x}\|_{\mathcal{L}^\infty(I_x)}\|H_{\infty,\lambda,\infty,x} - H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)}\\
&\le \frac{\mu(x)}{1-\mu(x)-x^2|\lambda|} + |\lambda|x^2\|H_{\infty,\lambda,\infty,x} - H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)}
\end{aligned}$$

using the inequalities $\|F_{0,\lambda,x}\|_{\mathcal{L}^\infty(I_x)} \le x^2|\lambda|$ and $\|H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)} \le (1-x^2|\lambda|)^{-1}$. This inequality for $\|H_{0,\lambda,\infty,x}\|_{\mathcal{L}^\infty(I_x)}$ implies the assertion. □

Lemma 4.7 *Assume $\lambda \in \mathbb{C}$ and $x \in [a,b] \cap [-b,-a] \setminus \{0\}$ such that $\mu(\pm x) + x^2|\lambda| < \frac{1}{2}$. Assume that $h_{j,\lambda,x}$, $j \in \{0,\infty\}$ are the numbers defined in Lemma 4.5. Then it holds*

$$|h_{\infty,\lambda,\pm x} - h_{0,\lambda,\pm x}| \le 2\,|(H_{\infty,0,\infty,\pm x_\mathfrak{b}}1)'(\pm x)| + 8\mu(x)|x\lambda|.$$

Proof. For $j \in \{0,\infty\}$ and $\xi \in I_x$, (3.11), (3.12), (3.8), and Corollary 3.19 imply

$$\begin{aligned}
(H_{j,\lambda,\infty,x}1)(\xi) &= 1 + (F_{j,\lambda,x}H_{j,\lambda,\infty,x}1)(\xi)\\
&= 1 + \int_0^\xi \int_x^\varsigma V_j(\tau)(H_{j,\lambda,\infty,x}1)(\tau)d\tau d\varsigma - \lambda \int_0^\xi \int_x^\varsigma (H_{j,\lambda,\infty,x}1)(\tau)d\tau d\varsigma\\
&= 1 + (F_{j,0,x}H_{j,\lambda,\infty,x}1)(\xi) + (F_{0,\lambda,x}H_{j,\lambda,\infty,x}1)(\xi)\\
&= 1 + (F_{j,0,x}H_{j,0,\infty,x}1)(\xi) + (F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x})1)(\xi) + (F_{0,\lambda,x}H_{j,\lambda,\infty,x}1)(\xi)\\
&= (H_{j,0,\infty,x}1)(\xi) + (F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x})1)(\xi) + (F_{0,\lambda,x}H_{j,\lambda,\infty,x}1)(\xi)\\
&= (H_{j,0,\infty,x_\mathfrak{b}}1)(\xi) - (H_{j,0,\infty,x_\mathfrak{b}}1)'(x)(H_{j,0,\infty,x}\iota)(\xi) \qquad (4.8)\\
&\quad + (F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x})1)(\xi) + (F_{0,\lambda,x}H_{j,\lambda,\infty,x}1)(\xi). \qquad (4.9)
\end{aligned}$$

For the terms in the line (4.8), the properties of the bilinear form $\lceil\cdot,\cdot\rfloor_\xi$, computation like in (4.7), and finally Lemma 3.22 imply

$$\begin{aligned}
&\left|\lim_{\xi\to 0\pm} \lceil H_{j,0,\infty,x_\mathfrak{b}}1 - (H_{j,0,\infty,x_\mathfrak{b}}1)'(x)H_{j,0,\infty,x}\iota, H_{j,0,\infty,x_\mathfrak{b}}1\rfloor_\xi\right|\\
&\quad = \left|(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)\lim_{\xi\to 0\pm}\lceil H_{j,0,\infty,x}\iota, H_{j,0,\infty,x_\mathfrak{b}}1\rfloor_\xi\right| = |(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)(H_{j,0,\infty,x}\iota)'(0)|\\
&\quad \le |(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)|\,(1+2\mu(x)) \le 2\,|(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)|\,.
\end{aligned}$$

Note that this expression vanishes if $j = 0$. For the left term in the line (4.9), the Lemmas 3.14 and 3.21 imply

$$\begin{aligned}
|(F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x}))'(\xi)| &\le 4\mu(x)|x\lambda|,\\
|(F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x}))(\xi)| &\le 4\mu(x)|\xi x\lambda|.
\end{aligned}$$

Hence, using Lemma 3.22, Corollary 3.23, and Lemma 3.5, it follows

$$\begin{aligned}
&\left|\lim_{\xi\to 0\pm}\lceil F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x})1, H_{j,0,\infty,x_\mathfrak{b}}1\rfloor_\xi\right|\\
&\quad = \Big|\lim_{\xi\to 0\pm}((F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x})1)(\xi)(H_{j,0,\infty,x_\mathfrak{b}}1)'(\xi)\\
&\qquad -(F_{j,0,x}(H_{j,\lambda,\infty,x} - H_{j,0,\infty,x})1)'(\xi)(H_{j,0,\infty,x_\mathfrak{b}}1)(\xi))\Big| \le 4\mu(x)|x\lambda|.
\end{aligned}$$

For the right term in the line (4.9), Lemma 4.6 and then integration imply

$$\begin{aligned}
&|(F_{0,\lambda,x}H_{\infty,\lambda,\infty,x}1)'(\xi) - (F_{0,\lambda,x}H_{0,\lambda,\infty,x}1)'(\xi)|\\
&\quad = \left|\lambda\int_x^\xi (H_{\infty,\lambda,\infty,x}1)(\varsigma) - (H_{0,\lambda,\infty,x}1)(\varsigma)d\varsigma\right| \le 4\mu(x)|x\lambda|,\\
&|(F_{0,\lambda,x}H_{\infty,\lambda,\infty,x}1)(\xi) - (F_{0,\lambda,x}H_{0,\lambda,\infty,x}1)(\xi)| \le 4\mu(x)|\xi x\lambda|.
\end{aligned}$$

Hence, again using Lemma 3.22, Corollary 3.23, and Lemma 3.5,

$$\left|\lim_{\xi\to 0\pm}\lceil F_{0,\lambda,x}H_{\infty,\lambda,\infty,x}1, H_{\infty,0,\infty,x_\mathfrak{b}}1\rfloor_\xi - \lim_{\xi\to 0\pm}\lceil F_{0,\lambda,x}H_{0,\lambda,\infty,x}1, H_{0,0,\infty,x_\mathfrak{b}}1\rfloor_\xi\right| \leq 4\mu(x)|x\lambda|.$$

Also compare the computation in the proof of Lemma 3.29. Using the above estimates, it follows

$$\begin{aligned}&\left|\lim_{\xi\to 0\pm}\lceil H_{\infty,\lambda,\infty,x}1, H_{\infty,0,\infty,x_\mathfrak{b}}1\rfloor_\xi - \lim_{\xi\to 0\pm}\lceil H_{0,\lambda,\infty,x}1, H_{0,0,\infty,x_\mathfrak{b}}1\rfloor_\xi\right|\\ &\quad\leq 2\left|(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)\right| + 4\mu(x)|x\lambda| + 4\mu(x)|x\lambda| = 2\left|(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)\right| + 8\mu(x)|x\lambda|.\end{aligned}$$

□

Lemma 4.8 *Assume $\lambda\in\mathbb{C}$ and $x\in[a,b]\cap[-b,-a]\setminus\{0\}$ such that $\mu(\pm x)+x^2|\lambda|<\frac{1}{4}$. Let $X_{j,\lambda,x}$, $j\in\{0,\infty\}$, be the matrices defined in Lemma 4.5. Then it holds*

$$\left\|L_\lambda^{-1}(X_{\infty,\lambda,x}-X_{0,\lambda,x})L_\lambda\right\|_{\mathbb{C}^2} \leq 454\mu(x) + 16\sqrt{|\lambda|}^{-1}\left|(H_{\infty,0,\infty,x_\mathfrak{b}}1)'(x)\right|.$$

Proof. Consider and bound all entries of the matrix $X_{\infty,\lambda,x}-X_{0,\lambda,x}$. Assume that x is positive. The proof for negative x is similar; only $x_\mathfrak{b}$ then must be replaced by $-x_\mathfrak{b}$ and the limits $\xi\to 0+$ become limits $\xi\to 0-$. As to the left and right upper entries, Lemmas 4.6 and 3.22 help to prove

$$\begin{aligned}&\left|\frac{1}{(H_{\infty,\lambda,\infty,x}1)(x)} - \frac{1}{(H_{0,\lambda,\infty,x}1)(x)}\right| = \left|\frac{(H_{0,\lambda,\infty,x}1)(x)-(H_{\infty,\lambda,\infty,x}1)(x)}{(H_{\infty,\lambda,\infty,x}1)(x)(H_{0,\lambda,\infty,x}1)(x)}\right|\\ &\quad\leq \frac{4\mu(x)}{(1-2(\mu(x)+x^2|\lambda|))^2} \leq 16\mu(x),\\ &\left|\frac{(H_{\infty,\lambda,\infty,x}\iota)(x)}{(H_{\infty,\lambda,\infty,x}1)(x)} - \frac{(H_{0,\lambda,\infty,x}\iota)(x)}{(H_{0,\lambda,\infty,x}1)(x)}\right|\\ &\quad= \left|\frac{(H_{\infty,\lambda,\infty,x}\iota)(x)(H_{0,\lambda,\infty,x}1)(x)-(H_{0,\lambda,\infty,x}\iota)(x)(H_{\infty,\lambda,\infty,x}1)(x)}{(H_{\infty,\lambda,\infty,x}1)(x)(H_{0,\lambda,\infty,x}1)(x)}\right|\\ &\quad\leq \left|\frac{((H_{\infty,\lambda,\infty,x}\iota)(x)-(H_{0,\lambda,\infty,x}\iota)(x))(H_{0,\lambda,\infty,x}1)(x)}{(1-2(\mu(x)+x^2|\lambda|))^2}\right|\\ &\qquad+\left|\frac{(H_{0,\lambda,\infty,x}\iota)(x)((H_{0,\lambda,\infty,x}1)(x)-(H_{\infty,\lambda,\infty,x}1)(x))}{(1-2(\mu(x)+x^2|\lambda|))^2}\right|\\ &\quad\leq 4((4\mu(x)|x|)2+2|x|(4\mu(x))) = 64\mu(x)|x|.\end{aligned}$$

As to the left lower entry, Lemmas 3.22, 4.7, and 4.6 help to prove

$$\begin{aligned}&\left|\frac{h_{\infty,\lambda,x}}{(H_{\infty,\lambda,\infty,x}1)(x)} - \frac{h_{0,\lambda,x}}{(H_{0,\lambda,\infty,x}1)(x)}\right|\\ &\quad\leq \left|\frac{(H_{0,\lambda,\infty,x}1)(x)h_{\infty,\lambda,x}1-(H_{\infty,\lambda,\infty,x}1)(x)h_{0,\lambda,x}}{(H_{\infty,\lambda,\infty,x}1)(x)(H_{0,\lambda,\infty,x}1)(x)}\right|\\ &\quad\leq \left|\frac{(H_{0,\lambda,\infty,x}1)(x)\,(h_{\infty,\lambda,x}-h_{0,\lambda,x})}{(1-2(\mu(x)+x^2|\lambda|))^2}\right|\\ &\qquad+\left|\frac{((H_{0,\lambda,\infty,x}1)(x)-(H_{\infty,\lambda,\infty,x}1)(x))h_{0,\lambda,x}}{(1-2(\mu(x)+x^2|\lambda|))^2}\right|\\ &\quad\leq 4(2(2\left|(H_{j,0,\infty,x_\mathfrak{b}}1)'(x)\right|+8\mu(x)|x\lambda|)+4\mu(x)2|x\lambda|)\\ &\quad= 16\left|(H_{\infty,0,\infty,x_\mathfrak{b}}1)'(x)\right|+96\mu(x)|x\lambda|.\end{aligned} \tag{4.10}$$

Here, the relation

$$\left|\lceil H_{0,\lambda,\infty,x}1, H_{0,0,\infty,x_{\mathfrak{b}}}1\rfloor_{\xi}\right| = |(H_{0,\lambda,\infty,x}1)'(0)| = \left|\lambda\int_x^{\xi}(H_{0,\lambda,\infty,x}1)(\varsigma)d\varsigma\right| \leq 2|x\lambda| \qquad (4.11)$$

has been used, which follows from Lemma 3.22. As to the right lower entry, (3.8), (3.11), (3.12), and the Lemmas 3.21, 4.6, and 3.22 yield

$$\begin{aligned}
&|(H_{\infty,\lambda,\infty,x}\iota)'(0) - (H_{0,\lambda,\infty,x}\iota)'(0)| \\
&\quad = |(F_{\infty,0,x}H_{\infty,\lambda,\infty,x}\iota)'(0) + (F_{0,\lambda,x}(H_{\infty,\lambda,\infty,x} - H_{0,\lambda,\infty,x})\iota)'(0)| \\
&\quad \leq 2\mu(x) + |x\lambda|4\mu(x)|x| = (2 + 4x^2|\lambda|)\mu(x).
\end{aligned}$$

Next, (4.10) and the Lemmas 4.6 and 3.22 help to prove

$$\begin{aligned}
&\left|\frac{(H_{\infty,\lambda,\infty,x}\iota)(x)h_{\infty,\lambda,x}}{(H_{\infty,\lambda,\infty,x}1)(x)} - \frac{(H_{0,\lambda,\infty,x}\iota)(x)h_{0,\lambda,x}}{(H_{0,\lambda,\infty,x}1)(x)}\right| \\
&\quad \leq \left|(H_{\infty,\lambda,\infty,x}\iota)(x)\left(\frac{h_{\infty,\lambda,x}}{(H_{\infty,\lambda,\infty,x}1)(x)} - \frac{h_{0,\lambda,x}}{(H_{0,\lambda,\infty,x}1)(x)}\right)\right| \\
&\qquad + \left|((H_{\infty,\lambda,\infty,x}\iota)(x) - (H_{0,\lambda,\infty,x}\iota)(x))\frac{h_{0,\lambda,x}}{(H_{0,\lambda,\infty,x}1)(x)}\right| \\
&\quad \leq 2|x|\left(16\left|(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| + 96\mu(x)|x\lambda|\right) + 4\mu(x)|x|4|x\lambda| \\
&\quad = 32\left|x(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| + 208\mu(x)x^2|\lambda|.
\end{aligned}$$

again using (4.11). Hence, since the assumptions imply $x^2|\lambda| < 1$ and using Lemma 3.24,

$$\begin{aligned}
&\left\|L_\lambda^{-1}(X_{\infty,\lambda,x} - X_{0,\lambda,x})L_\lambda\right\|_{\mathbb{C}^2} \\
&\quad \leq 16\mu(x) + 64\mu(x)|x|\sqrt{|\lambda|} \\
&\qquad + \sqrt{|\lambda|}^{-1}\left(16\left|(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| + 96\mu(x)|x\lambda|\right) \\
&\qquad + (2 + 4x^2|\lambda|)\mu(x) + 32\left|x(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| + 208\mu(x)x^2|\lambda| \\
&\quad \leq (16 + 64 + 96 + 6 + 64 + 208)\mu(x) + 16\sqrt{|\lambda|}^{-1}\left|(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| \\
&\quad = 454\mu(x) + 16\sqrt{|\lambda|}^{-1}\left|(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right|.
\end{aligned}$$

□

4.3 Conclusive comparison with the zero potential

Lemma 4.9 *Assume $\lambda \in \mathbb{C}$, $x \in [a,b] \cap [-b,-a] \setminus \{0\}$ such that $x^2|\lambda| < 1$, and $\xi \in I_x$. Then*

$$(H_{0,\lambda,\infty,x}1)(\xi) = \begin{cases} \cos\left(\sqrt{\lambda}\xi\right) + \tan\left(\sqrt{\lambda}x\right)\sin\left(\sqrt{\lambda}\xi\right) & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0 \end{cases}$$

$$(H_{0,\lambda,\infty,x}\iota)(\xi) = \begin{cases} \frac{\sin\left(\sqrt{\lambda}\xi\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)} & \text{if } \lambda \neq 0 \\ \xi & \text{if } \lambda = 0 \end{cases}$$

Proof. By Corollary 3.17, the functions $H_{0,\lambda,\infty,x}1$ and $H_{0,\lambda,\infty,x}\iota$ solve the differential equation $-y'' = \lambda y$. First consider the case $\lambda \neq 0$. Apply the ansatz

$$\begin{aligned}(H_{0,\lambda,\infty,x}1)(\xi) &= c_1 \cos\left(\sqrt{\lambda}\xi\right) + d_1 \sin\left(\sqrt{\lambda}\xi\right), \\ (H_{0,\lambda,\infty,x}\iota)(\xi) &= c_\iota \cos\left(\sqrt{\lambda}\xi\right) + d_\iota \sin\left(\sqrt{\lambda}\xi\right),\end{aligned}$$

and determine the constants from the boundary conditions stated in Corollary 3.17:

$$\begin{aligned}1 &= c_1, \\ 0 &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + d_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right), \\ 0 &= c_\iota, \\ 1 &= -c_\iota\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + d_\iota\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right).\end{aligned}$$

Procede similarly in the case $\lambda = 0$, using an ansatz as linear combination of the constant function 1 and the identity function ι. □

Corollary 4.10 *Assume $\lambda \in \mathbb{C}\setminus\{0\}$, $x \in [a, b]\cap[-b, -a]$ such that $x^2|\lambda| < 1$, and $\xi \in I_x$. Then*

$$X_{0,\lambda,x} = \begin{pmatrix} \cos\left(\sqrt{\lambda}x\right) & -\sqrt{\lambda}^{-1}\sin\left(\sqrt{\lambda}x\right) \\ -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) & -\cos\left(\sqrt{\lambda}x\right) \end{pmatrix}.$$

Proof. Compute all entries of the considered matrix with the help of Lemma 4.9. For simplicity assume $x > 0$. The case $x < 0$ is treated analogously; only some signs change.

$$\begin{aligned}
&\frac{1}{(H_{0,\lambda,\infty,x}1)(x)} = \frac{1}{\cos\left(\sqrt{\lambda}x\right) + \tan\left(\sqrt{\lambda}x\right)\sin\left(\sqrt{\lambda}x\right)} = \frac{\cos\left(\sqrt{\lambda}x\right)}{\cos^2\left(\sqrt{\lambda}x\right) + \sin^2\left(\sqrt{\lambda}x\right)} \\
&\quad = \cos\left(\sqrt{\lambda}x\right), \\
&-\frac{(H_{0,\lambda,\infty,x}\iota)(x)}{(H_{0,\lambda,\infty,x}1)(x)} = -\frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)}\cos\left(\sqrt{\lambda}x\right) = -\frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}}, \\
&\frac{\lim_{\xi\to 0+}\lceil H_{0,\lambda,\infty,x}1, H_{0,0,\infty,x_\flat}1\rfloor_\xi}{(H_{0,\lambda,\infty,x}1)(x)} \\
&\quad = \lim_{\xi\to 0+}\left\lceil\cos\left(\sqrt{\lambda}\cdot\right) + \tan\left(\sqrt{\lambda}x\right)\sin\left(\sqrt{\lambda}\cdot\right), 1\right\rfloor_\xi \cos\left(\sqrt{\lambda}x\right) \\
&\quad = \lim_{\xi\to 0+}\left(\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right) - \tan\left(\sqrt{\lambda}x\right)\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right)\right)\cos\left(\sqrt{\lambda}x\right) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right), \\
&-(H_{0,\lambda,\infty,x}\iota)'(0) - \frac{(H_{0,\lambda,\infty,x}\iota)(x)\lim_{\xi\to 0+}\lceil H_{0,\lambda,\infty,x}1, H_{0,0,\infty,x_\flat}1\rfloor_\xi}{(H_{0,\lambda,\infty,x}1)(x)} \\
&\quad = -\frac{\sqrt{\lambda}}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)} + \frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)}\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) = \frac{\sin^2\left(\sqrt{\lambda}x\right) - 1}{\cos\left(\sqrt{\lambda}x\right)} \\
&\quad = -\cos\left(\sqrt{\lambda}x\right).
\end{aligned}$$

□

Corollary 4.11 *Assume* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $x \in (0, x_{\mathfrak{b}}]$. *Then*

$$\begin{aligned} L_\lambda^{-1} X_{0,\lambda,-x} \Xi_{0,\lambda,a,-x} L_\lambda &= \begin{pmatrix} \cos\left(\sqrt{\lambda}a\right) & -\sin\left(\sqrt{\lambda}a\right) \\ -\sin\left(\sqrt{\lambda}a\right) & -\cos\left(\sqrt{\lambda}a\right) \end{pmatrix}, \\ L_\lambda^{-1} X_{0,\lambda,x} \Xi_{0,\lambda,b,x} L_\lambda &= \begin{pmatrix} \cos\left(\sqrt{\lambda}b\right) & -\sin\left(\sqrt{\lambda}b\right) \\ -\sin\left(\sqrt{\lambda}b\right) & -\cos\left(\sqrt{\lambda}b\right) \end{pmatrix}. \end{aligned}$$

Proof. Corollary 4.10 and (4.3) yield

$$\begin{aligned} & L_\lambda^{-1} X_{0,\lambda,x} \Xi_{0,\lambda,b,x} L_\lambda \\ &= \begin{pmatrix} \cos\left(\sqrt{\lambda}x\right) & -\sin\left(\sqrt{\lambda}x\right) \\ -\sin\left(\sqrt{\lambda}x\right) & -\cos\left(\sqrt{\lambda}x\right) \end{pmatrix} \begin{pmatrix} \cos\left(\sqrt{\lambda}(x-b)\right) & \sin\left(\sqrt{\lambda}(x-b)\right) \\ -\sin\left(\sqrt{\lambda}(x-b)\right) & \cos\left(\sqrt{\lambda}(x-b)\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\sqrt{\lambda}(-b)\right) & \sin\left(\sqrt{\lambda}(-b)\right) \\ \sin\left(\sqrt{\lambda}(-b)\right) & -\cos\left(\sqrt{\lambda}(-b)\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{\lambda}b\right) & -\sin\left(\sqrt{\lambda}b\right) \\ -\sin\left(\sqrt{\lambda}b\right) & -\cos\left(\sqrt{\lambda}b\right) \end{pmatrix}. \end{aligned}$$

This gives the second assertion. The proof for the first one is analogous. □

For $x \in [a, b] \setminus \{0\}$ define the function $\hat{\mu}$ by

$$\hat{\mu}(x) = \begin{cases} \left|x \, \|V_\infty\|_{\mathcal{L}^1([a,x])}\right| & \text{if } x < 0 \\ \left|x \, \|V_\infty\|_{\mathcal{L}^1([x,b])}\right| & \text{if } x > 0 \end{cases}.$$

The Lemmas 3.1 and 3.5 prove $\hat{\mu}(x) \to 0$ if $x \to 0$; hence set $\hat{\mu}(0) = 0$.

Lemma 4.12 *Assume* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $x \in (0, x_{\mathfrak{b}}]$ *with* $\mu(\pm x) + x^2|\lambda| < \frac{1}{4}$. *Then*

$$\begin{aligned} & \left\| L_\lambda^{-1} (X_{\infty,\lambda,-x} \Xi_{\infty,\lambda,a,-x} - X_{0,\lambda,-x} \Xi_{0,\lambda,a,-x}) L_\lambda \right\|_{\mathbb{C}^2} \\ & \quad \leq \left(\left(454\mu(-x) + 16\sqrt{|\lambda|}^{-1} \left|(H_{\infty,0,\infty,-x_{\mathfrak{b}}}1)'(-x)\right| \right) \exp \frac{\hat{\mu}(-x)}{|x|\sqrt{|\lambda|}} + \exp \frac{\hat{\mu}(-x)}{|x|\sqrt{|\lambda|}} - 1 \right) \\ & \qquad \cdot \exp \left| a \Im \sqrt{\lambda} \right| \\ & \left\| L_\lambda^{-1} (X_{\infty,\lambda,x} \Xi_{\infty,\lambda,b,x} - X_{0,\lambda,x} \Xi_{0,\lambda,b,x}) L_\lambda \right\|_{\mathbb{C}^2} \\ & \quad \leq \left(\left(454\mu(x) + 16\sqrt{|\lambda|}^{-1} \left|(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| \right) \exp \frac{\hat{\mu}(x)}{|x|\sqrt{|\lambda|}} + \exp \frac{\hat{\mu}(x)}{|x|\sqrt{|\lambda|}} - 1 \right) \\ & \qquad \cdot \exp \left| b \Im \sqrt{\lambda} \right|. \end{aligned}$$

Proof. The Lemmas 4.4, 4.8, 4.2, and Corollary 4.10 help to estimate

$$\begin{aligned} & \left\| L_\lambda^{-1} (X_{\infty,\lambda,x} \Xi_{\infty,\lambda,b,x} - X_{0,\lambda,x} \Xi_{0,\lambda,b,x}) L_\lambda \right\|_{\mathbb{C}^2} \\ & \quad \leq \left\| L_\lambda^{-1} (X_{\infty,\lambda,x} - X_{0,\lambda,x}) L_\lambda \right\|_{\mathbb{C}^2} \left\| L_\lambda^{-1} \Xi_{\infty,\lambda,b,x} L_\lambda \right\|_{\mathbb{C}^2} \\ & \qquad + \left\| L_\lambda^{-1} X_{0,\lambda,x} L_\lambda \right\|_{\mathbb{C}^2} \left\| L_\lambda^{-1} (\Xi_{\infty,\lambda,b,x} - \Xi_{0,\lambda,b,x}) L_\lambda \right\|_{\mathbb{C}^2} \\ & \quad \leq \left(454\mu(x) + 16\sqrt{|\lambda|}^{-1} \left|(H_{\infty,0,\infty,x_{\mathfrak{b}}}1)'(x)\right| \right) \exp \frac{\|V_\infty\|_{\mathcal{L}^1([x,b])}}{\sqrt{|\lambda|}} \exp \left| (x-b) \Im \sqrt{\lambda} \right| \end{aligned}$$

$$+\exp\left|x\Im\sqrt{\lambda}\right|\left(\exp\frac{\|V_\infty\|_{\mathcal{L}^1([x,b])}}{\sqrt{|\lambda|}}-1\right)\exp\left|(x-b)\Im\sqrt{\lambda}\right|$$

$$\leq\left(\left(454\mu(x)+16\sqrt{|\lambda|}^{-1}|(H_{\infty,0,\infty,x_\mathfrak{b}}1)'(x)|\right)\exp\frac{\hat{\mu}(x)}{|x|\sqrt{|\lambda|}}+\exp\frac{\hat{\mu}(x)}{|x|\sqrt{|\lambda|}}-1\right)$$

$$\cdot\exp\left|b\Im\sqrt{\lambda}\right|.$$

The proof for the first inequality is analogous. □

4.4 Resolvent, eigenvalues, and eigenvectors

Assume $z_\pm \in \overline{\mathbb{C}}$ and $B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & g & 1 \end{pmatrix}$ with $g \in \mathbb{C}$. Let the operator T_∞ be given by

$$T_\infty y = l_\infty[y], \quad y \in \mathcal{D}(T_\infty),$$

where the domain $\mathcal{D}(T_\infty)$ consists of all functions $y \in \mathcal{AC}^2_{\mathrm{loc}}([a,b]\setminus\{0\})$ such that $l_\infty[y] \in \mathcal{L}^p([a,b])$, $y'(a) = z_- y(a)$, $y'(b) = z_+ y(b)$, and $B\mathsf{b}_{\infty,0,x_\mathfrak{b}} y = 0$. Assume $\lambda \in \mathbb{C}$ and that $y_{\lambda,\pm} \in \mathcal{AC}^2_{\mathrm{loc}}([a,b]\setminus\{0\})$ are two solutions (3.4) for $j=\infty$ which vanish on $I_\mp$ and satisfy the boundary conditions

$$\begin{pmatrix} y_{\lambda,-}(a) \\ y'_{\lambda,-}(a) \end{pmatrix} = \vec{v}_-, \quad \begin{pmatrix} y_{\lambda,+}(b) \\ y'_{\lambda,+}(b) \end{pmatrix} = \vec{v}_+$$

in a and b, where the vectors $\vec{v}_\pm \in \mathbb{C}^2$ are $\vec{v}_\pm = \begin{pmatrix} 1 \\ z_\pm \end{pmatrix}$ if $z_\pm \in \mathbb{C}$ and $\vec{v}_\pm = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if $z_\pm = \infty$. Then $y_{\lambda,\pm}$ satisfy (3.28). Such functions have been used in the proof of Theorem 3.38. Since $\Xi_{\infty,\lambda,\cdot,\cdot}$ is a fundamental matrix for (3.4), it holds

$$\begin{pmatrix} y_{\lambda,-}(x) \\ y'_{\lambda,-}(x) \end{pmatrix} = \Xi_{\infty,\lambda,a,x} v-, \quad x \in [a,0), \quad \begin{pmatrix} y_{\lambda,+}(x) \\ y'_{\lambda,+}(x) \end{pmatrix} = \Xi_{\infty,\lambda,b,x} v+, \quad x \in (0,b]. \tag{4.12}$$

Define the characteristic matrix

$$M_\lambda = \begin{pmatrix} B\mathsf{b}_{\infty,0,x_\mathfrak{b}} y_{\lambda,-} & B\mathsf{b}_{\infty,0,x_\mathfrak{b}} y_{\lambda,+} \end{pmatrix}. \tag{4.13}$$

Lemma 4.13 *A number $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(T_\infty)$ if and only if M_λ is invertible. In this case, the resolvent $(T_\infty - \lambda)^{-1}$ can be computed using the formula*

$$((T_\infty - \lambda)^{-1} f)(\xi) = (K_{\infty,\lambda} f)(\xi) - \begin{pmatrix} y_-(\xi) & y_+(\xi) \end{pmatrix} M_\lambda^{-1} B\mathsf{b}_{\infty,0,x_\mathfrak{b}} K_{\infty,\lambda} f,$$

for all $f \in \mathcal{L}^p([a,b])$ and $\xi \in [a,b]$. Here $K_{\infty,\lambda}$ is the integral operator introduced in Lemma 3.31. Otherwise, λ is an eigenvalue of T_∞ and the eigenfunctions of T_∞ for λ are all functions

$$y = c_- y_{\lambda,-} + c_+ y_{\lambda,+}, \quad \begin{pmatrix} c_- \\ c_+ \end{pmatrix} \in \ker M_\lambda.$$

Proof. Assume $f \in \mathcal{L}^p([a,b])$. If $(T_\infty - \lambda)y = f$ for a vector $y \in \mathcal{D}(T_\infty)$, then Lemma 3.31 and the properties of $y_{\lambda,\pm}$ imply that y can be written as

$$y = c_- y_{\lambda,-} + c_+ y_{\lambda,+} + K_{\infty,\lambda} f, \tag{4.14}$$

where $c_\pm \in \mathbb{C}$. The condition $B\mathfrak{b}_{\infty,0,x_\mathfrak{b}} y = 0$ then translates to the equation

$$M_\lambda \begin{pmatrix} c_- \\ c_+ \end{pmatrix} + B\mathfrak{b}_{\infty,0,x_\mathfrak{b}} K_{\infty,\lambda} f = 0 \tag{4.15}$$

for the numbers $c_\pm$. If M_λ is invertible, this equation has a unique solution. Using (4.14), the function y then can be computed and the equality $(T_\infty - \lambda)y = f$ holds. Now assume that M_λ is singular. If $(c_-, c_+) \in \ker M_\lambda$, then (4.15) is fulfilled when $f = 0$. The properties of $y_{\lambda,\pm}$ imply that the function $c_- y_{\lambda,-} + c_+ y_{\lambda,+}$ then is an eigenfunction for λ and it is nonzero if $(c_-, c_+) \neq (0,0)$. □

Lemma 4.14 *The asymptotic relation*

$$\hat{\mu}(x) = o\left(\sqrt[\hat{q}]{x}\right)$$

holds if $x \to 0$.

Proof. The assumption (4.2) means that the function w given by (3.49) is contained in $\mathcal{L}^q([a,b])$. Since $w \in \mathcal{M}^q([a,b])$, Lemma 3.5 yields

$$\hat{\mu}(x) = |xw(x)| = |x|^{1-q^{-1}} \sqrt[q]{|x|} w(x) \leq |x|^{\hat{q}^{-1}} \|w\|_{\mathcal{L}^q(I_x)} = o\left(\sqrt[\hat{q}]{|x|}\right).$$

□

Lemma 4.15 *The following asymptotic relations hold if* $\lambda \to \infty$. *The left column of* $L_\lambda^{-1} M_\lambda$ *equals*

$$\begin{pmatrix} \cos\left(\sqrt{\lambda}a\right) \\ -\sin\left(\sqrt{\lambda}a\right) \end{pmatrix} + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|a\Im\sqrt{\lambda}\right|\right)$$

if $z_- \neq \infty$, *and*

$$\sqrt{\lambda}^{-1} \begin{pmatrix} -\sin\left(\sqrt{\lambda}a\right) \\ -\cos\left(\sqrt{\lambda}a\right) \end{pmatrix} + \sqrt{\lambda}^{-1} o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|a\Im\sqrt{\lambda}\right|\right)$$

if $z_- = \infty$. *Analogously, the right column of* $L_\lambda^{-1} M_\lambda$ *equals*

$$\begin{pmatrix} -\cos\left(\sqrt{\lambda}b\right) \\ \sin\left(\sqrt{\lambda}b\right) \end{pmatrix} + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|b\Im\sqrt{\lambda}\right|\right)$$

if $z_+ \neq \infty$, *and*

$$\sqrt{\lambda}^{-1} \begin{pmatrix} \sin\left(\sqrt{\lambda}b\right) \\ \cos\left(\sqrt{\lambda}b\right) \end{pmatrix} + \sqrt{\lambda}^{-1} o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|b\Im\sqrt{\lambda}\right|\right)$$

if $z_+ = \infty$.

Proof. Set

$$x = \left(8\sqrt{|\lambda|}\right)^{-1}. \tag{4.16}$$

Let $X_{j,\lambda,\cdot}$ be the matrix defined in Lemma 4.5. For $j \in \{0,\infty\}$, the properties of $X_{j,\lambda,\cdot}$, $\Xi_{j,\lambda,\cdot,\cdot}$, and $y_{j,\lambda,\pm}$, imply

$$\mathsf{b}_{j,0,x_\mathfrak{b}} y_{j,\lambda,-} = \begin{pmatrix} X_{j,\lambda,-x}\Xi_{j,\lambda,a,-x}\vec{v}_- \\ 0 \\ 0 \end{pmatrix}, \quad \mathsf{b}_{j,0,x_\mathfrak{b}} y_{j,\lambda,+} = \begin{pmatrix} 0 \\ 0 \\ -X_{j,\lambda,x}\Xi_{j,\lambda,b,x}\vec{v}_+ \end{pmatrix}.$$

This, the definition (4.13) of M_λ, and the structure of the matrix B imply

$$\begin{aligned} M_\lambda &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X_{\infty,\lambda,-x}\Xi_{\infty,\lambda,a,-x}\vec{v}_- \quad -\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} X_{\infty,\lambda,x}\Xi_{\infty,\lambda,b,x}\vec{v}_+ \right) \\ &= \left(L_\lambda L_\lambda^{-1} X_{\infty,\lambda,-x}\Xi_{\infty,\lambda,a,-x}\vec{v}_- \quad -L_\lambda L_\lambda^{-1}\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} X_{\infty,\lambda,x}\Xi_{\infty,\lambda,b,x}\vec{v}_+ \right) \\ &= L_\lambda \left(L_\lambda^{-1} X_{\infty,\lambda,-x}\Xi_{\infty,\lambda,a,-x} L_\lambda L_\lambda^{-1}\vec{v}_- \quad -\begin{pmatrix} 1 & 0 \\ \frac{g}{\sqrt{\lambda}} & 1 \end{pmatrix} L_\lambda^{-1} X_{\infty,\lambda,x}\Xi_{\infty,\lambda,b,x} L_\lambda L_\lambda^{-1}\vec{v}_+ \right). \end{aligned}$$

Consider the right column of the last matrix. The left column can be treated analogously up to the constant g, which must be replaced by zero. Together, (4.16) and Lemma 3.24 imply

$$\sqrt{|\lambda|}^{-1} \left|(H_{\infty,0,\infty,x_\mathfrak{b}}1)'(x)\right| = 8x\left|(H_{\infty,0,\infty,x_\mathfrak{b}}1)'(x)\right| \le 16\mu(x).$$

Thus, Lemma 4.12 and (4.16) and finally Lemmas 3.56 and 4.14 yield

$$\begin{aligned} &\left\| L_\lambda^{-1} X_{\infty,\lambda,x}\Xi_{\infty\lambda,b,x} L_\lambda - L_\lambda^{-1} X_{0,\lambda,x}\Xi_{0,\lambda,b,x} L_\lambda \right\|_{\mathbb{C}^2} \\ &\quad \le \left((454+256)\mu(x)\exp\left(8\hat{\mu}(x)\right) + \exp\left(8\hat{\mu}(x)\right) - 1\right)\exp\left|b\Im\sqrt{\lambda}\right| \\ &\quad = \left(710\mu\left(\left(8\sqrt{|\lambda|}\right)^{-1}\right)\exp\left(8\hat{\mu}\left(\left(8\sqrt{|\lambda|}\right)^{-1}\right)\right) + \exp\left(8\hat{\mu}\left(\left(8\sqrt{|\lambda|}\right)^{-1}\right)\right) - 1\right) \\ &\qquad \cdot \exp\left|b\Im\sqrt{\lambda}\right| \\ &\quad = O\left(\mu\left(\left(8\sqrt{|\lambda|}\right)^{-1}\right) + \hat{\mu}\left(\left(8\sqrt{|\lambda|}\right)^{-1}\right)\right)\exp\left|b\Im\sqrt{\lambda}\right| = o\left(\sqrt[-2\hat{q}]{\lambda}\exp\left|b\Im\sqrt{\lambda}\right|\right) \end{aligned} \tag{4.17}$$

for the limit $\lambda \to \infty$. The assumptions $q < \infty$ and hence $\hat{q} > 1$ imply the estimate

$$\sqrt{\lambda}^{-1} = o\left(\sqrt[-2\hat{q}]{\lambda}\right), \quad \lambda \to \infty. \tag{4.18}$$

This, Corollary 4.11, Lemma 4.2, and (4.17) yield

$$\begin{aligned} &\left\| \begin{pmatrix} 1 & 0 \\ g\sqrt{\lambda}^{-1} & 1 \end{pmatrix} L_\lambda^{-1} X_{\infty,\lambda,x}\Xi_{\infty\lambda,b,x} L_\lambda - L_\lambda^{-1} X_{0,\lambda,x}\Xi_{0,\lambda,b,x} L_\lambda \right\|_{\mathbb{C}^2} \\ &\quad = o\left(\sqrt[-2\hat{q}]{\lambda}|\exp\left|b\Im\sqrt{\lambda}\right|\right) + g\sqrt{\lambda}^{-1} O\left(\exp\left|b\Im\sqrt{\lambda}\right|\right) = o\left(\sqrt[-2\hat{q}]{\lambda}\exp\left|b\Im\sqrt{\lambda}\right|\right). \end{aligned}$$

If $z_+ \in \mathbb{C}$, Corollary 4.11, Lemma 4.2, and (4.18) yield

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ \frac{g}{\sqrt{\lambda}} & 1 \end{pmatrix} L_\lambda^{-1} X_{\infty,\lambda,x} \Xi_{\infty,\lambda,b,x} L_\lambda L_\lambda^{-1} \vec{v}_+ \\
&= \left(L_\lambda^{-1} X_{0,\lambda,x} \Xi_{0,\lambda,b,x} L_\lambda + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left| b \Im \sqrt{\lambda} \right| \right) \right) L_\lambda^{-1} \vec{v}_+ \\
&= \begin{pmatrix} \cos\left(\sqrt{\lambda} b\right) & -\sin\left(\sqrt{\lambda} b\right) \\ -\sin\left(\sqrt{\lambda} b\right) & -\cos\left(\sqrt{\lambda} b\right) \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{\lambda}^{-1} z_+ \end{pmatrix} + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left| b \Im \sqrt{\lambda} \right| \right) \\
&= \begin{pmatrix} \cos\left(\sqrt{\lambda} b\right) \\ -\sin\left(\sqrt{\lambda} b\right) \end{pmatrix} + z_+ \sqrt{\lambda}^{-1} O\left(\exp\left| b \Im \sqrt{\lambda} \right| \right) + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left| b \Im \sqrt{\lambda} \right| \right) \\
&= \begin{pmatrix} \cos\left(\sqrt{\lambda} b\right) \\ -\sin\left(\sqrt{\lambda} b\right) \end{pmatrix} + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left| b \Im \sqrt{\lambda} \right| \right).
\end{aligned}$$

If $z_+ = \infty$, then similarly

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ \frac{g}{\sqrt{\lambda}} & 1 \end{pmatrix} L_\lambda^{-1} X_{\infty,\lambda,x} \Xi_{\infty,\lambda,b,x} L_\lambda L_\lambda^{-1} \vec{v}_+ \\
&= \left(\begin{pmatrix} \cos\left(\sqrt{\lambda} b\right) & -\sin\left(\sqrt{\lambda} b\right) \\ -\sin\left(\sqrt{\lambda} b\right) & -\cos\left(\sqrt{\lambda} b\right) \end{pmatrix} + o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left| b \Im \sqrt{\lambda} \right| \right) \right) \begin{pmatrix} 0 \\ \sqrt{\lambda}^{-1} \end{pmatrix} \\
&= \sqrt{\lambda}^{-1} \begin{pmatrix} -\sin\left(\sqrt{\lambda} b\right) \\ -\cos\left(\sqrt{\lambda} b\right) \end{pmatrix} + \sqrt{\lambda}^{-1} o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left| b \Im \sqrt{\lambda} \right| \right).
\end{aligned}$$

□

For the following lemma also compare [BDL].

Lemma 4.16 *Assume that u is an entire function such that the asymptotic formula*

$$u(z) = \sin z + o\left(\sqrt[-\hat{q}]{z} \exp|\Im z| \right), \quad z \to \infty,$$

holds. Then the zeros z_k, $k \in \mathbb{Z}$, of u can be enumerated such that the asymptotic formula

$$z_k = k\pi + o\left(\sqrt[-\hat{q}]{k} \right), \quad k \to \pm\infty$$

holds. Moreover, all but finitely many zeros of u are simple.

Proof. For $z \in \mathbb{C}$, it holds

$$\begin{aligned}
\left| \exp|\Im z| \right| &\leq \left| e^{|\Im z|} + \left(1 - e^{-|\Im z|}\right) \right| = 1 + 2\sinh|\Im z|, \\
\sinh|\Im z| &= |\sinh \Im z| = \frac{1}{2} \left| e^{\Im z} - e^{-\Im z} \right| \\
&\leq \frac{1}{2} \left| e^{-i\Re z} e^{\Im z} - e^{i\Re z} e^{-\Im z} \right| = \frac{1}{2} \left| e^{-iz} - e^{iz} \right| = |\sin z|, \qquad (4.19)
\end{aligned}$$

which is seen from geometric consideration. The assumption on u implies that there is a constant $C \geq 1$ such that

$$|u(z) - \sin z| < \frac{1}{4} \exp|\Im z| \leq \frac{1}{4} + \frac{1}{2} |\sin z| \qquad (4.20)$$

for all $z \in \mathbb{C}$ with $|z| > C$. If $|\Im z| \geq 1$, (4.19) implies $|\sin z| > 1$. This implies

$$|u(z) - \sin z| < |\sin z|, \quad z \in \mathbb{C}, |z| > C, |\Im z| \geq 1. \tag{4.21}$$

Particularly, the function u has no zeros $z \in \mathbb{C}$ such that $|z| > C$ and $|\Im z| \geq 1$. Now consider numbers $k \in \mathbb{Z}$ and $z \in \mathbb{C}$ with $\Re z = \left(k + \frac{1}{2}\right)\pi$. For such numbers z, the properties of the sine function yield

$$|\sin z| = \left|\sin\left(\frac{\pi}{2} + i\Im z\right)\right| = |\cos i\Im z| = |\cosh \Im z| \geq 1.$$

This and (4.20) imply

$$|u(z) - \sin z| < |\sin z|, \quad z \in \mathbb{C}, |z| > C, \Re z = \left(k + \frac{1}{2}\right)\pi. \tag{4.22}$$

Rouché's theorem and the relations (4.21) and (4.22) prove that the function u has exactly one zero in the rectangle

$$R_k = \{z \in \mathbb{C} : \left(k - \frac{1}{2}\right)\pi < \Re z < \left(k + \frac{1}{2}\right)\pi, |\Im z| < 1\}$$

if $k \in \mathbb{Z}$ fulfills $|k| \geq k_0$, where $k_0 \in \mathbb{N}$ is some number such that $k_0 - \frac{1}{2} > \frac{C}{\pi}$. Moreover, this zero is simple. Denote the zero of u in R_k by z_k, $|k| \geq k_0$. Moreover, in the rectangular area

$$R_k = \{z \in \mathbb{C} : \left(-k_0 + \frac{1}{2}\right)\pi < \Re z < \left(k_0 - \frac{1}{2}\right)\pi, |\Im z| < C\}$$

there are exactly $2k_0 - 1$ zeros, counting multiplicity. Enumerate them in an arbitrary order to be the numbers z_k, $k \in \{-k_0 + 1, \ldots, 0, \ldots, k_0 - 1\}$. It is seen from (4.21) that the function u has no other zeros than the numbers z_k, $k \in \mathbb{Z}$.

To prove the asymptotic formula, it suffices to consider k such that $|k| \geq k_0$. Then $|\Im z_k| < 1$ and thus

$$\sup_{z \in R_k} |u(z) - \sin z| = o\left(\sqrt[-\hat{q}]{z}\right) = o\left(\sqrt[-\hat{q}]{k}\right) \tag{4.23}$$

if $k \to \pm\infty$. Since $k\pi$ is the only zero of the sine function in R_k, the minimum

$$\min_{z \in R_k, |z| \geq \frac{1}{2}} |\sin z|$$

is positive and independent of $k \in \mathbb{Z}$. So (4.23) yields that for all numbers $k \in \mathbb{Z}$ with sufficiently large absolute value, say $k \geq k_1 \geq k_0$, the function u has no zero in $R_k \setminus \left\{z \in \mathbb{C} : |z - k\pi| \geq \frac{1}{2}\right\}$. This proves $|z_k - k\pi| < \frac{1}{2}$ if $|k| \geq k_1$. The Taylor expansion of the sine function yields

$$|\sin z - z| \leq \frac{|z|^3}{6}\frac{1}{1 - |z^2|}, \quad z \in \mathbb{C}, |z| < 1,$$

and thus, using the properties of the sine function,

$$|\sin z| \geq \frac{1}{2}|z - \pi k|, \quad z \in \mathbb{C}, |z - \pi k| < \frac{1}{2}, k \in \mathbb{Z}.$$

This and (4.23) yield

$$|u(z)| \geq \frac{1}{2}|z - \pi k| + o\left(\sqrt[-\hat{q}]{k}\right), \quad z \in \mathbb{C}, |z - \pi k| < \frac{1}{2},$$

if $k \geq k_1$. Finally apply this estimate to $z = z_k$, which yields the statement. □

The following theorem only covers the case of Dirichlet boundary conditions in a and b, hence $z_- = z_+ = \infty$. Using a version of Lemma 4.16 also for the cosine function it is not too difficult to get similar statements for other boundary conditions of the form (3.28) in these points.

Theorem 4.17 *Assume that the function $V_\infty \in \mathcal{L}^1_{\text{loc}}([a,b] \setminus \{0\})$ fulfills (4.1) and (4.2) for some number $q \in [1, \infty)$. Let the operator T_∞ be given by*

$$T_\infty y = l_\infty[y], \quad y \in \mathcal{D}(T_\infty),$$

where the domain $\mathcal{D}(T_\infty)$ consists of all functions $y \in \mathcal{AC}^2_{\text{loc}}([a,b] \setminus \{0\})$ such that $l_\infty[y] \in \mathcal{L}^p([a,b])$, $y(a) = y(b) = 0$, and $B\mathsf{b}_{\infty,0,x_\mathsf{b}} y = 0$. Then the spectrum of T_∞ consists of isolated eigenvalues only. They can be enumerated such that the asymptotic formula

$$\lambda_n = \frac{\pi^2 n^2}{(b-a)^2} + o\left(n \sqrt[-\hat{q}]{n}\right)$$

holds, where λ_n, $n \in \mathbb{N}$, is the n-th eigenvalue of T_∞.

Proof. Lemma 4.15 implies

$$\begin{aligned}
\det M_\lambda &= \sqrt{\lambda} \det\left(L_\lambda^{-1} M_\lambda\right) \\
&= \sqrt{\lambda}^{-1} \begin{vmatrix} -\sin\left(\sqrt{\lambda}a\right) & \sin\left(\sqrt{\lambda}b\right) \\ -\cos\left(\sqrt{\lambda}a\right) & \cos\left(\sqrt{\lambda}b\right) \end{vmatrix} \\
&\quad + \sqrt{\lambda}^{-1} o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|(b-a)\Im\sqrt{\lambda}\right|\right) \\
&= \sqrt{\lambda}^{-1}\left(\cos\left(\sqrt{\lambda}a\right)\sin\left(\sqrt{\lambda}b\right) - \sin\left(\sqrt{\lambda}a\right)\cos\left(\sqrt{\lambda}b\right)\right) \\
&\quad + \sqrt{\lambda}^{-1} o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|(b-a)\Im\sqrt{\lambda}\right|\right) \\
&= \sqrt{\lambda}^{-1} \sin\left((b-a)\sqrt{\lambda}\right) + \sqrt{\lambda}^{-1} o\left(\sqrt[-2\hat{q}]{\lambda} \exp\left|(b-a)\Im\sqrt{\lambda}\right|\right).
\end{aligned}$$

Now the Lemmas 4.16 and 4.13 prove the assertion. □

Corollary 4.18 *Use the notation and assumptions of Theorem 4.17. Then the operator T_∞ has nonemtpy resolvent set.*

Remark 4.19 Let the differential operator T_∞ be given as in Theorem 4.17. Using a lemma presented in [BDL], it even follows that all but finitely many eigenvalues of T_∞ are simple and that all degenerate eigenvalues have finite algebraic multiplicity. In fact, the multiplicity of a zero λ of the characteristic determinant $\det M_\lambda$ equals the algebraic multiplicity of λ as an eigenvalue of T_∞.

Remark 4.20 In [BDL], the potential V_∞ given by $V_\infty(x) = \frac{1}{x}$ was considered with interface conditions in 0 as in Theorem 4.17. It was shown that the eigenvalues behave asymptotically as

$$\lambda_n = \frac{\pi^2 n^2}{(b-a)^2} + O(\ln n), \quad n \to \infty.$$

Since the potential V_∞ fulfills (4.2) for all $q < \infty$, Theorem 4.17 shows

$$\lambda_n = \frac{\pi^2 n^2}{(b-a)^2} + o\left(n^\epsilon\right), \quad n \to \infty,$$

with arbitrarily small $\epsilon > 0$. That the error term is in fact logarithmic, can be shown using a more careful treatment of the functions μ and $\hat{\mu}$ in the above analysis. Note that $\mu(x) = |x|$ and $\hat{\mu}(x) \sim |x \ln |x||$ if $x \to 0$ for that particular potential V_∞.

Corollary 4.21 *Assume that λ_n, $n \in \mathbb{Z}$, are the eigenvalues of T_∞, enumerated as in Theorem 4.17. Then*

$$\begin{aligned} \sup_{\xi \in \left[a, \frac{a-b}{8\pi n}\right]} \left\| L^{-1}_{\frac{\pi^2 n^2}{(b-a)^2}} \left(\Xi_{\infty,\lambda_n,a,\xi} - \Xi_{0,\frac{\pi^2 n^2}{(b-a)^2},a,\xi} \right) L_{\frac{\pi^2 n^2}{(b-a)^2}} \right\|_{\mathbb{C}^2} &= o\left(\sqrt[-\hat{q}]{n} \right), \\ \sup_{\xi \in \left[\frac{b-a}{8\pi n}, b\right]} \left\| L^{-1}_{\frac{\pi^2 n^2}{(b-a)^2}} \left(\Xi_{\infty,\lambda_n,b,\xi} - \Xi_{0,\frac{\pi^2 n^2}{(b-a)^2},b,\xi} \right) L_{\frac{\pi^2 n^2}{(b-a)^2}} \right\|_{\mathbb{C}^2} &= o\left(\sqrt[-\hat{q}]{n} \right) \end{aligned}$$

if $n \to \infty$.

Proof. Apply Lemma 4.4 for $\lambda = \frac{\pi n^2}{(b-a)^2}$ and the potential $V_\infty - \lambda_n + \lambda$ im place of V_∞. This and Theorem 4.17 for $x = \left(8\sqrt{|\lambda|}\right)^{-1} = \frac{b-a}{8\pi n}$ and the Lemmas 3.56 and 4.14 yield for all $\xi \in [x, b]$

$$\begin{aligned} &\left\| L_\lambda^{-1} \left(\Xi_{\infty,\lambda_n,b,\xi} - \Xi_{0,\lambda,b,\xi} \right) L_\lambda \right\|_{\mathbb{C}^2} \\ &\leq \left(\exp \frac{\|V_\infty\|_{\mathcal{L}^1([x,b])} + (b-x)|\lambda - \lambda_n|}{\sqrt{|\lambda|}} - 1 \right) \exp \left| (\xi - b) \Im \sqrt{\lambda_n} \right| \\ &\leq \left(\exp \frac{|x|^{-1} \hat{\mu}(x) + (b-x)|\lambda - \lambda_n|}{\sqrt{|\lambda|}} - 1 \right) \exp \left| (x - b) \Im \sqrt{\lambda_n} \right| \\ &= \left(\exp o\left(\sqrt[-\hat{q}]{n} \right) - 1 \right) \exp \left| (x-b) o\left(\sqrt[-\hat{q}]{n} \right) \right| = o\left(\sqrt[-\hat{q}]{n} \right), \end{aligned} \tag{4.24}$$

and the estimate evidently is uniform in ξ. The other assertion can be proved similarly. □

Lemma 4.22 *Assume $\lambda_j \in \mathbb{C}$, $x \in [a,b] \cap [-b,-a] \setminus \{0\}$ such that $\mu(\pm x) + x^2|\lambda_j| < \frac{1}{2}$, and that the functions y_j are solutions of (3.4) on I_x for $j \in \{0, \infty\}$. Then*

$$\begin{aligned} \|y_\infty - y_0\|_{\mathcal{L}^\infty(I_x)} \leq\ & 4|y_\infty(x) - y_0(x)| + 10|(y_\infty'(x) - y_0'(x))x| \\ & + (40|y_0(x)| + 100|y_0'(x)x|)(\mu(x) + x^2|\lambda_\infty - \lambda_0|). \end{aligned}$$

Proof. The equality (4.6) for $j \in \{0, \infty\}$ implies

$$\begin{aligned}
\|y_\infty - y_0\|_{\mathcal{L}^\infty(I_x)} \le & \left\| \frac{y_\infty(x)}{(H_{\infty,\lambda_\infty,\infty,x}1)(x)} H_{\infty,\lambda_\infty,\infty,x}1 - \frac{y_0(x)}{(H_{0,\lambda_0,\infty,x}1)(x)} H_{0,\lambda_0,\infty,x}1 \right\|_{\mathcal{L}^\infty(I_x)} \\
& + \left\| \frac{y_\infty'(x)(H_{\infty,\lambda_\infty,\infty,x}\iota)(x)}{(H_{\infty,\lambda_\infty,\infty,x}1)(x)} H_{\infty,\lambda_\infty,\infty,x}1 - \frac{y_0'(x)(H_{0,\lambda_0,\infty,x}\iota)(x)}{(H_{0,\lambda_0,\infty,x}1)(x)} H_{0,\lambda_0,\infty,x}1 \right\|_{\mathcal{L}^\infty(I_x)} \\
& + \|y_\infty'(x) H_{\infty,\lambda_\infty,\infty,x}\iota - y_0'(x) H_{0,\lambda_0,\infty,x}\iota\|_{\mathcal{L}^\infty(I_x)} .
\end{aligned}$$

As to the terms on the right side, the Lemmas 3.14, 4.6, and 3.22 yield

$$\begin{aligned}
& \left\| \frac{y_\infty(x)}{(H_{\infty,\lambda_\infty,\infty,x}1)(x)} H_{\infty,\lambda_\infty,\infty,x}1 - \frac{y_0(x)}{(H_{0,\lambda_0,\infty,x}1)(x)} H_{0,\lambda_0,\infty,x}1 \right\|_{\mathcal{L}^\infty(I_x)} \\
& \quad \le |y_\infty(x) - y_0(x)| \frac{\|H_{\infty,\lambda_\infty,\infty,x}1\|_{\mathcal{L}^\infty(I_x)}}{(H_{\infty,\lambda_\infty,\infty,x}1)(x)} \\
& \quad + |y_0(x)| \frac{|(H_{0,\lambda_0,\infty,x}1)(x) - (H_{\infty,\lambda_\infty,\infty,x}1)(x)|}{(H_{\infty,\lambda_\infty,\infty,x}1)(x)(H_{0,\lambda_0,\infty,x}1)(x)} \|H_{\infty,\lambda_\infty,\infty,x}1\|_{\mathcal{L}^\infty(I_x)} \\
& \quad + \frac{|y_0(x)|}{(H_{0,\lambda_0,\infty,x}1)(x)} \|H_{\infty,\lambda_\infty,\infty,x}1 - H_{0,\lambda_0,\infty,x}1\|_{\mathcal{L}^\infty(I_x)} \\
& \quad \le 4|y_\infty(x) - y_0(x)| \\
& \qquad + |y_0(x)| 4(4\mu(x) + 4x^2|\lambda_\infty - \lambda_0|)2 + |y_0(x)| 2(4\mu(x) + 4x^2|\lambda_\infty - \lambda_0|) \\
& \quad = 4|y_\infty(x) - y_0(x)| + 40|y_0(x)|(\mu(x) + x^2|\lambda_\infty - \lambda_0|), \\
& \left\| \frac{y_\infty'(x)(H_{\infty,\lambda_\infty,\infty,x}\iota)(x)}{(H_{\infty,\lambda_\infty,\infty,x}1)(x)} H_{\infty,\lambda_\infty,\infty,x}1 - \frac{y_0'(x)(H_{0,\lambda_0,\infty,x}\iota)(x)}{(H_{0,\lambda_0,\infty,x}1)(x)} H_{0,\lambda_0,\infty,x}1 \right\|_{\mathcal{L}^\infty(I_x)} \\
& \quad \le 4|y_\infty'(x)(H_{\infty,\lambda_\infty,\infty,x}\iota)(x) - y_0'(x)(H_{0,\lambda_0,\infty,x}\iota)(x)| \\
& \quad + 40|y_0'(x)(H_{0,\lambda_0,\infty,x}\iota)(x)|(\mu(x) + x^2|\lambda_\infty - \lambda_0|) \\
& \quad \le 4|(y_\infty'(x) - y_0'(x))(H_{\infty,\lambda_\infty,\infty,x}\iota)(x)| \\
& \qquad + 4|y_0'(0)((H_{\infty,\lambda_\infty,\infty,x}\iota)(x) - (H_{0,\lambda_0,\infty,x}\iota)(x))| \\
& \qquad + 80|y_0'(0)x|(\mu(x) + x^2|\lambda_\infty - \lambda_0|) \\
& \quad \le 8|(y_\infty'(x) - y_0'(x))x| \\
& \qquad + 4|y_0'(0)x|(4\mu(x) + 4x^2|\lambda_\infty - \lambda_0|) + 80|y_0'(0)x|(\mu(x) + x^2|\lambda_\infty - \lambda_0|) \\
& \quad = 8|(y_\infty'(x) - y_0'(x))x| + 96|y_0'(0)x|(\mu(x) + x^2|\lambda_\infty - \lambda_0|) \\
& \|y_\infty'(x) H_{\infty,\lambda_\infty,\infty,x}\iota - y_0'(x) H_{0,\lambda_0,\infty,x}\iota\|_{\mathcal{L}^\infty(I_x)} \\
& \quad \le |y_\infty'(x) - y_0'(x)| \, \|H_{\infty,\lambda_\infty,\infty,x}\iota\|_{\mathcal{L}^\infty(I_x)} + |y_0'(x)| \, \|H_{\infty,\lambda_\infty,\infty,x}\iota - H_{0,\lambda_0,\infty,x}\iota\|_{\mathcal{L}^\infty(I_x)} \\
& \quad \le 2|(y_\infty'(x) - y_0'(x))x| + |y_0'(x)x|(4\mu(x) + 4x^2|\lambda_\infty - \lambda_0|).
\end{aligned}$$

Now summation yields the statement. □

Theorem 4.23 *Use the notation and assumptions of Theorem 4.17. Then there is a sequence of root vectors y_n, $n \in \mathbb{N}$, of T_∞ such that for each $n \in \mathbb{N}$, y_n is a root vector for the eigenvalue λ_n. This sequence can be chosen such that*

$$\left\| y_n - \sin \frac{\pi n(\cdot - a)}{b - a} \right\|_{\mathcal{L}^\infty([a,b])} = o\left(\sqrt[-\hat{q}]{n} \right), \quad n \to \infty.$$

Proof. Consider $y_{\lambda,+}$, the arguments for $y_{\lambda,-}$ are similar, only b must be replaced by a. The equality (4.12), Corollary 4.21, and (4.3) for $x = \frac{b-a}{8\pi n}$ yield

$$\begin{aligned}
L^{-1}_{\frac{\pi n^2}{(b-a)^2}} \begin{pmatrix} y_{\lambda_n,+}(x) \\ y'_{\lambda_n,+}(x) \end{pmatrix} &= L^{-1}_{\frac{\pi n^2}{(b-a)^2}} \Xi_{\infty,\lambda_n,b,x} \vec{v}_+ \\
&= L^{-1}_{\frac{\pi n^2}{(b-a)^2}} \Xi_{0,\frac{\pi n^2}{(b-a)^2},b,x} \vec{v}_+ + o\left(\sqrt[-\hat{q}]{n}\right) L^{-1}_{\frac{\pi n^2}{(b-a)^2}} \vec{v}_+ \\
&= \begin{pmatrix} \cos\frac{\pi n(x-b)}{b-a} & \frac{b-a}{\pi n}\sin\frac{\pi n(x-b)}{b-a} \\ -\sin\frac{\pi n(x-b)}{b-a} & \frac{b-a}{\pi n}\cos\frac{\pi n(x-b)}{b-a} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + n^{-1} o\left(\sqrt[-\hat{q}]{n}\right) \\
&= \frac{b-a}{\pi n} \begin{pmatrix} \sin\frac{\pi n(x-b)}{b-a} \\ \cos\frac{\pi n(x-b)}{b-a} \end{pmatrix} + n^{-1} o\left(\sqrt[-\hat{q}]{n}\right), \qquad (4.25) \\
\left\| \frac{\pi n}{b-a} y_{\lambda_n,+} - \sin\frac{\pi n(\cdot - b)}{b-a} \right\|_{\mathcal{L}^\infty([x,b])} &= o\left(\sqrt[-\hat{q}]{n}\right).
\end{aligned}$$

As to the interval I_x, Lemma 4.22 and (4.25) yield

$$\begin{aligned}
\left\| y_{\lambda_n,+} - \sin\frac{\pi n(\cdot - b)}{b-a} \right\|_{\mathcal{L}^\infty(I_x)} &= 4o\left(\sqrt[-\hat{q}]{n}\right) + 10|x|\frac{\pi n}{b-a} o\left(\sqrt[-\hat{q}]{n}\right) \\
&\quad + \left(40 + 100|x|\frac{\pi n}{b-a}\right)\left(o\left(\sqrt[\hat{q}]{x}\right) + x^2 o\left(n\sqrt[-\hat{q}]{n}\right)\right) \\
&= o\left(\sqrt[-\hat{q}]{n}\right).
\end{aligned}$$

Put together the estimate and the analogous ones for $y_{\lambda_n,-}$. So it has been proved

$$\begin{aligned}
\left\| \frac{\pi n}{b-a} y_{\lambda_n,-} - \sin\frac{\pi n(\cdot - a)}{b-a} \right\|_{\mathcal{L}^\infty([a,0))} &= o\left(\sqrt[-\hat{q}]{n}\right), \\
\left\| \frac{\pi n}{b-a} y_{\lambda_n,+} - \sin\frac{\pi n(\cdot - b)}{b-a} \right\|_{\mathcal{L}^\infty((0,b])} &= o\left(\sqrt[-\hat{q}]{n}\right).
\end{aligned}$$

The eigenfunctions of T_∞ can be written as $y_n = c_{n,-} y_{\lambda_n,-} + c_{n,+} y_{\lambda_n,+}$. So it remains to determine the constants $c_{n,-}$ and $c_{n,+}$. By Lemma 4.13, the relation $M_\lambda \begin{pmatrix} c_- \\ c_+ \end{pmatrix} = 0$ must hold for the coefficients $c_\pm$. By Lemma 4.15, Theorem 4.17, and uniform boundedness of the derivatives of the trigonometric functions for all arguments with bounded imaginary part, this equation is

$$\begin{aligned}
0 &= \begin{pmatrix} -\sin\left(\sqrt{\lambda}_n a\right) & \sin\left(\sqrt{\lambda}_n b\right) \\ -\cos\left(\sqrt{\lambda}_n a\right) & \cos\left(\sqrt{\lambda}_n b\right) \end{pmatrix} \begin{pmatrix} c_- \\ c_+ \end{pmatrix} \\
&\quad + c_- o\left(\sqrt[-2\hat{q}]{\lambda_n} \exp|a\Im\lambda_n|\right) + c_+ o\left(\sqrt[-2\hat{q}]{\lambda_n} \exp|b\Im\lambda_n|\right) \\
&= \begin{pmatrix} -\sin\frac{\pi n a}{b-a} & \sin\frac{\pi n b}{b-a} \\ -\cos\frac{\pi n a}{b-a} & \cos\frac{\pi n b}{b-a} \end{pmatrix} \begin{pmatrix} c_- \\ c_+ \end{pmatrix} + (|c_-| + |c_+|) o\left(\sqrt[-\hat{q}]{n}\right).
\end{aligned}$$

Multiplication from the left side with the regular matrix $\begin{pmatrix} -\sin\frac{\pi n a}{b-a} & -\cos\frac{\pi n a}{b-a} \\ \cos\frac{\pi n a}{b-a} & -\sin\frac{\pi n a}{b-a} \end{pmatrix}$ yields the equivalent equation

$$0 = \begin{pmatrix} 1 & -\cos\frac{\pi n(b-a)}{b-a} \\ 0 & \sin\frac{\pi n(b-a)}{b-a} \end{pmatrix} \begin{pmatrix} c_- \\ c_+ \end{pmatrix} + (|c_-| + |c_+|) o\left(\sqrt[-\hat{q}]{n}\right)$$

$$= \begin{pmatrix} 1 & (-1)^{n+1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_- \\ c_+ \end{pmatrix} + (|c_-| + |c_+|) o\left(\sqrt[-\hat{q}]{n} \right).$$

Now set $c_- = 1$; then $c_+ = (-1)^n + o\left(\sqrt[-\hat{q}]{n} \right)$. Consider the corresponding eigenvector, which is given by

$$y(\xi) = \begin{cases} \sin \frac{\pi n(\xi - a)}{b-a} + o\left(\sqrt[-\hat{q}]{n} \right) & \text{if } \xi \in [a, 0) \\ (-1)^n \sin \frac{\pi n(\xi - b)}{b-a} + o\left(\sqrt[-\hat{q}]{n} \right) & \text{if } \xi \in (0, b] \end{cases}.$$

The properties of the sine function finally yield for $\xi \in (0, b]$

$$\begin{aligned} (-1)^n \sin \frac{\pi n(\xi - b)}{b - a} &= (-1)^n \sin \left(\frac{\pi n(\xi - a)}{b - a} + \frac{\pi n(b - a)}{b - a} \right) \\ &= (-1)^n \sin \left(\frac{\pi n(\xi - a)}{b - a} + \pi n \right) = \sin \frac{\pi n(\xi - a)}{b - a} \end{aligned}$$

and thus the statement is proved. □

For the definition of a Bari basis compare [GK, VI].

Definition 4.24 Assume that $\mathcal{H}$ is a separable Hilbert space. A basis of $\mathcal{H}$ is a sequence of vectors $y_n \in \mathcal{H}$, $n \in \mathbb{N}$, such that each vector $f \in \mathcal{H}$ can uniquely be written as series

$$f = \sum_{n \in \mathbb{N}} c_n y_n,$$

with coefficients $c_n \in \mathbb{C}$, $n \in \mathbb{N}$, such that this series converges in the norm of $\mathcal{H}$.
A basis of $\mathcal{H}$ is called a Riesz basis of $\mathcal{H}$ if it is the image of an orthonormal basis mapped by a bounded and boundedly invertible operator.
A basis of $\mathcal{H}$ is called a Bari basis of $\mathcal{H}$ if it is quadratically close to an orthonormal basis of $\mathcal{H}$. This means that there is an orthonormal basis of $\mathcal{H}$ consisting of vectors e_n, $n \in \mathbb{N}$, such that

$$\sum_{n \in \mathbb{N}} \|y_n - e_n\|_{\mathcal{H}}^2 < \infty.$$

Corollary 4.25 *Assume the conditions and use the notation of Theorem 4.23. If $p = 2$ and $q > 2$, then the operator T_∞ has a Bari basis of root vectors.*

Remark 4.26 It may well be that a refined treatment of the functions μ and $\hat{\mu}$ in the estimations (4.17) and (4.24), which have led to the Theorems 4.17 and 4.23, can yield a slightly stronger result. Among other things, the Hardy inequality must be applied to μ. In the end, as it seems, the widespread error terms of the form $o\left(\sqrt[-\hat{q}]{n} \right)$ in fact would lie in $\ell^q(\mathbb{N})$ if $q \geq 2$. Using this, for example Bari basisness could be obtained even in the case $q = 2$.

Conclude this section with the observation that the results obtained are formulated for interface conditions given by a 2×4-matrix of the structure $B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & g & 1 \end{pmatrix}$ with $g \in \mathbb{C}$. This covers almost all operators which have been obtained in Section 3 as limits of regular Sturm-Liouville operators. The only exception are Dirichlet interface conditions in 0 which are given by the matrix $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. However, it is not difficult to perform similar analysis also for this case. Even other 2×4-matrices B of rank 2 may be studied. Then, however, the behaviour of the eigenvalues may be more complicated, since the characteristic determinant $\det M_\lambda$ then may asymptotically equal the sum of two sine functions.

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Wie nur dem Kopf nicht alle Hoffnung schwindet,
Der immerfort an schalem Zeuge klebt,
Mit gier'ger Hand nach Schätzen gräbt,
Und froh ist, wenn er Regenwürmer findet!

JOHANN W. GOETHE