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A characterization of univalent functions on the complex unit disc by indefinite inner product spaces

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Introduction

It is a very strong property of a complex function to be univalent. The Riemann mapping theorem states the existence of a biholomorphic (hence univalent) mapping from any simply connected open proper subset of the complex plane to the unit circle. Nevertheless for a given mapping in many cases univalence is not easy to prove. There are a lot of necessary and sufficient conditions on functions to ensure univalence (see for example [Pom]).

In this master's thesis we focus on an approach based on the theory of indefinite inner product spaces. The so called Littlewood subordination principle or Littlewood subordination theorem states, that for a univalent self-mapping b of the complex unit disc, that fixes the origin, the composition operator is a contraction on various spaces of holomorphic functions. The composition operator induced by b is the linear operator, that maps any given function of the studied space to the composition with b. It was introduced in 1925 by John Edensor Littlewood [Lit25] and holds for example for the Bergman, Hardy and Dirichlet space. Unfortunately this criterion is not sufficient for any of the mentioned spaces. Nevertheless it is possible to expand the subordination principle to a certain Krein space, such that it is sufficient. This will be the goal of the Master's thesis at hand.

In Chapter 1 we start by introducing the concept of formal power series and how the contraction operator may be defined for such series. It is loosely based on [Hen74] and [GK02]. We continue by discussing the theory of indefinite inner product spaces and how a topology can be defined on Krein spaces, a certain class of indefinite inner product spaces. For a more detailed discussion of indefinite inner product spaces see for example [Bog74]. Moreover, we briefly describe the concept of defect spaces and defect operators of contraction operators, since it is an important ingredient in some proofs of Chapter 4. We conclude the first chapter with an introduction to the theory of reproducing kernel Hilbert spaces, a class of Hilbert spaces, that includes the Bergman and Hardy space. A more detailed survey of such spaces was written by Aronszajn in 1950 [Aro50].

The second chapter consists of the introduction of two certain reproducing kernel Hilbert spaces, namely the Bergman and the Dirichlet space.

In Chapter 3, we prove the Littlewood subordination principle for the Dirichlet space. Further we introduce generalized Dirichlet spaces and present a proof for the subordination principle based on [RR94].

The last chapter is dedicated to proving the main result of this work, Theorem 4.0.6, which states, that the composition operator induced by a function b being a contraction on a certain Krein space is already sufficient for b to be univalent. This theorem first appeared in the proof of the Bieberbach conjecture by L. de Branges [dB85] in 1985.

The proof in hand is based on an article by N. Nikolski and V. Vasyunin [NV92], which we tried to supplement with many details to make it an intelligible read.

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Chapter 1

Preliminaries

1.1 Formal Power Series

Defintion 1.1.1. Let $(a_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ be an arbitrary complex series. Then we call

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

a formal power series and denote the family of all such formal power series by $\mathcal{S}_0^+$. If we consider only series $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, starting with index one, we denote the resulting family of formal power series by $\mathcal{S}^+$. For $k \in \mathbb{N}$ the symbol $\mathcal{P}^k$ refers to the set of all polynomials with complex coefficients of degree less or equal than k, i.e.

$$\mathcal{P}^k := \{\sum_{n=0}^{\infty} a_n z^n : \, a_n = 0, \, n > k\}$$

and $\mathcal{P} := \bigcup_{k=0}^{\infty} \mathcal{P}^k$ the set of all Polynomials. Furthermore we denote the set of all Polynomials $p \in \mathcal{P}^k$ with $p(0) = 0$ by $\mathcal{P}_0^k$ and $\mathcal{P}_0 := \bigcup_{k=0}^{\infty} \mathcal{P}_0^k$.

The radius of convergence $R(f) \in [0, +\infty]$ of a formal power series f is defined as

$$R(f) := \begin{cases} \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}, & \limsup_{n\to\infty} |a_n|^{1/n} > 0 \\ +\infty, & \limsup_{n\to\infty} |a_n|^{1/n} = 0 \end{cases} \tag{1.2}$$

Remark 1.1.2.

(i) The term formal refers to the fact, that up to this point, we do not make any assumptions about convergence of the series or which values can be substituted for z. Technically until now every formal power series is nothing else, but the sequence of its coefficients.

(ii) Note that $R(f)$ is non-negative, but in general not positive. Consider for example the formal power series

$$f(z) := \sum_{n=0}^{\infty} n^n z^n.$$

Then, because of

$$\limsup_{n\to\infty} |n^n|^{1/n} = \lim_{n\to\infty} n = +\infty$$

the radius of convergence $R(f)$ is 0.

Provided with addition and scalar multiplication on $\mathcal{S}_0^+$ defined by

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n &:= \sum_{n=0}^{\infty} (a_n + b_n) z^n \\ \lambda \sum_{n=0}^{\infty} a_n z^n &:= \sum_{n=0}^{\infty} \lambda a_n z^n,\end{aligned} \tag{1.3}$$

for all $\sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=0}^{\infty} b_n z^n \in \mathcal{S}_0^+$ and $\lambda \in \mathbb{C}$, the space $\mathcal{S}_0^+$ is a complex vector space. The neutral element of the addition is the formal power series with only zero coefficients $\sum_{n=0}^{\infty} 0 \cdot z^n =: 0$.

The following theorem states some well-known facts about power series. Proofs can be found in any basic analysis book. Notation is mostly based on [RS02]

Theorem 1.1.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{S}_0^+$ be a formal power series. Then*

(i) $f(z)$ converges absolutely for $z \in B_{R(f)}(0)$ and is divergent for $|z| > R(f)$, where $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ denotes the complex open disc with center z_0 and radius r.

(ii) in case that $R(f) > 0$, the function $f : z \mapsto f(z)$ is holomorphic on $B_{R(f)}(0)$. Its derivative is given by $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$. Further $R(f') = R(f)$.

(iii) for a second power series $g(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{S}_0^+$ the product $(f \cdot g)(z)$ has the radius of convergence $R(f \cdot g) \geq \min\{R(f), R(g)\}$ and

$$(f \cdot g)(z) = \sum_{n=0}^{\infty} c_n z^n$$

with

$$c_n = \sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=n}} a_i b_j.$$

(iv) Every holomorphic function $f : D \to \mathbb{C}$ on an open set D with $0 \in D$ allows a unique power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R(f) \geq \sup\{r > 0 : B_r(0) \subseteq D\}$.

(v) $f(z)$ converges uniformly on $B_r(0)$ for any $r \in \mathbb{R}$ with $0 < r < R(f)$.

By Theorem 1.1.3 *(i)*, we can interpret every formal power series $f(z)$ with $R(f) > 0$ as a function

$$f : \begin{cases} B_{R(f)}(0) & \to & \mathbb{C} \\ z & \mapsto & f(z). \end{cases}$$

For $z \in B_{R(f)}(0) \cap B_{R(g)}(0)$ addition and scalar multiplication as defined in (1.3) coincide with the point-wise addition and scalar multiplication in the vector space of functions.

Defintion 1.1.4. We define the formal differential operator on $\mathcal{S}_0^+$ by

$$d : \begin{cases} \mathcal{S}_0^+ & \to & \mathcal{S}_0^+ \\ \sum_{n=0}^\infty a_n z^n & \mapsto & \sum_{n=0}^\infty (n+1) a_{n+1} z^n. \end{cases}$$

Theorem 1.1.3 *(ii)* shows that we can interpret every formal power series f, with $R(f) > 0$ as a holomorphic function $f : B_{R(f)}(0) \to \mathbb{C}$, where for $z \in B_{R(f)}(0)$

$$(df)(z) = f'(z).$$

Defintion 1.1.5. The product of two formal power series can be defined as

$$\left(\sum_{n=0}^\infty a_n z^n\right) \cdot \left(\sum_{n=0}^\infty b_n z^n\right) := \sum_{n=0}^\infty \left(\sum_{i=0}^n a_i b_{n-i}\right) z^n.$$

For $k \in \mathbb{N}$ we are able to introduce the k-th power of a formal power series by induction

$$\left(\sum_{n=0}^\infty b_n z^n\right)^1 := \sum_{n=0}^\infty b_n z^n$$

$$\left(\sum_{n=0}^\infty b_n z^n\right)^k := \left(\sum_{n=0}^\infty b_n z^n\right)^{k-1} \cdot \left(\sum_{n=0}^\infty b_n z^n\right).$$

We write

$$\left(\sum_{n=0}^\infty b_n z^n\right)^k = \sum_{n=0}^\infty b_n^{(k)} z^n \tag{1.4}$$

for the corresponding coefficients $b_n^{(k)}$. Further we define the zeroth power of a formal power series by

$$\left(\sum_{n=0}^\infty b_n z^n\right)^0 := 1.$$

By Theorem 1.1.3 *(iii)* we know that $R(b^k) \geq R(b)$. The coefficients $b_n^{(k)}$ can be computed explicitly in the following way:

Theorem 1.1.6. *For $k \in \mathbb{N}$, the coefficients in (1.4) satisfy*

$$b_n^{(k)} = \sum_{\substack{n_1+n_2+\ldots+n_k=n \\ (n_1,\ldots,n_k)\in\mathbb{N}_0^k}} b_{n_1} b_{n_2} \ldots b_{n_k}. \tag{1.5}$$

Proof. For $k = 1$ we have

$$\left(\sum_{n=0}^\infty b_n z^n\right)^1 = \sum_{n=0}^\infty b_n^{(1)} z^n = \sum_{n=0}^\infty b_n z^n.$$

The coefficients $b_n^{(1)} = b_n$ fulfill equation (1.5), since the sum only consists of the summand b_n.

Now assume that (1.5) holds for $k \in \mathbb{N}$. By Definition 1.1.5, we have

$$\begin{aligned}\left(\sum_{n=0}^{\infty} b_n z^n\right)^{k+1} &= \left(\sum_{n=0}^{\infty} b_n z^n\right)^{k} \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right) \\ &= \left(\sum_{n=0}^{\infty} b_n^{(k)} z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} b_i^{(k)} b_{n-i}\right) z^n.\end{aligned}$$

From

$$\begin{aligned}\sum_{i=0}^{n} b_i^{(k)} b_{n-i} &= \sum_{i=0}^{n} \left(\sum_{\substack{n_1+n_2+\ldots+n_k=i \\ (n_1,\ldots,n_k)\in\mathbb{N}_0^k}} b_{n_1} b_{n_2} \ldots b_{n_k} \right) b_{n-i} \\ &= \sum_{i=0}^{n} \sum_{\substack{n_1+n_2+\ldots+n_k+n-i=n \\ (n_1,\ldots,n_k,n-i)\in\mathbb{N}_0^{k+1}}} b_{n_1} b_{n_2} \ldots b_{n_k} b_{n-i} \\ &= \sum_{\substack{n_1+n_2+\ldots+n_{k+1}=n \\ (n_1,\ldots,n_{k+1})\in\mathbb{N}_0^{k+1}}} b_{n_1} b_{n_2} \ldots b_{n_{k+1}} = b_n^{(k+1)}\end{aligned}$$

we obtain

$$\left(\sum_{n=0}^{\infty} b_n z^n\right)^{k+1} = \sum_{n=0}^{\infty} b_n^{(k+1)} z^n,$$

which proves (1.5) by complete induction. □

Now we are able to define the composition of two formal power series.

Defintion 1.1.7. Let $b(z) = \sum_{n=0}^{\infty} b_n z^n$ be a formal power series. Then we define the composition operator C_b by

$$C_b : \begin{cases} \operatorname{dom} C_b & \rightarrow \quad \mathcal{S}_0^+ \\ \sum_{n=0}^{\infty} a_n z^n & \mapsto \quad \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k b_n^{(k)}\right) z^n, \end{cases}$$

where the domain of C_b is defined by

$$\operatorname{dom} C_b := \{\sum_{n=0}^{\infty} a_n z^n \in \mathcal{S}_0^+ : \sum_{k=0}^{\infty} a_k b_n^{(k)} \text{ converges, } n \in \mathbb{N}_0\}.$$

Remark 1.1.8.

(i) Let $b(z) = \sum_{n=0}^{\infty} b_n z^n$ be arbitrary and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} c_n z^n$ such that $f, g \in \operatorname{dom} C_b$. Then

$$\sum_{n=0}^{\infty} (a_k + \lambda c_k) b_n^{(k)} = \sum_{n=0}^{\infty} a_k b_n^{(k)} + \lambda \sum_{n=0}^{\infty} c_k b_n^{(k)}$$

exists for $\lambda \in \mathbb{C}$. Hence, $\operatorname{dom} C_b$ is a linear subspace of $\mathcal{S}_0^+$.

(ii) For $f(z) = z^k$, $k \in \mathbb{N}$,

$$(C_b f)(z) = \sum_{n=0}^{\infty} b_n^{(k)} z^n = b(z)^k$$

shows, that the composition of z^k with a formal power series $b(z)$ coincides with the k-th power of the formal power series $b(z)$, as defined in Definition 1.1.5.

The following example shows, that in general $\operatorname{dom} C_b \neq \mathcal{S}_0^+$:

Example 1.1.9. We define two formal power series by $f(z) := \sum_{n=0}^{\infty} z^n$, and $b(z) := 1 + z$. Then $b(z)^k = \sum_{n=0}^{k} \binom{k}{n} z^n$. But the sum $\sum_{k=0}^{\infty} \binom{k}{n}$ is divergent for all $n \in \mathbb{N}_0$. Hence, $f \notin \operatorname{dom} C_b$.

Nevertheless, in the special case, that $b(0) = 0$, the domain of C_b is the whole space, as the following theorem shows:

Theorem 1.1.10. *For $b(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=1}^{\infty} b_n z^n$ (i.e. $b_0 = 0$) we have $\operatorname{dom} C_b = \mathcal{S}_0^+$.*

Proof. Let $k \in \mathbb{N}$ $k > n$. Then for every tuple $(n_1, n_2, \ldots, n_k) \in \mathbb{N}_0^k$ with $n_1 + n_2 + \ldots + n_k = n$, there exists at least one $j \in \{1, 2, \ldots, k\}$ such that $n_j = 0$. It follows from Theorem 1.1.6, that

$$b_n^{(k)} = \sum_{\substack{n_1+n_2+\ldots+n_k=n \\ (n_1,n_2,\ldots,n_k)\in\mathbb{N}_0^k}} b_{n_1} b_{n_2} \ldots b_{n_k} = \sum_{\substack{n_1+n_2+\ldots+n_k=n \\ (n_1,n_2,\ldots,n_k)\in\mathbb{N}_0^k}} 0 = 0,$$

since $b_0 = 0$. Hence, for any arbitrary complex sequence $(a_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$, the sum $\sum_{k=0}^{\infty} a_k b_n^{(k)}$ has only a finite number of non-zero summands. Therefore, it converges and its sum coincides with $\sum_{k=0}^{n} a_k b_n^{(k)}$. This shows that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \operatorname{dom} C_b$ and we obtain $\operatorname{dom} C_b = \mathcal{S}_0^+$. □

Lemma 1.1.11. *Let $b(z) = \sum_{n=0}^{\infty} b_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \operatorname{dom} C_b$. Then $|b_0| \leq R(f)$.*

Proof. Since $f \in \operatorname{dom} C_b$, the series $\sum_{n=0}^{\infty} b_k^{(n)} a_n$ converges for every $k \in \mathbb{N}_0$ by definition of $\operatorname{dom} C_b$. In particular, this is true for $k = 0$. Because of $b_0^{(n)} = b_0^n$, we find that

$$\sum_{n=0}^{\infty} b_0^{(n)} a_n = \sum_{n=0}^{\infty} b_0^n a_n$$

converges. By Theorem 1.1.3, *(i)* this can only be the case if $|b_0| \leq R(f)$. □

Theorem 1.1.12. *An arbitrary complex series $\sum_{n,k=0}^{\infty} a_{n,k}$ converges absolutely if and only if $\sum_{k=0}^{\infty} |a_{n,k}| < +\infty$, for all $n \in \mathbb{N}_0$ and $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{n,k}| < +\infty$. In this case*

$$\sum_{n,k=0}^{\infty} a_{n,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n,k}.$$

Proof. A proof can be found in [Rud70]. □

Lemma 1.1.13. *Let $b(z) = \sum_{n=0}^{\infty} b_n z^n$, $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{S}_0^+$, such that $R(b) > 0$ and $|b_0| < R(f)$. Then $f \in \operatorname{dom} C_b$ and there exists a real number $\delta > 0$, with $\delta \leq R(C_b f)$, such that $(C_b f)(z) = f(b(z))$ for all $z \in B_\delta(0)$.*

In this case for all real constants $r > 0$ such that $r \leq R(b)$ and $b(B_r(0)) \subseteq B_{R(f)}(0)$, we have $R(C_b f) \geq r$ and $(C_b f)(z) = f(b(z))$ for all $z \in B_r(0)$.

Proof. We define a formal power series by

$$c(z) := \sum_{n=0}^{\infty} c_n z^n := \sum_{n=0}^{\infty} |b_n| z^n.$$

By Theorem 1.1.6, we have

$$\begin{aligned} |b_n^{(k)}| &= \left| \sum_{\substack{n_1+\ldots+n_k=n \\ (n_1,n_2,\ldots,n_k)\in\mathbb{N}^k}} b_{n_1} \cdot \ldots \cdot b_{n_k} \right| \\ &\leq \sum_{\substack{n_1+\ldots+n_k=n \\ (n_1,n_2,\ldots,n_k)\in\mathbb{N}^k}} |b_{n_1}| \cdot \ldots \cdot |b_{n_k}| \\ &= \sum_{\substack{n_1+\ldots+n_k=n \\ (n_1,n_2,\ldots,n_k)\in\mathbb{N}^k}} c_{n_1} \cdot \ldots \cdot c_{n_k} = c_n^{(k)}, \end{aligned}$$

for $k \in \mathbb{N}$.

If we recall the definition of the radius of convergence (1.2), it is clear, that $R(c) = R(b) > 0$. Since $c(z)$ is continuous on $B_{R(b)}(0)$ and $c(0) = |b_0| < R(f)$ by assumption, there exists a $\delta \in \mathbb{R}$, $0 < \delta < R(b)$ such that $|c(z)| < R(f)$ for all $z \in B_\delta(0)$.

Using this, we can calculate for $z_0 \in B_\delta(0)$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|a_n b_k^{(n)} z_0^k\right| = \sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{\infty}|b_k^{(n)}||z_0|^n \leq$$

$$\leq \sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{\infty}c_k^{(n)}|z_0|^n = \sum_{n=0}^{\infty}|a_n|c(|z_0|)^n < +\infty.$$

since $c(|z_0|) = |c(|z_0|)| < R(f)$.

Furthermore, since $|z_0| < R(b)$, the series $b\,(z_0)^n = \sum_{k=0}^{\infty} b_k^{(n)} z_0^k$ converges absolutely for every $n \in \mathbb{N}$. So we can apply Theorem 1.1.12 to the series $\sum_{n,k=0}^{\infty} a_n b_k^{(n)} z_0^k$ and obtain that $\sum_{n=0}^{\infty} a_n b_k^{(n)}$ converges for all $k \in \mathbb{N}$. Thus, $f \in \operatorname{dom} C_b$. Moreover,

$$f(b(z_0)) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} b_k^{(n)} z_0^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_n b_k^{(n)} z_0^k = (C_b f)(z_0).$$

Now let $r > 0$ be such that $b(B_r(0)) \subseteq B_{R(f)}(0)$. Then $f(b(z))$ is a holomorphic function on $B_r(0)$, since it is the composition of two holomorphic functions. Hence, by Theorem 1.1.3, *(iv)*, it has a unique power series expansion $f(b(z)) = \sum_{n=0}^{\infty} d_n z^n$. The radius of convergence of $\sum_{n=0}^{\infty} d_n z^n$ is at least r.

Let δ be as above. Since $C_b f$ is holomorphic on $B_\delta(0)$, it has a unique power series expansion $\sum_{n=0}^{\infty} e_n z^n$. We conclude that

$$\sum_{n=0}^{\infty} d_n z^n = f(b(z)) = (C_b f)(z) = \sum_{n=0}^{\infty} e_n z^n$$

for all $z \in B_\delta(0)$. Since the power series expansion is unique, $d_n = e_n$ follows for all $n \in \mathbb{N}_0$. Hence, $C_b f$ has a radius of convergence of, at least, r and $(C_b f)(z) = f(b(z))$ for all $z \in B_r(0)$. □

Example 1.1.14. Let $b(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{S}^+$ be an arbitrary power series with $R(b) > 0$ and

$$B_\mu(z) = \sum_{n=0}^{\infty}\binom{\mu}{n} z^n \tag{1.6}$$

denote the Binomial series, for an arbitrary complex number μ. The binomial coefficients in (1.6) are defined by

$$\binom{\mu}{n} := \begin{cases} \frac{\mu(\mu-1)(\mu-2)\cdots(\mu-n+1)}{n!}, & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}$$

It is well known, that

$$R(B_\mu) = \begin{cases} +\infty, & \mu \in \mathbb{N}_0 \\ 1, & \mu \in \mathbb{C}\setminus\mathbb{N}_0 \end{cases}$$

and $B_\mu(z) = (1+z)^\mu$ for all $z \in B_{R(B_\mu)}(0)$. Since $b(0) = 0$, we know by Theorem 1.1.10, that $\operatorname{dom} C_b = \mathcal{S}_0^+$ and hence $B_\mu(z) \in \operatorname{dom} C_b$.

Now let $r > 0$ be such that $|b\,(z)\,| < 1$ for all $z \in B_r(0)$. Then due to Lemma 1.1.13 we have $R(C_b B_\mu) \geq r$ and

$$(C_b B_\mu)\,(z) = B_\mu\,(b(z)) = (1 + b(z))^\mu\,,$$

for all $z \in B_r(0)$.

This motivates the following definition.

Defintion 1.1.15. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{S}_0^+$ be an arbitrary power series with $a_0 \neq 0$ and $\mu \in \mathbb{C}$. Then we define the μ-th power of $f(z)$ by

$$f(z)^\mu := a_0^\mu (C_b B_\mu)(z)$$

where

$$b(z) := \sum_{n=1}^{\infty} \frac{a_n}{a_0} z^n.$$

Remark 1.1.16. Let f, b and μ be as before with $R(f) > 0$. Then because of $R(b) = R(f) > 0$ and $b(0) = 0$ Lemma 1.1.13 asserts, that $R(f^\mu) > 0$.

1.2 Spaces with indefinite inner product

Defintion 1.2.1. Let $\mathcal{X}$ be a complex vector space, and $[\cdot,\cdot] : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ a hermitian mapping (i.e. $[x + \lambda y, z] = [x, z] + \lambda[y, z]$ and $[x, y] = \overline{[y, x]}$ for $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$). Then we call $[\cdot,\cdot]$ an inner product and the pair $(\mathcal{X}, [\cdot,\cdot])$ an inner product space.

Remark 1.2.2. Note, that for $x \in \mathcal{X}$ from $[x, x] = \overline{[x, x]}$ follows that $[x, x] \in \mathbb{R}$.

Defintion 1.2.3. An element $x \in \mathcal{X}$ is called

$$\begin{array}{rcl} \text{positive} & \Leftrightarrow & [x,x] > 0 \\ \text{negative} & \Leftrightarrow & [x,x] < 0 \\ \text{neutral} & \Leftrightarrow & [x,x] = 0 \\ \text{isotropic} & \Leftrightarrow & [x,y] = 0,\ y \in \mathcal{X}. \end{array}$$

We denote the set of all isotropic elements by $\mathcal{X}^\circ$ and call it the isotropic part of $(\mathcal{X}, [\cdot,\cdot])$.

An inner product space is called positive (negative) definite if all elements except for the zero vector are positive (negative) elements. It is called positive (negative) semidefinite if it has no negative (positive) elements. Otherwise it is called indefinite.

$\mathcal{X}$ is called degenerated if $\mathcal{X}^\circ \neq \{0\}$. Otherwise it is called non-degenerated.

Remark 1.2.4. If $\mathcal{Y} \leq \mathcal{X}$ is a linear subspace of an inner product space $(\mathcal{X}, [\cdot,\cdot])$, then $(\mathcal{Y}, [\cdot,\cdot])$ is an inner product space itself. We call the subspace Y positive/negative (semi)definite if $(\mathcal{Y}, [\cdot,\cdot])$ is positive/negative (semi)definite.

Lemma 1.2.5. *The isotropic part $\mathcal{X}^\circ$ of an inner product space $(\mathcal{X}, [\cdot,\cdot])$ is a linear subspace of $\mathcal{X}$. Every element of $\mathcal{X}^\circ$ is neutral.*

Proof. For $x, y \in \mathcal{X}^\circ$ and arbitrary $z \in \mathcal{X}$, $\lambda \in \mathbb{C}$ we have

$$[x + \lambda y, z] = [x, z] + \lambda[y, z] = 0.$$

Hence $\mathcal{X}^\circ$ is a linear subspace. The second assertion is clear. □

Example 1.2.6. Every Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a positive definite inner product space.

Remark 1.2.7. Since $\mathcal{X}^\circ$ is a linear subspace of $\mathcal{X}$, the factor space $\mathcal{X}/_{\mathcal{X}^\circ}$ is again a vector space. If we endow $\mathcal{X}/_{\mathcal{X}^\circ}$ with the inner product

$$[x + \mathcal{X}^\circ, y + \mathcal{X}^\circ]_{/\mathcal{X}^\circ} := [x, y] \tag{1.7}$$

the space $(\mathcal{X}/_{\mathcal{X}^\circ}, [\cdot,\cdot]_{/\mathcal{X}^\circ})$ is again an inner product space. Note, that $[\cdot,\cdot]_{/\mathcal{X}^\circ}$ is hermitian, since $[\cdot,\cdot]$ is hermitian. It is well-defined, since for $x_1, x_2, y_1, y_2 \in \mathcal{X}$ such that $x_1 - x_2$, $y_1 - y_2 \in \mathcal{X}^\circ$, we have $[x_2 - x_1, y_1] = 0$ and there follows

$$\begin{aligned}[x_1 + \mathcal{X}^\circ, y_1 + \mathcal{X}^\circ]_{/\mathcal{X}^\circ} - [x_2 + \mathcal{X}^\circ, y_2 + \mathcal{X}^\circ]_{/\mathcal{X}^\circ} &= \\ = [x_1, y_1] - [x_2, y_2] = [x_1, y_1] - [x_2, y_2] - [x_1 - x_2, y_1] &= \\ = [x_2, y_1 - y_2] &= 0.\end{aligned}$$

Let $x + \mathcal{X}^\circ \in \mathcal{X}/_{\mathcal{X}^\circ}$ such that $[x + \mathcal{X}^\circ, y + \mathcal{X}^\circ]_{/\mathcal{X}^\circ} = 0$ for all $y + \mathcal{X}^\circ \in \mathcal{X}/_{\mathcal{X}^\circ}$. Then

$$0 = [x + \mathcal{X}^\circ, y + \mathcal{X}^\circ]_{/\mathcal{X}^\circ} = [x, y]$$

for all $y \in \mathcal{X}$ shows, that $x \in \mathcal{X}^\circ$ and hence $(\mathcal{X}/_{\mathcal{X}^\circ})^\circ = 0$. Thus $\mathcal{X}/_{\mathcal{X}^\circ}$ is non-degenerated.

Example 1.2.8. Let $(\mathcal{X}_1, [\cdot,\cdot]_1)$, $(\mathcal{X}_2, [\cdot,\cdot]_2)$ be positive semidefinite inner product spaces. Then we define $(\mathcal{X}, [\cdot,\cdot])$ by $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$ and

$$[(x_1, x_2), (y_1, y_2)] := [x_1, y_1]_1 - [x_2, y_2]_2 \,.$$

Since $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ are inner products, $[\cdot,\cdot]$ is an inner product as well. Let ι_1, ι_2 denote the embeddings of $\mathcal{X}_1$, $\mathcal{X}_2$ into $\mathcal{X}$. Then, since $\mathcal{X}_1$, $\mathcal{X}_2$ are positive semidefinite,

$$[\iota_1(x_1), \iota_1(x_1)] = [x_1, x_1]_1 - [0, 0]_2 = [x_1, x_1]_1 \geq 0$$

and

$$[\iota_2(x_2), \iota_2(x_2)] = [0, 0]_1 - [x_2, x_2]_2 = -[x_2, x_2]_2 \leq 0$$

for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$. Hence $\iota_1(\mathcal{X}_1)$ is a positive semidefinite and $\iota_2(\mathcal{X}_2)$ a negative semidefinite subspace of $\mathcal{X}$.

If $(x_1, x_2) \in \mathcal{X}$ is isotropic, we have

$$0 = [(x_1, x_2), (y_1, y_2)] = [x_1, y_1]_1 - [x_2, y_2]_2 \tag{1.8}$$

for all $(y_1, y_2) \in \mathcal{X}$. Hence $x_1 \in \mathcal{X}_1^\circ$ and $x_2 \in \mathcal{X}_2^\circ$. If, on the other hand $x_1 \in \mathcal{X}_1^\circ$ and $x_2 \in \mathcal{X}_2^\circ$, equation (1.8) holds for all $(y_1, y_2) \in \mathcal{H}$. Thus,

$$\mathcal{X}^\circ = \mathcal{X}_1^\circ \times \mathcal{X}_2^\circ.$$

Defintion 1.2.9. We call two elements $x, y \in \mathcal{X}$ of an inner product space $(\mathcal{X}, [\cdot,\cdot])$ orthogonal if $[x,y] = 0$ and denote this by $x[\perp]y$. Two subspaces $\mathcal{X}_1, \mathcal{X}_2 \leq \mathcal{X}$ are called orthogonal if $x_1[\perp]x_2$ for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, and we denote this fact by $\mathcal{X}_1[\perp]\mathcal{X}_2$.

The orthogonal complement $M^{[\perp]}$ of a set $M \subseteq \mathcal{X}$ is defined as

$$M^{[\perp]} := \{x \in \mathcal{X} : x[\perp]m,\ m \in M\}.$$

A linear operator $P : \mathcal{X} \to \mathcal{X}$ is called a projection if it is idempotent (i.e. $P^2 = P$). A projection is called orthogonal (with respect to $[\cdot,\cdot]$) if $\operatorname{ran} P[\perp]\ker P$.

Lemma 1.2.10. *Let $(\mathcal{X}, [\cdot,\cdot])$ be an inner product space and $\mathcal{X}_1, \mathcal{X}_2 \leq \mathcal{X}$ subspaces of $\mathcal{X}$, such that*

$$\mathcal{X}_1 \dot{+} \mathcal{X}_2 = \mathcal{X} \tag{1.9}$$

(the symbol $\dot{+}$ denotes a direct sum i.e. $\mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}$ and $\mathcal{X}_1 \cap \mathcal{X}_2 = \{0\}$). Then there exist unique linear projection operators $P_i : \mathcal{X} \to \mathcal{X}_i$, $i = 1, 2$, with $P_i\mathcal{X}_i = \mathcal{X}_i$ and $\ker P_i = \mathcal{X}_{1-i}$

If the spaces are such that

$$\mathcal{X}_1[\dot{+}]\mathcal{X}_2 = \mathcal{X} \tag{1.10}$$

(i.e. $\mathcal{X}_1 \dot{+} \mathcal{X}_2 = \mathcal{X}$ and $\mathcal{X}_1[\perp]\mathcal{X}_2$), the projectors P_i are orthogonal.

Proof. For $x \in \mathcal{X}$ exist $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, such that $x = x_1 + x_2$. This decomposition is unique: For $y_1 \in \mathcal{X}_1$, $y_2 \in \mathcal{X}_2$ with $y_1 + y_2 = x$, we have

$$0 = x_1 - y_1 + x_2 - y_2.$$

Since $x_1 - y_1 \in \mathcal{X}_1$ and $x_2 - y_2 \in \mathcal{X}_2$ and $\mathcal{X}_1 \cap \mathcal{X}_2 = \{0\}$,

$$x_1 - y_1 = x_2 - y_2 = 0.$$

Now we define $P_i x := x_i$ for $i = 1, 2$. Because of

$$P_i(x_1 + x_2 + \lambda(y_1 + y_2)) = x_i + \lambda y_i = P_i(x_1 + x_2) + \lambda P_i(y_1 + y_2)$$

for $x_i, y_i \in \mathcal{X}_i$, $\lambda \in \mathbb{C}$ they are linear, and since $P_i^2 x = P_i x_i = x_i = P_i x$, they are projections. The kernel of P_1 is

$$\ker\ P_1 = \{x \in \mathcal{X} : P_1 x = 0\} = \mathcal{X}_2$$

and vice versa.

Let $Q \neq 0$ be linear a projection on $\mathcal{X}_i$. Then

$$P_i(x_1 + x_2) - Q(x_1 + x_2) = x_i - x_i = 0$$

shows that $Q = P_i$. Hence, if $\mathcal{X}_1[\perp]\mathcal{X}_2$, the projectors are orthogonal. □

Defintion 1.2.11. Let $(\mathcal{X}, [\cdot,\cdot])$ be an inner product space and $\mathcal{X}_+$, $\mathcal{X}_-$ subspaces of $\mathcal{X}$ such that

$$\mathcal{X} = \mathcal{X}_+[\dot{+}]\mathcal{X}_-[\dot{+}]X_\circ.$$

Then we call the space $\mathcal{X}$ decomposable and the pair $(\mathcal{X}_+, \mathcal{X}_-)$ a fundamental decomposition of $\mathcal{X}$. The linear operator

$$J : \begin{cases} \mathcal{X} & \to & \mathcal{X} \\ x & \mapsto & P_+x - P_-x, \end{cases}$$

where $P_\pm$ denote the projections onto $\mathcal{X}_\pm$, is called fundamental symmetry or metric operator. If we want to emphasize, that a fundamental symmetry J belongs to the fundamental decomposition $(\mathcal{X}_+, \mathcal{X}_-)$, we will also call the triplet $(\mathcal{X}_+, \mathcal{X}_-, J)$ a fundamental decomposition of $\mathcal{X}$.

Lemma 1.2.12. *For a fundamental decomposition $(\mathcal{X}_+, \mathcal{X}_-, J)$ of an indefinite inner product space $(\mathcal{H}, [\cdot,\cdot])$,*

(i) $[x, y] = [P_+x, y] + [P_-x, y]$

(ii) $[P_\pm x, y] = [x, P_\pm y] = [P_\pm x, P_\pm y]$

(iii) $[Jx, y] = [x, Jy]$

(iv) $[x, y] = [Jx, Jy] = [JJx, y]$

hold for $x, y \in \mathcal{X}$

Proof. (i): Let $x_\circ \in \mathcal{X}^\circ$ such that $x = (P_+ + P_-)x + x_\circ$. Then

$$[x, y] = [P_+x, y] + [P_-x, y] + [x_\circ, y] = [P_+x, y] + [P_-x, y].$$

(ii): Follows from $\mathcal{X}_+ \perp \mathcal{X}_-$.
(iii):

$$[Jx, y] = [P_+x, y] - [P_-x, y] = [x, P_+y] - [x, P_-y] = [x, Jy]$$

(iv):

$$\begin{aligned}[Jx, Jy] = [P_+x, P_+y] - [P_+x, P_-y] - [P_-x, P_+y] + [P_-x, P_-y] = \\ = [P_+x, P_+y] + [P_-x, P_-y] = [P_+x, y] + [P_-x, y] = [x, y].\end{aligned}$$

The second equality follows from *(iii)*. □

Example 1.2.13. Recall the indefinite inner product space from Example 1.2.8 and assume this time, that $\mathcal{X}_1, \mathcal{X}_2$ are positive definite. Then, since $\mathcal{X}^\circ = \{0\}$ and

$$[\iota_1(x), \iota_2(y)] = [(x, 0), (0, y)] = [x, 0]_1 - [0, y]_2 = 0,$$

$(\iota_1(\mathcal{X}_1), \iota_2(\mathcal{X}_2))$ is a fundamental decomposition. The action of J is given by $J(x_1, x_2) = (x_1, -x_2)$.

Defintion 1.2.14. Let $(\mathcal{X}, [\cdot,\cdot])$ be an inner product space and $(\mathcal{X}_+, \mathcal{X}_-, J)$ a fundamental decomposition. Then we define a mapping by

$$[\cdot,\cdot]_J : \begin{cases} \mathcal{X} \times \mathcal{X} & \to & \mathbb{C} \\ (x, y) & \mapsto & [Jx, y] \end{cases}$$

and call it the inner product induced by the fundamental decomposition $(\mathcal{X}_+, \mathcal{X}_-, J)$.

Lemma 1.2.15. *Let $(\mathcal{X}, [\cdot,\cdot])$ and $(\mathcal{X}_+, \mathcal{X}_-, J)$, be again an inner product space and an arbitrary fundamental decomposition. Then $[\cdot,\cdot]_J$ is a positive semidefinite inner product. Moreover*

$$(\mathcal{X}, [\cdot,\cdot])^\circ = (\mathcal{X}, [\cdot,\cdot]_J)^\circ$$

and $(\mathcal{X}, [\cdot,\cdot]_J)$ is positive definite if and only if $(\mathcal{X}, [\cdot,\cdot])$ is non degenerated.

Proof. Since J is linear, $[\cdot,\cdot]_J$ is linear in the first argument and because of

$$[x, y]_J = [Jx, y] = [x, Jy] = \overline{[Jy, x]} = \overline{[y, x]_J}$$

by Lemma 1.2.12, (*iii*), for all $x, y \in \mathcal{X}$, it is hermitian. It is also positive semidefinite, since for $x \in \mathcal{X}$

$$[x, x]_J = [Jx, x] = [P_+x, x] - [P_-x, x] = [P_+x, P_+x] - [P_-x, P_-x] \geq 0.$$

Now let $x \in (\mathcal{X}, [\cdot,\cdot])^\circ$. Then by Lemma 1.2.12, (*iii*),

$$[x, y]_J = [Jx, y] = [x, Jy] = 0$$

for all $y \in \mathcal{X}$ and hence $(\mathcal{X}, [\cdot,\cdot])^\circ \subseteq (\mathcal{X}, [\cdot,\cdot]_J)^\circ$.

For $x \in (\mathcal{X}, [\cdot,\cdot]_J)^\circ$ we have

$$[x, y] = [Jx, Jy] = [x, Jy]_J = 0$$

for all $y \in \mathcal{X}$ due to Lemma 1.2.12, (*iv*), and thus $(\mathcal{X}, [\cdot,\cdot])^\circ \supseteq (\mathcal{X}, [\cdot,\cdot]_J)^\circ$. The last assertion is clear since the isotropic parts coincide. □

Remark 1.2.16. Since the inner product $[\cdot,\cdot]_J$ is positive semidefinite, it induces a semi-norm $\|\cdot\|_J$ on $\mathcal{X}$ by

$$\|x\|_J^2 := [x, x]_J$$

for all $x \in \mathcal{X}$. By Lemma 1.2.15 $\|\cdot\|_J$ is a norm if and only if $(X, [\cdot,\cdot])$ is non-degenerated.

Defintion 1.2.17. Let $(\mathcal{K}, [\cdot,\cdot])$ be a decomposable, non-degenerated inner product space and assume that there exists a fundamental symmetry J such that $(\mathcal{K}, [\cdot,\cdot]_J)$ is complete. Then we call $(\mathcal{K}, [\cdot,\cdot])$ a Krein space.

Example 1.2.18. Consider again the inner product space from Examples 1.2.8 and 1.2.13. This time assume, that $\mathcal{X} = \mathcal{H}_1 \times \mathcal{H}_2 = \iota_1(\mathcal{H}_1)[\dot{+}]\iota_2(\mathcal{H}_2)$, where $(\mathcal{H}_1, \langle\cdot,\cdot\rangle_1)$, $(\mathcal{H}_2, \langle\cdot,\cdot\rangle_2)$ are Hilbert spaces. Now let $((x_n, y_n))_{n\in\mathbb{N}}$ be a Cauchy series in $\mathcal{X}$. Then because of

$$\begin{aligned}\|(x_n, y_n) - (x_m, y_m)\|_J^2 &= [(x_n - x_m, y_n - y_m), (x_n - x_m, y_n - y_m)]_J \\ &= [x_n - x_m, x_n - x_m]_1 + [y_n - y_m, y_n - y_m]_2 \\ &= \|x_n - x_m\|_1^2 + \|y_n - y_m\|_2^2,\end{aligned}$$

the series $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are Cauchy in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are complete, there exist $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ such that $\lim_{n\to\infty} \|x_n - x\|_1 = 0$ and $\lim_{n\to\infty} \|y_n - y\|_2 = 0$. Thus,

$$\lim_{n\to\infty} \|(x_n, y_n) - (x, y)\|_J^2 = \lim_{n\to\infty} \|x_n - x\|_1^2 + \lim_{n\to\infty} \|y_n - y\|_1^2 = 0.$$

Hence, $(\mathcal{X}, [\cdot,\cdot]_J)$ is complete and $(\mathcal{X}, [\cdot,\cdot])$ is a Krein space.

Lemma 1.2.19. *Let $(\mathcal{X}, [\cdot,\cdot])$ be a semidefinite inner product space. Then*

(i) $\mathcal{X}^\circ = \{x \in \mathcal{X} : x \text{ is neutral}\}$

(ii) for $x, y \in \mathcal{X}$ the Cauchy-Schwartz inequality

$$|[x, y]| \leq \sqrt{[x, x][y, y]} \tag{1.11}$$

holds.

Proof. If $(\mathcal{X}, [\cdot,\cdot])$ is negative semidefinite, we just consider the positive semidefinite inner product space $(\mathcal{X}, -[\cdot,\cdot])$. So we can assume without loss of generality, that $(\mathcal{X}, [\cdot,\cdot])$ is positive semidefinite.

Let $x, y \in \mathcal{X}$ be arbitrary such that $[x, y] \neq 0$. For $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}0 &\leq \left[x - \lambda y \frac{[x, y]}{|[x, y]|}, x - \lambda y \frac{[x, y]}{|[x, y]|}\right] \\ &= [x, x] - \lambda \frac{[x, y]}{|[x, y]|}[y, x] - \lambda \frac{[y, x]}{|[x, y]|}[x, y] + \lambda^2 [y, y] \\ &= [x, x] - 2\lambda |[x, y]| + \lambda^2 [y, y].\end{aligned} \tag{1.12}$$

We already know that $\mathcal{X}^\circ \subseteq \{x \in \mathcal{X} : x \text{ is neutral}\}$. Assume that y is neutral and $[x, y] \neq 0$ for some $x \in \mathcal{X}$. Then, because of $[y, y] = 0$, choosing $\lambda > \frac{[x,x]}{2|[x,y]|}$ in inequality (1.12) leads to

$$0 \leq [x, x] - 2\lambda |[x, y]| < [x, x] - [x, x] = 0.$$

Hence, $[x, y] = 0$ for all $x \in \mathcal{X}$ and therefore $y \in \mathcal{X}^\circ$, which shows (i).

If x or y are isotropic, inequality (1.11) is fulfilled. For $x, y \notin \mathcal{X}^\circ$, setting $\lambda = \frac{|[x,y]|}{[y,y]}$ in inequality (1.12) asserts

$$0 \leq [x, x] - \frac{|[x, y]|^2}{[y, y]},$$

which is equivalent to (1.11). □

Lemma 1.2.20. *Let $(\mathcal{K}, [\cdot,\cdot])$ be an inner product space and $(\mathcal{K}_+, \mathcal{K}_-, J)$ some fundamental decomposition. Then for $M \subseteq \mathcal{K}$, the set $M^{[\perp]}$ is a linear subspace of $\mathcal{K}$ and $M^{[\perp]}$ is closed with respect to $\|\cdot\|_J$.*

Proof. For $x \in M$, we define a linear mapping by

$$f_x : \begin{cases} \mathcal{K} & \to & \mathbb{C} \\ y & \mapsto & [x, y]. \end{cases}$$

Then we can write $M^{[\perp]}$ as

$$M^{[\perp]} = \bigcap_{x \in M} \ker f_x.$$

Because of

$$|f_x(y)|^2 = |[x, y]|^2 = |[Jx, y]_J|^2 \leq \|Jx\|_J \|y\|_J$$

for all $x, y \in \mathcal{K}$, the mappings f_x are continuous with respect to $\|\cdot\|_J$ and each $\ker f_x$ is a closed subspace of $(\mathcal{K}, [\cdot,\cdot]_J)$. Hence, $M^{[\perp]}$ is a closed subspace as well. □

Remark 1.2.21. The proof in Lemma 1.2.20 works also for for an arbitrary inner product space $(\mathcal{X}, [\cdot,\cdot])$ and a norm $\|\cdot\|$ on $\mathcal{X}$, such that f_x is continuous for all $x \in \mathcal{X}$.

Since we will use it in the proof of the following lemma, we present a formulation of the closed graph theorem (without proof).

Theorem 1.2.22 (Closed graph theorem)**.** *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a linear operator. Then T is continuous if and only if the graph of T (i.e. the set $\{(x, y) \in \mathcal{X} \times \mathcal{Y} : y = Tx\}$) is closed in $\mathcal{X} \times \mathcal{Y}$, endowed with the product topology.*

Proof. Can be found in [Rud70]. □

Lemma 1.2.23. *Let $\mathcal{K}$ be a Krein space and $(\mathcal{K}_+, \mathcal{K}_-, J)$ be the fundamental decomposition such that $(\mathcal{K}, [\cdot,\cdot]_J)$ is complete. Further, let $(\mathcal{K}'_+, \mathcal{K}'_-, J')$ be an arbitrary fundamental decomposition. Then there exists a real constant $C > 0$, such that*

$$\|JJ'x\|_J \leq C\|x\|_J \tag{1.13}$$

and

$$\|J'Jx\|_{J'} \leq C\|x\|_{J'} \tag{1.14}$$

for all $x \in \mathcal{X}$

Proof. To prove inequality (1.13), we first note that, because of

$$\|Jx\|_J^2 = [JJx, Jx] = [Jx, x] = [x, x]_J = \|x\|_J^2,$$

the operator J is an isometry with respect to $\|\cdot\|_J$ and therefore continuous. Now let $(x_n)_{n\in\mathbb{N}}$ be a series in $\mathcal{K}$ such that $\lim_{n\to\infty} \|x_n - x\|_J = 0$ and $\lim_{n\to\infty} \|P'_+x_n - y\|_J = 0$

for some $x, y \in \mathcal{K}$. Then, since by Lemma 1.2.20 $\mathcal{K}'_+ = \left(\mathcal{K}'_-\right)^{[\perp]}$ is closed, we have $y \in \mathcal{K}'_+$. Moreover, because $P'_+ x_n - x_n \in \mathcal{K}'_-$, and $\mathcal{K}'_- = \left(\mathcal{K}'_+\right)^{[\perp]}$ is closed as well, we obtain

$$\lim_{n\to\infty} P'_+ x_n - x_n = y - x \in \mathcal{K}'_-.$$

Now

$$0 = P'_+(y - x) = y - P'_+ x$$

shows, that $P'_+ x = y$. Thus P'_+ is closed with respect to $\|\cdot\|_J$. By the closed graph theorem, it is continuous. It can be shown analogously, that P'_- is continuous with respect to $\|\cdot\|_J$ as well. Since $JJ' = J(P'_+ - P'_-)$ is the composition of continuous mappings, it is continuous itself. Hence, there exists a real constant $C > 0$, such that (1.13) holds.

To prove inequality (1.14), we first show by complete induction, that

$$\left\|J'Jx\right\|_{J'}^{2^n} \leq \left\|(J'J)^{2^n} x\right\|_{J'} \|x\|_{J'}^{2^n-1} \tag{1.15}$$

holds for every $n \in \mathbb{N}$. By the Cauchy-Schwartz inequality and Lemma 1.2.12, we find

$$\begin{aligned}
\left\|J'Jx\right\|_{J'}^2 &= \left[J'Jx, J'Jx\right]_{J'} \\
&= \left[J'Jx, Jx\right] \\
&= \left[J'J'JJ'Jx, x\right] \\
&= \left[(J'J)^2 x, x\right]_{J'}^2 \leq \left\|(J'J)^2 x\right\|_{J'} \|x\|_{J'},
\end{aligned}$$

for all $x \in \mathcal{K}$. But this is nothing else, but inequality (1.15) with $n = 1$.

Now assume, that (1.15) holds for some $n \in \mathbb{N}$. Then for $x \in \mathcal{K}$

$$\begin{aligned}
\left\|J'Jx\right\|_{J'}^{2^{n+1}} &= \left(\left\|J'Jx\right\|_{J'}^{2^n}\right)^2 \\
&\leq \left(\left\|(J'J)^{2^n} x\right\| \|x\|_{J'}^{2^n-1}\right)^2 \\
&= \left\|(J'J)^{2^n} x\right\|^2 \|x\|_{J'}^{2^{n+1}-2} \\
&= \left[(J'J)^{2^n} x, (J'J)^{2^n} x\right]_{J'} \|x\|_{J'}^{2^{n+1}-2} \\
&= \left[\left(JJ'\right)^{2^n} J' \left(J'J\right)^{2^n} x, x\right] \|x\|_{J'}^{2^{n+1}-2} \\
&= \left[(J'J)^{2^{n+1}} x, x\right]_{J'} \|x\|_{J'}^{2^{n+1}-2} \\
&\leq \left\|(J'J)^{2^{n+1}} x\right\|_{J'} \|x\|_{J'} \|x\|_{J'}^{2^{n+1}-2} = \left\|(J'J)^{2^{n+1}} x\right\|_{J'} \|x\|_{J'}^{2^{n+1}-1},
\end{aligned}$$

shows that (1.15) holds for all $n \in \mathbb{N}$. From the previously proven inequality (1.13), we derive

$$\begin{aligned}
\left\|(J'J)^{2^n} x\right\|_{J'}^2 &= \left[(J'J)^{2^n} x, (J'J)^{2^n} x\right]_{J'} \\
&= [(J'J)^{2^n} x, J(J'J)^{2^n-1} x] \\
&= [J(J'J)^{2^n} x, J(J'J)^{2^n-1} x]_J \\
&\leq \|(JJ')^{2^n} Jx\|_J \|(JJ')^{2^n-1} Jx\|_J \\
&\leq C^{2^n} \|Jx\|_J C^{2^n-1} \|Jx\|_J = C^{2^{n+1}-1} \|x\|_J^2.
\end{aligned}$$

Combining this with (1.15) yields

$$\left\|J'Jx\right\|_{J'} \leq \left(\left\|(J'J)^{2^n}x\right\| \|x\|_{J'}^{2^n-1}\right)^{2^{-n}} \leq$$
$$\leq \left(C^{2^n-\frac{1}{2}}\|x\|_J \|x\|_{J'}^{2^n-1}\right)^{2^{-n}} = C^{1-2^{-n-1}}\|x\|_J^{2^{-n}}\|x\|_{J'}^{1-2^{-n}}$$

for all $n \in \mathbb{N}$. For $n \to \infty$ on the right hand side we obtain inequality (1.14). □

Theorem 1.2.24. *Let $\mathcal{K}$ be a Krein space. Then two norms induced by fundamental symmetries are equivalent.*

Proof. Let $(\mathcal{K}_+, \mathcal{K}_-, J)$ be the fundamental decomposition such that $(\mathcal{K}, [\cdot,\cdot]_J)$ is complete and $(\mathcal{K}'_+, \mathcal{K}'_-, J')$ an arbitrary fundamental decomposition. Then it is sufficient to show that $\|\cdot\|_J$ and $\|\cdot\|_{J'}$ are equivalent, i.e. there exist real constants $\lambda_1, \lambda_2 > 0$, such that

$$\lambda_1\|x\|_{J'} \leq \|x\|_J \leq \lambda_2\|x\|_{J'}$$

for all $x \in \mathcal{X}$.

Since $(\mathcal{X}, [\cdot,\cdot]_J)$ is positive definite, by the Cauchy-Schwartz inequality, Lemma 1.2.12 and Lemma 1.2.23 we have

$$\|x\|_{J'}^2 = [J'x, x] = [JJ'x, x]_J \leq \|JJ'x\|_J\|x\|_J \leq C\|x\|_J^2. \tag{1.16}$$

Inequality (1.16) also holds if we switch J and J'. Thus, we obtain

$$\frac{1}{\sqrt{C}}\|x\|_{J'} \leq \|x\|_J \leq \sqrt{C}\|x\|_{J'}.$$

□

Corollary 1.2.25. *Let $(\mathcal{X}, [\cdot,\cdot])$ be an inner product space. Then the following statements are equivalent:*

- *(i) $(\mathcal{X}, [\cdot,\cdot])$ is a Krein space.*
- *(ii) There exists a fundamental decomposition $(\mathcal{H}_1, \mathcal{H}_2, J)$ such that $(\mathcal{H}_1, [\cdot,\cdot])$ and $(\mathcal{H}_2, -[\cdot,\cdot])$ are Hilbert spaces.*

Proof. *(ii)* ⇒ *(i)*:
We've already established that in Examples 1.2.8, 1.2.13 and 1.2.18.

(i) ⇒ *(ii)*:
Let $(\mathcal{X}_+, \mathcal{X}_-, J)$ be a fundamental decomposition, such that $(\mathcal{X}, [\cdot,\cdot]_J)$ is complete. Since for $x \in \mathcal{X}_+$, $y \in \mathcal{X}_-$ we have

$$[x, x] = [Jx, x] \geq 0$$
$$-[y, y] = [Jy, y] \geq 0$$

the spaces $(\mathcal{X}_+, [\cdot,\cdot])$ and $(\mathcal{X}_-, -[\cdot,\cdot])$ are positive semidefinite. Moreover, because by definition $(\mathcal{X}, [\cdot,\cdot])$ is non-degenerated and all neutral elements are isotropic by Lemma 1.2.19, $(\mathcal{X}; [\cdot,\cdot])$ and $(\mathcal{X}_-, -[\cdot,\cdot])$ are positive definite. Since $\mathcal{X}_+ = (\mathcal{X}_-)^\perp$ and vice versa, they are closed with respect to $\|\cdot\|_J$ and, therefore, Hilbert spaces. □

Since we will need it later on, we define contraction operators on (indefinite) inner product spaces as follows:

Defintion 1.2.26. Let $(\mathcal{X}, [\cdot,\cdot])$ be an inner product space, and $T : \mathcal{X} \to \mathcal{X}$ be a linear operator. Then we call T a contraction if

$$[Tx, Tx] \leq [x, x]$$

for all $x \in \mathcal{X}$.

Note that, if $[\cdot,\cdot]$ is positive definite, T being a contraction is equivalent to T being a bounded linear operator with norm smaller or equal 1.

Defintion 1.2.27. A Krein space $\mathcal{H}_1[\dot{+}]\mathcal{H}_2$ such that $\min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\} < +\infty$ is called Pontryagin space.

1.3 Defect operators

Defintion 1.3.1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})_{\mathcal{H}}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator. Then we denote the family of all such operators by $\mathcal{B}(\mathcal{H})$. We call $T \in \mathcal{B}(\mathcal{H})$ a contraction if

$$\|Tx\|_{\mathcal{H}} \leq \|x\|_{\mathcal{H}}$$

for all $x \in \mathcal{H}$ (i.e. $\|T\| \leq 1$).

Further we define the space $\mathfrak{H}(T)$ by

$$\mathfrak{H}(T) := \mathrm{ran} T = \{x \in \mathcal{H} :\ \exists y \in \mathcal{H} : x = Ty\}\,.$$

Since for every $x \in \mathcal{H}$

$$T^{-1}x = \{y\} + \ker T,$$

for any $y \in T^{-1}x$ and $\ker T$ is a closed subspace of $\mathcal{H}$, the set $T^{-1}x$ is closed in $\mathcal{H}$. Hence there exists a unique $y_x \in \mathcal{H}$ with

$$\|y_x\|_{\mathcal{H}} = \min\{\|y\|_{\mathcal{H}} :\ x = Ty\}$$

and we can define an inner product and norm on $\mathfrak{H}(T)$ by

$$\langle x, z\rangle_{\mathfrak{H}(T)} := \langle y_x, y_z\rangle_{\mathcal{H}}\,, \qquad \|x\|^2_{\mathfrak{H}(T)} := \|y_x\|^2_{\mathcal{H}} = \langle y_x, y_x\rangle_{\mathcal{H}}\,.$$

Remark 1.3.2. Note that $y_x = y - Py$ for any $y \in T^{-1}x$, where P denotes the orthogonal projection on $\ker T$. Therefore y_x is the unique element of $T^{-1}x \cap (\ker T)^{\perp}$.

Since $\mathrm{ran}\, T^* \subseteq (\ker T)^{\perp}$

$$\|TT^*x\|_{\mathfrak{H}(T)} = \|T^*x\|_{\mathcal{H}}$$

follows for all $x \in \mathcal{H}$.

It is easy to check, that $\left(\mathfrak{H}(T), \langle\cdot,\cdot\rangle_{\mathfrak{H}(T)}\right)$ is again a Hilbert space.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})_{\mathcal{H}}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a contraction operator on $\mathcal{H}$, then because of (I denotes the identity on $\mathcal{H}$)

$$\langle (I - TT^*)x, x\rangle_{\mathcal{H}} = \|x\|^2_{\mathcal{H}} - \langle TT^*x, x\rangle_{\mathcal{H}} = \|x\|^2_{\mathcal{H}} - \|T^*x\|^2_{\mathcal{H}} \geq$$
$$\geq \|x\|^2_{\mathcal{H}}\left(1 - \|T^*\|^2_{\mathcal{H}}\right) = \|x\|^2_{\mathcal{H}}\left(1 - \|T\|^2_{\mathcal{H}}\right) \geq 0,$$

the operator $I - TT^*$ is positive. In particular there exists a unique, self-adjoint and non-negative square-root $(I - TT^*)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$.

Defintion 1.3.3. Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ a linear contraction operator. Then we call the operators

$$D_T := (I - T^*T)^{\frac{1}{2}}, \qquad D_{T^*} := (I - TT^*)^{\frac{1}{2}}$$

the defect operators and the Hilbert spaces

$$\mathcal{D}_T := \mathfrak{H}(D_T), \qquad \mathcal{D}_{T^*} := \mathfrak{H}(D_{T^*})$$

the defect spaces of the operator T.

Theorem 1.3.4. *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ a linear contraction operator. Then $x \in \mathcal{H}$ is an element of $\mathcal{D}_{T^*}$ if and only if*

$$\sup_{y\in\mathcal{H}} \left(\|x + Ty\|^2_{\mathcal{H}} - \|y\|^2_{\mathcal{H}}\right) < +\infty$$

In this case we can calculate the $\mathcal{D}_{T^*}$-norm by

$$\|x\|^2_{\mathcal{D}_{T^*}} = \sup_{y\in\mathcal{H}} \left(\|x + Ty\|^2_{\mathcal{H}} - \|y\|^2_{\mathcal{H}}\right)$$

Proof. See [NV91], Chapter 1. □

Remark 1.3.5.

(i) Since for $x \in \mathcal{D}_{T^*}$

$$\|x\|_{\mathcal{D}^*_T} = \sup_{y\in\mathcal{H}} \left(\|x + Ty\|^2_{\mathcal{H}} - \|y\|^2_{\mathcal{H}}\right) \geq \|x + T0\|^2_{\mathcal{H}} - \|0\|^2_{\mathcal{H}} = \|x\|^2_{\mathcal{H}}$$

the embedding

$$\iota : \begin{cases} \mathcal{D}_{T^*} & \to & \mathcal{H} \\ x & \mapsto & x \end{cases}$$

is a contraction.

(ii) Because of

$$\begin{aligned}\|D_{T^*}^2 x\|_{\mathcal{D}_{T^*}}^2 &= \|D_{T^*} D_{T^*}^* x\|_{\mathcal{D}_{T^*}}^2 \\ &= \|D_{T^*}^* x\|_{\mathcal{H}}^2 \\ &= \left\langle (I - TT^*)^{\frac{1}{2}} x, (I - TT^*)^{\frac{1}{2}} x \right\rangle_{\mathcal{H}}^2 \\ &= \langle (I - TT^*) x, x \rangle_{\mathcal{H}} \\ &= \|x\|_{\mathcal{H}}^2 - \|T^* x\|_{\mathcal{H}}^2 \le \|x\|_{\mathcal{H}}^2\end{aligned}$$

for all $x \in \mathcal{H}$, the operator $D_{T^*}^2 : \mathcal{H} \to \mathcal{D}_{T^*}$ is a contraction as well.

Lemma 1.3.6. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and $\mathcal{P}$ an arbitrary dense subset of $\mathcal{H}$. Then the set $D_{T^*}^2 \mathcal{P}$ is dense in $\mathcal{D}_{T^*}$.*

Proof. Let $f = Tg \in \mathfrak{H}(T)$ be arbitrary. Since T is continuous and self-adjoint, we can decompose the space $\mathcal{H}$ into the closed linear subspaces $\operatorname{ran} T \dotplus \ker T = \operatorname{ran} T \dotplus (\operatorname{ran} T)^\perp$, and denote the orthogonal projections on $\ker T$ and $\operatorname{ran} T$ by P_k and P_r respectively. Then since $P_r g \in \operatorname{ran} T$ there exists $h \in \mathcal{H}$ such that $Th = P_r g$. Moreover since $h \in \mathcal{H}$ and $\mathcal{P}$ is dense in $\mathcal{H}$, there exists a sequence $(p_n)_{n\in\mathbb{N}}$, such that $\lim_{n\in\mathbb{N}} \|p_n - h\|_{\mathfrak{D}} = 0$. Now we define $(f_n)_{n\in\mathbb{N}} := \left(T^2 p_n\right)_{n\in\mathbb{N}} \subseteq T^2\mathcal{P}$.

Because of

$$\begin{aligned}\|f_n - f\|_{\mathfrak{H}(T)} &= \|T^2 p_n - Tg\|_{\mathfrak{H}(T)} = \|T(Tp_n - g)\|_{\mathfrak{H}(T)} = \\ &= \min_{k\in\ker T} \|Tp_n - g + k\|_{\mathfrak{D}} = \min_{k\in\ker T} \|Tp_n - P_r g - P_k g + k\|_{\mathfrak{D}} \le \\ &\le \|Tp_n - Th\|_{\mathcal{H}} + \min_{k\in\ker T} \|P_k g + k\|_{\mathcal{H}} \le \|T\|\|p_n - h\|_{\mathcal{H}}\end{aligned}$$

$\lim_{n\to\infty} \|f_n - f\|_{\mathfrak{H}(T)} = 0$. Hence $T^2\mathcal{P}$ is dense in $\mathfrak{H}(T)$, since f was arbitrary. □

1.4 Reproducing Kernel Hilbert spaces

For an arbitrary, non-empty set X, we denote the set of all functions from X to $\mathbb{C}$ by $\mathbb{C}^X$. If we define the addition of $f, g \in \mathbb{C}^X$ by $(f+g)(x) := f(x) + g(x)$ and the scalar multiplication of $f \in \mathbb{C}^X$ with $\lambda \in \mathbb{C}$ as $(\lambda f)(x) := \lambda f(x)$ for all $x \in X$, then $\mathbb{C}^X$ is a vector space.

For every $x \in X$ we can define the mapping

$$\iota_x : \begin{cases} \mathbb{C}^X & \to & \mathbb{C} \\ f & \mapsto & f(x). \end{cases}$$

Because of $\iota_x(\lambda f + g) = \lambda f(x) + g(x) = \lambda \iota_x f + \iota_x g$ for $f, g \in \mathbb{C}^X$, $\lambda \in \mathbb{C}$, the mapping ι_x is linear.

Defintion 1.4.1. Let X be an arbitrary set and $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a Hilbert space, such that $\mathcal{H} \leq \mathbb{C}^X$ i.e. $\mathcal{H}$ is a linear subspace of $\mathbb{C}^X$. The space $\mathcal{H}$ is called a reproducing kernel Hilbert space (or RKHS) over X if $\iota_x \in \mathcal{H}'$ for all $x \in X$, where the symbol $\mathcal{H}'$ denotes the topological dual space of $\mathcal{H}$, defined by

$$\mathcal{H}' := \{h : \mathcal{H} \to \mathbb{C} \,\big|\, h \text{ is linear and continuous}\}.$$

Lemma 1.4.2. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})_{\mathcal{H}}$ be a RKHS over X, then there exists a kernel function $K_{\mathcal{H}} : X \times X \to \mathbb{C}$, such that for all $w \in X$ the function $K_{\mathcal{H}}(\cdot, w) \in \mathcal{H}$ and $K_{\mathcal{H}}$ has the reproducing property:*

$$f(w) = \langle f, K_{\mathcal{H}}(\cdot, w)\rangle_{\mathcal{H}} \tag{1.17}$$

for all $f \in \mathcal{H}$.

Proof. Let w be an arbitrary element of X. Since $\mathcal{H}$ is a RKHS, $\iota_w \in \mathcal{H}'$ and by the Riesz representation theorem there exists a unique $k_w \in \mathcal{H}$, such that $\langle f, k_w\rangle_{\mathcal{H}} = \iota_w f = f(w)$ for all $f \in \mathcal{H}$. Now we define our kernel function as

$$K_{\mathcal{H}}(z, w) := k_w(z).$$

□

The existence of a kernel function is already a sufficient condition for a Hilbert space to be a RKHS.

Lemma 1.4.3. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})_{\mathcal{H}}$ be an arbitrary Hilbert space, such that $\mathcal{H} \leq \mathbb{C}^X$ for a non-empty set X. Assume that there exists a kernel function $K_{\mathcal{H}} : X \times X \to \mathbb{C}$ with $K_{\mathcal{H}}(\cdot, w) \in \mathcal{H}$, for all $w \in X$ such that $K_{\mathcal{H}}$ fulfills the reproducing property (1.17). Then $\mathcal{H}$ is a RKHS over X.*

Proof. If we have $\langle f, K_{\mathcal{H}}(\cdot, w)\rangle_{\mathcal{H}} = f(w) = \iota_w f$ for $f \in \mathcal{H}$, $w \in X$, the functional ι_w is bounded by the Cauchy-Schwartz inequality.

□

If $\mathcal{H}$ is a RKHS, the following lemma shows that $\mathfrak{H}(T)$ has the same structure.

Theorem 1.4.4. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})_{\mathcal{H}}$ be a RKHS over a set X with reproducing kernel K and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then the space $\left(\mathfrak{H}(T), \langle\cdot,\cdot\rangle_{\mathfrak{H}(T)}\right)$ is again a RKHS with reproducing kernel $K_{\mathfrak{H}(T)}(\cdot, w) := TT^*K(\cdot, w)$.*

Proof. For fixed w, the function $K_{\mathfrak{H}(T)}(\cdot, w)$ is an element of $\mathfrak{H}(T)$. So it is only left to show, that $K_{\mathfrak{H}(T)}$ has indeed the reproducing property. For this purpose let $w \in X$, $f \in \mathfrak{H}(T)$ and $g \in \mathcal{H}$ such that $f = Tg$ and $g \in (\ker T)^{\perp}$. Now we calculate

$$\left\langle f, K_{\mathfrak{H}(T)}(\cdot, w)\right\rangle_{\mathfrak{H}(T)} = \langle Tg, TT^*K(\cdot, w)\rangle_{\mathfrak{H}(T)} = \langle g, T^*K(\cdot, w)\rangle_{\mathcal{H}}$$

since $\mathrm{ran} T^* \subseteq \ker T^{\perp}$, and further

$$\langle g, T^*K(\cdot, w)\rangle_{\mathcal{H}} = \langle Tg, K(\cdot, w)\rangle_{\mathcal{H}} = \langle f, K(\cdot, w)\rangle_{\mathcal{H}} = f(w).$$

□

Defintion 1.4.5. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then we call $T \leq \mathcal{H}\times\mathcal{H}$ a linear relation. If T is a closed subspace of $\mathcal{H}\times\mathcal{H}$ it is called a closed linear relation. The adjoint of T is defined by

$$T^* := \left\{(x,y) \in \mathcal{H}\times\mathcal{H} \,\middle|\, \langle u,y\rangle = \langle x,v\rangle\,,(u,v)\in T\right\}. \tag{1.18}$$

Remark 1.4.6.

(i) Every linear operator $T : \mathcal{H} \to \mathcal{H}$ can be viewed as a linear relation if we identify T with its graph.

(ii) The adjoint of the graph of a linear operator T as defined in (1.18) coincides with the graph of the adjoint operator T^*.

(iii) For any linear relation R

$$R^{**} = \overline{R},\ \overline{R}^* = R^*.$$

For a proof and more on linear relations and their adjoint see [Kal14].

Lemma 1.4.7. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})_{\mathcal{H}}$ be a RKHS over a set X with reproducing kernel K and $b : X \to X$ be a function such that the linear operator C defined by $Cf := f\circ b$ maps into $\mathcal{H}$. Then $C^* = \overline{\operatorname{span}}\{(K(\cdot,w); K(\cdot,b(w)) : w\in X)\}$ if we identify C with its graph.*

If in addition C is a contraction

$$K_{\mathfrak{H}(T)}(z,w) := K(z,w) - K(b(z),b(w))$$

is the reproducing kernel of the space $\mathfrak{H}(T)$, with $T := (I-CC^*)^{\frac{1}{2}}$.

Proof. We define $R := \operatorname{span}\{(K(\cdot,w), K(\cdot,b(w)) : w\in X)\}$. Then $\overline{R}$ is a closed subspace of $\mathcal{H}\times\mathcal{H}$ and therefore a closed linear relation. A pair $(f;g)\in\mathcal{H}\times\mathcal{H}$ by definition of the adjoint linear relation is an element of $R^* = \left(\overline{R}\right)^*$ if and only if

$$\langle g,u\rangle_{\mathcal{H}} = \langle f,v\rangle_{\mathcal{H}} \tag{1.19}$$

for all $(u;v)\in R$. By definition of R, we have

$$(u;v) = \left(\sum_{n=1}^{N}\lambda_n K(\cdot,w_n); \sum_{n=1}^{N}\lambda_n K(\cdot,b(w_n))\right)$$

for some $N\in\mathbb{N}$, $\lambda_n\in\mathbb{C}$, $w_n\in X$. Hence, if $(f;g)\in R^*$

$$g(w) = \langle g, K(\cdot,w)\rangle_{\mathcal{H}} = \langle f, K(\cdot,b(w))\rangle_{\mathcal{H}} = f(b(w)) = (Cf)(w)$$

for all $w\in X$. Thus we have $R^*\subseteq C$. On the other hand $f\circ b = Cf$ together with the reproducing property yields

$$\langle f, K(\cdot,b(w))\rangle_{\mathcal{H}} = f(b(w)) = \langle f\circ b, K(\cdot,w)\rangle_{\mathcal{H}}.$$

By linearity (1.19) is fulfilled for all $(u;v) \in R$, and therefore $R^* = C$. We conclude

$$C^* = (R^*)^* = \left((\overline{R})^*\right)^* = \overline{R}.$$

To prove the second assertion we calculate the kernel of $\mathfrak{H}(T)$ by Theorem 1.4.4 and obtain

$$\begin{aligned} K_{\mathfrak{H}(T)}(z,w) &= \left(TT^*K(\cdot,w)\right)(z) = ((I - CC^*)K(\cdot,w))(z) = \\ &= K(z,w) - (CK(\cdot,b(w)))(z) = K(z,w) - K(b(z),b(w)). \end{aligned}$$

□

Chapter 2

The Bergman and the Dirichlet space

2.1 The Bergman Space

Lemma 2.1.1. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a (complex) Hilbert space, X an arbitrary (complex) vector space and $\psi : \mathcal{H} \to X$ a linear bijection. Then $(X, \langle \cdot, \cdot \rangle_X)$ is a Hilbert space, where the inner product is defined as*

$$\langle \cdot, \cdot \rangle_X : \begin{cases} X \times X & \to \mathbb{C} \\ (x, y) & \mapsto \left\langle \psi^{-1}(x), \psi^{-1}(y) \right\rangle_{\mathcal{H}} . \end{cases}$$

The mapping ψ is isometric.

Proof. First note, that $\langle \cdot, \cdot \rangle_X$ is sesquilinear, conjugate symmetric and non-negative, since it is the composition of a conjugate symmetric, non-negative sesquilinear form and a linear mapping.

Let $x \in X$, such that $\langle x, x \rangle_X = 0$. Since

$$0 = \langle x, x \rangle_X = \left\langle \psi^{-1}(x), \psi^{-1}(x) \right\rangle_{\mathcal{H}}$$

and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is positive definite, we get that $\psi^{-1}(x) = 0$ and hence $x \in \ker \psi$. Due to the fact that ψ is one-to-one, we have that $x = 0$. This shows that $\langle \cdot, \cdot \rangle_X$ is positive definite.

Because for arbitrary $x \in X$

$$\|\psi(x)\|_X^2 = \langle \psi(x), \psi(x) \rangle_X = \left\langle \psi^{-1}(\psi(x)), \psi^{-1}(\psi(x)) \right\rangle_{\mathcal{H}} = \langle x, x \rangle_{\mathcal{H}} = \|x\|_{\mathcal{H}}^2,$$

the mapping ψ is in fact an isometric isomorphism.

Now let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy series in X, i.e.

$$\forall \varepsilon > 0 \, \exists N_0 \in \mathbb{N} : \|x_n - x_m\|_X < \varepsilon, \, \forall m, n \geq N_0.$$

Since ψ is isometric, the series $\left(\psi^{-1}(x_n)\right)_{n \in \mathbb{N}}$ is a Cauchy series in $\mathcal{H}$. Because of the completeness of $\mathcal{H}$ it converges to some $h \in \mathcal{H}$. Now

$$\|x_n - \psi(h)\|_X = \|\psi^{-1}(x_n) - h\|_{\mathcal{H}}$$

shows that $x_n \to \psi(h)$, and hence X is complete. $\square$

Let $\mathbb{C}^{\mathbb{N}_0} = \left\{(a_n)_{n\in\mathbb{N}_0} : a_n \in \mathbb{C}, n \in \mathbb{N}_0\right\}$ denote the set of all complex series. With addition and scalar multiplication defined by

$$(a_n)_{n\in\mathbb{N}_0} + (b_n)_{n\in\mathbb{N}_0} := (a_n + b_n)_{n\in\mathbb{N}_0} \quad (2.1)$$
$$\lambda (a_n)_{n\in\mathbb{N}_0} := (\lambda a_n)_{n\in\mathbb{N}_0} \quad (2.2)$$

for $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$, $\lambda \in \mathbb{C}$, the set $\mathbb{C}^{\mathbb{N}_0}$ is a complex vector space. The neutral element of the addition is the series $(0)_{n\in\mathbb{N}_0}$.

It is a well known fact, that the linear subspace $l_2 \leq \mathbb{C}^{\mathbb{N}_0}$, defined by

$$l_2 := \left\{ (a_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\},$$

equipped with the inner product

$$\langle \cdot, \cdot \rangle_{l_2} : \begin{cases} l_2 \times l_2 & \to \mathbb{C} \\ \left((a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}\right) & \mapsto \sum_{n=0}^{\infty} a_n \bar{b}_n \end{cases}$$

is a Hilbert space.

Defintion 2.1.2. Let the mapping ψ be defined by

$$\psi : \begin{cases} \mathbb{C}^{\mathbb{N}_0} & \to \mathcal{S}_0^+ \\ (a_n)_{n\in\mathbb{N}_0} & \mapsto \sum_{n=0}^{\infty} \sqrt{n+1}\, a_n z^n. \end{cases}$$

The space $\left(A^2, \langle \cdot, \cdot \rangle_{A^2}\right)$, with $A^2 := \psi(l_2)$ and $\langle \psi(a), \psi(b) \rangle_{A^2} := \langle a, b \rangle_{l^2}$ is called Bergman space.

Theorem 2.1.3. *The Bergman space $\left(A^2, \langle \cdot, \cdot \rangle_{A^2}\right)$, is a RKHS over the open unit disc $\mathbb{D} := R_1(0)$ with kernel function*

$$K_{A^2}(z, w) := \sum_{n=0}^{\infty} (n+1) \bar{w}^n z^n.$$

Proof. To show, that A^2 is in fact a Hilbert space, we'd like to apply Lemma 2.1.1. So we have to verify its assumptions. We already know, that l_2 is a Hilbert space, and $\mathcal{S}_0^+$ is a vector space.

For $\lambda \in \mathbb{C}$, $a := (a_n)_{n\in\mathbb{N}_0}, b := (b_n)_{n\in\mathbb{N}_0} \in l_2$ and $z \in \mathbb{D}$, we have

$$\begin{aligned} \psi(a + \lambda b)(z) &= \sum_{n=0}^{\infty} \sqrt{n+1}(a_n + \lambda b_n) z^n \\ &= \sum_{n=0}^{\infty} \sqrt{n+1}(a_n) z^n + \lambda \sum_{n=0}^{\infty} \sqrt{n+1}(b_n) z^n = \psi(a)(z) + \lambda \psi(b)(z). \end{aligned}$$

For $a = (a_n)_{n\in\mathbb{N}_0}$ such that $\psi(a) = 0 = \sum_{n=0}^{\infty} 0 \cdot z^n$, immediately follows $a_n = 0$ for all $n \in \mathbb{N}_0$. Hence ker $\psi(a) = \{(0)_{n\in\mathbb{N}_0}\}$.

This shows, that $\psi(l_2)$ is a vector space, and ψ is an isomorphism between l_2 and $\psi(l_2)$. So all the requirements of Lemma 2.1.1 are fulfilled. Thus A^2 is a Hilbert space.

Next we show that $A^2 \leq \mathbb{C}^{\mathbb{D}}$. To do so, we define a function $f : [0,1) \to \mathbb{C}$ by $f(x) := \frac{x}{1-x}$. Because of

$$f'(x) = \frac{d}{dx}\frac{x}{1-x} = \frac{d}{dx}\sum_{n=1}^{\infty} x^n = \frac{d}{dx}\sum_{n=0}^{\infty} x^{n+1} = \sum_{n=0}^{\infty}(1+n)x^n < +\infty \tag{2.3}$$

for all $x \in [0,1)$, it follows for $z \in \mathbb{D}$, that

$$f'\left(|z|^2\right) = \sum_{n=0}^{\infty}(n+1)|z|^{2n} < +\infty.$$

This shows, that for $z \in \mathbb{D}$, the series $\left(\sqrt{n+1}\,|z|^n\right)_{n\in\mathbb{N}_0}$ is an element of l_2.

Let $(a_n)_{n\in\mathbb{N}_0}$ be an arbitrary l_2-sequence. Then $(|a_n|)_{n\in\mathbb{N}_0}$ is an element of l_2 as well and we may calculate

$$\sum_{n=0}^{\infty}|\sqrt{n+1}\,a_n z^n| = \left\langle (|a_n|)_{n\in\mathbb{N}_0}, \left(\sqrt{n+1}\,|z|^n\right)_{n\in\mathbb{N}_0}\right\rangle_{l_2} < +\infty$$

for all $z \in \mathbb{D}$. Because of

$$\psi((a_n)_{n\in\mathbb{N}_0}) = \sum_{n=0}^{\infty}\sqrt{n+1}\,a_n z^n,$$

this shows, that the radius of convergence of $\psi((a_n)_{n\in\mathbb{N}_0})$ is indeed greater or equal to one and hence $\psi((a_n)_{n\in\mathbb{N}_0}) \in \mathbb{C}^{\mathbb{D}}$. By Theorem 1.1.3 (*i*) we can interpret the elements of A^2 as functions on $\mathbb{D}$, where addition and scalar multiplication on A^2 coincides with the point wise addition and scalar multiplication on $\mathbb{C}^{\mathbb{D}}$, i.e. $A^2 \leq \mathbb{C}^{\mathbb{D}}$.

It remains to show, that A^2 is indeed a RKHS with reproducing kernel K_{A^2}. Since for $w \in \mathbb{D}$

$$K_{A^2}(\cdot, w) = \psi((\sqrt{n+1}\,\bar{w}^n)_{n\in\mathbb{N}_0}),$$

$K_{A^2}(\cdot, w) \in A^2$ and because for $\psi((a_n)_{n\in\mathbb{N}_0}) \in A^2$ we have

$$\langle \psi((a_n)_{n\in\mathbb{N}_0}), K_{A^2}(\cdot, w)\rangle_{A^2} = \left\langle (a_n)_{n\in\mathbb{N}_0}, (\sqrt{1+n}\,\bar{w}^n)_{n\in\mathbb{N}_0}\right\rangle_{l^2} = \sum_{n=0}^{\infty}\sqrt{1+n}\,a_n w^n = \psi((a_n)_{n\in\mathbb{N}_0})(w),$$

K_{A^2} has the reproducing property (1.17). By Lemma 1.4.3 this is sufficient for A^2 to be a RKHS. □

Remark 2.1.4. If we substitute $\bar{w}z$ for x in equation (2.3), we obtain

$$K_{A^2}(z, w) = \sum_{n=0}^{\infty}(n+1)\bar{w}^n z^n = \frac{d}{dx}\frac{x}{1-x}\bigg|_{x=\bar{w}z} = \frac{1}{(1-\bar{w}z)^2}$$

for $|z| < \frac{1}{|w|}$. Hence $K_{A^2}(\cdot, w)$ has a radius of convergence of $\frac{1}{|w|}$.

Theorem 2.1.5. *A formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to A^2 if and only if*

$$\sum_{n=0}^{\infty} \frac{1}{n+1}|a_n|^2 < \infty.$$

In this case

$$\|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{1}{n+1}|a_n|^2.$$

For two functions $f, g \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ we can calculate the inner product by

$$\langle f, g\rangle_{A^2} = \sum_{n=0}^{\infty} \frac{1}{1+n} a_n \bar{b}_n.$$

Proof. If we define $(\tilde{a}_n)_{n\in\mathbb{N}} := \psi^{-1}(f) = \left(\frac{1}{\sqrt{n+1}} a_n\right)_{n\in\mathbb{N}_0}$, we have

$$\sum_{n=0}^{\infty} |\tilde{a}_n|^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2$$

in the sense, that the sum on the left hand side is finite, if and only if the sum on the right hand side is finite. Hence $f \in A^2$ if and only if $\sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 < +\infty$. In this case

$$\|f\|_{A^2}^2 = \|\psi^{-1}(f)\|_{l_2}^2 = \|(\tilde{a}_n)_{n\in\mathbb{N}}\|_{l_2}^2 = \sum_{n=0}^{\infty} |\tilde{a}_n|^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 .$$

We can calculate the inner product by

$$\langle f, g\rangle_{A^2} = \left\langle \psi^{-1}(f), \psi^{-1}(g)\right\rangle_{l_2} =$$
$$= \left\langle \left(\frac{1}{\sqrt{1+n}} a_n\right)_{n\in\mathbb{N}}, \left(\frac{1}{\sqrt{1+n}} b_n\right)_{n\in\mathbb{N}} \right\rangle_{l^2} = \sum_{n=0}^{\infty} \frac{1}{1+n} a_n \bar{b}_n.$$

□

The Bergman space is a subspace of the vector space of all analytic functions on the open unit disc $\mathbb{D}$. The following example shows, that it is a proper subspace.

Example 2.1.6. Consider the function $f(z) := \frac{1}{1-z}$. It is analytic on the open unit disc and allows a power series expansion $f(z) = \sum_{n=0}^{\infty} z^n$. But since

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is not finite, we know by Theorem 2.1.5, that $f \notin A^2$.

Lemma 2.1.7. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a formal power series with $R(f) \geq 1$. Then $f \in A^2$ if and only if*

$$\int_{\mathbb{D}} |f(z)|^2 \, dA(z) < +\infty,$$

where dA means integration by the normalized area measure on $\mathbb{D}$ (i.e. $dA = \frac{1}{\pi} d\lambda_2$).

In this case

$$\|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z).$$

Furthermore for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A^2$, we can calculate the inner product by

$$\langle f, g \rangle_{A^2} = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z).$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be arbitrary analytic functions on $\mathbb{D}$ and R be an arbitrary real number, $0 < R < 1$. Since the involved series converge uniformly on $B_R(0)$, we can calculate

$$\begin{aligned}
\int_{B_R(0)} f(z)\overline{g(z)} \, dA(z) &= \int_{B_R(0)} \sum_{n=0}^{\infty} a_n z^n \overline{\sum_{k=0}^{\infty} b_k z^k} \, dA(z) \\
&= \sum_{n,k=0}^{\infty} a_n \bar{b}_k \int_{B_R(0)} z^n \bar{z}^k \, dA(z) \\
&= \frac{1}{\pi} \sum_{n,k=0}^{\infty} a_n \bar{b}_k \int_0^R \int_0^{2\pi} r^{n+k+1} e^{i\phi(n-k)} \, d\phi \, dr \qquad (2.4) \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} a_n \bar{b}_n \, 2\pi \int_0^R r^{2n+1} \, dr \\
&= \sum_{n=0}^{\infty} R^{2n+2} \frac{1}{n+1} a_n \bar{b}_n.
\end{aligned}$$

We denote by $\chi_A(z)$ the characteristic function of the set $A \subseteq \mathbb{C}$, defined as

$$\chi_A := \begin{cases} \mathbb{C} & \to \quad \{0,1\} \\ z & \mapsto \quad \begin{cases} 1, & z \in A \\ 0, & z \notin A \end{cases} \end{cases}$$

If we substitute f for g in equation (2.4) and calculate the limit $R \nearrow 1$, we obtain by the monotone convergence theorem

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 = \lim_{R \nearrow 1} R^{2n+2} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 =$$
$$= \lim_{R \nearrow 1} \int_{\mathbb{D}} \chi_{B_R(0)}(z) |f(z)|^2 \, dA(z) = \int_{\mathbb{D}} |f(z)|^2 \, dA(z),$$

which shows the first statement. For $f, g \in A^2$ we have

$$\langle f, g\rangle_{A^2} = \lim_{R\nearrow 1} R^{2n+2} \langle f, g\rangle_{A^2} = \lim_{R\nearrow 1} R^{2n+2} \sum_{n=0}^{\infty} \frac{1}{n+1} a_n \overline{b_n} = \\ = \lim_{R\nearrow 1} \int_{\mathbb{D}} \chi_{B_R(0)}(z) f(z) \overline{g(z)}\, dA(z). \quad (2.5)$$

Since

$$\left(\int_{\mathbb{D}} |\chi_{B_R(0)}(z) f(z)\overline{g(z)}|\, dA(z)\right)^2 \leq \left(\int_{\mathbb{D}} |f(z)\overline{g(z)}|\, dA(z)\right)^2 \leq \\ \leq \int_{\mathbb{D}} |f(z)|^2\, dA(z) \int_{\mathbb{D}} |g(z)|^2\, dA(z) < +\infty$$

by the Cauchy-Schwarz inequality, we can apply the dominated convergence theorem to the right hand side of equation (2.5) and obtain

$$\langle f, g\rangle_{A^2} = \lim_{R\nearrow 1} \int_{\mathbb{D}} \chi_{B_R(0)}(z) f(z)\overline{g(z)}\, dA(z) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z).$$

□

Lemma 2.1.8. *Let f be a formal power series, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that its formal derivative $df \in A^2$. Then f is an element of A^2.*

If $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is a second function fulfilling the same assumptions, then

$$\langle f', g'\rangle_{A^2} = \sum_{n=1}^{\infty} n a_n \bar{b}_n.$$

Proof. By assumption we have $(df)(z) = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n \in A^2$. Theorem 2.1.5 asserts

$$\sum_{n=1}^{\infty} \frac{1}{n+1}|a_n|^2 \leq \sum_{n=1}^{\infty} n|a_n|^2 = \sum_{n=0}^{\infty}(n+1)|a_{n+1}|^2 = \\ = \sum_{n=0}^{\infty} \frac{1}{n+1}|(n+1)a_{n+1}|^2 < +\infty.$$

Hence $f \in A^2$. Since $f, g \in A^2 \leq \mathbb{C}^{\mathbb{D}}$, they are holomorphic on $\mathbb{D}$ and we can write f', g' instead of the formal derivative. By the second part of Theorem 2.1.5 we have

$$\langle f', g'\rangle_{A^2} = \left\langle \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n, \sum_{n=0}^{\infty}(n+1)b_{n+1}z^n \right\rangle_{A^2} = \\ \sum_{n=0}^{\infty} \frac{1}{n+1}(n+1)a_{n+1}(n+1)\bar{b}_{n+1} = \sum_{n=1}^{\infty} n a_n \bar{b}_n.$$

□

The following example shows, that in general the converse of the assertion in Lemma 2.1.8 is not true.

Example 2.1.9. We define a formal power series f by $f(z) := \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} z^n$. Because of

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left| \frac{1}{\sqrt{n}} \right|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$$

f is an element of A^2. The derivative f' is given by

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n}} z^{n-1} = \sum_{n=0}^{\infty} \sqrt{n+1}\, z^n.$$

But $f' \notin A^2$, since

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\sqrt{n+1}|^2 = \sum_{n=0}^{\infty} 1$$

is not finite.

Because for $\lambda \in \mathbb{C}$ and two analytic functions $f, g : \mathbb{D} \to \mathbb{C}$ with $f', g' \in A^2$

$$(f + \lambda g)' = f' + \lambda g' \in A^2,$$

the set $\left\{ f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, } f' \in A^2 \right\}$ is a subspace of A^2, and Example 2.1.9 shows, that it is a proper subspace. Unfortunately, the following example proves, that it is not closed with respect to $\|\cdot\|_{A^2}$.

Example 2.1.10. We define a series of functions $(f_k)_{k\in\mathbb{N}}$ by $f_k(z) := \sum_{n=1}^{k} \frac{1}{\sqrt{n}} z^n$. Since for all $k \in \mathbb{N}$, the function f_k is a polynomial, it is analytic on $\mathbb{D}$. The derivative is given by $(df_k)(z) = f_k'(z) = \sum_{n=1}^{k} \frac{n}{\sqrt{n}} z^{n-1} = \sum_{n=0}^{k-1} \sqrt{n+1}\, z^n$. Because of

$$\sum_{n=0}^{k-1} \frac{1}{n+1}(n+1) = k < +\infty$$

$f_k' \in A^2$ holds for all $k \in \mathbb{N}$ by Lemma 2.1.8. If we remember the function f defined in Example 2.1.9 we have

$$\|f - f_k\|_{A^2}^2 = \left\| \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} z^n - \sum_{n=1}^{k} \frac{1}{\sqrt{n}} z^n \right\|_{A^2}^2 = \left\| \sum_{n=k+1}^{\infty} \frac{1}{\sqrt{n}} z^n \right\|_{A^2} =$$
$$= \sum_{n=k+1}^{\infty} \frac{1}{n+1} \frac{1}{n} \leq \sum_{n=k+1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\lim_{k\to\infty} \|f - f_k\|_{A^2} = 0$. Hence the limit of $(f_k)_{k\in\mathbb{N}}$ with respect to $\|\cdot\|_{A^2}$ is f. But we know from Example 2.1.9, that $f' \notin A^2$.

So if we want the set of all formal power series f, such that $df \in A^2$ to be a Hilbert space, we have to choose a different norm.

2.2 The Dirichlet Space

Defintion 2.2.1. Let ϕ be the mapping

$$\phi : \begin{cases} s & \to \mathcal{S}^+ \\ (a_n)_{n\in\mathbb{N}_0} & \mapsto \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_{n-1} z^n. \end{cases}$$

Then we define the Dirichlet space $(\mathfrak{D}, \langle\cdot,\cdot\rangle_{\mathfrak{D}})$ by

$$\mathfrak{D} := \phi(l_2), \qquad \langle\phi(a), \phi(b)\rangle_{\mathfrak{D}} := \langle a, b\rangle_{l_2}$$

for $\phi(a), \phi(b) \in \mathfrak{D}$.

Theorem 2.2.2. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be a formal power series. Then the following statements are equivalent:*

(i) $f \in \mathfrak{D}$,

(ii) $df \in A^2$,

(iii) $\sum_{n=1}^{\infty} n|a_n|^2 < +\infty$,

(iv) $R(f) \geq 1$ *and* $\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty$

Proof. We shall prove the implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, $(ii) \Rightarrow (iv)$, $(iii) \Rightarrow (i)$ and $(iv) \Rightarrow (ii)$.

$(i) \Rightarrow (ii)$:

We assume that $f \in \mathfrak{D}$. Then there exists a series $(b_n)_{n\in\mathbb{N}_0} \in l_2$, such that

$$f(z) = \phi\left((b_n)_{n\in\mathbb{N}_0}\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} b_{n-1} z^n.$$

The formal derivative of f is given by

$$(df)(z) = \sum_{n=1}^{\infty} n \frac{1}{\sqrt{n}} b_{n-1} z^{n-1} = \sum_{n=0}^{\infty} \sqrt{n+1} b_n z^n.$$

Due to

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |\sqrt{n+1}\, b_n|^2 = \sum_{n=0}^{\infty} |b_n|^2 = \|(b_n)_{n\in\mathbb{N}_0}\|_{l_2}^2 < +\infty$$

by Theorem 2.1.5 the formal power series df is an element of A^2.

$(ii) \Rightarrow (iii)$, $(ii) \Rightarrow (iv)$:

If we assume, that the formal power series df belongs to A^2, then due to Lemma 2.1.8, $f \in A^2$ and f is holomorphic on $\mathbb{D}$, i.e. $R(f) \geq 1$ and satisfies $(df)(z) = f'(z)$. We also know by Lemma 2.1.8, that

$$\sum_{n=1}^{\infty} n|a_n|^2 = \|f'\|_{A^2}^2 < +\infty,$$

since $f' \in A^2$. Due to Lemma 2.1.7 $f' \in A^2$ also implies

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty.$$

$(iv) \Rightarrow (ii)$: If f is analytic on $\mathbb{D}$, then f' is also analytic on $\mathbb{D}$ by Theorem 1.1.3 (iv). Lemma 2.1.7 tells us that this, together with the assumption $\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty$ gives $f' \in A^2$.

$(iii) \Rightarrow (i)$

We assume that $\sum_{n=1}^{\infty} n|a_n|^2 < +\infty$ and have to show that $\phi^{-1}(f) \in l_2$. Since

$$\phi^{-1}\left(\sum_{n=1}^{\infty} a_n z^n\right) = \left(\sqrt{n+1}\, a_{n+1}\right)_{n\in\mathbb{N}_0},$$

we have

$$+\infty > \sum_{n=1}^{\infty} n|a_n|^2 = \sum_{n=0}^{\infty} (n+1)|a_{n+1}|^2 = \left\| \left(\sqrt{n+1}\, a_{n+1}\right)_{n\in\mathbb{N}_0} \right\|_{l_2}^2 = \|\phi^{-1}(f)\|_{l_2}.$$

□

Theorem 2.2.3. *The Dirichlet space is a RKHS over $\mathbb{D}$ with kernel function*

$$K_{\mathfrak{D}}(z,w) := \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} z^n.$$

For $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathfrak{D}$ the inner product satisfies

$$\langle f, g \rangle_{\mathfrak{D}} = \sum_{n=1}^{\infty} n a_n \bar{b}_n = \int_{\mathbb{D}} f'(z)\overline{g'(z)} \, dA(z) = \left\langle f', g' \right\rangle_{A^2}.$$

Proof. For the first part, we use again Lemma 2.1.1. We already know, that l_2 is a Hilbert space. Since for $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \in l_2$, $\lambda \in \mathbb{C}$

$$\begin{aligned}\phi\left((a_n)_{n\in\mathbb{N}_0} + \lambda(b_n)_{n\in\mathbb{N}_0}\right) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(a_{n-1} + \lambda b_{n-1}\right) z^n = \\ &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_{n-1} z^n + \lambda \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} b_{n-1} z^n = \phi((a_n)_{n\in\mathbb{N}_0}) + \lambda\phi((b_n)_{n\in\mathbb{N}_0}),\end{aligned}$$

the mapping $\phi|_{l_2} : l_2 \to \mathfrak{D}$ is linear, and by definition of $\mathfrak{D}$ onto. Since

$$\ker\, \phi = \phi^{-1}(0) = (0)_{n\in\mathbb{N}_0}$$

it is also one-to-one and therefore a bijection. According to Lemma 2.1.1 $(\mathfrak{D}, \langle\cdot,\cdot\rangle_{\mathfrak{D}})$ is a Hilbert space.

By Theorem 2.2.2 (*iv*), we know that $\mathfrak{D} \leq \mathbb{C}^{\mathbb{D}}$. Let $w \in \mathbb{D}$ be arbitrary. Because of

$$\sum_{n=1}^{\infty} n \left| \frac{\bar{w}^n}{n} \right|^2 = \sum_{n=1}^{\infty} \frac{|w|^{2n}}{n} \leq \sum_{n=1}^{\infty} |w|^{2n} = \frac{|w|^2}{1-|w|^2} < +\infty$$

Theorem 2.2.2 (*iii*) yields $\sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} z^n \in \mathfrak{D}$. Since for an arbitrary $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathfrak{D}$

$$\left\langle f, \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} z^n \right\rangle_{\mathfrak{D}} = \left\langle \sum_{n=1}^{\infty} a_n z^n, \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} z^n \right\rangle_{\mathfrak{D}} = \sum_{n=1}^{\infty} n a_n \frac{\overline{\bar{w}^n}}{n} = \sum_{n=1}^{\infty} a_n w^n = f(w),$$

the function $K_{\mathfrak{D}}(z,w)$ is the reproducing kernel of the RKHS $\mathfrak{D}$.

For two arbitrary functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathfrak{D}$ we calculate the inner product by

$$\begin{aligned} \langle f, g \rangle_{\mathfrak{D}} &= \left\langle \phi^{-1}(f), \phi^{-1}(g) \right\rangle_{l_2} = \\ &= \left\langle \left(\sqrt{n+1}\, a_{n+1}\right)_{n\in\mathbb{N}_0}, \left(\sqrt{n+1}, b_{n+1}\right)_{n\in\mathbb{N}_0} \right\rangle_{l_2} = \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} \bar{b}_{n+1} = \sum_{n=1}^{\infty} n a_n \bar{b}_n \end{aligned}$$

which shows the first equality. By Theorem 2.2.2 (*ii*) and (*iv*), we know that $df = f'$, $dg = g' \in A^2$. Hence we can use the second part of Lemma 2.1.8 and Lemma 2.1.7 and obtain

$$\sum_{n=1}^{\infty} n a_n \bar{b}_n = \langle f', g' \rangle_{A^2} = \int_{\mathbb{D}} f'(z) \overline{g'(z)}\, dA(z).$$

□

Remark 2.2.4. Because of

$$\log \left(\frac{1}{1-z} \right) = \sum_{n=1}^{\infty} \frac{z}{n}$$

for all $z \in \mathbb{D}$, we can calculate the Dirichlet Kernel by

$$K_{\mathfrak{D}}(z,w) = \log \left(\frac{1}{1-\bar{w}z} \right)$$

for all $|z| < \frac{1}{|w|}$. Hence the radius of convergence of the function $K_{\mathfrak{D}}(\cdot, w)$ is $\frac{1}{|w|}$.

2.3 Generalized Dirichlet spaces

Defintion 2.3.1. Let λ be an arbitrary real number and $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Then we call

$$f(z) = \sum_{n=1}^{\infty} a_n z^{n+\lambda}$$

a generalized power series and denote the set of all such generalized power series by $\mathcal{S}_\lambda$.

Remark 2.3.2.

(i) With addition and scalar multiplication defined similar as on $\mathcal{S}_0^+$, the space $\mathcal{S}_\lambda$ is a vector space.

(ii) We can write every $f(z) = \sum_{n=1}^\infty a_n z^{n+\lambda} \in \mathcal{S}_\lambda$ as $f(z) = z^\lambda g(z)$ with $g(z) = \sum_{n=1}^\infty a_n z^n \in \mathcal{S}^+$.

(iii) For $\lambda \in \mathbb{Z}_{\geq -1}$, we have $\mathcal{S}_\lambda \subseteq \mathcal{S}_0^+$. Hence, we can interpret $f(z) = \sum_{n=1}^\infty a_n z^{n+\lambda} \in S_\lambda$ as analytic function on $B_{R(f)}(0)$.

(iv) For $\lambda \in \mathbb{Z}_{\leq -2}$, a function $f(z) = z^\lambda g(z) := z^\lambda \sum_{n=1}^\infty a_n z^n$ with $R(g) > 0$ and $a_1 \neq 0$ has a pole of order $\lambda - 1$ at the origin. Thus f is analytic on $B_{R(g)}(0) \setminus \{0\}$.

(v) For $\lambda \in \mathbb{R} \setminus \mathbb{Z}$ a function $f(z) = z^\lambda g(z) := z^\lambda \sum_{n=1}^\infty a_n z^n$ can be interpreted as a function on $B_{R(g)}(0)$. In fact, fixing $\theta \in \mathbb{R}$ and defining z^λ by $\left(re^{i\phi}\right)^\lambda := r^\lambda e^{i\lambda\phi}$, where we choose $\phi \in [\theta, \theta + 2\pi)$ we can view $f(z)$ as function $z \mapsto z^\lambda g(z)$. This function is analytic on $B_{R(g)}(0) \setminus e^{i\theta} \cdot [0, +\infty)$.

$g(z)$ is an analytic continuation on $B_{R(g)}(0)$ of $\frac{f(z)}{z^\lambda}$.

Defintion 2.3.3. Let λ a real number. Then we define the generalized Dirichlet space $(\mathfrak{D}_\lambda, [\cdot,\cdot]_{\mathfrak{D}_\lambda})$ by

$$\mathfrak{D}_\lambda := \left\{ f(z) = \sum_{n=1}^\infty a_n z^{n+\lambda} \in S_\lambda : \sum_{n=1}^\infty a_n z^n \in \mathfrak{D} \right\}$$

with the corresponding inner product

$$[f,g]_{\mathfrak{D}_\lambda} := \sum_{n=1}^\infty (n+\lambda)\, a_n \bar{b}_n,$$

for $f(z) = \sum_{n=1}^\infty a_n z^{n+\lambda}$, $g(z) = \sum_{n=1}^\infty b_n z^{n+\lambda} \in \mathfrak{D}_\lambda$.

Remark 2.3.4.

(i) Note, that the sum in the definition of the inner product converges, since the $\sum_{n=1}^\infty n a_n \bar{b}_n$ and hence also $\sum_{n=1}^\infty a_n \bar{b}_n$ converge absolutely.

(ii) For $f(z) = \sum_{n=1}^\infty a_n z^{n+\lambda} \in \mathfrak{D}_\lambda$, the function $\sum_{n=1}^\infty a_n z^n \in \mathfrak{D}$ is an analytic continuation of $\frac{f(z)}{z^\lambda}$ to the whole unit disc. The mapping

$$\psi : \begin{cases} \mathfrak{D} & \to & \mathfrak{D}_\lambda \\ \sum_{n=1}^\infty a_n z^n & \mapsto & \sum_{n=1}^\infty a_n z^{n+\lambda} \end{cases}$$

is linear and bijective, and thus an isomorphism.

(iii) It is easy to check, that for $\lambda \in \mathbb{R}$ the space $\mathfrak{D}_\lambda$ is a linear subspace of S_λ and that $[\cdot,\cdot]_{\mathfrak{D}_\lambda}$ is an inner product. Therefore $(\mathfrak{D}_\lambda, [\cdot,\cdot]_{\mathfrak{D}_\lambda})$ is an inner product space as defined in Definition 1.2.1. Since for $f(z) = \sum_{n=1}^{\infty}(n+\lambda)a_n z^{n+\lambda} \in \mathfrak{D}_\lambda$

$$[f(z), f(z)]_{\mathfrak{D}_\lambda} = \sum_{n=1}^{\infty}(n+\lambda)|a_n|^2,$$

it is positive definite if $\lambda > -1$, positive semidefinite if $\lambda = -1$ and indefinite if $\lambda < -1$. If $\lambda \in \mathbb{R} \setminus \mathbb{Z}_{\leq -1}$ it is also non-degenerated. For $\lambda \in \mathbb{Z}_{\leq -1}$ the isotropic part consists of all constant functions.

Lemma 2.3.5. *For arbitrary $\lambda \in \mathbb{R}$, let $f(z) = \sum_{n=1}^{\infty} a_n z^{n+\lambda} \in \mathcal{S}_\lambda$. Then f is an element of $\mathfrak{D}_\lambda$ if and only if the sum*

$$\sum_{n=1}^{\infty}(n+\lambda)|a_n|^2 \tag{2.6}$$

converges. In this case

$$[f,f]_{\mathfrak{D}_\lambda}^2 = \sum_{n=1}^{\infty}(n+\lambda)|a_n|^2.$$

Proof. $f(z) \in \mathfrak{D}_\lambda$ by definition implies $\sum_{n=1}^{\infty} a_n z^n \in \mathfrak{D}$, and hence

$$\begin{aligned}\left|\sum_{n=1}^{\infty}(n+\lambda)|a_n|^2\right| &\leq \sum_{n=1}^{\infty} n|a_n|^2 + |\lambda|\sum_{n=1}^{\infty}|a_n|^2 \leq \\ &\leq \sum_{n=1}^{\infty} n|a_n|^2 + |\lambda|\sum_{n=1}^{\infty} n|a_n|^2 \leq (1+|\lambda|)\left\|\sum_{n=1}^{\infty} a_n z^n\right\|_{\mathfrak{D}}^2 < +\infty. \end{aligned} \tag{2.7}$$

If on the other hand (2.6) converges, we get

$$\sum_{n=1}^{\infty} n|a_{n+N-1}|^2 \leq \sum_{n=1}^{\infty}(n+N+\lambda-1)|a_{n+N-1}|^2 = \sum_{n=N}^{\infty}(n+\lambda)|a_n|^2 < +\infty$$

with $N := \max\{1, -\lfloor\lambda\rfloor + 1\}$. Thus

$$\sum_{n=1}^{\infty}(n+N-1)|a_{n+N-1}|^2 = (N-1)\sum_{n=1}^{\infty}|a_{n+N-1}|^2 + \sum_{n=1}^{\infty} n|a_{n+N-1}|^2 < +\infty$$

and moreover

$$\sum_{n=1}^{\infty} n|a_n|^2 = \sum_{n=1}^{N-1} n|a_n|^2 + \sum_{n=1}^{\infty}(n+N-1)|a_{n+N-1}|^2 < +\infty.$$

Therefore $\sum_{n=1}^{\infty} a_n z^n \in \mathfrak{D}$. The last assertion is clear. □

Remark 2.3.6. Let $k \in \mathbb{N}$ and $f(z) = \sum_{n=1}^{\infty} a_n z^{n+k} \in \mathcal{S}_k$. Then because of

$$f(z) = \sum_{n=1}^{\infty} a_n z^{n+k} = \sum_{n=k+1}^{\infty} c_n z^n,$$

with $c_n = a_{n-k}$ and

$$\sum_{n=k+1}^{\infty} n|c_n|^2 = \sum_{n=1}^{\infty} (n+k)|a_n|^2,$$

f is an element of $\mathfrak{D}_k$ if and only if $f \in \mathfrak{D}$. Also the inner products on $\mathfrak{D}_k$ and $\mathfrak{D}$ coincide. Since $\mathfrak{D}_k = (\mathcal{P}_k)^{\perp}$, it is a closed subspace of $\mathfrak{D}$ and therefore a Hilbert space itself.

Theorem 2.3.7. *For an arbitrary real number $\lambda > -1$, the generalized Dirichlet space $\mathfrak{D}_\lambda$ is a Hilbert space.*

Proof. We have already established, that for such λ, the space $(\mathfrak{D}_\lambda, [\cdot,\cdot]_{\mathfrak{D}_\lambda})$ is a positive definite inner product space and hence $\|\cdot\|_{\mathfrak{D}_\lambda} := \sqrt{[\cdot,\cdot]_{\mathfrak{D}_\lambda}}$ is a norm.

Let $(f_k(z))_{k\in\mathbb{N}} = \left(\sum_{n=1}^{\infty} a_n^k z^{n+\lambda}\right)_{k\in\mathbb{N}} \subseteq \mathfrak{D}_\lambda$ be a Cauchy series. Then because for arbitrary $g(z) = \sum_{n=1}^{\infty} c_n z^{n+\lambda} \in \mathfrak{D}_\lambda$

$$\begin{aligned}\|\sum_{n=1}^{\infty} c_n z^n\|_{\mathfrak{D}}^2 &= \sum_{n=1}^{\infty} n|c_n|^2 \le \sum_{n=1}^{\infty}(n+\lambda+1)|c_n|^2 = \\ &= \|g(z)\|_{\mathfrak{D}_\lambda}^2 + \sum_{n=1}^{\infty}|c_n|^2 \le \|g(z)\|_{\mathfrak{D}_\lambda}^2 + \frac{1}{1+\lambda}\sum_{n=1}^{\infty}(n+\lambda)|c_n|^2 = \frac{2+\lambda}{1+\lambda}\|g(z)\|_{\mathfrak{D}_\lambda}^2\end{aligned}$$

we have that the series

$$\left(\tilde{f}_k(z)\right)_{k\in\mathbb{N}} := \left(\sum_{n=1}^{\infty} a_n^k z^n\right)_{k\in\mathbb{N}}$$

is a Cauchy series in $\mathfrak{D}$ and since $\mathfrak{D}$ is complete, it converges to some function $f \in \mathfrak{D}$ with respect to $\|\cdot\|_{\mathfrak{D}}$. Combining this with equation (2.7) in all we obtain

$$\lim_{k\to\infty} \|f_k(z) - f(z)\|_{\mathfrak{D}_\lambda}^2 \le (1+\lambda) \lim_{k\to\infty} \|\tilde{f}_k(z) - f(z)\|_{\mathfrak{D}}^2 = 0.$$

Thus $(\mathfrak{D}_\lambda, [\cdot,\cdot]_{\mathfrak{D}_\lambda})$ is complete and therefore a Hilbert space.

□

Theorem 2.3.8. *For an arbitrary real number $\lambda < -1$, $\lambda \notin \mathbb{Z}$ the generalized Dirichlet space $\mathfrak{D}_\lambda$ is a Pontryagin space.*

Proof. We already mentioned in Remark 2.3.4, (i), that $\mathfrak{D}_\lambda$ is a non-degenerated inner product space. We define

$$\mathcal{X}_+ := \left\{\sum_{n=1}^{\infty} a_n z^{n+\lambda} : \, a_n = 0, \, n+\lambda < 0\right\}$$

$$\mathcal{X}_- := \left\{ \sum_{n=1}^{\infty} a_n z^{n+\lambda} : \, a_n = 0, \, n+\lambda > 0 \right\}.$$

Then $\mathcal{X}_+$, ($\mathcal{X}_-$) are positive (negative) definite linear subspaces of $\mathfrak{D}_\lambda$ and since $\mathfrak{D}_\lambda = \mathcal{X}_+[\dot{+}]_{\mathfrak{D}_\lambda}\mathcal{X}_-$, $(\mathcal{X}_+, \mathcal{X}_-)$ is a fundamental decomposition of $\mathfrak{D}_\lambda$. The according fundamental symmetry J is defined by

$$\begin{aligned}(Jf)(z) := f_+(z) - f_-(z) := \sum_{n=\lceil|\lambda|\rceil}^{\infty} a_n z^{n+\lambda} - \sum_{n=1}^{\lfloor|\lambda|\rfloor} a_n z^{n+\lambda} = \\ = \sum_{n=1}^{\infty} c_n z^{n+\lambda+\lceil|\lambda|\rceil} - \sum_{n=1}^{\lfloor|\lambda|\rfloor} a_n z^{n+\lambda}\end{aligned}$$

for $f(z) = \sum_{n=1}^{\infty} a_n z^{n+\lambda} \in \mathfrak{D}_\lambda$ and $c_n := a_{n+\lceil|\lambda|\rceil}$. Hence $(\mathcal{X}_+, [\cdot,\cdot])$ is nothing else, but $\mathfrak{D}_{\lambda+\lceil|\lambda|\rceil}$ and therefore a Hilbert space.

For $f(z) = \sum_{n=1}^{\lfloor|\lambda|\rfloor} a_n z^{n+\lambda} \in \mathcal{X}_-$ we can calculate the J-norm by

$$\|f(z)\|_J^2 = -\sum_{n=1}^{\lfloor|\lambda|\rfloor} (n+\lambda)|a_n|^2 \geq 0.$$

Thus $\|\cdot\|_J$ on $\mathcal{X}_-$ is equivalent to the Euclidean norm on $\mathbb{C}^{\lfloor|\lambda|\rfloor}$ and therefore $(\mathcal{X}_-, -[\cdot,\cdot])$ is complete as well. Hence by Corollary 1.2.25 $(\mathfrak{D}_\lambda, [\cdot,\cdot]_{\mathfrak{D}_\lambda})$ is a Krein space and since $\mathcal{X}_-$ has finite dimension it is a Pontryagin space. □

Remark 2.3.9. Let $N \in \mathbb{N}$. Then $(\mathcal{X}_+, \mathcal{X}_-)$ with

$$\mathcal{X}_+ := \left\{ \sum_{n=1}^{\infty} a_n z^{n-N} : \, a_n = 0, \, n < N \right\}$$

$$\mathcal{X}_- := \left\{ \sum_{n=1}^{\infty} a_n z^{n-N} : \, a_n = 0, \, n > N \right\}$$

is a fundamental decomposition of $\mathfrak{D}_{-N}$. By the same means as in the proof of Theorem 2.3.8, one shows, that $(\mathcal{X}_\pm, [\cdot,\cdot]_J)$ are Hilbert spaces. But since $(\mathfrak{D}_{-N})^\circ = \mathcal{C} := \{f(z) = c : \, c \in \mathbb{C}\} \neq \{0\}$ it is not a Krein space. Nevertheless, we can define the factor space $\left(\mathfrak{D}_{-N}/_{\mathcal{C}}, [\cdot,\cdot]_{/\mathcal{C}}\right)$ as in Remark 1.2.7 to obtain a Krein space.

Chapter 3

Littlewood's subordination principle

3.1 The subordination principle for the Dirichlet space

Defintion 3.1.1. Let $b(z) := \sum_{n=0}^{\infty} b_n z^n \in \mathcal{S}_0^+$ be an arbitrary formal power series. Since the Bergman and the Dirichlet space are linear subspaces of $\mathcal{S}_0^+$, we can apply the composition operator C_b from Definition 1.1.7 to all functions $f \in A^2\,(\mathfrak{D})$, such that $f \in \operatorname{dom} C_b$. We define

$$C_b^{A^2} : \begin{cases} \operatorname{dom} C_b^{A^2} & \to & A^2 \\ f & \mapsto & C_b f \end{cases}$$

$$C_b^{\mathfrak{D}} : \begin{cases} \operatorname{dom} C_b^{\mathfrak{D}} & \to & \mathfrak{D} \\ f & \mapsto & C_b f, \end{cases}$$

where $\operatorname{dom} C_b^{A^2} := \{f \in \operatorname{dom} C_b : f, C_b f \in A^2\}$ and $\operatorname{dom} C_b^{\mathfrak{D}} := \{f \in \operatorname{dom} C_b : f, C_b f \in \mathfrak{D}\}$.

Lemma 3.1.2. *Let $b(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{S}_0^+$ be a formal power series with $R(b) > 0$ and $|b_0| < 1$. Then there exists a real constant $\delta > 0$ such that for $f \in \operatorname{dom} C_b^{\mathfrak{D}}$ ($\operatorname{dom} C_b^{A^2}$), $f(b(z)) = \left(C_b^{\mathfrak{D}} f\right)(z)$ ($f(b(z)) = \left(C_b^{A^2} f\right)(z)$) for all $z \in B_\delta(0)$.*

Proof. Since b is continuous on $B_{R(b)}(0)$, there exists $0 < \delta \leq R(b)$ such that $|b(z)| < 1$ for all $z \in B_\delta(0)$. Thus for every $f \in \operatorname{dom} C_b^{\mathfrak{D}}$ ($\operatorname{dom} C_b^{A^2}$) we can apply the second part of Lemma 1.1.13 with $r = 1$. □

Remark 3.1.3. For every $f \in \operatorname{dom} C_b^{\mathfrak{D}}$ ($\operatorname{dom} C_b^{A^2}$), the function $C_b^{\mathfrak{D}} f$ ($C_b^{A^2} f$) is an analytic continuation of $f \circ b$ on the whole unit disc.

Lemma 3.1.4. *Let $b(z) = \sum_{n=0}^{\infty} b_n z^n$ be a formal power series such that $\operatorname{dom} C_b^{\mathfrak{D}} = \mathfrak{D}$ ($\operatorname{dom} C_b^{A^2} = A^2$) and $|b_0| < 1$. Then $R(b) \geq 1$ and $|b(z)| < 1$ for all $z \in \mathbb{D}$.*

Proof. Since the function $f(z) = z$ is an element of $\mathfrak{D}$ (A^2), we have $(C_b f)(z) = b(z) \in \mathfrak{D}$ (A^2). Hence $R(b) \geq 1$ by Theorem 2.2.2, (iv).

Now assume that there exists $z_0 \in \mathbb{D}$ such that $|b(z_0)| > 1$. Then the function $g(z) := K_{\mathfrak{D}}\left(z, \overline{b(z_0)}^{-1}\right)$ is an element of $\mathfrak{D} = \operatorname{dom} C_b^{\mathfrak{D}}$. Hence the composition $C_b^{\mathfrak{D}} g$ is again in $\mathfrak{D}$ and by Lemma 3.1.2 there exists some $\delta > 0$ such that

$$(C_b^{\mathfrak{D}} g)(z) = g(b(z)) = K_{\mathfrak{D}}\left(b(z), \overline{b(z_0)}^{-1}\right) = \sum_{n=1}^{\infty} \frac{1}{n} b(z)^n b(z_0)^{-n} = \log\left(\frac{1}{1 - b(z)b(z_0)^{-1}}\right)$$

for all $z \in B_\delta(0)$. But the right hand side has no analytic continuation to the point z_0, which is a contradiction to the fact that $C_b^{\mathfrak{D}} g$ is analytic on $\mathbb{D}$.

Now assume, that $b(z_1) = 1$ for some $z_1 \in \mathbb{D}$. Then, by the maximum modulus principle we have that $b \equiv 1$ on $\mathbb{D}$, which is a contradiction to the assumption $|b_0| = |b(0)| < 1$.

The same line of proof works for the Bergman space. □

Defintion 3.1.5. Let $b : \mathbb{D} \to \mathbb{D}$ a holomorphic, injective function, such that $b(0) = 0$. We call such a function a normalized, univalent mapping of $\mathbb{D}$ into $\mathbb{D}$ and denote the family of all such functions by $\mathfrak{B}$.

Remark 3.1.6. Every $b \in \mathfrak{B}$ has a unique power series expansion of the form $b(z) = \sum_{n=1}^{\infty} b_n z^n$ with radius of convergence $R(b) \geq 1$.

Theorem 3.1.7 (Littlewood subordination theorem for the Dirichlet space)**.** *Let $b \in \mathfrak{B}$ be a normalized, univalent mapping from $\mathbb{D}$ into $\mathbb{D}$. Then the domain of the composition operator* $\operatorname{dom} C_b^{\mathfrak{D}}$ *is the whole space $\mathfrak{D}$ and*

$$(C_b^{\mathfrak{D}} f)(z) = f(b(z))$$

for all $z \in \mathbb{D}$. Furthermore $C_b^{\mathfrak{D}}$ is a contraction operator (i.e. $C_b^{\mathfrak{D}}$ is bounded with norm $\|C_b^{\mathfrak{D}}\| \leq 1$).

Proof. According to Theorem 1.1.10, the domain of the composition operator C_b is the whole space of formal power series $\mathcal{S}_0^+$.

Let $f(z) \in \mathfrak{D}$. Then by Theorem 2.2.2, *(iv)* we know that $R(f) \geq 1$. Since $|b(z)| < 1$ for all $z \in \mathbb{D}$, we can apply the second part of Lemma 1.1.13 with $r = 1$ and obtain, that $R(C_b f) \geq 1$ and $(C_b f)(z) = f(b(z))$ for all $z \in \mathbb{D}$. Hence, $C_b f = f \circ b$ is holomorphic on $\mathbb{D}$.

If we split b in its real and imaginary part, $b(x + iy) = u(x, y) + i\, v(x, y)$ and use the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}, \end{aligned}$$

we get

$$\det |\mathrm{D}b| = \left| \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |b'|^2 .$$

Using the transformation rule and the fact that $b(\mathbb{D}) \subseteq \mathbb{D}$, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \left|\frac{d}{dz}(C_b f)(z)\right|^2 dA &= \int_{\mathbb{D}} \left|\frac{d}{dz} f(b(z))\right|^2 dA \\ &= \int_{\mathbb{D}} \left|f'(b(z))\right|^2 \left|b'(z)\right|^2 dA \\ &= \int_{b(\mathbb{D})} \left|f'(z)\right|^2 dA \\ &\leq \int_{\mathbb{D}} \left|f'(z)\right|^2 dA < +\infty \end{aligned} \tag{3.1}$$

by Theorem 2.2.2, *(iv)*.

Using again Theorem 2.2.2, *(iv)*, inequality (3.1) yields $C_b f \in \mathfrak{D}$. Hence, $\operatorname{dom} C_b^{\mathfrak{D}} = \mathfrak{D}$. Furthermore by Theorem 2.2.3

$$\|C_b^{\mathfrak{D}} f\|_{\mathfrak{D}}^2 = \|C_b f\|_{\mathfrak{D}}^2 = \int_{\mathbb{D}} \left|\frac{d}{dz}(C_b f)(z)\right|^2 dA \leq \int_{\mathbb{D}} \left|f'(z)\right|^2 dA = \|f\|_{\mathfrak{D}}^2$$

which shows $\|C_b^{\mathfrak{D}}\| \leq 1$. □

Although the condition of Theorem 3.1.7 is necessary for a function $b : \mathbb{D} \to \mathbb{D}$ with $b(0) = 0$ to be univalent, unfortunately it is not sufficient. Our goal throughout the remainder of this section will be, to expand the Littlewood subordination theorem to a superset of $\mathfrak{D}$, that this condition is sufficient.

3.2 The subordination principle for generalized Dirichlet spaces

Defintion 3.2.1. Let $\lambda \in \mathbb{R}$ and $b(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{S}^+$ be a formal power series such that $b_1 \neq 0$. Then we define the composition operator C_b^λ on $\mathcal{S}_\lambda$ by

$$C_b^\lambda : \begin{cases} \mathcal{S}_\lambda & \to \mathcal{S}_\lambda \\ z^\lambda g(z) & \mapsto z^\lambda \left(\frac{b(z)}{z}\right)^\lambda (C_b g)(z) \end{cases}$$

where $\frac{b(z)}{z} := \sum_{n=1}^{\infty} b_n z^{n-1} = \sum_{n=0}^{\infty} b_{n+1} z^n$. Note, that the expression $\left(\frac{b(z)}{z}\right)^\lambda$ is well defined and an element of $\mathcal{S}_0^+$ by Definition 1.1.15, since $b_1 \neq 0$. Moreover $g(z) \in \operatorname{dom} C_b$, since due to Theorem 1.1.10 $\operatorname{dom} C_b = \mathcal{S}_0^+$. Because of $g(z) \in \mathcal{S}^+$ the formal power series $\left(\frac{b(z)}{z}\right)^\lambda (C_b g)(z)$ is also an element of $\mathcal{S}^+$ and therefore C_b^λ maps indeed into the space S_λ.

Since $\mathfrak{D}_\lambda \subseteq S_\lambda$, we can apply the composition operator C_b^λ to functions in $\mathfrak{D}_\lambda$.

Remark 3.2.2. Let $b \in \mathfrak{B}$. For $\lambda \in \mathbb{Z}$ the elements from $\mathfrak{D}_\lambda$ can be interpreted as analytic functions on $\mathbb{D} \setminus \{0\}$, and the operator C_b^λ acts just as $f \mapsto f \circ b$.

For $\lambda \in \mathbb{R} \setminus \mathbb{Z}$ we can interpret the elements from $\mathfrak{D}_\lambda$ as functions on $\mathbb{D} \setminus \{0\}$ as in Remark 2.3.2, *(v)*, with a fixed $\theta \in \mathbb{R}$. Here C_b^λ no longer acts as $f \mapsto f \circ b$. Nevertheless, for $f \in \mathfrak{D}_\lambda$ the quotient of $(C_b^\lambda f)(z)$ and $f(b(z))$ is the quotient of two possibly different branches of the function z^λ. Hence, $(C_b^\lambda f)(z) = f(b(z)) \cdot \psi(z)$ with an unimodular function $\psi(z)$.

Our goal throughout this section will be to prove the following theorem.

Theorem 3.2.3 (Littlewood subordination principle for generalized Dirichlet spaces)**.** *Let λ be any real number and $b \in \mathfrak{B}$. Then the composition operator C_b^λ is a contraction from $\mathfrak{D}_\lambda$ to itself.*

For proving this theorem we need the following classical result from real analysis:

Theorem 3.2.4 (Green's theorem)**.** *Let $\gamma : [a,b] \to \mathbb{R}^2$ be a positive oriented, piecewise smooth, simple closed curve, Ω the region bounded by γ and D an open set containing $\Omega \cup \gamma$. Let further p, $q \in C(D;\mathbb{R}) \cap C^1(\Omega;\mathbb{R})$, then*

$$\int_\Omega \frac{\partial}{\partial x} p(x,y) + \frac{\partial}{\partial y} q(x,y)\, d\lambda_2(x,y) = \int_\gamma p(x,y)\, dy - q(x,y)\, dx, \tag{3.2}$$

where

$$\int_\gamma q(x,y)\, dx := \int_a^b q\left(\gamma\left(t\right)\right) \cdot \gamma_1'(t)\, dt, \int_\gamma p(x,y)\, dy := \int_a^b p\left(\gamma(t)\right) \cdot \gamma_2'(t)\, dt,$$

holds.

Proof. A proof can be found in [Rud70]. □

Remark 3.2.5. By splitting p and q into their real and imaginary parts, one checks quickly that Theorem 3.2.4 also holds for complex valued p and q. By identifying the complex plane with $\mathbb{R}^2$ equation (3.2) holds, where

$$\int_\gamma q\, dx := \int_a^b q\left(\gamma\left(t\right)\right) \cdot \mathrm{Re}\gamma(t)\, dt, \qquad \int_\gamma p\, dy := \int_a^b p\left(\gamma(t)\right) \cdot \mathrm{Im}\gamma(t)\, dt,$$

for a positive oriented, piecewise smooth, simple closed curve γ with interior Ω and p, q and their partial derivatives are holomorphic in Ω and continuous on an open set $D \supseteq \Omega$.

Corollary 3.2.6 (Green's theorem, complex version)**.** *Let $\gamma \subseteq \mathbb{C}$ be a piecewise smooth, positive oriented, simple closed curve and $\Omega \subseteq \mathbb{C}$ its interior. Let further f, g be holomorphic functions on Ω which are continuous on an open set D containing $\Omega \cup \gamma$ such that f', g' are continuous on Ω. Then*

$$\int_\Omega f\left(z\right) \overline{g'\left(z\right)}\, d\lambda_2(z) = \frac{1}{2i} \oint_\gamma f(z) \overline{g(z)}\, dz$$

holds.

Proof. If we use the Wirtinger derivatives

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

then by using the Cauchy-Riemann equations we obtain for any holomorphic function $h(x+iy) = u(x,y) + iv(x,y)$

$$\begin{aligned}\frac{\partial h(z)}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial h(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right) = 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \overline{h(z)}}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial \overline{h(z)}}{\partial x} + i\frac{\partial \overline{h(z)}}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial x} = \overline{h'(z)}.\end{aligned}$$

Because for holomorphic functions h, k

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}\left(h(z)k(z)\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(h(z)k(z)\right) + \frac{\partial}{\partial y}\left(h(z)k(z)\right)\right)\\ &= \frac{1}{2}\left(\frac{\partial h(z)}{\partial x}k(z) + h(z)\frac{\partial k(z)}{\partial x} + i\frac{\partial h(z)}{\partial y}k(z) + ih(z)\frac{\partial k(z)}{\partial y}\right)\\ &= \frac{\partial h(z)}{\partial \bar{z}}k(z) + h(z)\frac{\partial k(z)}{\partial \bar{z}}\end{aligned}$$

the product rule also holds for the the Wirtinger derivative $\frac{\partial}{\partial \bar{z}}$ and

$$\begin{aligned}\frac{1}{2i}\oint_\gamma f(z)\overline{g(z)}\,dz &= \frac{1}{2}\int_\gamma f(z)\overline{g(z)}\,dy - if(z)\overline{g(z)}\,dx\\ &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial x}f(z)\overline{g(z)} + i\frac{\partial}{\partial y}f(z)\overline{g(z)}\,d\lambda_2(z)\\ &= \int_\Omega \frac{\partial}{\partial \bar{z}}f(z)\overline{g(z)}\,d\lambda_2(z)\\ &= \int_\Omega \frac{\partial f(z)}{\partial \bar{z}}\overline{g(z)} + \frac{\partial \overline{g(z)}}{\partial \bar{z}}f(z)\,d\lambda_2(z)\\ &= \int_\Omega \overline{g'(z)}f(z)\,d\lambda_2(z)\end{aligned}$$

follows. □

Now the proof of Theorem 3.2.3 is basically contained in the following lemma:

Lemma 3.2.7. *Let λ be a real number, $f(z) = \sum_{n=1}^{\infty} a_n z^{n+\lambda} \in \mathfrak{D}_\lambda$ and $b \in \mathfrak{B}$. Then*

$$\sum_{n=1}^{\infty}(n+\lambda)|a_n|^2 - \sum_{n=1}^{\infty}(n+\lambda)|c_n|^2 = \int_{\mathbb{D}\setminus b(\mathbb{D})} |f'|^2 dA \tag{3.3}$$

holds, where $\sum_{n=1}^{\infty} c_n z^{n+\lambda} = (C_b^\lambda f)(z)$.

Proof. If $b(z) = \phi z$ with $\phi \in \mathbb{T}$ both sides of equation (3.3) are zero. So we may assume by Schwarz's Lemma, that $|b(z)| < |z|$ for all $z \in \mathbb{D}$.

Let δ be a real number with $0 < \delta < 1$ and $C_\delta := \{z \in \mathbb{C} : |z| = \delta\}$. Then $|b(C_\delta)| < \delta$ and because b is one to one and continuous $b(C_\delta)$ is again a closed curve located in the interior of C_δ. Furthermore let $m \in C_\delta$ be a point satisfying $r := |b(m)| = \max_{z\in C_\delta} |b(z)|$, θ such that $b(m) = re^{i\theta}$, and $\varepsilon > 0$ arbitrary, we define a curve $\gamma_{\delta,\varepsilon}$ by

$$\gamma_{\delta,\varepsilon} := \gamma_1 - \gamma_2 - \gamma_3 + \gamma_4.$$

with

$$\begin{aligned} \gamma_1(t) &:= \delta e^{it}, & t &\in [\theta+\varepsilon, \theta+2\pi-\varepsilon] \\ \gamma_2(t) &:= te^{i(\theta+2\pi-\varepsilon)}, & t &\in [r_2, \delta] \\ \gamma_3(t) &:= b\left(\delta e^{it}\right), & t &\in [\theta_1, \theta_2] \\ \gamma_4(t) &:= te^{i(\theta+\varepsilon)}, & t &\in [r_1, \delta]. \end{aligned}$$

Here θ_1, θ_2, r_1, r_2 are chosen such that $b(\delta e^{i\theta_1}) = r_1 e^{i(\theta+\varepsilon)}$ and $b(\delta e^{i\theta_2}) = r_2 e^{i(\theta+2\pi-\varepsilon)}$. We call the interior of this curve $\Omega_{\delta,\varepsilon}$.

We interpret z^λ on $\mathbb{D}\setminus\{0\}$ as $(re^{i\phi})^\lambda := r^\lambda e^{i\lambda\phi}$, where $\phi \in [\theta, \theta+2\pi)$; see Remark 2.3.2. Interpreting $f(z)$ accordingly, f and f' are holomorphic on $\Omega_{\delta,\epsilon} \cup \gamma_{\delta,\epsilon}$. With Theorem 3.2.6 we obtain

$$\begin{aligned} \pi \int_{\Omega_{\delta,\varepsilon}} \left|f'(z)\right|^2 dA(z) &= \int_{\Omega_{\delta,\varepsilon}} \left|f'(z)\right|^2 d\lambda_2(z) \\ &= \frac{1}{2i}\oint_{\gamma_{\delta,\varepsilon}} f'(z)\overline{f(z)}dz \\ &= \frac{1}{2i}\oint_{\gamma_1} f'(z)\overline{f(z)}dz - \frac{1}{2i}\oint_{\gamma_2} f'(z)\overline{f(z)}dz - \\ &- \frac{1}{2i}\oint_{\gamma_3} f'(z)\overline{f(z)}dz + \frac{1}{2i}\oint_{\gamma_4} f'(z)\overline{f(z)}dz. \end{aligned} \tag{3.4}$$

Note that for $f(z) = z^\lambda g(z) \in \mathcal{S}_\lambda$ with $R(g) \geq 1$, the function $z \mapsto f'(z)\overline{f(z)}$ does not depend on the branch of z^λ that we chose above and has therefore a continuous continuation to $\mathbb{D} \setminus \{0\}$. In fact,

$$\begin{aligned} f'(re^{i\phi})\overline{f(re^{i\phi})} &= \sum_{n=1}^{\infty}(n+\lambda)a_n(re^{i\phi})^{\lambda+n-1}\sum_{m=1}^{\infty}\overline{a}_m\overline{re^{i\phi}}^{\lambda+m} \\ &= r^{2\lambda}\sum_{n=1}^{\infty}(n+\lambda)a_n r^{n-1}e^{i(n-1)\phi}\sum_{m=1}^{\infty}\overline{a}_m r^m e^{-im\phi} \\ &= r^{2\lambda}\sum_{n=1}^{\infty}(n+\lambda)a_n r^{n-1}e^{i(n-1)(\phi+2\pi)}\sum_{m=1}^{\infty}\overline{a}_m r^m e^{-im(\phi+2\pi)} \\ &= f'(re^{i(\phi+2\pi)})\overline{f(re^{i(\phi+2\pi)})}, \end{aligned} \tag{3.5}$$

for all $\phi \in \mathbb{R}$, $0 < r < 1$.

For fixed $\delta > 0$ the functions $f(z)$, $f'(z)$ are uniformly bounded on $\overline{B_\delta(0)} \setminus b(B_\delta(0)) \supseteq \Omega_{\delta,\varepsilon}$. Thus by the dominated convergence theorem, equation (3.4) becomes

$$\begin{aligned} \pi\int_{\Omega_\delta}\left|f'(z)\right|^2 dA(z) &= \int_{\Omega_\delta}\left|f'(z)\right|^2 d\lambda_2(z) = \frac{1}{2i}\oint_{\gamma_\delta} f'(z)\overline{f(z)}dz = \\ &= \frac{1}{2i}\oint_{\gamma_1} f'(z)\overline{f(z)}dz - \frac{1}{2i}\oint_{\gamma_2} f'(z)\overline{f(z)}dz - \\ &\quad - \frac{1}{2i}\oint_{\gamma_3} f'(z)\overline{f(z)}dz + \frac{1}{2i}\oint_{\gamma_4} f'(z)\overline{f(z)}dz. \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} \gamma_1(t) &:= \delta e^{it}, & t &\in [\theta, \theta+2\pi] \\ \gamma_2(t) &:= te^{i(\theta+2\pi)}, & t &\in [r, \delta] \\ \gamma_3(t) &:= b\left(\delta e^{it}\right), & t &\in [\theta, \theta+2\pi] \\ \gamma_4(t) &:= te^{i\theta}, & t &\in [r, \delta]. \end{aligned}$$

Since $f'(z)\overline{f(z)}$ is continuous on $\mathbb{D} \setminus \{0\}$, we get

$$\frac{1}{2i}\oint_{\gamma_4} f'(z)\overline{f(z)}\,dz - \frac{1}{2i}\oint_{\gamma_2} f'(z)\overline{f(z)}\,dz = 0.$$

We calculate

$$\begin{aligned} \frac{1}{2i}\oint_{\gamma_1} f'(z)\overline{f(z)}\,dz &= \frac{1}{2i}\int_\theta^{\theta+2\pi}\delta^{2\lambda}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n+\lambda)a_n\overline{a}_m\delta^{n-1}e^{it(n-1)}\delta^m e^{-itm}i\delta e^{it}\,dt \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n+\lambda)a_n\overline{a}_m\delta^{m+n+2\lambda}\int_\theta^{\theta+2\pi}e^{i(n-m)t}dt \\ &= \pi\sum_{n=1}^{\infty}(n+\lambda)|a_n|^2\delta^{2n+2\lambda}. \end{aligned}$$

As $C_b^\lambda f$ also belongs to $\mathcal{S}_\lambda$, we also have

$$\pi\sum_{n=1}^{\infty}(\lambda+n)|c_n|^2\delta^{2n+2\lambda} = \frac{1}{2i}\oint_{\gamma_1}(C_b^\lambda f)'(z)\overline{(C_b^\lambda f)(z)}\,dz.$$

The function $z \mapsto f(b(z))$ is holomorphic on $\mathbb{D}\setminus b^{-1}(e^{i\theta}\cdot[0,+\infty))$, which is dense in $\mathbb{D}$. Moreover, by Remark 3.2.2, $z\mapsto f(b(z))$ coincides with $z\mapsto C_b^\lambda f(z)$ up to a multiplicative unimodular function. As both functions are holomorphic on $\mathbb{D}\setminus(b^{-1}(e^{i\theta}\cdot[0,+\infty))\cup e^{i\theta}\cdot[0,+\infty))$ this unimodular function is constant on the components of this open subset of $\mathbb{D}$. Hence, for z belonging to this open set we have

$$(C_b^\lambda f)'(z)\overline{(C_b^\lambda f)(z)} = e^{i\mu}f(b(z))'\overline{e^{i\mu}f(b(z))} = f'(b(z))\overline{f(b(z))}b'(z).$$

Since both sides of this equation have a continuous continuation to $\mathbb{D}\setminus\{0\}$ and as an intersection of two open and dense sets $\mathbb{D}\setminus(b^{-1}(e^{i\theta}\cdot[0,+\infty))\cup e^{i\theta}\cdot[0,+\infty))$ is dense in $\mathbb{D}\setminus\{0\}$, we have

$$(C_b^\lambda f)'(z)\overline{C_b^\lambda f(z)} = f'(b(z))\overline{f(b(z))}b'(z),$$

on $\mathbb{D}\setminus\{0\}$. Thus, we obtain

$$\begin{aligned}\pi\sum_{n=1}^{\infty}(\lambda+n)|c_n|^2\delta^{2n+2\lambda} &= \frac{1}{2i}\oint_{\gamma_1} f'(b(z))\overline{f(b(z))}b'(z)\,dz\\ &= \frac{1}{2i}\oint_{b(\gamma_1)} f'(z)\overline{f(z)}\,dz\\ &= \frac{1}{2i}\oint_{\gamma_3} f'(z)\overline{f(z)}\,dz.\end{aligned}$$

All in all (3.6) becomes

$$\pi\int_{\Omega_\delta}\left|f'(z)\right|^2 dA = \pi\sum_{n=1}^{\infty}(\lambda+n)|a_n|^2\delta^{2n+2\lambda} - \pi\sum_{n=1}^{\infty}(\lambda+n)|c_n|^2\delta^{2n+2\lambda}.$$

Since

$$\int_{\Omega_\delta}\left|f'(z)\right|^2 dA = \int_{\mathbb{D}}\chi_{\Omega_\delta}(z)\left|f'(z)\right|^2 dA \le$$
$$\le \delta^{2+2\lambda}\left(\sum_{n=1}^{\infty}(\lambda+n)|a_n|^2 + \sum_{n=1}^{\infty}(\lambda+n)|b_n|^2\right) < +\infty$$

we can apply the dominated convergence theorem and by calculating the limit $\delta\nearrow 1$ we obtain equation (3.3). □

After this preliminaries the proof of Theorem 3.2.3 is no longer difficult:

Proof of Theorem 3.2.3. For $f(z) = \sum_{n=1}^{\infty} a_n z^{n+\lambda} \in \mathfrak{D}_\lambda$ Lemma 3.2.7 asserts

$$\begin{aligned}\left[C_b^\lambda f, C_b^\lambda f\right]_{\mathfrak{D}_\lambda} &\leq \left[C_b^\lambda f, C_b^\lambda f\right]_{\mathfrak{D}_\lambda} + \int_{\mathbb{D}\setminus b(\mathbb{D})} |f'|^2 dA \\ &= \sum_{n=1}^{\infty} (n+\lambda)|c_n|^2 + \int_{\mathbb{D}\setminus b(\mathbb{D})} |f'|^2 dA \\ &= \sum_{n=1}^{\infty} (n+\lambda)|a_n|^2 = [f, f]_{\mathfrak{D}_\lambda} < +\infty.\end{aligned}$$

Hence, $C_b^\lambda f \in \mathfrak{D}_\lambda$ and C_b^λ is a contraction.

□

Chapter 4

De Branges' univalence criterion

In the previous section we have seen, that for an analytic function $b : \mathbb{D} \to \mathbb{D}$ with $b(0) = 0$, $b'(0) \neq 0$ to be univalent, it is a necessary condition that the composition operator $C_b^\lambda : \mathfrak{D}_\lambda \to \mathfrak{D}_\lambda$ is a contraction. In the following section we will see, that the composition operator being a contraction on a certain space is already sufficient for such a function b to be univalent.

Defintion 4.0.1. We denote the vector space of all formal power series with also negative powers by $\mathcal{S} := \left\{\sum_{k\in\mathbb{Z}} a_k z^k : a_k \in \mathbb{C}\right\}$ and define operators $P_\pm$, P_0 and S on $\mathcal{S}$ by

$$\begin{aligned} (P_+ f)(z) &:= \sum_{k=1}^{\infty} a_k z^k, \\ (P_0 f)(z) &:= a_0, \\ (P_- f)(z) &:= \sum_{k=1}^{\infty} a_{-k} z^{-k} \end{aligned}$$

and

$$(Sf)(z) := f\left(z^{-1}\right).$$

P_+ and P_- are the projections on the subspaces of power series $\mathcal{S}^+$ ($\mathcal{S}^-$) with only positive (negative) exponents. $P_+ + P_0$ is the projection on the space

$$\mathfrak{D}^0 := \left\{\sum_{n=0}^{\infty} a_n z^n : \sum_{n=1}^{\infty} a_n z^n \in \mathfrak{D}\right\}$$

.

Further we define the space $\mathcal{I}\mathfrak{D}$ by

$$\mathcal{I}\mathfrak{D} := \bigcup_{N\in\mathbb{N}} \mathfrak{D}_{-N}.$$

We equip $\mathcal{I}\mathfrak{D}$ with the indefinite inner product

$$\begin{aligned} \left[\sum_{k=-N_1}^{\infty} a_k z^k, \sum_{k=-N_2}^{\infty} b_k z^k\right]_{\mathcal{I}\mathfrak{D}} &:= \left[\sum_{k=-N_1}^{\infty} a_k z^k, \sum_{k=-N_2}^{\infty} b_k z^k\right]_{\mathfrak{D}_{\min\{-N_1,-N_2\}}} = \\ &= \sum_{k=\min\{-N_1,-N_2\}}^{\infty} k a_k \bar{b}_k. \end{aligned}$$

Remark 4.0.2. The elements of $\mathcal{I}\mathfrak{D}$ are Laurent series with only finite negative powers. They can be interpreted as analytic functions on $\mathbb{D} \setminus \{0\}$ and can have a pole of finite order at 0.

Defintion 4.0.3. For a given function $b \in \mathcal{S}^+$, $b'(0) \neq 0$, we define the composition operator on $\mathcal{I}\mathfrak{D}$ in the following way.

$$C_b^{\mathcal{I}\mathfrak{D}} : \begin{cases} \operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}} & \to \quad \mathcal{I}\mathfrak{D} \\ f(z) := \sum_{n=-N}^{\infty} a_n z^n & \mapsto \quad C_b^{-N} f, \end{cases}$$

where

$$\operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}} := \left\{ f \in \mathcal{I}\mathfrak{D} : \, C_b^{\mathcal{I}\mathfrak{D}} f \in \mathcal{I}\mathfrak{D} \right\}.$$

Remark 4.0.4.

(i) Since for $k \in \mathbb{Z}$ by Remark 3.2.2 $C_b^{\mathfrak{D}_k} f = f \circ b$, we have $\left(C_b^{\mathcal{I}\mathfrak{D}} f\right)(z) = f(b(z))$, for $z \in \mathbb{D} \setminus \{0\}$.

(ii) If $b \in \mathcal{S}^+$ is such that $\operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}} = \mathcal{I}\mathfrak{D}$, then because $z \mapsto z^{-1} \in \mathcal{I}\mathfrak{D}$, the function b has no zeros in $\mathbb{D} \setminus \{0\}$. Otherwise $C_b^{\mathcal{I}\mathfrak{D}} \left(z \mapsto z^{-1}\right)$ would have a pole of at least order one in $\mathbb{D} \setminus \{0\}$.

(iii) For $b \in \mathfrak{B}$ the operator $C_b^{\mathcal{I}\mathfrak{D}}$ is a contraction from $\mathcal{I}\mathfrak{D}$ to itself, since for $\sum_{n=-N}^{\infty} a_n z^n \in \mathcal{I}\mathfrak{D}$

$$\left[C_b^{\mathcal{I}\mathfrak{D}} f, C_b^{\mathcal{I}\mathfrak{D}} f\right]_{\mathcal{I}\mathfrak{D}} = \left[C_b^{-N} f, C_b^{-N} f\right]_{\mathfrak{D}_{-N}} \leq [f, f]_{\mathfrak{D}_{-N}} = [f, f]_{\mathcal{I}\mathfrak{D}} \, ,$$

by Theorem 3.2.3.

Defintion 4.0.5. For any formal power series $f(z) = \sum_{k \in \mathbb{Z}} a_k z^k \in \mathcal{S}$, we define

$$f^\sharp(z) := \sum_{k \in \mathbb{Z}} \bar{a}_k z^k.$$

Note that f and $f^\sharp$ have the same domain of analyticity and for z such that f is analytic in z

$$f^\sharp(z) = \overline{f(\bar{z})}.$$

Our goal for the remainder of this section will be to prove the following result:

Theorem 4.0.6. *Let $b : \mathbb{D} \to \mathbb{D}$ be holomorphic with $b'(0) \neq 0$. Then the following three statements are equivalent:*

(i) $b \in \mathfrak{B}$.

(ii) The operator $C_b^{\mathcal{I}\mathfrak{D}} : \mathcal{I}\mathfrak{D} \to \mathcal{I}\mathfrak{D}$ is a contraction with respect to $[\cdot, \cdot]_{\mathcal{I}\mathfrak{D}}$.

(iii) $C_b^{\mathfrak{D}}$ is bounded and there exists a well defined contraction operator $G_b : \mathfrak{D} \to \mathfrak{D}$ such that

$$G_b|_{\mathcal{P}} = P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* \Big|_{\mathcal{P}} . \tag{4.1}$$

Defintion 4.0.7. We define the indefinite inner product space $\mathcal{K}\mathfrak{D}$ by

$$\mathcal{K}\mathfrak{D} := \left\{ \sum_{k\in\mathbb{Z}} a_k z^k : \sum_{k=1}^{\infty} a_k z^k, \sum_{k=1}^{\infty} a_{-k} z^k \in \mathfrak{D} \right\}$$

and endow it with the inner product

$$\left[\sum_{k\in\mathbb{Z}} a_k z^k, \sum_{k\in\mathbb{Z}} b_k z^k \right]_{\mathcal{K}\mathfrak{D}} := \sum_{k\in\mathbb{Z}} k a_k \bar{b}_k + a_0 \bar{b}_0.$$

Remark 4.0.8.

(i) If we endow $\mathfrak{D}^0 \subseteq \mathcal{K}\mathfrak{D}$ with $[\cdot,\cdot]_{\mathcal{K}\mathfrak{D}}$, it can be interpreted as a one dimensional extension of $\mathfrak{D}$. Thus, $\left(\mathfrak{D}^0, [\cdot,\cdot]_{\mathcal{K}\mathfrak{D}}\right)$ is again a Hilbert space.

(ii) $((P_+ + P_0)(\mathcal{K}\mathfrak{D}), P_-(\mathcal{K}\mathfrak{D})) = \left(\mathfrak{D}^0, S\mathfrak{D}\right)$ is a fundamental decomposition of $\mathcal{K}\mathfrak{D}$. For $f = Sg \in S\mathfrak{D}$

$$-[f,f]_{\mathcal{K}\mathfrak{D}} = \|Sf\|_{\mathfrak{D}}^2 = \|SSg\|_{\mathfrak{D}}^2 = \|g\|_{\mathfrak{D}}^2,$$

and the spaces $(\mathfrak{D}^0, [\cdot,\cdot]_{\mathcal{K}\mathfrak{D}})$, $(S\mathfrak{D}, -[\cdot,\cdot]_{\mathfrak{D}})$ are Hilbert spaces. Therefore, $\mathcal{K}\mathfrak{D}$ is a Krein space.

The J-norm of $f(z) = \sum_{k\in\mathbb{Z}}^{\infty} a_k z^k \in \mathcal{K}\mathfrak{D}$ can be calculated by

$$\|f\|_J^2 = [P_+ f, f]_{\mathcal{K}\mathfrak{D}} + [P_0, f]_{\mathcal{K}\mathfrak{D}} - [P_- f, f]_{\mathcal{K}\mathfrak{D}} = \\ = \sum_{n=1}^{\infty} n|a_n|^2 + |a_0|^2 - \sum_{n=1}^{\infty} n|a_{-n}|^2 = \sum_{k\in\mathbb{Z}} |k||a_k|^2 + |a_0|^2.$$

(iii) Let $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k \in \mathcal{K}\mathfrak{D}$ be arbitrary. Then the functions $f_n(z) := \sum_{k=-n}^{\infty} a_k z^k$ form a sequence in $\mathcal{I}\mathfrak{D}$ with

$$\lim_{n\to\infty} \|f - f_n\|_J^2 = \lim_{n\to\infty} \left\| \sum_{k\le -n-1} a_k z^k \right\|_J^2 = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} k|a_{-k}|^2 = 0,$$

since $SP_- f = \sum_{k=1}^{\infty} a_k z^k \in \mathfrak{D}$. Hence, $\mathcal{I}\mathfrak{D}$ is dense in $\mathcal{K}\mathfrak{D}$ with respect to $\|\cdot\|_J$.

Defintion 4.0.9. Let $M \subseteq \mathbb{C}$ be a one dimensional manifold. Then we denote the surface measure of M by μ_M and use the notation $L_2(M)$ for the space of all square integrable functions on M with respect to μ_M.

Remark 4.0.10.

(i) If we set $M = \mathbb{T}$, the $L_2(\mathbb{T})$-norm of $f \in L_2(\mathbb{T})$ can be calculated by

$$\|f\|_{L_2(\mathbb{T})}^2 = \int_{\mathbb{T}} |f(z)|^2 \, d\mu_{\mathbb{T}}(z) = \int_0^{2\pi} |f(e^{it})|^2 \, dt$$

(ii) Let $b : \mathbb{T} \to \mathbb{C}$ be analytic and one to one. Then the $L_2(b(\mathbb{T}))$-norm of $f \in L_2(b(\mathbb{T}))$ can be calculated by

$$\|f\|^2_{L_2(b(\mathbb{T}))} = \int_{b(\mathbb{T})} |f(z)|^2 \, d\mu_{b(\mathbb{T})}(z) =$$
$$= \int_0^{2\pi} |f(b(e^{it}))|^2 |b'(e^{it})| \, dt = \int_{\mathbb{T}} |(f \circ b)(\zeta)|^2 |b'(\zeta)| \, d\mu_{\mathbb{T}}(\zeta)$$

(iii) If there exist $C_1, C_2 > 0$ such that $C_1 \leq |b'(\zeta)| \leq C_2$ for all $\zeta \in \mathbb{T}$, we have

$$C_1 \|f \circ b\|^2_{L_2(\mathbb{T})} \leq \|f\|^2_{L_2(b(\mathbb{T}))} \leq C_2 \|f \circ b\|^2_{L_2(\mathbb{T})}.$$

Lemma 4.0.11. *Let $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k \in \mathcal{K}\mathfrak{D}$ such that f is analytic on an annulus containing $\mathbb{T}$. Then $f \in L_2(\mathbb{T})$ and there exists a real constant $C \geq 0$ such that*

$$\|f(z)\|_{L_2(\mathbb{T})} \leq C \|f(z)\|_J.$$

Proof. We can compute the $L_2(\mathbb{T})$-norm squared of f by

$$\begin{aligned}
\int_{\mathbb{T}} |f(z)|^2 \, d\mu_{\mathbb{T}}(z) &= \int_{\mathbb{T}} \sum_{k\in\mathbb{Z}} a_k z^k \sum_{j\in\mathbb{Z}} \bar{a}_j \bar{z}^j \, d\mu_{\mathbb{T}}(z) \\
&= \sum_{k,j\in\mathbb{Z}} a_k \bar{a}_j \int_{\mathbb{T}} z^k \bar{z}^j \, d\mu_{\mathbb{T}}(z) \\
&= \sum_{k,j\in\mathbb{Z}} a_k \bar{a}_j \int_0^{2\pi} e^{i\varphi(k-j)} \, d\varphi \\
&= 2\pi \sum_{k\in\mathbb{Z}} |a_k|^2 \\
&\leq 2\pi \left(\sum_{k\in\mathbb{Z}} |k||a_k|^2 + |a_0|^2 \right) = 2\pi \|f(z)\|_J^2 < +\infty
\end{aligned}$$

since the involved series converge uniformly. □

Lemma 4.0.12. *Let $b : B_{1+\varepsilon}(0) \to \mathbb{C}$ be an analytic, univalent function for some $\varepsilon > 0$, such that $b(0) = 0$ and there exists $r \in \mathbb{R}$, $0 < r < 1$ with $b(\mathbb{D}) \subseteq B_r(0)$. Let further p_n be a sequence of polynomials in z^{-1} such that $p_n(\frac{1}{z}) \in \mathcal{P}_0$ and $\lim_{n\to\infty} \|p_n\|_J = 0$. If the sequence $C_b^{\mathcal{K}\mathfrak{D}} p_n$ converges to some $g \in \mathcal{K}\mathfrak{D}$ with respect to $\|\cdot\|_J$, then $g = 0$.*

Proof. Since convergence in $\|\cdot\|_J$ dominates convergence in $\|\cdot\|_{L_2(\mathbb{T})}$, we have

$$\lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} p_n - g\|_{L_2(\mathbb{T})} = 0.$$

Thus there exists a subsequence p_{n_k} such that

$$\lim_{n\to\infty} \left(C_b^{\mathcal{I}\mathfrak{D}} p_{n_k} \right)(z) = g(z)$$

for almost every $z \in \mathbb{T}$.

Since b is analytic and univalent on $B_{1+\varepsilon}$ its derivative b' is bounded and bounded away from zero on $\mathbb{T}$. Therefore, the sequence p_{n_k} is a Cauchy-sequence in the space $L_2(b(\mathbb{T}))$ by Remark 4.0.10. Thus there exists $h \in L_2(b(\mathbb{T}))$ such that

$$\lim_{n\to\infty} \|p_{n_k} - h\|_{L_2(b(\mathbb{T}))} = 0.$$

Again there exists a subsequence of p_{n_k} which we will call p_{n_i} such that $\lim_{n_i\to\infty} p_{n_i}(b(z)) = h(b(z))$ for almost every $z \in \mathbb{T}$. By uniqueness of the limit, we have $h \circ b = g$ almost everywhere on $\mathbb{T}$.

Now let $z \in \mathbb{C} \setminus \overline{b(\mathbb{D})}$. Then we obtain for $R > 1$

$$2\pi i p_n(z) = \oint_{\partial B_R(0)} \frac{p_n(\zeta)}{z-\zeta}\, d\zeta - \oint_{b(\mathbb{T})} \frac{p_n(\zeta)}{z-\zeta}\, d\zeta. \tag{4.2}$$

by Cauchy's integral formula. Because of

$$\begin{aligned}
\lim_{R\to\infty} \left| \oint_{\partial B_R(0)} \frac{p_n(\zeta)}{z-\zeta}\, d\zeta \right| &= \lim_{R\to\infty} \left| \int_0^{2\pi} \frac{p_n(Re^{it})}{z - Re^{it}} iRe^{it}\, dt \right| \\
&\leq \lim_{R\to\infty} \int_0^{2\pi} \frac{|p_n(Re^{it})|}{|\frac{z}{Re^{it}} - 1|}\, dt \\
&= 0,
\end{aligned}$$

we can write (4.2) as

$$\begin{aligned}
p_n(z) &= -\frac{1}{2\pi i} \oint_{b(\mathbb{T})} \frac{p_n(\zeta)}{z-\zeta}\, d\zeta \\
&= -\frac{1}{2\pi i} \int_0^{2\pi} \frac{p_n(b(e^{it}))}{z - b(e^{it})} b'(e^{it}) ie^{it}\, dt \\
&= -\frac{1}{2\pi i} \int_{b(\mathbb{T})} \frac{p_n(\eta)}{z-\eta} ib^{-1}(\eta) \frac{b'(b^{-1}(\eta))}{|b'(b^{-1}(\eta))|}\, d\mu_{b(\mathbb{T})}(\eta) \\
&= -\frac{1}{2\pi i} \int_{b(\mathbb{T})} \frac{p_n(\eta)}{z-\eta} i\phi_z(\eta)\, d\mu_{b(\mathbb{T})}(\eta)
\end{aligned}$$

if we define

$$\phi_z(\eta) := b^{-1}(\eta) \frac{b'(b^{-1}(\eta))}{|b'(b^{-1}(\eta))|}.$$

The assumption $\lim_{n\to\infty} \|p_n\|_J = 0$ implies the existence of a subsequence p_{n_j}, such that $\lim_{j\to\infty} p_{n_j}(z) = 0$ for almost every $z \in \mathbb{T}$. Thus calculating the limit $n_j \to \infty$ yields

$$0 = -\frac{1}{2\pi i} \int_{b(\mathbb{T})} \frac{h(\eta)}{z-\eta} i\phi_z(\eta)\, d\mu_{b(\mathbb{T})}(\eta). \tag{4.3}$$

Since the right hand side is holomorphic in z, this holds for every $z \in \mathbb{C} \setminus \overline{b(\mathbb{D})}$.

Next we study the expression

$$\left[z^k, \overline{h\phi_z}\right]_{L_2(b(\mathbb{T}))} = \int_{b(\mathbb{T})} h(\eta)\phi_z(\eta)\eta^k \, d\mu_{b(\mathbb{T})}(\eta) = \oint_{b(\mathbb{T})} h(\eta)\eta^k \, d\eta. \tag{4.4}$$

For $k < 0$ the right hand side is zero, since by the residue theorem

$$\lim_{n\to\infty} \oint_{b(\mathbb{T})} p_n(\eta)\eta^k \, d\eta = 0.$$

To obtain the same for $k \geq 0$ we rewrite (4.3) by

$$\begin{aligned} 0 &= \int_{b(\mathbb{T})} \frac{h(\eta)}{z-\eta} i\phi_z(\eta)\, d\mu_{b(\mathbb{T})}(\eta) \\ &= \int_{b(\mathbb{T})} \sum_{k=0}^{\infty} \frac{\eta^k}{z^{k+1}} h(\eta) i\phi_z(\eta)\, d\mu_{b(\mathbb{T})}(\eta) \\ &= \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} i \int_{b(\mathbb{T})} \eta^k h(\eta)\phi_z(\eta)\, d\mu_{b(\mathbb{T})}(\eta), \end{aligned} \tag{4.5}$$

where we used

$$\frac{1}{z-\eta} = \frac{1}{z}\frac{1}{1-\frac{\eta}{z}} = \sum_{k=0}^{\infty} \frac{\eta^k}{z^{k+1}}.$$

Since (4.5) is the unique Laurent series expansion of the zero function, all the coefficients are zero. Thus, (4.4) is zero for all $k \in \mathbb{Z}$. Because the set $\{z^k : k \in \mathbb{Z}\}$ is dense in $L_2(b(\mathbb{T}))$ and $\phi_z \neq 0$, this implies $h = 0$ as an element of $L_2(b(\mathbb{T}))$. Therefore, $g = 0$ as an element of $L_2(\mathbb{T})$ and, since it is analytic also as an element of $\mathcal{K}\mathfrak{D}$. ☐

Lemma 4.0.13. *Let f_n, f be holomorphic functions on A_δ such that $f_n \in \mathcal{K}\mathfrak{D}$ and $\lim_{n\to\infty} f_n = f$ uniformly on A_δ. Then $f \in \mathcal{K}\mathfrak{D}$ and $\lim_{n\to\infty} \|f_n - f\|_J = 0$.*

Proof. By Cauchy's integral formula, we can write the derivative of $P_+ f_n$ for $z \in A_\delta$ and $0 < \varepsilon < \delta$ as

$$\begin{aligned} 2\pi i \left(P_+ f_n\right)'(z) &= \oint_{\partial B_{1+\varepsilon}(0)} \frac{f_n(\zeta) - \left(P_- f_n\right)(\zeta)}{(\zeta - z)^2}\, d\zeta = \\ &= \oint_{\partial B_{1+\varepsilon}(0)} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta - \oint_{\partial B_{1+\varepsilon}(0)} \frac{\left(P_- f_n\right)(\zeta)}{(\zeta - z)^2}\, d\zeta. \end{aligned} \tag{4.6}$$

Since $P_- f_n$ is holomorphic in $\mathbb{C} \setminus \overline{B_{1-\delta}(0)}$, we have

$$\oint_{\partial B_{1+\varepsilon}(0)} \frac{\left(P_- f_n\right)(\zeta)}{(\zeta - z)^2}\, d\zeta = \oint_{\partial B_R(0)} \frac{\left(P_- f_n\right)(\zeta)}{(\zeta - z)^2}\, d\zeta = \int_0^{2\pi} \frac{\left(P_- f_n\right)(Re^{it})}{(1 - \frac{z}{Re^{it}})^2} \frac{i}{Re^{it}}\, dt$$

for all $R > 1 - \delta$, $z \in A\delta$. By calculating the limit $R \to +\infty$ we obtain

$$\oint_{\partial B_{1+\varepsilon}(0)} \frac{\left(P_- f_n\right)(\zeta)}{(\zeta - z)^2}\, d\zeta = 0.$$

Thus, (4.6) becomes

$$2\pi i \left(P_+ f_n\right)'(z) = \oint_{\partial B_{1+\varepsilon}(0)} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Since $P_+ f_n$ is holomorphic on $B_{1+\delta}(0)$, this holds in particular for $z \in \mathbb{D}$. Repeating the same argument with f instead of f_n yields

$$\begin{aligned} 2\pi i \sup_{z\in\mathbb{D}} \left|\left(P_+ f_n\right)'(z) - \left(P_+ f\right)'(z)\right| &= \sup_{z\in\mathbb{D}} \left| \oint_{\partial B_{1+\varepsilon}(0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\, d\zeta\right| \\ &\le \sup_{z\in\mathbb{D}} \oint_{\partial B_{1+\varepsilon}(0)} \frac{|f_n(\zeta) - f(\zeta)|}{\varepsilon^2}\, d\zeta \\ &\le 2(1+\varepsilon)\pi \sup_{\zeta\in\partial B_{1+\varepsilon}(0)} |f_n(\zeta) - f(\zeta)|. \end{aligned}$$

Therefore $(P_+ f_n)' \to (P_+ f)'$ uniformly in $\mathbb{D}$ and from this it follows that

$$\begin{aligned} \lim_{n\to\infty} \|\left(P_+ f_n\right) - \left(P_+ f\right)\|_{\mathfrak{D}}^2 = \lim_{n\to\infty} \int_{\mathbb{D}} \left|\left(P_+ f_n\right)'(z) - \left(P_+ f\right)'(z)\right|^2 dA(z) \le \\ \le \lim_{n\to\infty} \sup_{z\in\mathbb{D}} \left|\left(P_+ f_n\right)'(z) - \left(P_+ f\right)'(z)\right|^2 = 0. \quad (4.7) \end{aligned}$$

We can repeat the same argument with the functions $g_n(z) := (Sf_n)(z) = f_n(z^{-1})$, $g(z) := (Sf)(z) = f(z^{-1})$ and obtain

$$\begin{aligned} 0 = \lim_{n\to\infty} \|\left(P_+ g_n\right) - \left(P_+ g\right)\|_{\mathfrak{D}}^2 = \\ = \lim_{n\to\infty} \|\left(P_+ S f_n\right) - \left(P_+ S f\right)\|_{\mathfrak{D}}^2 = \lim_{n\to\infty} \|\left(S P_- f_n\right) - \left(S P_- f\right)\|_{D}^2. \quad (4.8) \end{aligned}$$

For $P_0 f_n$ there holds

$$\begin{aligned} 2\pi i \lim_{n\to\infty} |P_0 f_n - P_0 f| = \lim_{n\to\infty} \left| \oint_{\partial B_{1+\varepsilon}(0)} \frac{f_n(\zeta) - f(\zeta)}{\zeta} d\zeta \right| \le \\ \le \lim_{n\to\infty} \sup_{\zeta\in\partial B_{1+\varepsilon}(0)} |f_n(\zeta) - f(\zeta)| = 0. \quad (4.9) \end{aligned}$$

By combining (4.7), (4.8) and (4.9) we finally obtain

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{J}}^2 = \lim_{n\to\infty} \|P_+ f_n - P_+ f\|_{\mathfrak{D}}^2 + \|SP_- f_n - SP_- f\|_{\mathfrak{D}}^2 + |P_0 f_n - P_0 f|^2 = 0.$$

□

On $\mathcal{K}\mathfrak{D}$ the composition operator is not as easy to define as on $\mathcal{I}\mathfrak{D}$, nevertheless the following lemma shows under which conditions on b it admits a closure:

Theorem 4.0.14. *Let $b : B_{1+\varepsilon}(0) \to \mathbb{C}$ be an analytic, univalent function for some $\varepsilon > 0$, such that $b(0) = 0$ and there exists $r \in \mathbb{R}$, $0 < r < 1$ with $b(\mathbb{D}) \subseteq B_r(0)$. Then there exists a linear operator $C_b^{\mathcal{K}\mathfrak{D}} : \operatorname{dom} C_b^{\mathcal{K}\mathfrak{D}} \to \mathcal{K}\mathfrak{D}$ such that $C_b^{\mathcal{K}\mathfrak{D}}$ is the closure of the operator $C_b^{\mathcal{I}\mathfrak{D}}$ with respect to the norm $\|\cdot\|_J$.*

Let $f \in \mathcal{K}\mathfrak{D}$ and $\delta \in \mathbb{R}$, $\delta > 0$ such that f is analytic on A_δ, where $A_\delta := \{z \in \mathbb{C} : 1 - \delta < |z| < 1 + \delta\}$ and $b(A_\varepsilon) \subseteq A_\delta$. Then $f \in dom C_b^{\mathcal{K}\mathfrak{D}}$ and

$$(C_b^{\mathcal{K}\mathfrak{D}} f)(z) = f(b(z))$$

holds for all $z \in A_\varepsilon$.

Proof. To show that the operator $C_b^{\mathcal{I}\mathfrak{D}}$ allows closure, we have to show, that for every sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{I}\mathfrak{D}$, such that $\lim_{n\to\infty} \|f_n\|_J = 0$ and $\lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} f_n - g\|_J = 0$ for some $g \in \mathcal{K}\mathfrak{D}$, it follows that $g = 0$.

Now let f_n be such a sequence. Because of

$$\|f_n\|_J^2 = \|P_+ f_n\|_J + \|P_0 f_n\|_J + \|P_- f_n\|_J^2 = \|P_+ f_n\|_{\mathfrak{D}}^2 + |P_0 f_n|^2 + \|SP_- f_n\|_{\mathfrak{D}}^2,$$

we know, that

$$\lim_{n\to\infty} \|P_+ f_n\|_{\mathcal{K}\mathfrak{D}} = \lim_{n\to\infty} |P_0 f_n| = \lim_{n\to\infty} \|P_- f_n\|_J = 0. \tag{4.10}$$

Therefore, since $C_b^{\mathfrak{D}}$ is a contraction

$$\lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} P_+ f_n\|_{\mathfrak{D}}^2 = \lim_{n\to\infty} \|C_b^{\mathfrak{D}} P_+ f_n\|_{\mathfrak{D}}^2 \leq \lim_{n\to\infty} \|P_+ f_n\|_{\mathfrak{D}}^2 = 0. \tag{4.11}$$

Combining (4.10) and (4.11) and the fact that $C_b^{\mathcal{I}\mathfrak{D}} P_0 f_n = P_0 f_n$ yields

$$\begin{aligned}
\lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} P_- f_n - g\|_J^2 &= \lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} P_- f_n + C_b^{\mathcal{I}\mathfrak{D}} P_+ f_n + C_b^{\mathcal{I}\mathfrak{D}} P_0 f_n \\
&- g - C_b^{\mathcal{I}\mathfrak{D}} P_+ f_n - C_b^{\mathcal{I}\mathfrak{D}} P_0 f_n\|_J^2 \\
&= \lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} f_n - g - C_b^{\mathcal{I}\mathfrak{D}} P_+ f_n - C_b^{\mathcal{I}\mathfrak{D}} P_0 f_n\|_J^2 \\
&\leq \lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} f_n - g\|_J^2 + \lim_{n\to\infty} \|C_b^{\mathcal{I}\mathfrak{D}} P_+ f_n\|_J^2 \\
&+ \lim_{n\to\infty} |C_b^{\mathcal{I}\mathfrak{D}} P_0 f_n|^2 = 0.
\end{aligned}$$

Therefore, the sequence $C_b^{\mathcal{I}\mathfrak{D}} P_- f_n$ converges to g with respect to $\|\cdot\|_J$. Thus, we can apply Lemma 4.0.12 for $p_n := P_- f_n$ to conclude that $g = 0$.

To show the second part, write $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ and define

$$f_n(z) := \sum_{k=-n}^{\infty} a_k z^k \in \mathcal{I}\mathfrak{D}.$$

The sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on any compact subset of A_δ. Therefore $\lim_{n\to\infty} f_n(b(z)) = f(b(z))$ uniformly on any compact subset of A_ε, particularly on A_{ε_1}, for $\varepsilon_1 < \varepsilon$. Since uniform convergence on A_{ε_1} implies convergence with respect to $\|\cdot\|_J$ by Lemma 4.0.13, we have that $f \in \operatorname{dom} C_b^{\mathcal{K}\mathfrak{D}}$ and $(C_b^{\mathcal{K}\mathfrak{D}} f)(z) = f(b(z))$ for all $z \in A_\varepsilon$. □

Next we list some useful facts about the operator $f \mapsto f^\sharp$:

Remark 4.0.15.

(i) Because of
$$\sum_{n=1}^{\infty} n|\bar{a}_n|^2 = \sum_{n=1}^{\infty} n|a_n|^2, \tag{4.12}$$
a function f is in $\mathfrak{D}$ if and only if $f^\sharp \in \mathfrak{D}$. In this case $\|f\|_{\mathfrak{D}} = \|f^\sharp\|_{\mathfrak{D}}$. Due to this fact, and the fact that
$$\overline{f(\bar{z}) + \lambda g(\bar{z})} = f^\sharp(z) + \bar{\lambda} g^\sharp(z),$$
for $g, f \in \mathfrak{D}$, the mapping $f \mapsto f^\sharp$ is an anti-linear, isometric bijection on $\mathfrak{D}$.

(ii) A function $f \in \mathcal{S}$ is an element of $\mathcal{I}\mathfrak{D}$ if and only if $f^\sharp \in \mathcal{I}\mathfrak{D}$. In this case
$$\left[f^\sharp, f^\sharp\right]_{\mathcal{I}\mathfrak{D}} = [f, f]_{\mathcal{I}\mathfrak{D}} \, .$$

(iii) Let $b(z) = \sum_{n=1}^{\infty} b_n z^n \in \mathcal{S}^+$, $b'(0) \neq 0$ and $f(z) = \sum_{k=-N}^{\infty} a_k z^k \in \operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}}$. Then
$$\begin{aligned} (C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} f)(z) &= f(\overline{b(\bar{z})}) \\ &= \sum_{k=-N} a_k \overline{\left(\sum_{n=1}^{\infty} b_n \bar{z}^n\right)}^k \\ &= \overline{\sum_{k=-N} \bar{a}_k \left(\sum_{n=1}^{\infty} b_n \bar{z}^n\right)^k} \\ &= \overline{f^\sharp(b(\bar{z})} = \overline{(C_b^{\mathcal{I}\mathfrak{D}} f^\sharp)(\bar{z})} = (C_b^{\mathcal{I}\mathfrak{D}} f^\sharp)^\sharp(z). \end{aligned}$$

(iv) Let $b : \mathbb{D} \to \mathbb{D}$, $b'(0) \neq 0$ be such that $\operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}} = \mathcal{I}\mathfrak{D}$. Then because of *(iv)* and *(iii)*
$$\left[C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} f, C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} f\right]_{\mathcal{I}\mathfrak{D}} = \left[(C_b^{\mathcal{I}\mathfrak{D}} f^\sharp)^\sharp, (C_b^{\mathcal{I}\mathfrak{D}} f^\sharp)^\sharp\right]_{\mathcal{I}\mathfrak{D}} = \left[C_b^{\mathcal{I}\mathfrak{D}} f^\sharp, C_b^{\mathcal{I}\mathfrak{D}} f^\sharp\right]_{\mathcal{I}\mathfrak{D}}$$
for all $f \in \mathcal{I}\mathfrak{D}$. In particular this implies, that
$$\operatorname{dom} C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} = \operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}}$$
and that $C_b^{\mathcal{I}\mathfrak{D}}$ is a contraction if and only if $C_{b^\sharp}^{\mathcal{I}\mathfrak{D}}$ is a contraction.

Moreover if b satisfies the assumptions of Lemma 4.0.14, we have
$$\operatorname{dom} C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} = \operatorname{dom} C_b^{\mathcal{K}\mathfrak{D}}.$$

Lemma 4.0.16. *Let $b(z) = \sum_{n=1}^{\infty} b_n z^n$ be holomorphic on $\mathbb{D}$, with $b(\mathbb{D}) \subseteq \mathbb{D}$, such that $C_b^{\mathfrak{D}}$ is bounded and let $p \in \mathcal{P}_0^k$. Then $\left(C_b^{\mathfrak{D}}\right)^* p \in \mathcal{P}_0^k$ and $\left(C_b^{\mathfrak{D}}\right)^* p$ depends only on $b_1, \ldots, b_k$.*

Proof. As for $j > k$

$$\left\langle \left(C_b^{\mathfrak{D}}\right)^* p, (z \mapsto z^j) \right\rangle_{\mathfrak{D}} = \left\langle p, \left(C_b^{\mathfrak{D}}\right)(z \mapsto z^j) \right\rangle_{\mathfrak{D}} = \langle p, b^j \rangle_{\mathfrak{D}} = 0,$$

$\left(C_b^{\mathfrak{D}}\right)^* p(z)$ is a polynomial of degree k as well.

Let $b^{(k)} := \sum_{n=1}^{k} b_n z^n$. For arbitrary $f \in \mathfrak{D}$ we have

$$\left\langle \left(C_b^{\mathfrak{D}}\right)^* p, f \right\rangle_{\mathfrak{D}} = \langle p, f \circ b \rangle_{\mathfrak{D}} = \left\langle p, f \circ b^{(k)} \right\rangle_{\mathfrak{D}} = \left\langle \left(C_{b^{(k)}}^{\mathfrak{D}}\right)^* p, f \right\rangle_{\mathfrak{D}},$$

since, writing $f(b(z)) = \sum_{n=1}^{\infty} c_n z^n$, the coefficients c_n for $n \leq k$ only depend on $b_1, b_2, \ldots, b_k$. □

Remark 4.0.17. Recall the following fact about a closed operator. Let $\mathcal{H}$ be a Hilbert space and $T : \operatorname{dom} T \to \mathcal{H}$ a closed linear operator. Let further $a_n \in \operatorname{dom} T$ be a sequence, such that $\lim_{n\to\infty} a_n \in \mathcal{H}$ and $\lim_{n\to\infty} T a_n \in \mathcal{H}$. Then $\lim_{n\to\infty} a_n \in \operatorname{dom} T$ and

$$T\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} T a_n.$$

Lemma 4.0.18. *Let $b(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic and univalent on $B_{1+\varepsilon}(0)$ for some $\varepsilon > 0$, such that $b(0) = 0$ and there exists $r \in \mathbb{R}$, $0 < r < 1$ with $b(\mathbb{D}) \subseteq B_r(0)$. Then*

$$P_- C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p = Sp \tag{4.13}$$

holds for all $p \in \mathcal{P}_0$.

Proof. The operator $C_b^{\mathfrak{D}}$ is bounded and due to Lemma 1.4.7 we have, that

$$\left(S\left(C_b^{\mathfrak{D}}\right)^* K_{\mathfrak{D}}(\cdot, w)\right)(z) = \left(S K_{\mathfrak{D}}(\cdot, b(w))\right)(z) = K_{\mathfrak{D}}(z^{-1}, b(w)) \tag{4.14}$$

for all $w \in \mathbb{D}$, $|z| > 1$. Since for fixed w the right hand side of equation (4.14) is analytic on $\mathbb{C} \setminus \overline{B_{|b(w)|}(0)}$, by the uniqueness theorem equation (4.14) even holds for all $z > |b(w)|$.

Since b is univalent and continuous on $B_{1+\varepsilon}$ and $b(0) = 0$ there exist $\delta, r > 0$, such that $|b(w)| < 1 - \delta < |b(z)|$ for all $z \in A_\varepsilon$ and $|w| \leq r$. Therefore $b(A_\varepsilon) \subseteq A_\delta$ and the functions

$$f_w(z) := K_{\mathfrak{D}}(z^{-1}, b(w))$$

are analytic on A_δ, since the radius of convergence of $K_{\mathfrak{D}}(z, w)$ is $\frac{1}{|w|}$ by Remark 2.2.4. Thus, Lemma 4.0.14 yields

$$f_w \in \operatorname{dom} C_b^{\mathcal{K}\mathfrak{D}} = \operatorname{dom} C_{b^\sharp}^{\mathcal{K}\mathfrak{D}}$$

for all $|w| \leq r$. Hence, we can apply $C_{b^\sharp}^{\mathcal{K}\mathfrak{D}}$ and obtain for $z \in A_\varepsilon$

$$\begin{aligned}
\left(C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* K_{\mathfrak{D}}(\cdot, w)\right)(z) &= \left(C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} f_w\right)(z) \\
&= K_{\mathfrak{D}}(b^\sharp(z)^{-1}, b(w)) \\
&= \log\left(\frac{1}{1 - \overline{b(w)} b^\sharp(z)^{-1}}\right) \\
&= \log\left(\frac{b^\sharp(z)(z - \bar{w})}{z\left(b^\sharp(z) - b^\sharp(\bar{w})\right)}\right) + \log\left(\frac{1}{1 - \bar{w} z^{-1}}\right) \\
&= h(z, w) + K_{\mathfrak{D}}(z^{-1}, w), \qquad (4.15)
\end{aligned}$$

where we used the notation

$$h(z, w) := \log\left(\frac{b^\sharp(z)(z - \bar{w})}{z\left(b^\sharp(z) - b^\sharp(\bar{w})\right)}\right).$$

Since $b^\sharp$ is univalent on $B_{1+\varepsilon}(0)$ by assumption and therefore $(b^\sharp)'(0), (b^\sharp)'(\bar{w}) \neq 0$, the function $h(\cdot, w)$ is analytic on $B_{1+\varepsilon}(0)$. The function $K_{\mathfrak{D}}(z^{-1}, w)$ on the other hand is analytic on $\mathbb{C} \setminus \overline{B_{|w|}(0)}$ as function in z.

Now let $k \in \mathbb{N}$ arbitrary. By multiplying (4.15) with w^k and integrating over $\partial B_r(0)$ with respect to $\nu := \mu_{\partial B_r(0)}$, we obtain for fixed $z \in B_{1+\varepsilon}(0)$

$$\begin{aligned}
&\int_{\partial B_r(0)} \left(C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* w^k K_{\mathfrak{D}}(\cdot, w)\right)(z)\, d\nu(w) = \\
&\qquad = \int_{\partial B_r(0)} w^k h(z, w)\, d\nu(w) + \int_{\partial B_r(0)} w^k K_{\mathfrak{D}}(z^{-1}, w)\, d\nu(w). \qquad (4.16)
\end{aligned}$$

Because of

$$\begin{aligned}
\int_{\partial B_r(0)} w^k K_{\mathfrak{D}}(y, w)\, d\nu(w) &= \int_{\partial B_r(0)} w^k \sum_{n=1}^{\infty} \frac{\bar{w}^n y^n}{n}\, d\nu(w) = \\
&= \sum_{n=1}^{\infty} \frac{y^n r^{k+n}}{n} \int_0^{2\pi} e^{it(k-n)}\, dt = 2\pi \frac{r^{2k} y^k}{k},
\end{aligned}$$

for all $|y| < \frac{1}{|w|}$, (4.16) becomes

$$\begin{aligned}
&\int_{\partial B_r(0)} \left(C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* w^k K_{\mathfrak{D}}(\cdot, w)\right)(z)\, d\nu(w) = \\
&\qquad = \int_{\partial B_r(0)} w^k h(z, w)\, d\nu(w) + 2\pi \frac{r^{2k}}{k} z^{-k}, \qquad (4.17)
\end{aligned}$$

for $|z| > |w|$.

Since $z \mapsto h_w(z) := h(z,w)$ is holomorphic on $B_{1+\varepsilon}(0)$ we can write its $\|\cdot\|_J$-norm for $w \in \partial B_r(0)$ as

$$\|h_w\|_J^2 = \|h_w - h_w(0)\|_{\mathfrak{D}}^2 + |h_w(0)|^2 = \int_{\mathbb{D}} |h_w'(\zeta)|^2 \, dA(\zeta) + |h_w(0)|^2 \leq \\ \leq \sup_{\zeta\in\mathbb{D}} |h_w'(\zeta)|^2 + |h_w(0)|^2 = |h_w'(\zeta_w)|^2 + |h_w(0)|^2$$

for some $\zeta_w \in \mathbb{T}$, by the maximum modulus principle for holomorphic functions. Thus,

$$\sup_{w\in\partial B_r(0)} \|h(\cdot,w)\|_J^2 \leq \sup_{w\in\partial B_r(0)} |h_w'(\zeta_w)|^2 + |h_w(0)|^2 = \\ = \sup_{w\in\partial B_r(0)} \left|\frac{1}{\zeta_w - \bar{w}} - \frac{(b^\sharp)'(\zeta_w)}{b^\sharp(\zeta_w) - b^\sharp(\bar{w})}\right|^2 + \left|\log\left(\left(b^\sharp\right)'(0)\right) + \log\left(\frac{\bar{w}}{b^\sharp(\bar{w})}\right)\right|^2 < +\infty$$

shows, that the function on the right hand side of (4.17) is again an element of $\mathcal{K}\mathfrak{D}$. Moreover, since we can interpret the integral as limit of a Riemann sum, the bounded operator $S\left(C_b^{\mathfrak{D}}\right)^*$ commutes with the integral and we have

$$\int_{\partial B_r(0)} \left(S\left(C_b^{\mathfrak{D}}\right)^* w^k K_{\mathfrak{D}}(\cdot,w)\right)(z)\, d\nu(w) = S\left(C_b^{\mathfrak{D}}\right)^* 2\pi \frac{r^{2k} z^k}{k} \in \mathcal{K}\mathfrak{D}$$

Since $C_{b^\sharp}^{\mathcal{K}\mathfrak{D}}$ is closed it commutes with the integral as well (see Remark 4.0.17) and we obtain

$$\int_{\partial B_r(0)} \left(C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* w^k K_{\mathfrak{D}}(\cdot,w)\right)(z)\, d\nu(w) = C_{b^\sharp}^{\mathcal{K}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* 2\pi \frac{r^{2k} z^k}{k}. \tag{4.18}$$

Combining equations (4.17) and (4.18), we have for $p(z) := z^k$

$$\frac{2\pi r^{2k}}{k} \left(C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p\right)(z) = \int_{\partial B_r(0)} w^k h(z,w)\, d\nu(w) + \frac{2\pi r^{2k}}{k} z^{-k}.$$

Since $h(\cdot,w)$ is analytic and uniformly bounded on $\mathbb{D}$, the function $\int_{\partial B_r(0)} w^k h(\cdot,w)\, d\nu(w)$ is analytic on $\mathbb{D}$. Thus

$$P_- C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p = Sp.$$

Since $C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^*$ is linear, this holds for all $p \in \mathcal{P}_0$. □

Lemma 4.0.19. *Let $b(z) = \sum_{n=1}^\infty b_n z^n \in \mathcal{S}_+$, $b_1 \neq 0$ be arbitrary and $p \in \mathcal{P}_0^k$. Then $P_- C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} Sp$ only depends on $b_1, \ldots, b_k$*

Proof. We define $(b^\sharp)^{(k)}(z) := \sum_{n=1}^{k} \bar{b}_n z^n$. Then for $j \leq k$, $q(z) = z^{-j}$

$$\begin{aligned}
\left(P_- C^{\mathcal{I}\mathfrak{D}}_{b^\sharp} q\right)(z) &= P_-\left(b^\sharp(z)^{-j}\right) \\
&= P_-\left(\left((b^\sharp)^{(k)}(z) + z^k c^k(z)\right)^{-j}\right) \\
&= P_-\left((b^\sharp)^{(k)}(z)^{-j}\left(1 + \frac{z^k c^k(z)}{(b^\sharp)^{(k)}(z)}\right)^{-j}\right) \\
&= P_-\left((b^\sharp)^{(k)}(z)^{-j}\left(1 + \sum_{n=1}^{\infty}\binom{-j}{n} z^{k+n-1} c^k(z)^n \left(\frac{z}{(b^\sharp)^{(k)}(z)}\right)^n\right)\right) \\
&= P_-(b^\sharp)^{(k)}(z)^{-j} = P_-\left(C^{\mathcal{I}\mathfrak{D}}_{(b^\sharp)^{(k)}} q\right)(z),
\end{aligned}$$

where we used the notation $c^k(z) := \sum_{n=1}^{\infty} \bar{b}_{n+k} z^n$. □

Corollary 4.0.20. *Let $b : \mathbb{D} \to \mathbb{D}$, $b(z) := \sum_{n=1}^{\infty} b_n z^n$, with $b_1 \neq 0$, such that $C^{\mathfrak{D}}_b$ is bounded. Then*

$$P_- C^{\mathcal{I}\mathfrak{D}}_{b^\sharp} S \left(C^{\mathfrak{D}}_b\right)^* p = Sp \tag{4.19}$$

holds for all polynomials $p \in \mathcal{P}_0$.

Proof. For $k \in \mathbb{N}$, let $p \in \mathcal{P}_0^k$ be arbitrary. Then $b^{(k)}(z) := \sum_{n=1}^{k} b_n z^n \in \mathcal{P}_0^k$ and because of $(b^{(k)})'(0) \neq 0$ there exist some $\varepsilon > 0$ and $1 > r, \delta > 0$, such that $b_\delta(z) := b^{(k)}(\delta z)$ is univalent on $B_{1+\varepsilon}(0)$ and $b_\delta(\mathbb{D}) \subseteq B_r(0)$. Because of $\delta z \in \mathfrak{B}$ and

$$\begin{aligned}
\left\langle C^{\mathfrak{D}}_{\delta z} \sum_{n=1}^{\infty} a_n z^n, \sum_{n=1}^{\infty} c_n z^n \right\rangle_{\mathfrak{D}} &= \left\langle \sum_{n=1}^{\infty} a_n (\delta z)^n, \sum_{n=1}^{\infty} c_n z^n \right\rangle_{\mathfrak{D}} = \\
&= \sum_{n=1}^{\infty} n a_n \delta^n \bar{c}_n = \left\langle \sum_{n=1}^{\infty} a_n z^n, C^{\mathfrak{D}}_{\delta z} \sum_{n=1}^{\infty} c_n z^n \right\rangle_{\mathfrak{D}}
\end{aligned}$$

for $\sum_{n=1}^{\infty} c_n z^n \in \mathfrak{D}$ arbitrary, the operator $C^{\mathfrak{D}}_{\delta z}$ is self-adjoint.

Furthermore $C^{\mathcal{I}\mathfrak{D}}_{\delta z} S = S C^{\mathcal{I}\mathfrak{D}}_{\delta^{-1} z}$ and $C^{\mathcal{I}\mathfrak{D}}_{b_\delta} = C^{\mathcal{I}\mathfrak{D}}_{\delta z} C^{\mathcal{I}\mathfrak{D}}_{b^{(k)}}$. Hence, we obtain

$$\begin{aligned}
p &= C^{\mathcal{I}\mathfrak{D}}_{\delta z} C^{\mathcal{I}\mathfrak{D}}_{\delta^{-1} z} p \\
&= C^{\mathcal{I}\mathfrak{D}}_{\delta z} S P_- C^{\mathcal{I}\mathfrak{D}}_{b^\sharp_\delta} S \left(C^{\mathfrak{D}}_{b_\delta}\right)^* C^{\mathfrak{D}}_{\delta^{-1} z} p \\
&= S P_- C^{\mathcal{I}\mathfrak{D}}_{\delta^{-1} z} C^{\mathcal{I}\mathfrak{D}}_{b^\sharp_\delta} S \left(C^{\mathfrak{D}}_{\delta^{-1} z} C^{\mathfrak{D}}_{b_\delta}\right)^* p \\
&= S P_- C^{\mathcal{I}\mathfrak{D}}_{(b^{(k)})^\sharp} S \left(C^{\mathfrak{D}}_{b^{(k)}}\right)^* p = S P_- C^{\mathcal{I}\mathfrak{D}}_{b^\sharp} S \left(C^{\mathfrak{D}}_b\right).
\end{aligned}$$

□

With this preliminaries we are able to show that the Grunsky operator satisfying (4.1) indeed exists.

Theorem 4.0.21. *Let $b : \mathbb{D} \to \mathbb{D}$, $b(z) = \sum_{n=1}^{\infty} b_n z^n$, $b_1 \neq 0$, such that* $\operatorname{dom} C_b^{\mathcal{I}\mathfrak{D}} = \mathcal{I}\mathfrak{D}$ *and $C_b^{\mathcal{I}\mathfrak{D}}$ is a contraction with respect to $[\cdot,\cdot]_{\mathcal{I}\mathfrak{D}}$. Then there exists a contraction operator $G_b : \mathfrak{D} \to \mathfrak{D}$ that satisfies (4.1).*

Proof. By Remark 4.0.15, *(iv)* we know, that $C_{b^\sharp}^{\mathcal{I}\mathfrak{D}}$ is a contraction, since $C_b^{\mathcal{I}\mathfrak{D}}$ is a contraction by assumption. Let $f \in \mathfrak{D}$ and $p \in P$ be arbitrary. Then $f + S\left(C_b^{\mathfrak{D}}\right)^* p \in \mathcal{I}\mathfrak{D}$ and thus

$$\|C_{b^\sharp}^{\mathcal{I}\mathfrak{D}}\left(f + S\left(C_b^{\mathfrak{D}}\right)^* p\right)\|_{\mathcal{I}\mathfrak{D}}^2 \leq \|f + S\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathcal{I}\mathfrak{D}}^2. \tag{4.20}$$

By using Corollary 4.0.20 on the left hand side, we obtain

$$\begin{aligned}
\|C_{b^\sharp}^{\mathcal{I}\mathfrak{D}}\left(f + S\left(C_b^{\mathfrak{D}}\right)^* p\right)\|_{\mathcal{I}\mathfrak{D}}^2 &= \|C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} f + (P_+ + P_-)\, C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathcal{I}\mathfrak{D}}^2 \\
&= \|C_{b^\sharp}^{\mathfrak{D}} f + P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p + Sp\|_{\mathcal{I}\mathfrak{D}}^2 \qquad (4.21)\\
&= \|C_{b^\sharp}^{\mathfrak{D}} f + P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathfrak{D}}^2 - \|p\|_{\mathfrak{D}}^2
\end{aligned}$$

since $C_{b^\sharp}^{\mathfrak{D}} f + P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p \in \mathfrak{D}$ and $Sp \in S\mathfrak{D}$. On the right hand side, because of $f \in \mathfrak{D}$ and $S\left(C_b^{\mathfrak{D}}\right)^* p \in S\mathfrak{D}$, we have

$$\|f + S\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathcal{I}\mathfrak{D}}^2 = \|f\|_{\mathfrak{D}}^2 - \|\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathfrak{D}}^2. \tag{4.22}$$

Combining (4.20), (4.21) and (4.22) yields

$$\|C_{b^\sharp}^{\mathfrak{D}} f + P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathfrak{D}}^2 - \|f\|_{\mathfrak{D}}^2 \leq \|p\|_{\mathfrak{D}}^2 - \|\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathfrak{D}}^2. \tag{4.23}$$

If we take the supremum over all $f \in \mathfrak{D}$ and use Theorem 1.3.4, the left hand side is nothing else but the $\mathcal{D}_{\left(C_{b^\sharp}^{\mathfrak{D}}\right)^*}$-norm of $P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p$ squared. The right hand side can be written as

$$\begin{aligned}
|p\|_{\mathfrak{D}}^2 - \|\left(C_b^{\mathfrak{D}}\right)^* p\|_{\mathfrak{D}}^2 &= \langle p, p\rangle_{\mathfrak{D}} - \left\langle \left(C_b^{\mathfrak{D}}\right)^* p, \left(C_b^{\mathfrak{D}}\right)^* p \right\rangle_{\mathfrak{D}} \\
&= \left\langle p - C_b^{\mathfrak{D}}\left(C_b^{\mathfrak{D}}\right)^* p, p \right\rangle_{\mathfrak{D}} \\
&= \left\langle \left(I - C_b^{\mathfrak{D}}\left(C_b^{\mathfrak{D}}\right)^*\right)^{\frac{1}{2}} p, \left(I - C_b^{\mathfrak{D}}\left(C_b^{\mathfrak{D}}\right)^*\right)^{\frac{1}{2}} p \right\rangle_{\mathfrak{D}} \\
&= \|D_{\left(C_b^{\mathfrak{D}}\right)^*} p\|_{\mathfrak{D}}^2 = \|D^2_{\left(C_b^{\mathfrak{D}}\right)^*} p\|^2_{\mathcal{D}_{\left(C_b^{\mathfrak{D}}\right)^*}}.
\end{aligned}$$

Thus we have

$$\|P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^* p\|^2_{\mathcal{D}_{\left(C_{b^\sharp}^{\mathfrak{D}}\right)^*}} \leq \|D^2_{\left(C_b^{\mathfrak{D}}\right)^*} p\|^2_{\mathcal{D}_{\left(C_b^{\mathfrak{D}}\right)^*}}$$

for all polynomials p.

Therefore the operator F defined by

$$F : \begin{cases} D^2_{\left(C_b^{\mathfrak{D}}\right)^*}\mathcal{P} & \to \quad \mathcal{D}_{\left(C_{b^\sharp}^{\mathfrak{D}}\right)^*} \\ D^2_{\left(C_b^{\mathfrak{D}}\right)^*}p & \mapsto \quad P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p \end{cases}$$

is well defined, and a contraction if we endow $D^2_{\left(C_b^{\mathfrak{D}}\right)^*}\mathcal{P}$ with the $\mathcal{D}_{\left(C_b^{\mathfrak{D}}\right)^*}$-norm. Since $D^2_{\left(C_b^{\mathfrak{D}}\right)^*}\mathcal{P}$ is dense in $\mathcal{D}_{\left(C_b^{\mathfrak{D}}\right)^*}$ by Lemma 1.3.6, we can extended F to a contraction operator $H : \mathcal{D}_{\left(C_b^{\mathfrak{D}}\right)^*} \to \mathcal{D}_{\left(C_{b^\sharp}^{\mathfrak{D}}\right)^*}$.

Now we define the Grunsky operator G_b by

$$G_b : \begin{cases} \mathfrak{D} & \to \quad \mathfrak{D} \\ f & \mapsto \quad \iota H D^2_{\left(C_b^{\mathfrak{D}}\right)^*} f \end{cases}$$

where ι is the embedding of $\mathcal{D}_{\left(C_b^{\mathfrak{D}}\right)^*}$ into $\mathfrak{D}$. Then G_b fulfills (4.1) by construction and, since H, ι and $D^2_{\left(C_b^{\mathfrak{D}}\right)^*}$ are contractions (see Remark 1.3.5), the Grunsky operator is a contraction as well. ☐

Lemma 4.0.22. *Let $b : \mathbb{D} \to \mathbb{D}$, $b(z) = \sum_{n=1}^{\infty} b_n z^n$, with $b_1 \neq 0$, such that $C_b^{\mathcal{I}\mathfrak{D}}$ is a contraction. Then for arbitrary $w \in \mathbb{D}$ the function*

$$g_{\bar{w}}(z) := G_b K_{\mathfrak{D}}(\cdot, w)(z)$$

is the analytic continuation on $\mathbb{D}$ of the function

$$f_{\bar{w}}(z) := \log\left(\frac{\overline{b(w)}b^\sharp(z)}{\overline{b'(0)}\bar{w}z}\right) + \log\left(\frac{z - \bar{w}}{b^\sharp(z) - \overline{b(w)}}\right).$$

Proof. For $k \in \mathbb{N}$, $p(z) \in \mathcal{P}_k$ we already established in Lemma 4.0.16, that $\left(C_b^{\mathfrak{D}}\right)^* p$ is a polynomial of degree k with $\left(\left(C_b^{\mathfrak{D}}\right)^* p\right)(0) = 0$. Thus we may define

$$p_k(z) := \sum_{n=0}^{k} p_n^{(k)} z^n := \left(C_b^{\mathfrak{D}}\right)^* (\zeta \mapsto \zeta^k) + k c_k,$$

where the c_k are the coefficients of the power series expansion of

$$\log\left(\frac{b^\sharp(z)}{\overline{b'(0)}z}\right) = \sum_{k=1}^{\infty} c_k z^k$$

which has a positive radius of convergence r_0.

By Theorem 4.0.21 the Grunsky operator G_b exists and by Corollary 4.0.20 fulfills for $p \in \mathcal{P}$, $p(0) = 0$

$$\begin{aligned} G_b p &= P_+ C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p \\ &= C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p - P_- C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p - P_0 C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p \\ &= C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p - Sp - P_0 C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S \left(C_b^{\mathfrak{D}}\right)^* p. \end{aligned}$$

Hence for $z \in \mathbb{D}$, $k \in \mathbb{N}$

$$G_b\left(\zeta \mapsto \zeta^k\right)(z) = p_k\left(\frac{1}{b^\sharp(z)}\right) - nc_k - \frac{1}{z^k} - P_0 C_{b^\sharp}^{\mathcal{I}\mathfrak{D}} S\left(C_b^{\mathfrak{D}}\right)^*(\zeta \mapsto \zeta^k) =$$
$$=: p_k\left(\frac{1}{b^\sharp(z)}\right) - \frac{1}{z^k} + d_k.$$

Further we calculate for arbitrary $w \in \mathbb{D}$

$$g_{\bar{w}}(z) = \left(G_b K_{\mathfrak{D}}(\cdot, w)\right)(z) = \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n}\left(G_b\left(\zeta \mapsto \zeta^n\right)\right)(z) =$$
$$= \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n}\left(p_n\left(\frac{1}{b^\sharp(z)}\right) - \frac{1}{z^n} + d_n\right). \quad (4.24)$$

By using Lemma 1.4.7 we have

$$\log\left(\frac{1}{1-\overline{b(w)}z}\right) = K_{\mathfrak{D}}(z, b(w)) = \left(\left(C_b^{\mathfrak{D}}\right)^* K_{\mathfrak{D}}(\cdot, w)\right)(z) = \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n}(p_n(z) - nc_n)$$

for all $z, w \in \mathbb{D}$. For $|w| < r_0$ the series $\sum_{n=1}^{\infty} c_n \bar{w}^n$ converges and we can write

$$\sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} p_n(z) = \log\left(\frac{1}{1-\overline{b(w)}z}\right) + \sum_{n=1}^{\infty} \bar{w}^n c_n = \log\left(\frac{1}{1-\overline{b(w)}z}\right) + \log\left(\frac{\overline{b(w)}}{\overline{b'(0)}\bar{w}}\right).$$

The series on the left hand side converges if $|w| < r_0$ and $|b(w)| < |z|^{-1}$. For $\lambda \in \mathbb{D}\backslash\{0\}$ let $\delta_\lambda > 0$ be such that $|w| < r_0$, $|w| < |\lambda|$, $|b(w)| < |b^\sharp(\lambda)|$ for all $|w| < \delta_\lambda$. Then for arbitrary $\lambda \in \mathbb{D}$ and $w \in B_{\delta_\lambda}(0)$ we have

$$\begin{aligned} f_{\bar{w}}(\lambda) &= \log\left(\frac{\overline{b(w)} b^\sharp(\lambda)}{\overline{b'(0)}\bar{w}\lambda} \frac{\lambda - \bar{w}}{b^\sharp(\lambda) - \overline{b(w)}}\right) \\ &= \log\left(\frac{1}{1 - \frac{\overline{b(w)}}{b^\sharp(\lambda)}}\right) + \log\left(\frac{\overline{b(w)}}{\overline{b'(0)}\bar{w}}\right) - \log\left(\frac{1}{1-\frac{\bar{w}}{\lambda}}\right) \quad (4.25) \\ &= \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} p_n\left(\frac{1}{b^\sharp(\lambda)}\right) - \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} \frac{1}{\lambda^n}. \end{aligned}$$

Since for fixed $z \in \mathbb{D}$, the right hand side of (4.24) converges, this shows, that the series $\sum_{n=1}^{\infty} \frac{\bar{w}}{n} d_n$ has a radius of convergence of at least δ_z and from (4.24) and (4.25) we obtain

$$g_{\bar{w}}(\lambda) - \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} d_n = f_{\bar{w}}(\lambda) \quad (4.26)$$

for $\lambda \in \mathbb{D} \setminus \{0\}$, $w \in B_{\delta_\lambda}(0)$.

Since $g_{\bar{w}} \in \mathfrak{D}$ for all $w \in \mathbb{D}$, the functions $g_{\bar{w}}$ are analytic on $\mathbb{D}$. Moreover, because of

$$g_{\bar{w}}(z) = G_b K_{\mathfrak{D}}(\cdot, w)(z) = [G_b K_{\mathfrak{D}}(\cdot, w), K_{\mathfrak{D}}(\cdot, z)]_{\mathfrak{D}} = \overline{[(G_b)^* K_{\mathfrak{D}}(\cdot, z), K_{\mathfrak{D}}(\cdot, w)]_{\mathfrak{D}}} = \overline{((G_b)^* K_{\mathfrak{D}}(\cdot, z))(w)}$$

the function $w \mapsto g_w(z)$ is analytic on $\mathbb{D}$ for fixed $z \in \mathbb{D}$. Hence the left hand side of (4.26) is analytic as a function in $\bar{w}$ for $|w| < R\left(\sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} d_n\right)$. For small w, the right hand side is analytic in λ on a disc containing the origin. Hence on this disc equation (4.26) is valid. In particular, we have

$$0 = f_{\bar{w}}(0) = g_{\bar{w}}(0) - \sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} d_n = -\sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} d_n$$

for all w in this disc. Hence $\sum_{n=1}^{\infty} \frac{\bar{w}^n}{n} d_n = 0$ for all $w \in \mathbb{D}$ and thus $g_{\bar{w}}(z)$ is an analytic continuation of $f_{\bar{w}}(z)$ to the whole bidisc. □

Now we are able to prove Theorem 4.0.6:

proof (of Theorem 4.0.6). (i) ⇒ (ii) follows immediately from Remark 4.0.2, *(iii)*
(ii) ⇒ (iii) is exactly the result of Theorem 4.0.21.

Last we prove that *(iii)* ⇒ (*i*): Suppose that b is not univalent. Since we already know, that $b(z) \neq 0$ for all $z \neq 0$ by Remark 4.0.4, (ii), there exist $\bar{w}_1, z_1 \in \mathbb{D}$, $\bar{w}_1 \neq z_1$ with $b(\bar{w}_1) = b(z_1) \neq 0$. Hence the function $f_{\bar{w}_1}$ has no analytic continuation to the point z_1 which is a contradiction to the statement of Lemma 4.0.22. □

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