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Joint Spectral Theorem for definitizable self-adjoint operators on Krein spaces

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Introduction

The purpose of the master thesis is to develop a joint spectral theorem for a tuple of pairwise commuting definitizable self-adjoint operators on a Krein space, cf. Theorem 3.4.6. This is inspired by [5], where a functional calculus for normal definitizable operators on Krein spaces is developed.

In the first section we start with a introduction to Krein spaces. Then we will show that we can find a Hilbert space $\mathcal{H}$ and a injective and linear bounded mapping $T : \mathcal{H} \to \mathcal{K}$ for every positive operator P on a Krein space $\mathcal{K}$ such that $TT^+ = P$. Additionally, we define a meaningful concept of joint spectrum for a tuple $\boldsymbol{a} = (a_i)_{i=1}^n$ in a commutative unital Banach algebra. This concept will be extended to the unital Banach algebra of bounded and linear operators on a Krein space $L_\mathrm{b}(\mathcal{K})$. We also show that the joint spectrum of a tuple is non-empty. Moreover, we state the concept of a joint spectral measure for a tuple of commuting self-adjoint operators on a Hilbert space.

In Section 2 we will give a short introduction to linear relation. Furthermore we will present the $*$-homomorphism Θ from [6]. This $*$-homomorphism drags the Krein space setting into a Hilbert space setting.

In Section 3 we present the joint spectral theorem for a tuple of pairwise commuting definitizable self-adjoint operators on a Krein space. For every definitizable A_i we choose a real definitizing polynomial p_i. According to the first section there exists a Hilbert space $\mathcal{H}$ and a injective and linear bounded $T : \mathcal{H} \to \mathcal{K}$ for the positive operator $\sum_{i=1}^n p_i(A_i)$ on the Krein space $\mathcal{K}$ such that $TT^+ = \sum_{i=1}^n p_i(A_i)$. We introduce a proper function class $\mathcal{F}_{\boldsymbol{A}}$ for which we can define the functional calculus $\phi \mapsto \phi(\boldsymbol{A})$. This will be done by decomposing ϕ into a polynomial s and a remainder g which vanishes at every critical point. We then define $\phi(\boldsymbol{A}) = s(\boldsymbol{A}) + T \int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \,\mathrm{d}E\, T^+$, where E is the joint spectral measure of $\Theta(\boldsymbol{A})$. We will show that this constitutes a $*$-homomorphism. Furthermore, we will endow the function class $\mathcal{F}_{\boldsymbol{A}}$ with a norm and proof that $\phi \mapsto \phi(\boldsymbol{A})$ is continuous in ϕ with respect to this norm. Since every entry A_i in the tuple $\boldsymbol{A}$ has its own functional calculus, if we regard one entry as a one-tuple, we will give a connections between the functional calculus of one entry A_i and the spectral calculus of the tuple $\boldsymbol{A}$.

In Section 4 we derive a spectral calculus for normal definitizing operators. This will be done by splitting a normal operator N into its real and imaginary part A_1 and A_2 and using the spectral calculus for $\boldsymbol{A} = (A_1, A_2)$.

Notation

Symbol	Meaning
$\mathbb{N}$	natural numbers starting with 1
$\mathbb{N}_0$	natural numbers starting with 0 ($\mathbb{N} \cup \{0\}$)
$\mathbb{Z}$	the set of all integers
$[n,m]_{\mathbb{Z}}$	$\{k \in \mathbb{Z} \mid n \leq k \leq m\}$
i	imaginary unit
$L_{\mathrm{b}}(M,X)$	Set of all bounded linear mappings $f: M \to X$
$L_{\mathrm{b}}(X)$	Set of all bounded linear mappings $f: X \to X$
$B_r^X(x)$	open ball with center x and radius r in X
$B_r(x)$	open ball with center x and radius r if the space is clear
$\delta_{i,j}$	Kronecker delta ($\delta_{i,j} = 1$ if $i = j$ and 0 else)

1 Preliminaries

1.1 Krein space

Definition 1.1.1. Let X be vector space over $\mathbb{C}$. We call a mapping $[.,.]_X : X \times X \to \mathbb{C}$, which fulfills

$$\begin{array}{lll} (a) & [\lambda x + \mu y, z]_X = \lambda[x,z]_X + \mu[y,z]_X, & \text{(linerarity)} \\ (b) & [x,y]_X = \overline{[y,x]_X}, & \text{(conjugate symmetry)} \end{array}$$

for $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$ an *inner product* and $(X, [.,.]_X)$ an *inner product space.*

An element $x \in X$ is called *positiv/negativ/neutral* if the real number $[x,x]_X$ is *positiv/negativ/zero.* A linear subspace Y of X is called *positiv (semi)definite* if the equality $[y,y]_X > (\geq) 0$ holds for all $0 \neq y \in Y$. Accordingly, Y can be *negative (semi)definite* or (*neutral*). The inner product is called *positiv/negativ (semi)definite* if $X \leq X$ has the corresponding property.

Two elements $x, y \in X$ are called *orthogonal*, if $[x,y]_X = 0$, we will write $x[\perp]_X y$. Two subsets A, B of X are called orthogonal if $[x,y]_X = 0$ for all $x \in A$ and all $y \in B$, this will be denoted by $A[\perp]_X B$. For a subset A of X we set $A^{[\perp]_X} := \{x \in X : [x,y]_X = 0 \text{ for all } y \in A\}$, and call $A^{[\perp]_X}$ the *orthogonal companion* of A.

An element $x \in X$ is called *isotropic* if $\{x\}[\perp]_X X$. By $(X, [.,.]_X)^\circ$ we denote the set of all isotropic elements, called the *isotropic part* of $(X, [.,.]_X)$. If $(X, [.,.]_X)^\circ \neq \{0\}$, then we call the inner product *degenerated*, otherwise we call it *nondegenerated.* We call $(X, [.,.]_X)$ *degenerated*, if its inner product is degenerated. Accordingly, $(X, [.,.]_X)$ is *nondegenerated* if its inner product is nondegenerated.

If M, N are orthogonal subspaces of X such that $M \cap N = \{0\}$, then we denote the direct sum by $M[\dot{+}]_X N$ and call it the *direct and orthogonal sum.*

If no confusions are possible we will write $[.,.]$ instead of $[.,.]_X$, X° instead of $(X, [.,.]_X)^\circ$, $[\dot{+}]$ instead of $[\dot{+}]_X$, and $[\perp]$ instead of $[\perp]_X$ or even just $\perp$.

Example 1.1.2. Let us regard the vector space $X = \mathbb{C}^2$ endowed with

$$[x,y] = x_1 y_1 - x_2 y_2.$$

It is straightforward to check that $(X, [.,.])$ is an inner product space. The orthogonal companion of $M := \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ is again M. We want to recall that in a Hilbert space $(\mathcal{H}, [.,.]_\mathcal{H})$ we have $\mathcal{H} = U[\dot{+}]_\mathcal{H} U^{[\perp]_\mathcal{H}}$ for a closed subspace U. Contrary to these expectations, we neither have $M \cap M^{[\perp]} = \{0\}$ nor $M + M^{[\perp]} = X$.

Definition 1.1.3. Let $(X, [.,.])$ be a inner product space, X_+ a positive definite and X_- a negative definite subspace of X.

If we can express X as the direct and orthogonal sum

$$X = X_+ [\dot{+}]\, X^\circ [\dot{+}]\, X_-,$$

then we call (X_+, X_-) *fundamental decomposition* of $(X, [.,.])$. The space $(X, [.,.])$ is called *decomposable*, if there exists a fundamental decomposition.

The orthogonal projections P_+ along $X_-[\dot{+}]\,X^\circ$ onto X_+ and P_- along $X_+[\dot{+}]\,X^\circ$ onto X_- are called *fundamental projections.*

The linear mapping $J := P_+ - P_-$ is called *fundamental symmetry.* Furthermore we set $(x, y)_J := [Jx, y]$ for $x, y \in X$.

Facts 1.1.4. Let $(X, [.,.])$ be a decomposable inner product space, (X_+, X_-) a fundamental decomposition, P_+, P_- the corresponding fundamental projections, and J the fundamental symmetry.

- $(X_+, [.,.])$ and $(X_-, -[.,.])$ are a pre-Hilbert spaces.
- For $x, y \in X_+$, we have $(x, y)_J = [x, y]$.
- For $x, y \in X_-$, we have $(x, y)_J = -[x, y]$.
- X_+ and X_- are also orthogonal with respect to $(.,.)_J$, i.e. $X_+(\perp)_J\, X_-$.

Lemma 1.1.5. *Let $(X, [.,.])$ be a decomposable inner product space with fundamental symmetry J. Then the following assertions hold true:*

(i) $[Jx, y] = [x, Jy]$, $(Jx, y)_J = (x, Jy)_J$ *for all* $x, y \in X$.

(ii) $[x, y] = (Jx, y)_J$ *for all* $x, y \in X$.

(iii) $(.,.)_J$ *is a positive semidefinite inner product on* X.

(iv) If X *is nondegenerated, then* $(.,.)_J$ *induces the norm* $\|x\|_J := \sqrt{(x, x)_J}$.

(v) If X *is nondegenerated,* $J^2 = I$.

(vi) If X *is nondegenerated,* $X_+^{[\perp]} = X_-$ *and* $X_-^{[\perp]} = X_+$.

Proof. Since X is decomposable, every $x \in X$ can be written as $x = P_+x + P_-x + x_0$ for some $x_0 \in X^\circ$. Since the isotropic part x_0 does not change the value of the inner product, we have

$$\begin{aligned}[Jx, y] &= [P_+x, y] - [P_-x, y] = [P_+x, P_+y + P_-y] - [P_-x, P_+y + P_-y] \\ &= [P_+x, P_+y] - [P_-x, P_-y] = [(P_+ + P_-)x, (P_+ - P_-)y] = [x, Jy].\end{aligned}$$

From the already shown, we obtain

$$(Jx, y)_J = [J(Jx), y] = [Jx, Jy] = (x, Jy)_J.$$

By the definition of the fundamental symmetry J, we have

$$J^2 = (P_+ - P_-)(P_+ - P_-) = P_+^2 - P_+P_- - P_-P_+ + P_-^2 = P_+ + P_-. \tag{1.1}$$

Again by writing x as $P_+x + P_-x + x_0$ and mind that the isotropic part x_0 does not change the value of the inner product, we have

$$(Jx, y)_J = [JJx, y] = [P_+x + P_-x, y] = [P_+x + P_-x + x_0, y] = [x, y].$$

The linearity of J yields that $(.,.)_J$ is linear in the first argument. Moreover, $(.,.)_J$ is even a inner product, since

$$(x, y)_J = [Jx, y] = \overline{[y, Jx]} = \overline{[Jy, x]} = \overline{(y, x)_J}.$$

By the definition of the fundamental projections, we obtain

$$(x,x)_J = \underbrace{[P_+x, P_+x]}_{\geq 0} - \underbrace{[P_-x, P_-x]}_{\leq 0} \geq 0.$$

Hence, $(.,.)_J$ is a positive semidefinite inner product. Moreover, by the Cauchy-Schwarz inequality $(x,x)_J = 0$, if and only if $x \in X^\circ$. Consequently, if X is nondegenerated, $(.,.)_J$ is positive definite and $\|.\|_J$ is a norm on X.

If X is nondegenerated, then $x = P_+x + P_-x$ and consequently (1.1) implies $J^2 = I$.

By definition we have that $X = X_+ [\dot{+}]\, X^\circ [\dot{+}]\, X_-$. If X is nondegenerated, then it is easy to see that $X_- \subseteq X_+^{[\perp]}$. Moreover, if $0 \neq x \in X_+$, then we have $[x,x] > 0$. For $x \in X_+^{[\perp]}$ we obtain

$$0 = [x, P_+x] = [P_+x + P_-x, P_+x] = [P_+x, P_+x].$$

This yields that $P_+x = 0$ and in consequence $x = P_-x \in X_-$. Hence, $X_+^{[\perp]} \subseteq X_-$.

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Facts 1.1.6. Let $(\mathcal{K}, [.,.])$ be a nondegenerated and decomposable inner product space and $(\mathcal{K}_+, \mathcal{K}_-)$ a fundamental decomposition. Furthermore, let P_+, P_- be the corresponding fundamental projections and J the fundamental symmetry.

- For $x \in \mathcal{K}$ we have

$$\|Jx\|_J^2 = (Jx, Jx)_J = (\underbrace{JJ}_{=I} x, x)_J = \|x\|_J^2, \quad \text{and}$$

$$\|P_\pm x\|_J^2 = [\underbrace{JP_\pm}_{=P_\pm} x, P_\pm x] \leq \pm[P_\pm x, P_\pm x] \mp [P_\mp x, P_\mp x] = [Jx, x] = \|x\|_J^2.$$

  Hence, J, P_+, P_- are continuous with respect to $\|.\|_J$.

- The functions $f_y : x \mapsto [x,y] = (Jx, y)_J$ are linear and bounded. Hence, for $M \subseteq \mathcal{K}$

$$M^{[\perp]} = \bigcap_{y \in M} \ker f_y$$

  is closed with respect to $\|.\|_J$.

- Let $(\hat{\mathcal{K}}_+, \hat{\mathcal{K}}_-)$ be an arbitrary fundamental decomposition. Since $\hat{\mathcal{K}}_+ = \hat{\mathcal{K}}_-^{[\perp]}$ and $\hat{\mathcal{K}}_- = \hat{\mathcal{K}}_+^{[\perp]}$, both $\hat{\mathcal{K}}_+$ and $\hat{\mathcal{K}}_-$ are closed with respect to $\|.\|_J$.

Definition 1.1.7. An inner product space $(\mathcal{K}, [.,.]_\mathcal{K})$ is called *Krein space*, if it is nondegenerated and decomposable, such that $(\mathcal{K}_+, [.,.]_\mathcal{K})$ and $(\mathcal{K}_-, -[.,.]_\mathcal{K})$ are Hilbert spaces for a some fundamental decomposition $(\mathcal{K}_+, \mathcal{K}_-)$.

Remark 1.1.8. Every Hilbert space $(\mathcal{H}, [.,.]_\mathcal{H})$ is also a Krein space.

Lemma 1.1.9. *If $(\mathcal{K}, [.,.]_\mathcal{K})$ is a Krein space and J denotes the fundamental symmetry of the fundamental decomposition $(\mathcal{K}_+, \mathcal{K}_-)$, which justifies that $(\mathcal{K}, [.,.]_\mathcal{K})$ is a Krein space, then $(\mathcal{K}, (.,.)_J)$ is a Hilbert space.*

Proof. Clearly, $(\mathcal{K}, (.,.)_J)$ is a pre-Hilbert space. By Facts 1.1.4, we have

$$\mathcal{K} = \mathcal{K}_+ (\dot{+})_J \mathcal{K}_-.$$

Since $(\mathcal{K}_+, [.,.]_\mathcal{K}) = (\mathcal{K}_+, (.,.)_J)$ and $(\mathcal{K}_-, -[.,.]_\mathcal{K}) = (\mathcal{K}_-, (.,.)_J)$ are Hilbert spaces, $(\mathcal{K}, (.,.)_J)$ is also complete.

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Theorem 1.1.10. *Let $(\mathcal{K}, [.,.]_\mathcal{K})$ be a Krein space, $(\mathcal{K}_+, \mathcal{K}_-)$ the fundamental decomposition from Definition 1.1.7, and $(\hat{\mathcal{K}}_+, \hat{\mathcal{K}}_-)$ another fundamental decomposition. Furthermore, let J be the fundamental symmetry of $(\mathcal{K}_+, \mathcal{K}_-)$ and $\hat{J}$ be the fundamental symmetry of $(\hat{\mathcal{K}}_+, \hat{\mathcal{K}}_-)$. Then $(\hat{\mathcal{K}}_+, [.,.])$ and $(\hat{\mathcal{K}}_-, -[.,.])$ are also Hilbert spaces. Moreover $\|.\|_J$ and $\|.\|_{\hat{J}}$ are equivalent.*

Proof. Let J, P_+, P_- denote the fundamental symmetry and the fundamental projections according to $(\mathcal{K}_+, \mathcal{K}_-)$, and $\hat{J}, \hat{P}_+, \hat{P}_-$ denote the fundamental symmetry and the fundamental projections according to $(\hat{\mathcal{K}}_+, \hat{\mathcal{K}}_-)$.

As a first step we will show that $\hat{J}, \hat{P}_+, \hat{P}_-$ are continuous as mappings from $(\mathcal{K}, (.,.)_J)$ to $(\mathcal{K}, (.,.)_J)$. We will apply the closed graph theorem: Let $\big((x_n; \hat{P}_+ x_n)\big)_{n\in\mathbb{N}}$ a sequence in the graph of $\hat{P}_+$ which converges to $(x; y) \in \mathcal{K} \times \mathcal{K}$. Since $\hat{\mathcal{K}}_+$ and $\hat{\mathcal{K}}_-$ are closed and $x_n - \hat{P}_+ x_n = \hat{P}_- x_n \in \hat{\mathcal{K}}_-$, we have $y \in \hat{\mathcal{K}}_+$ and $x - y \in \hat{\mathcal{K}}_-$. Hence, $y = \hat{P}_+ y = \hat{P}_+ x$. Consequently, the graph of $\hat{P}_+$ is closed. In the same manner it can be shown that $\hat{P}_-$ is also continuous. From $\hat{J} = \hat{P}_+ - \hat{P}_-$, we conclude the continuity of $\hat{J}$.

By the continuity of $\hat{J}$ and J, we obtain

$$\|x\|_{\hat{J}}^2 = [\hat{J}x, x] = (J\hat{J}x, x)_J \le \|J\hat{J}x\|_J \|x\|_J \le C^2 \|x\|_J^2$$

for some $C > 0$. This proves

$$\|x\|_{\hat{J}} \le C \|x\|_J \tag{1.2}$$

As a next step we will show that the mapping $\hat{P}_+\big|_{\mathcal{K}_+} : (\mathcal{K}_+, \|.\|_J) \to (\hat{\mathcal{K}}_+, \|.\|_{\hat{J}})$ is bijective, bounded and boundedly invertible. For $x \in \mathcal{K}_+$, we have

$$\|x\|_J^2 = [x, x] = [\hat{P}_+ x, \hat{P}_+ x] + [\hat{P}_- x, \hat{P}_- x] \le [\hat{P}_+ x, \hat{P}_+ x] = \big\|\hat{P}_+ x\big\|_{\hat{J}}^2.$$

This yields

$$\|x\|_J \le \big\|\hat{P}_+ x\big\|_{\hat{J}} \overset{(1.2)}{\le} C\big\|\hat{P}_+ x\big\|_J \le C\|\hat{P}_+\| \, \|x\|_J \quad \text{for} \quad x \in \mathcal{K}_+.$$

Hence, $\hat{P}_+\big|_{\mathcal{K}_+}$ is injective and $(\operatorname{ran} \hat{P}_+\big|_{\mathcal{K}_+}, [.,.])$ is a Hilbert space. In order to show that $\hat{P}_+\big|_{\mathcal{K}_+}$ is surjective, we assume that $\operatorname{ran} \hat{P}_+\big|_{\mathcal{K}_+} \neq \hat{\mathcal{K}}_+$. Then there exists a $0 \neq y \in \hat{\mathcal{K}}_+$ such that $y[\perp] \operatorname{ran} \hat{P}_+\big|_{\mathcal{K}_+}$. For an arbitrary $x \in \mathcal{K}_+$ we have

$$[x, y] = \underbrace{[\hat{P}_+ x, y]}_{=0} + \underbrace{[\hat{P}_- x, y]}_{=0} = 0.$$

This yields $y \in \mathcal{K}_+^{[\perp]} = \mathcal{K}_-$ and consequently $y \in \mathcal{K}_- \cap \hat{\mathcal{K}}_+$, which is only possible for $y = 0$. This contradicts our assumption. Consequently, $\hat{P}_+\big|_{\mathcal{K}_+}$ is surjective and $(\hat{\mathcal{K}}_+, [.,.])$ is a Hilbert space.

By the same argument we can show that $(\hat{\mathcal{K}}_-, -[.,.])$ is also a Hilbert space. Therefore, we have justified that we can switch the roles of $(\mathcal{K}_+, \mathcal{K}_-)$ and $(\hat{\mathcal{K}}_+, \hat{\mathcal{K}}_-)$. Hence, (1.2) gives us the equivalence of $\|.\|_J$ and $\|.\|_{\hat{J}}$.

❑

Theorem 1.1.10 tells us that, if there exists one fundamental decomposition which makes $(\mathcal{K}, [.,.])$ a Krein space, then every fundamental decomposition does so.

In the following we will equip every Krein space $(\mathcal{K}, [.,.]_\mathcal{K})$ with the norm topology of $\|.\|_J$ for an arbitrary fundamental symmetry J, if not other stated.

Lemma 1.1.11. *Let $(\mathcal{K}, [.,.])$ be a Krein space and $M \subseteq \mathcal{K}$. Then $M^{[\perp][\perp]} = \overline{M}$.*

Proof. Let J be a arbitrary fundamental symmetry of $(\mathcal{K}, [.,.])$. Since $[x,y] = (Jx, y)_J = (x, Jy)_J$ for $x, y \in \mathcal{K}$, we have

$$x[\perp]M \quad \Leftrightarrow \quad Jx(\perp)_J M \quad \Leftrightarrow \quad x(\perp)_J JM.$$

Therefore, $M^{[\perp]} = J(M^{(\perp)_J}) = (JM)^{(\perp)_J}$. This identity yields

$$M^{[\perp][\perp]} = (J(M^{(\perp)_J}))^{[\perp]} = (JJ(M^{(\perp)_J}))^{(\perp)_J} = M^{(\perp)_J(\perp)_J} = \overline{M}.$$

❑

Remark 1.1.12. If $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$ and $(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ are Krein spaces, then we can endow $\mathcal{K}_1 \times \mathcal{K}_2$ with an inner product

$$[(x;y),(u;v)]_{\mathcal{K}_1 \times \mathcal{K}_2} := [x,u]_{\mathcal{K}_1} + [y,v]_{\mathcal{K}_2}$$

and obtain the Krein space $(\mathcal{K}_1 \times \mathcal{K}_2, [.,.]_{\mathcal{K}_1 \times \mathcal{K}_2})$. In fact, it is straightforward to check that $[.,.]_{\mathcal{K}_1 \times \mathcal{K}_2}$ is an inner product. Let $(\mathcal{K}_{1+}, \mathcal{K}_{1-})$ be a fundamental decomposition of $\mathcal{K}_1$ and $(\mathcal{K}_{2+}, \mathcal{K}_{2-})$ be a fundamental decomposition of $\mathcal{K}_2$. Then $(\mathcal{K}_{1+} \times \mathcal{K}_{2+}, \mathcal{K}_{1-} \times \mathcal{K}_{2-})$ is a fundamental decomposition of $\mathcal{K}_1 \times \mathcal{K}_2$. Since $(\mathcal{K}_{1\pm}, [.,.]_{\mathcal{K}_1})$ and $(\mathcal{K}_{2\pm}, [.,.]_{\mathcal{K}_2})$ are Hilbert spaces, $(\mathcal{K}_{1+} \times \mathcal{K}_{2+}, [.,.]_{\mathcal{K}_1 \times \mathcal{K}_2})$ and $(\mathcal{K}_{1-} \times \mathcal{K}_{2-}, [.,.]_{\mathcal{K}_1 \times \mathcal{K}_2})$ are also Hilbert spaces.

1.2 Operators on Krein spaces

For two Krein spaces $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$ and $(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ we can equip $L_\mathrm{b}(\mathcal{K}_1, \mathcal{K}_2)$ with the operator norm

$$\|A\| := \sup_{x \in \mathcal{K}_1 \setminus \{0\}} \frac{\|Ax\|_{J_2}}{\|x\|_{J_1}} \qquad \text{for} \quad A \in L_\mathrm{b}(\mathcal{K}_1, \mathcal{K}_2),$$

where J_1 is a fundamental symmetry of $\mathcal{K}_1$ and J_2 is a fundamental symmetry of $\mathcal{K}_2$. If we choose different fundamental symmetries, then we obtain an equivalent norm.

Lemma 1.2.1. *Let $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1}), (\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ be Krein spaces, and let $A \in L_\mathrm{b}(\mathcal{K}_1, \mathcal{K}_2)$. Then there exists a unique operator $A^+ \in L_\mathrm{b}(\mathcal{K}_2, \mathcal{K}_1)$, which satisfies*

$$[Ax, y]_{\mathcal{K}_2} = [x, A^+ y]_{\mathcal{K}_1} \quad \textit{for} \quad x \in \mathcal{K}_1, y \in \mathcal{K}_2.$$

Moreover, we have $\|A\| = \|A^+\|$. We will call the operator A^+ the Krein space adjoint of A.

Proof. Let J_1 and J_2 be a fundamental symmetry of $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$ and $(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ respectively. Furthermore, let A^* the Hilbert space adjoint of A, when $\mathcal{K}_1$ is endowed with $(.,.)_{J_1}$ and $\mathcal{K}_2$ is endowed with $(.,.)_{J_2}$. Due to

$$[Ax, y]_{\mathcal{K}_2} = (Ax, J_2 y)_{J_2} = (x, A^* J_2 y)_{J_1} = [x, \underbrace{J_1 A^* J_2}_{=:A^+} y]_{\mathcal{K}_1}$$

we can be certain of the existence of A^+. Since J_1, J_2 are boundedly invertible, the uniqueness follows from the uniqueness of A^*. Since $\|A^*\| = \|A\|$, we obtain

$$\left\|A^+\right\| = \|J_1 A^* J_2\| \leq \|J_1\| \, \|A^*\| \, \|J_2\| = \|A^*\| = \|A\| \tag{1.3}$$

The uniqueness of A^+ implies $A^{++} = A$. Hence, we can switch the roles of A^+ and A in (1.3) and obtain $\|A\| = \|A^+\|$.

❑

Remark 1.2.2. If $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1}), (\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ are even Hilbert spaces, then the Krein space adjoint coincides with the Hilbert space adjoint.

Facts 1.2.3. Let $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1}), (\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ and $(\mathcal{K}_3, [.,.]_{\mathcal{K}_3})$ be Krein spaces, $A, B \in L_{\mathrm{b}}(\mathcal{K}_1, \mathcal{K}_2)$, and $C \in L_{\mathrm{b}}(\mathcal{K}_2, \mathcal{K}_3)$. Then

- $(A + \lambda B)^+ = A^+ + \overline{\lambda} B^+$,
- $(CA)^+ = A^+ C^+$.

Definition 1.2.4. Let $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be a Krein space and $A \in L_{\mathrm{b}}(\mathcal{K})$. Then we call A

- *normal*, if it commutes with its adjoint A^+,
- *self-adjoint*, if $A = A^+$.

Remark 1.2.5. Clearly, every self-adjoint operator is normal.

Definition 1.2.6. Let $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be a Krein space. Then we call a self-adjoint operator $P \in L_{\mathrm{b}}(\mathcal{K})$ *positive*, if P satisfies

$$[Px, x]_{\mathcal{K}} \geq 0 \quad \text{for all} \quad x \in \mathcal{K}.$$

Definition 1.2.7. Let $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be a Krein space and $A \in L_{\mathrm{b}}(\mathcal{K})$ be a self-adjoint Operator. We will call A *definitizable* if there exists a polynomial $p \in \mathbb{C}[x] \setminus \{0\}$ such that $p(A)$ is a positive operator. Any $p \in \mathbb{C}[x] \setminus \{0\}$ which satisfies this condition will be called a *definitizing polynomial* for A.

Lemma 1.2.8. *If $(\mathcal{K}, [.,.]_{\mathcal{K}})$ is a Krein space and $A \in L_{\mathrm{b}}(\mathcal{K})$ is definitizable, then there exists a definitizing polynomial $p \in \mathbb{R}[z] \setminus \{0\}$.*

Proof. Let $q \in \mathbb{C}[z] \setminus \{0\}$ be a definitizing polynomial for A. Then we define $q^{\#}(z) := \overline{q(\overline{z})} \in \mathbb{C}[z]$ and $p(z) := q^{\#}(z) + q(z)$. Clearly, we have $p \in \mathbb{R}[z]$. Since $q(A)$ is self-adjoint, we have

$$q(A) = q(A)^{+} = q^{\#}(A),$$

and therefore the operator $p(A) = 2q(A)$ is positive. If $p \neq 0$, then we are done.

For $p = 0$ we conclude that $-q(z) = q^{\#}(z)$ and that the coefficients of q are purely imaginary. Hence,

$$-q(A) = q^{\#}(A) = q(A)^{+} = q(A),$$

and in consequence $q(A) = 0 = \mathrm{i}q(A)$. Since q's coefficients are purely imaginary, iq is a definitizing polynomial for A in $\mathbb{R}[z] \setminus \{0\}$.

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According to the previous Lemma we will always choose definitizing polynomials in $\mathbb{R}[z] \setminus \{0\}$.

Lemma 1.2.9. *Let* $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$ *and* $(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ *be Krein spaces. For every* $A \in L_{\mathrm{b}}(\mathcal{K}_1, \mathcal{K}_2)$ *we have*

$$(\operatorname{ran} A)^{[\perp]_{\mathcal{K}_2}} = \ker A^{+}.$$

Proof. By definition we can write the orthogonal companion of $\operatorname{ran} A$ as

$$\begin{aligned}(\operatorname{ran} A)^{[\perp]_{\mathcal{K}_2}} &= \{x \in \mathcal{K}_2 : [x, Ay]_{\mathcal{K}_2} = 0 \text{ for all } y \in \mathcal{K}_1\} \\ &= \{x \in \mathcal{K}_2 : [A^{+}x, y]_{\mathcal{K}_2} = 0 \text{ for all } y \in \mathcal{K}_1\}.\end{aligned}$$

Since ever Krein space is nondegenerated, we have

$$(\operatorname{ran} A)^{[\perp]_{\mathcal{K}_2}} = \{x \in \mathcal{K}_2 : A^{+}x = 0\} = \ker A^{+}.$$

❑

Lemma 1.2.10. *Let* $(\mathcal{K}, [.,.]_{\mathcal{K}})$ *be a Krein space and* $P \in L_{\mathrm{b}}(\mathcal{K})$ *a positive Operator. Then there exists a Hilbert space* $(\mathcal{H}, [.,.]_{\mathcal{H}})$ *and an injective and bounded linear mapping* $T : \mathcal{H} \to \mathcal{K}$ *such that* $TT^{+} = P$.

Proof. Since P is positive $\langle .,. \rangle := [P.,.]_{\mathcal{K}}$ defines a positive semidefinite inner product on $\mathcal{K}$. Factorizing $\mathcal{K}$ by its isotropic part $\mathcal{K}^{\langle\circ\rangle}$ relating to $\langle .,. \rangle$ we obtain the pre-Hilbert space $\mathcal{K}/\mathcal{K}^{\langle\circ\rangle}$ with the canonical projection

$$\iota : \left\{ \begin{array}{ccc} \mathcal{K} & \to & \mathcal{K}/\mathcal{K}^{\langle\circ\rangle}, \\ x & \mapsto & x + \mathcal{K}^{\langle\circ\rangle}, \end{array} \right.$$

and the scalar product $\langle x + \mathcal{K}^{\langle\circ\rangle}, y + \mathcal{K}^{\langle\circ\rangle} \rangle := \langle x, y \rangle$. We define $\mathcal{H}$ as the Hilbert space completion of $\mathcal{K}/\mathcal{K}^{\langle\circ\rangle}$. We can regard ι as a mapping into $\mathcal{H}$. From

$$\|\iota x\|^2 = \langle \iota x, \iota x \rangle = [Px, x]_{\mathcal{K}} \leq \|P\| \, \|x\|^2,$$

we conclude the continuity of ι. Therefore, we can define $T : \mathcal{H} \to \mathcal{K}$ as $T := \iota^{+}$. Since ι is bounded, T is also bounded. Due to the continuity of the inner product

$(\operatorname{ran}\iota)^\perp = (\overline{\operatorname{ran}\iota})^\perp$. Hence, the density of $\operatorname{ran}\iota$ in $\mathcal{H}$ implies $\ker\iota^+ = \{0\}$ and consequently the injectivity of T. By definition, for $x, y \in \mathcal{K}$ we have

$$[TT^+x, y]_\mathcal{K} = \langle T^+x, T^+y\rangle = \langle \iota x, \iota y\rangle = \langle x, y\rangle = [Px, y]_\mathcal{K}$$

and consequently $TT^+ = P$.

❑

Remark 1.2.11. It is possible that the Hilbert space $\mathcal{H}$ in the previous Lemma is the zero-dimensional space $\{0\}$. This will happen, if and only if $P = 0$.

Corollary 1.2.12. *Let $\mathcal{K}$ be a Krein space and $A \in L_\mathrm{b}(\mathcal{K})$ self-adjoint and definitizable. Then there exists a Hilbert space $\mathcal{H}$ and an injective and bounded linear mapping $T : \mathcal{H} \to \mathcal{K}$ such that $TT^+ = p(A)$.*

Proof. Let $p \in \mathbb{C}[x]$ be a definitizing polynomial for A. By definition $p(A)$ is a positive operator. Lemma 1.2.10 will do the rest.

❑

1.3 Gelfand space

Definition 1.3.1. Let $A \neq \{0\}$ be a vector space over $\mathbb{C}$.

(*i*) If A is equipped with a bilinear mapping

$$\begin{cases} A \times A & \to & A, \\ (a, b) & \mapsto & ab, \end{cases}$$

which is additionally associative, i.e.

$$a(bc) = (ab)c \quad \text{for all} \quad a, b, c \in A,$$

then we will call A an *algebra* over $\mathbb{C}$. This mapping is called the multiplication in A.

(*ii*) An algebra A is said to be *commutative*, if

$$ab = ba \quad \text{for all} \quad a, b \in A.$$

(*iii*) A *subalgebra* B of an algebra A is a linear subspace of A such that

$$ab \in B \quad \text{for} \quad a, b \in B.$$

(*iv*) An element $e \in A$ is called *unit element* of A, if

$$ea = ae = a \quad \text{for all} \quad a \in A.$$

If A contains a unit element, A is said to be *unital*. In the following we will denote the unit element always by e.

(*v*) An element a in a unital algebra A is said to be *invertible* if there exists an element $b \in A$, such that

$$ab = ba = e,$$

where e is the unit element. The set of all invertible elements of A will be denoted by $\operatorname{Inv}(A)$

(*vi*) For every a in a unital algebra A the set

$$\rho_A(a) := \{\lambda \in \mathbb{C} : (a - \lambda e) \in \mathrm{Inv}(A)\}$$

is called the *resolvent set* of a. The set

$$\sigma_A(a) := \mathbb{C} \setminus \rho(a) = \{\lambda \in \mathbb{C} : (a - \lambda e) \notin \mathrm{Inv}(A)\}$$

is called the *spectrum* of a. We will just write $\sigma(a)\,, \rho(a)$ if no confusions about the algebra is possible.

(*vii*) If A is equipped with a norm $\|.\|$, such that $\|.\|$ is *submultiplicative*, i.e.

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for all} \quad a, b \in A,$$

then A is a *normed algebra*. If A equipped with $\|.\|$ additionally is a Banach space, then we call A a *Banach algebra*.

(*viii*) If a normed algebra A contains a unital element e, then e is said to be *normed* if $\|e\| = 1$. If A additionally is a Banach algebra and contains a normed unital element, we call A a *unital Banach algebra*.

(*ix*) If there is a mapping

$$(.)^* : \begin{cases} A & \rightarrow & A, \\ a & \mapsto & a^*, \end{cases}$$

such that

- $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$,
- $(a^*)^* = a$,
- $(ab)^* = b^* a^*$,

then we call A a $*$-*algebra*.

Lemma 1.3.2. *Let X be unital Banach algebra. Then the set* $\mathrm{Inv}(X)$ *is open and the mapping $a \mapsto a^{-1}$ is continuous on* $\mathrm{Inv}(X)$.

Proof. As first step we will show that if $\|a\| < 1$ for an $a \in X$, then $e - a \in \mathrm{Inv}(X)$ and $(e-a)^{-1} = \sum_{n=0}^{\infty} a^n$: Since $\|a^n\| \leq \|a\|^n$ we have

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} < +\infty.$$

Hence, $\sum_{n=0}^{\infty} a^n$ converges absolutely. The continuity of $c \mapsto cb$ yields

$$(e - a) \sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty} a^n - \sum_{n=1}^{\infty} a^n = a^0 = e. \tag{1.4}$$

In the same way $\sum_{n=0}^{\infty} a^n (e - a) = e$ can be shown. Hence, $(e - a)$ is invertible.

Let $a \in \mathrm{Inv}(X)$ and $\|b\| \leq \frac{1}{\|a^{-1}\|}$. Then we can write $a + b = a(e - a^{-1}(-b))$ where $\left\|a^{-1}(-b)\right\| < 1$. Hence, $(e - a^{-1}(-b))$ is invertible by the first step.

Consequently $a+b$ has $(e-a^{-1}(-b))^{-1}a^{-1}$ as its inverse. We showed that $B_{\frac{1}{\|a^{-1}\|}}(a) = a + B_{\frac{1}{\|a^{-1}\|}}(0) \subseteq \mathrm{Inv}(X)$ which implies that $\mathrm{Inv}(X)$ is open.

Let again $a \in \mathrm{Inv}(X)$ and $\|b\| \leq \frac{1}{\|a^{-1}\|}$. By the already shown we have

$$\begin{aligned}\left\|(a+b)^{-1} - a^{-1}\right\| &= \left\|\sum_{i=0}^{\infty} \big(a^{-1}(-b)\big)^n a^{-1} - a^{-1}\right\| = \left\|\sum_{i=0}^{\infty} \big(a^{-1}(-b)\big)^n a^{-1}\right\| \\ &\leq \left\|a^{-1}\right\| \sum_{i=1}^{\infty} \left\|a^{-1}b\right\|^n = \frac{\left\|a^{-1}\right\| \left\|a^{-1}b\right\|}{1-\|a^{-1}b\|} \leq \frac{\left\|a^{-1}\right\|^2}{1-\|a^{-1}b\|} \|b\| .\end{aligned}$$

Therefore, $\left\|(a+b)^{-1} - a^{-1}\right\|$ converges to 0, if $\|b\| \to 0$. Consequently, the mapping $a \mapsto a^{-1}$ is continuous.

❑

Lemma 1.3.3. *Let X be a unital Banach algebra and $a \in X$. Then $\rho(a)$ is open subset of $\mathbb{C}$ and the mapping*

$$R_{(.)}(a) : \begin{cases} \rho(a) & \to & X, \\ \lambda & \mapsto & (a-\lambda e)^{-1}. \end{cases}$$

is continuous. Moreover, $\lim_{|\lambda|\to\infty} \|R_\lambda(a)\| = 0$.

Proof. Consider the mapping $\Phi : \mathbb{C} \to X, \lambda \mapsto a - \lambda e$. This mapping is clearly continuous. Hence, $\rho(a)$ is open as the preimage of the open set $\mathrm{Inv}\, X$. Since we have $R_\lambda(a) = \big(\Phi\big|_{\rho(a)}(\lambda)\big)^{-1}$, we conclude that $R_{(.)}(a)$ is a composition of continuous mappings.

If $|\zeta| < \frac{1}{\|a\|}$ we can calculate the inverse of $(e - \zeta a)$ as we did in (1.4). Hence,

$$R_{\frac{1}{\zeta}}(a) = \big(a - \frac{1}{\zeta}e\big)^{-1} = -\zeta(e-\zeta a)^{-1} = -\zeta \sum_{n=0}^{\infty} \zeta^n a^n = -\sum_{n=0}^{\infty} \zeta^{n+1} a^n.$$

Since the series on the right-hand-side converges uniformly for $|\zeta| \leq \frac{1}{2\|a\|}$, we obtain

$$\begin{aligned}\lim_{|\lambda|\to\infty} \|R_\lambda(a)\| &= \lim_{|\zeta|\to 0} \left\|R_{\frac{1}{\zeta}}(a)\right\| = \lim_{|\zeta|\to 0} \left\|\sum_{n=0}^{\infty} \zeta^{n+1} a^n\right\| \\ &\leq \sum_{n=0}^{\infty} \lim_{|\zeta|\to 0} \left\|\zeta^{n+1} a^n\right\| = 0.\end{aligned}$$

❑

Theorem 1.3.4. (Liouville) *Let $\phi : \mathbb{C} \to \mathbb{C}$ be holomorphic. If ϕ is bounded, then ϕ has to be constant.*

Theorem 1.3.5. *Let X be a unital Banach algebra and $x \in X$. Then $\sigma(x) \neq \emptyset$.*

Proof. Let us assume that $x - \lambda e$ is invertible for every $\lambda \in \mathbb{C}$, i.e. $\sigma(x) = \emptyset$. For $\alpha, \beta \in \mathbb{C}$ such that $\alpha \neq \beta$ we have

$$\begin{aligned}(x-\alpha e)^{-1}(\alpha-\beta)(x-\beta e)^{-1} &= (x-\alpha e)^{-1}\big((x-\beta e)-(x-\alpha e)\big)(x-\beta e)^{-1}\\ &= (x-\alpha e)^{-1}-(x-\beta e)^{-1}.\end{aligned}$$

Applying any $f \in A'$ (continuous dual space of A) on this equation yields

$$\frac{f((x-\alpha e)^{-1})-f((x-\beta e)^{-1})}{\alpha-\beta} = f\big((x-\alpha e)^{-1}(x-\beta e)^{-1}\big).$$

Since the limit on the right hand side exists for $\alpha \to \beta$, the limit on the left hand side also exists. Hence, $\alpha \mapsto f((x-\alpha e)^{-1})$ is a holomorphic function with domain $\mathbb{C}$. Since $\lim_{|\alpha|\to\infty} \left\|(x-\alpha e)^{-1}\right\| = 0$ and $f((x-\alpha e)^{-1})$ is bounded for α in a compact set, we conclude by Liouville that $\alpha \mapsto f((x-\alpha e)^{-1})$ has to be constant 0. The seperating property of A' yields $(x-\alpha e)^{-1} = 0$ which is not possible for an invertible element.

❑

Theorem 1.3.6. (Gelfand-Mazur) *Let X be a unital Banach algebra. If* $\mathrm{Inv}(X) = X \setminus \{0\}$*, then X is one-dimensional.*

Proof. By Theorem 1.3.5 for every $x \in X$ there exists a $\lambda_x \in \sigma(x)$. Since 0 is the only not invertible element we conclude that $x - \lambda_x e = 0$ and consequently $x = \lambda_x e$. Hence, $\{e\}$ spans X.

❑

Definition 1.3.7. Let A be an algebra over $\mathbb{C}$.

- A subalgebra I of A is called *ideal*, if $ai, ia \in I$ for all $a \in A$ and $i \in I$. If additionally $I \neq A$, we call I a *proper ideal.*
- A proper ideal I is called *maximal ideal* if there is no proper ideal J such that $I \subsetneq J$ (i.e $I \subseteq J$ and $I \neq J$).
- A linear functional $m : A \to \mathbb{C}$ is said to be *multiplicative* if $m \neq 0$ and

$$m(ab) = m(a)m(b) \quad \text{for all} \quad a, b \in A.$$

Lemma 1.3.8. *Let A be a unital algebra.*

⇝ *A proper ideal does not contain any invertible elements.*

⇝ *Every proper ideal is contained in a maximal ideal.*

⇝ *Ever ideal with codimension one is a maximal ideal.*

⇝ *If A is a normed algebra, then the closure of an ideal is again an ideal.*

⇝ *If A is a unital Banach algebra, then every maximal ideal is closed.*

Proof.

- ⇝ If $a \in I \cap \mathrm{Inv}(A)$, then $e = a^{-1}a \in I$. Hence, $A = eA \subseteq I$, which is a contradiction.

- ⇝ Let I be a proper ideal and $\mathcal{I}$ the set of all proper ideals J satisfying $I \subseteq J$. Let $\mathcal{J}$ be an arbitrary chain (totally ordered subset) of $\mathcal{I}$ with respect to $\subseteq$. It is easy to check that

$$\bigcup_{J \in \mathcal{J}} J$$

  is also an ideal. Furthermore, it is a proper ideal since no $J \in \mathcal{J}$ contains the unit element e.

  By the Lemma of Zorn $\mathcal{I}$ has a maximal element, which is a maximal ideal containing I.

- ⇝ Let I be an ideal with codimension one. Then it certainly is a hyperspace. Hence, I is a proper ideal. Since every strictly greater subspace has to be already A, I is a maximal ideal.

- ⇝ If I is an ideal, then $\overline{I}$ is a subspace of A. By the submultiplicativity of the norm it is easy to check that the mapping $(a, b) \mapsto ab$ is continuous in the second argument. Hence, we have that $a\overline{I} \subseteq \overline{(aI)} \subseteq \overline{I}$. Analogously, we obtain $\overline{I}a = \overline{I}$. Consequently, $\overline{I}$ is an ideal.

- ⇝ Let I be a maximal ideal in the unital Banach algebra A. By the first statement of the present Lemma $I \subseteq \mathrm{Inv}(A)^{\mathsf{c}}$. By Lemma 1.3.2 the subset $\mathrm{Inv}(A)^{\mathsf{c}}$ is closed. Hence, $\overline{I} \subseteq \mathrm{Inv}(A)^{\mathsf{c}} \subsetneq A$. By the fourth statement of this Lemma $\overline{I}$ is a proper ideal. Since I is a maximal ideal, we conclude $I = \overline{I}$.

❑

Lemma 1.3.9. *Let A be a commutative unital algebra. Then $a \in A$ is invertible, if and only if $a \in A$ is not contained in any maximal ideal.*

Proof. If $a \in A$ is invertible, then a is by the first statement of Lemma 1.3.8 not contained in any proper ideal.

Since A is commutative the set $aA := \{ab \in A : b \in A\}$ is an ideal. If a is not invertible, then $e \notin aA$. Consequently, aA is a proper ideal. By the second statement of Lemma 1.3.8 there exists a maximal ideal J such that $aA \subseteq J$.

❑

Definition 1.3.10. Let A, B be algebras. We call a mapping $\Phi : A \to B$ an *algebra homomorphism*, if it satisfies

- $\Phi(\lambda a + \mu b) = \lambda\Phi(a) + \mu\Phi(b)$,
- $\Phi(ab) = \Phi(a)\Phi(b)$,

for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. If Φ is additionally bijective, then we call it an *algebra isomorphism.*

If A, B are even $*$-algebras, then we call an algebra homomoporphism Φ $*$-*homomorphism*, if it additionally satisfies

$$\Phi(a^*) = \Phi(a)^* \quad \text{for all} \quad a \in A.$$

Lemma 1.3.11. *Let I be an ideal of an algebra A. Then the mapping*

$$((a+I),(b+I)) \mapsto (a+I)(b+I) := (ab+I) \tag{1.5}$$

is well-defined and satisfies all condition of Definition 1.3.1 (i), i.e A/I is an algebra. Moreover the canonical projection $\pi_{A/I} : A \to A/I, a \mapsto a+I$ is an algebra homomorphism.

If A is a unital algebra, then A/I is also one.

Proof. Let $a_1 + I = a_2 + I$ and $b_1 + I = b_2 + I$. Then

$$a_1b_1 - a_2b_2 = a_1b_1 - (a_1+i)(b_1+j) = 0 - \underbrace{a_1j - b_1i - ij}_{\in I}$$

implies $a_1b_1 + I = a_2b_2 + I$. Hence, the mapping in (1.5) is well-defined. The bilinearity and associativity can be in a straightforward manner derived from the corresponding properties of $(a,b) \mapsto ab$.

If e is the unit element of A, then it can easily be seen that $e+I$ is the unit element of A/I.

It is also straightforward to check that $\pi_{A/I}$ is compatible with all algebra operation. We will exemplarily show the compatibility with the multiplication:

$$\pi_{A/I}(ab) = ab + I = (a+I)(b+I) = \pi_{A/I}(a)\pi_{A/I}(b).$$

❑

Corollary 1.3.12. *Let A be a unital Algebra and I an ideal with codimension one. Then the mapping $\beta_I : \lambda \mapsto \lambda e + I$ is an isomorphism from $\mathbb{C}$ to A/I. Moreover the mapping $m_I := \beta_I^{-1} \circ \pi_{A/I} : A \to \mathbb{C}$ is multiplicative functional with $\ker m_I = I$.*

Proof. Since A/I is by assumption one-dimensional and $e+I$ is not the 0 element in A/I, the set $\{e+I\}$ is a basis of A/I. Consequently the mapping $\beta_I : \lambda \mapsto \lambda(e+I) = \lambda e + I$ is bijective. It is straightforward to show that β_I is even a homomorphism and therefore an isomorphism.

As a composition of homomorphisms the mapping m_I is also a homomorphism and homomorphisms into $\mathbb{C}$ are multiplicative functionals.

❑

Proposition 1.3.13. *Let $(X, \|.\|)$ be a Banach space and N a closed subspace of X. Then X/N equipped with*

$$\|x+N\|_{X/N} := \inf_{z\in N} \|x+z\|$$

is also a Banach space

Proof. Let $x, y \in X$ and $z_1, z_2 \in N$.

$$\|(x+N)+(y+N)\|_{X/N} \leq \|x+y+z_1+z_2\| \leq \|x+z_1\| + \|y+z_2\|$$

Since $z_1, z_2 \in N$ were arbitrary, we obtain the triangular inequality for $\|.\|_{X/N}$. For $\lambda \in \mathbb{C} \setminus \{0\}$ we have that $\lambda N = N$. We will apply $\inf_{z \in N}$ on the following equation

$$\|\lambda x + z\| = |\lambda| \left\|x + \lambda^{-1} z\right\| \quad z \in N$$

on both sides in a different order. This yields

$$\begin{aligned} \|\lambda x + N\|_{X/N} &\leq |\lambda| \left\|x + \lambda^{-1} z\right\| & \|\lambda x + z\| &\geq |\lambda| \, \|x + N\|_{X/N} \\ \|\lambda x + N\|_{X/N} &\leq |\lambda| \, \|x + N\|_{X/N} & \|\lambda x + N\|_{X/N} &\geq |\lambda| \, \|x + N\|_{X/N} \end{aligned}$$

and in consequence $\|\lambda x + N\|_{X/N} = |\lambda| \, \|x + N\|_{X/N}$. This is even true for $\lambda = 0$. Clearly $0 \leq \|0 + N\|_{X/N} \leq \|0 + 0\| = 0$. If $\|x + N\|_{X/N} = 0$, then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that $z_n \in N$ for all $n \in \mathbb{N}$ and $\|x + z_n\| \to 0$. This means that $\lim_{n \in \mathbb{N}} z_n = -x$ and $-x \in N$, since N is closed. Hence, $x + N = 0 + N$.

Let $(x_n + N)_{n \in \mathbb{N}}$ be a Cauchy-sequence in X/N. We choose a subsequence $(x_{n_k} + N)_{k \in \mathbb{N}}$ such that $\left\|(x_{n_{k+1}} + N) - (x_{n_k} + N)\right\|_{X/N} \leq 2^{-k}$. We will recursively define $y_k \in (x_{n_k} + N)$ such that $\|y_{k+1} - y_k\| < 2^{-k}$:

We set $y_1 := x_{n_1}$. Let $y_1, \dots, y_k$ have the claimed properties. Then by

$$\begin{aligned} 2^{-k} &> \left\|(x_{n_{k+1}} + N) - (x_{n_k} + N)\right\|_{X/N} = \left\|x_{n_{k+1}} - y_k + N\right\|_{X/N} \\ &= \inf_{z \in N} \left\|x_{n_{k+1}} - y_k + z\right\| \end{aligned}$$

there exists a $z_k \in N$ such that $\left\|(x_{n_{k+1}} + z_k) - y_k\right\| < 2^{-k}$. Hence, we set $y_{k+1} := x_{n_{k+1}} + z_k$.

If $l \leq m$, then

$$\|y_m - y_l\| = \left\| \sum_{k=l}^{m-1} (y_{k+1} - y_k) \right\| \leq \sum_{k=l}^{m-1} \|y_{k+1} - y_k\| \leq \sum_{k=l}^{\infty} 2^{-k} \leq 2^{-l+1}$$

implies that $(y_k)_{k \in \mathbb{N}}$ is Cauchy-sequence in X. Since X is Banach space there exists a $y \in X$ such that $y_k \to y$. By

$$\|(y + N) - (x_{n_k} + N)\| = \|(y + N) - (y_k + N)\| \leq \|y - y_k\| \to 0$$

we conclude that $x_{n_k} + N$ converges to $y + N$ and since $x_n + N$ is a Cauchy-sequence, $x_n + N$ has the same limit.

❑

Proposition 1.3.14. *Let X be a commutative unital Banach algebra. Then every maximal ideal I of X has codimension one.*

Proof. Let I be a maximal ideal of X. Then I is closed and, by Proposition 1.3.13, X/I equipped with the factor norm is a Banach space. By Lemma 1.3.11, X/I is also an algebra. From

$$\|(xy + I)\|_{X/I} \leq \|xy + \underbrace{ix + jy + ij}_{\in I}\| = \|(x + j)(y + i)\| \leq \|x + j\| \, \|y + i\| \, ,$$

we conclude $\|(x+I)(y+I)\|_{X/I} \leq \|x+I\|_{X/I} \|y+I\|_{X/I}$. Clearly $e+I$ is the unit element in X/I and $0 < \|e+I\|_{X/I} \leq \|e+0\| = 1$. On the other hand $\|e+I\|_{X/I} = \|(e+I)(e+I)\|_{X/I} \leq \|e+I\|_{X/I}^2$, which gives us the missing inequality for $\|e+I\|_{X/I} = 1$. Hence, X/I is also a commutative unital Banach algebra.

Let $y+I \neq 0+I$ and J be an arbitrary ideal of X/I containing $y+I$. Furthermore, let $\pi_{X/I}$ denote the projection $x \mapsto x+I$. Then it is straightforward to show that $K := \pi_{X/I}^{-1}(J)$ is an ideal of X. Clearly $I = \pi_{X/I}^{-1}(\{0+I\}) \subseteq K$ and $x \in K \setminus I$, where $x \in X$ is such that $\pi_{X/I}(x) = y+I$. Since I is a maximal ideal, we conclude that $K = X$ and $J = X/I$. Therefore, there exists no proper ideal of X/I that contains $y+I$. By Lemma 1.3.9 every element of $(X/I) \setminus \{0+I\}$ is invertible. By Theorem 1.3.6 (Gelfand-Mazur) X/I is one-dimensional. Hence, the codimension of I is one.

❑

Definition 1.3.15. Let X be a commutative unital Banach algebra. Then we will call the set M_X of all multiplicative functionals on X the *Gelfand space* of X.

Theorem 1.3.16. *If X is a commutative unital Banach algebra, then the Gelfand space M_X is non-empty.*

Proof. If $X \setminus \{0\}$ does not contain any not invertible elements, then due to Theorem 1.3.6 (Gelfand-Mazur) we have $\mathbb{C}e = X$. Hence, for every element $x \in X$ there exists a unique $\lambda_x \in \mathbb{C}$ such that $x = \lambda_x e$. Consequently, the mapping

$$m : \begin{cases} X & \to & \mathbb{C}, \\ x & \mapsto & \lambda_x, \end{cases}$$

is as an element of M_X.

If $X \setminus \{0\}$ contains an element x which is not invertible, then by Lemma 1.3.9 x is contained in a maximal ideal J. By Proposition 1.3.14 J has codimension one. Hence, the mapping m_J from Corollary 1.3.12 is an element of M_X.

❑

Definition 1.3.17. Let X be a commutative unital Banach algebra and $\boldsymbol{a} = (a_i)_{i=1}^n \in X^n$ a n-tuple.

- Then $\boldsymbol{a}$ is said to be *invertible*, if there exists a $\boldsymbol{b} \in X^n$ such that

$$\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{i=1}^n a_i b_i = e.$$

  The set of all invertible elements of X^n will be denoted by $\operatorname{Inv}(X^n)$.

- We will interpret a $\boldsymbol{\lambda} \in \mathbb{C}^n$ as an element of X^n by $\boldsymbol{\lambda} = (\lambda_i e)_{i=1}^n \in X^n$.

- We will call the set

$$\rho_X(\boldsymbol{a}) := \left\{ \boldsymbol{\lambda} \in \mathbb{C}^n : (\boldsymbol{a} - \boldsymbol{\lambda}) \in \operatorname{Inv}(X^n) \right\}$$

the resolvent set of $\boldsymbol{a}$, where $\boldsymbol{a} - \boldsymbol{b} := (a_i - b_i)_{i=1}^n$. When we want to emphasize that we are talking about the resolvent set of a tuple, we will use the term *joint resolvent set*. We will just write $\rho(\boldsymbol{a})$ if no confusions about the algebra is possible.

- We will call the set

$$\sigma_X(\boldsymbol{a}) := \mathbb{C}^n \setminus \rho_X(\boldsymbol{a}) = \left\{\boldsymbol{\lambda} \in \mathbb{C}^n : (\boldsymbol{a} - \boldsymbol{\lambda}) \notin \mathrm{Inv}(X^n)\right\}$$

spectrum of $\boldsymbol{a}$. When we want to emphasize that we are talking about the spectrum of a tuple, we will use the term *joint spectrum*. We will just write $\sigma(\boldsymbol{a})$ if no confusions about the algebra is possible.

- Let Y be a commutative unital Banach algebra and $\psi : X \to Y$ an algebra homomorphism. Then we set

$$\psi(\boldsymbol{a}) := (\psi(a_i))_{i=1}^n.$$

Remark 1.3.18. If there exists an entry a_j in $\boldsymbol{a} = (a_i)_{i=1}^n$, such that a_j is invertible, then $\boldsymbol{a}$ is also invertible.

Proposition 1.3.19. *Let X be a commutative unital Banach algebra, $\boldsymbol{a} = (a_i)_{i=1}^n \in X^n$ and $\boldsymbol{\lambda} \in \mathbb{C}^n$. Then the following statements are equivalent*

(i) *$(\boldsymbol{a} - \boldsymbol{\lambda})$ is not invertible.*

(ii) *$I := \left\{(\boldsymbol{a} - \boldsymbol{\lambda}) \cdot \boldsymbol{b} : \boldsymbol{b} \in X^n\right\}$ is a proper ideal of X.*

(iii) *$\boldsymbol{\lambda} \in \{\phi(\boldsymbol{a}) : \phi \in M_X\}$.*

Proof. It is straightforward to check that in any case I is an ideal of X.

(i) $\Leftrightarrow$ *(ii)*: The fact that I is a proper ideal is equivalent to $e \notin I$ which is equivalent to $(\boldsymbol{a} - \boldsymbol{\lambda})$ being not invertible.

(ii) $\Rightarrow$ *(iii)*: If I is a proper ideal, it is contained in a maximal ideal J which has codimension one. Therefore, $I \subseteq \ker m_J$ where $m_J \in M_X$ is the mapping from Corollary 1.3.12. If we choose $\boldsymbol{b} = (\delta_{i,k} e)_{i=1}^n$, then

$$m_J(a_k - \lambda_k) = m_J((\boldsymbol{a} - \boldsymbol{\lambda}) \cdot \boldsymbol{b}) = 0.$$

Since this is true for $k \in [1, n]_{\mathbb{Z}}$, we obtain $m_J(\boldsymbol{a}) = \boldsymbol{\lambda}$.

(iii) $\Rightarrow$ *(ii)*: If $\phi \in M_X$ is such that $\phi(\boldsymbol{a}) = \boldsymbol{\lambda}$, then $\phi(a_k - \lambda_k) = 0$ for all $k \in [1, n]_{\mathbb{Z}}$. Hence, $I \subseteq \ker \phi$ and consequently I cannot contain e.

❑

Corollary 1.3.20. *Let X be a commutative unital Banach algebra and $\boldsymbol{a} = (a_i)_{i=1}^n \in X^n$. Then the spectrum $\sigma(\boldsymbol{a})$ is not empty.*

Proof. By Theorem 1.3.16 the Gelfand space M_X is not empty. Hence, there exists a $\phi \in M_X$. By Proposition 1.3.19 $(\boldsymbol{a} - \phi(\boldsymbol{a}))$ is not invertible and consequently $\phi(\boldsymbol{a}) \in \sigma(\boldsymbol{a})$.

❑

1.4 Joint Spectrum in Krein spaces

We already defined the term joint spectrum for a tuple of elements in commutative unital Banach algebra. Unfortunately, the space $L_b(\mathcal{K})$ is just a unital Banach algebra, but not commutative.

Definition 1.4.1. Let A be an algebra and $C \subseteq A$. Then we define the *commutant* C' of C by

$$C' := \{a \in A : ac = ca \text{ for all } c \in C\}.$$

If $\boldsymbol{a} \in A^n$, then we set $\boldsymbol{a}' := \{a_i : i \in [1, n]_{\mathbb{Z}}\}'$. The set $C'' := (C')'$ will be called the *bicommutant* of C.

Facts 1.4.2.

1. C' is the intersection of the kernels of the linear mappings ψ_c, $c \in C$, where

$$\psi_c : \begin{cases} A & \to & A, \\ x & \mapsto & xc - cx. \end{cases}$$

   Hence, C' is linear subspace of A. If $x, y \in C'$ and $c \in C$, then

$$(xy)c = x(yc) = x(cy) = (xc)y = (cx)y = c(xy),$$

   and consequently $xy \in C'$. Hence, C' is a subalgebra of A.

2. If A is normed algebra then all ψ_c are continuous. Hence, C' is closed as intersection of closed sets.

3. If $C_1 \subset C_2$, then ${C_1}' \supseteq {C_2}'$.

4. Since $xc = cx$ for all $x \in C'$ and all $c \in C$, we conclude $C \subseteq C''$.

5. From $C \subseteq C''$ we derive from Statement 3, $C' \supseteq (C'')'$. On the other hand Statement 4 combined with Statement 3 yields $C' \subseteq (C')''$. Hence, $C' = C'''$ and $C'' = C''''$.

6. $C \subseteq C'$ means nothing else than $cd = dc$ for all $c, d \in C$. This implies by Statement 3, $C' \supseteq C''$. Since $C' = C'''$, we have $C'' \subseteq C'''$. Therefore, C'' is a commutative algebra.

7. If A contains a unit element e, then $e \in C'$. Furthermore for $c \in C \cap \text{Inv}(A)$ we conclude from $xc = cx$ for all $x \in C'$, that also $xc^{-1} = c^{-1}x$ for all $x \in C'$ holds true. Hence, $c^{-1} \in C''$.

Proposition 1.4.3. *Let X be a unital Banach algebra and $C \subseteq X$ be such that $xy = xy$ for all $x, y \in C$. Then C'' is a commutative unital Banach algebra. Moreover,* $\text{Inv}(C'') = \text{Inv}(X) \cap C''$ *and* $\sigma_{C''}(x) = \sigma_X(x)$.

Proof. By Facts 1.4.2, C'' is commutative unital Banach algebra. If $x \in C'' \cap \text{Inv}(X)$, then $x^{-1} \in C'''' = C''$. Therefore, $\text{Inv}(C'') = \text{Inv}(X) \cap C''$, and in turn $\sigma_{C''}(x) = \sigma_X(x)$ for $x \in C''$.

❑

Definition 1.4.4. Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be a n-tuple of normal commuting operators in $L_b(\mathcal{K})$ where $(\mathcal{K}, [.,.]_{\mathcal{K}})$ is a Krein space.

(i) We call $\boldsymbol{A}$ *invertible* if $\boldsymbol{A}$ is invertible as an element of the commutative unital algebra $\boldsymbol{A}''$ in the sense of Definition 1.3.17.

(ii) The spectrum $\sigma(\boldsymbol{A})$ is defined by $\sigma_{\boldsymbol{A}''}(\boldsymbol{A})$ and the resolvent set $\rho(\boldsymbol{A})$ is defined by $\rho_{\boldsymbol{A}''}(\boldsymbol{A})$

Corollary 1.4.5. *If $\boldsymbol{A} = (A_i)_{i=1}^n$ is a n-tuple of normal commuting operators in $L_b(\mathcal{K})$, where $(\mathcal{K}, [.,.]_{\mathcal{K}})$ is a Krein space, then the spectrum $\sigma(\boldsymbol{A})$ is not empty.*

Proof. This follows directly from Corollary 1.3.20.

❑

1.5 Spectral theory in Hilbert spaces

In Hilbert spaces we can find for every self-adjoint operator A a spectral measure E, which gives us the functional calculus

$$f(A) = \int f \, \mathrm{d}E,$$

where f is measurable and bounded on $\sigma(A)$. In [1] the authors introduce a product spectral measure for commuting spectral measure $(E_i)_{i=1}^n$ (i.e. $E_i(\Delta_i)E_j(\Delta_j) = E_j(\Delta_j)E_i(\Delta_i)$). As a consequence it is possible to construct a joint spectral measure for a tuple $\boldsymbol{A} = (A_i)_{i=1}^n$ of pairwise commuting self-adjoint operators. The following theorem from [1, Theorem 6.5.1] explains how this joint spectral measure has to be understood.

Theorem 1.5.1. *Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be a tuple of self-adjoint commuting operators in $L_b(\mathcal{H})$ where $(\mathcal{H}, [.,.]_{\mathcal{H}})$ is a Hilbert space. Then there exists a unique spectral measure E on the Borel sets of $\mathbb{R}^n$, such that*

$$A_i = \int \pi_i \, \mathrm{d}E,$$

where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection on the i-th coordinate. We will call E the joint spectral measure *of $\boldsymbol{A}$.*

Remark 1.5.2. We can and will regard every spectral measure E on the Borel sets of $\mathbb{R}^n$ as a measure on the Borel sets of $\mathbb{C}^n$, if we set

$$E(A) = E(A \cap \mathbb{R}^n).$$

For the next theorem recall the definition of the support of a spectral measure E:

$$\operatorname{supp} E := \{\boldsymbol{x} \in \mathbb{C}^n : \epsilon > 0 \Rightarrow E(B_\epsilon(\boldsymbol{x})) \neq 0\}.$$

Theorem 1.5.3. *Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be a tuple of pairwise communting self-adjoint operators in $L_b(\mathcal{H})$ where $(\mathcal{H}, [.,.]_{\mathcal{H}})$ is a Hilbert space and let E denote the joint spectral measure of $\boldsymbol{A}$. Then*

$$\sigma(\boldsymbol{A}) = \operatorname{supp} E.$$

Proof. If $\boldsymbol{\lambda} \in \operatorname{supp} E$, then $E(B_\epsilon(\boldsymbol{\lambda})) \neq 0$ for every $\epsilon > 0$. Hence, for every $\epsilon > 0$ there exists a $f_\epsilon \in \operatorname{ran} E(B_\epsilon(\boldsymbol{\lambda}))$ such that $\|f_\epsilon\| = 1$. We obtain

$$\begin{aligned}\|(A_i - \lambda_i) f_\epsilon\|^2 &= \int |x_i - \lambda_i|^2 \,\mathrm{d}(E(\boldsymbol{x}) f_\epsilon, f_\epsilon) = \int_{B_\epsilon(\boldsymbol{\lambda})} |x_i - \lambda_i|^2 \,\mathrm{d}(E(\boldsymbol{x}) f_\epsilon, f_\epsilon) \\ &\leq \epsilon^2 \|f_\epsilon\|^2\end{aligned}$$

for all $i \in [1, n]_{\mathbb{Z}}$. Let us assume that $\boldsymbol{A} - \boldsymbol{\lambda}$ is invertible. Then there exists a tuple $\boldsymbol{B}$ such that $\boldsymbol{B} \cdot (\boldsymbol{A} - \boldsymbol{\lambda}) = I$, and in turn

$$\|f_\epsilon\| = \Big\| \underbrace{\sum_{i=1}^n B_i (A_i - \lambda_i)}_{=I} f_\epsilon \Big\| \leq \sum_{i=1}^n \|B_i\| \, \|(A_i - \lambda_i) f_\epsilon\| \leq \epsilon \|f_\epsilon\| \sum_{i=1}^n \|B_i\| .$$

Hence,

$$1 \leq \epsilon \sum_{i=1}^n \|B_i\| ,$$

which gives us a contradiction for $\epsilon < \frac{1}{\sum_{i=1}^n \|B_i\|}$. Consequently, $\boldsymbol{A} - \boldsymbol{\lambda}$ in not invertible and $\boldsymbol{\lambda} \in \sigma(\boldsymbol{A})$.

On the other hand if $\boldsymbol{\lambda} \in \mathbb{C}^n \setminus \operatorname{supp} E$, then we can define

$$\boldsymbol{B} := \int_{\operatorname{supp} E} \frac{1}{\|\boldsymbol{x} - \boldsymbol{\lambda}\|_2^2} \overline{(\boldsymbol{x} - \boldsymbol{\lambda})} \,\mathrm{d}E = \Big(\int_{\operatorname{supp} E} \frac{1}{\|\boldsymbol{x} - \boldsymbol{\lambda}\|_2^2} \overline{(x_i - \lambda_i)} \,\mathrm{d}E \Big)_{i=1}^n ,$$

because $\frac{1}{\|\boldsymbol{x} - \boldsymbol{\lambda}\|_2^2}$ is bounded on $\operatorname{supp} E$. The following calculation verifies that $\boldsymbol{\lambda}$ belongs to $\rho(\boldsymbol{A}) = \mathbb{C}^n \setminus \sigma(\boldsymbol{A})$:

$$\begin{aligned}(\boldsymbol{A} - \boldsymbol{\lambda}) \cdot \boldsymbol{B} &= \int (\boldsymbol{x} - \boldsymbol{\lambda}) \,\mathrm{d}E \cdot \int \frac{1}{\|\boldsymbol{x} - \boldsymbol{\lambda}\|_2^2} \overline{(\boldsymbol{x} - \boldsymbol{\lambda})} \,\mathrm{d}E \\ &= \sum_{i=1}^n \int (x_i - \lambda_i) \,\mathrm{d}E \int \frac{1}{\|\boldsymbol{x} - \boldsymbol{\lambda}\|_2^2} \overline{(x_i - \lambda_i)} \,\mathrm{d}E \\ &= \int \frac{1}{\|\boldsymbol{x} - \boldsymbol{\lambda}\|_2^2} (\boldsymbol{x} - \boldsymbol{\lambda}) \cdot \overline{(\boldsymbol{x} - \boldsymbol{\lambda})} \,\mathrm{d}E = \int 1 \,\mathrm{d}E = I.\end{aligned}$$

❑

Remark 1.5.4. We want to recall the polarization identity for a symmetric sesquilinear form:

$$\begin{aligned}[Ax, y] = \frac{1}{4} \Big(&[A(x + y), x + y] - [A(x - y), x - y] \\ &+ \mathrm{i}[A(x + \mathrm{i}y), x + \mathrm{i}y] - \mathrm{i}[A(x + \mathrm{i}y), x + \mathrm{i}y] \Big).\end{aligned}$$

Lemma 1.5.5. *Let $(\Omega, \mathfrak{S})$ and $(\Upsilon, \mathfrak{A})$ be measurable spaces, $(\mathcal{H}, [.,.]_{\mathcal{H}})$ a Hilbert space and E be a spectral measure on $(\Omega, \mathfrak{S}, \mathcal{H})$. If $T : \Omega \to \Upsilon$ is measurable mapping, then $E^T(\Delta) := (E \circ T^{-1})(\Delta)$ is a spectral measure on $(\Upsilon, \mathfrak{A}, \mathcal{H})$ and*

$$\int_\Delta \phi \,\mathrm{d}E^T = \int_{T^{-1}(\Delta)} \phi \circ T \,\mathrm{d}E$$

for all bounded and measurable ϕ.

Proof. It is straightforward to check that E^T is a spectral measure.

For arbitrary $f, g \in \mathcal{H}$ we have that $(E^T)_{f,g} = (E_{f,g})^T$. Since $E_{f,f}$ is a non-negative measure on $\mathfrak{S}$, The general transformation theorem for measures yields

$$\int_\Delta \phi \,\mathrm{d}(E^T)_{f,f} = \int_\Delta \phi \,\mathrm{d}(E_{f,f})^T = \int_{T^{-1}(\Delta)} \phi \circ T \,\mathrm{d}E_{f,f}$$

for all $f \in \mathcal{H}$ and for all $\Delta \in \mathfrak{A}$. By the polarization identity we also have $\int_\Delta \phi \,\mathrm{d}(E^T)_{f,g} = \int_{T^{-1}(\Delta)} \phi \circ T \,\mathrm{d}E_{f,g}$. Hence,

$$\int_\Delta \phi \,\mathrm{d}E^T = \int_{T^{-1}(\Delta)} \phi \circ T \,\mathrm{d}E$$

holds true.

❑

Corollary 1.5.6. *Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be tuple of pairwise commuting self-adjoint operators in $L_\mathrm{b}(\mathcal{H})$, where $(\mathcal{H}, [.,.]_\mathcal{H})$ is a Hilbert space. Furthermore, let E_i denote the spectral measure corresponding to E_i for fixed $i \in [1,n]_\mathbb{Z}$ and let E denote the joint spectral measure of $\boldsymbol{A}$. Then $E_i = E^{\pi_i}$ and*

$$\int_\Delta \phi \,\mathrm{d}E_i = \int_{\pi_i^{-1}(\Delta)} \phi \circ \pi_i \,\mathrm{d}E, \tag{1.6}$$

where $\pi_i : \mathbb{R}^n \mapsto \mathbb{R}$ is the projection on the i-th coordinate, Δ is a Borel set of $\mathbb{R}$ and ϕ is measurable function.

Proof. By Theorem 1.5.1 and Lemma 1.5.5 E^{π_i} is a spectral measure of A. Since the spectral measure of A is unique, E^{π_i} coincides with E_i. Hence,

$$\int_\Delta \phi \,\mathrm{d}E_i = \int_\Delta \phi \,\mathrm{d}E^{\pi_i} = \int_{\pi_i^{-1}(\Delta)} \phi \circ \pi_i \,\mathrm{d}E.$$

❑

2 Diagonal Transform of Linear Relations

2.1 Linear Relations

Definition 2.1.1. Let X, Y be two vector spaces over the same scalar field. Then we will call a subspace T of $X \times Y$ a *linear relation* between X and Y. A linear relation between X and X will be called a linear relation on X.

Remark 2.1.2. Every linear operator $T : X \to Y$ can be identified by a linear relation by considering the graph of T. In fact, if we consider mappings from X to Y as subsets of $X \times Y$ then T is already a linear relation. On the other hand not every linear relation comes from an operator as $\{0\} \times Y$ demonstrates the most degenerated example.

Definition 2.1.3. For a linear relation T between the vector spaces X and Y we define

- $\operatorname{dom} T := \{x \in X : \exists\, y \in Y \text{ such that } (x; y) \in T\}$ the *domain* of T,
- $\operatorname{ran} T := \{y \in Y : \exists\, x \in X \text{ such that } (x; y) \in T\}$ the *range* of T,
- $\ker T := \{x \in X : (x; 0) \in T\}$ the *kernel* of T,
- $\operatorname{mul} T := \{y \in Y : (0; y) \in T\}$ the *multi-value-part* of T.

Remark 2.1.4. Every linear relation T which satisfies $\operatorname{mul} T = \{0\}$ can be regarded as a linear mapping T on $\operatorname{dom} T$, where $Tx = y$ is well defined by $(x; y) \in T$.

Definition 2.1.5. Let X, Y, Z vector spaces and S, T linear relations between X and Y, and R a linear relation between Y and Z.

- $S + T := \{(x; y_1 + y_2) \in X \times Y : (x; y_1) \in S \text{ and } (x; y_2) \in T\}$,
- $\lambda T := \{(x; \lambda y)\} \in X \times Y : (x; y) \in T\}$,
- $T^{-1} := \{(y; x) \in Y \times X : (x; y) \in T\}$,
- $RS := \{(x; z) \in X \times Z : \exists\, y \in Y \text{ such that } (x; y) \in S \text{ and } (y; z) \in R\}$.

It is easy to check that the sets defined in the previous definition are also linear relations.

Definition 2.1.6. For a Banach space $(X, \|.\|)$ and a linear relation A on X, we define

- $\rho(A) := \{\lambda \in \mathbb{C} \cup \{\infty\} : (A - \lambda)^{-1} \in L_{\mathrm{b}}(X)\}$ as the *resolvent set*,
- $\sigma(A) := (\mathbb{C} \cup \{\infty\}) \setminus \rho(A)$ as the *spectrum*,
- $\sigma_p(A) := \{\lambda \in \mathbb{C} \cup \{\infty\} : \ker(A - \lambda)^{-1} \neq \{0\}\}$ as *point spectrum*, and
- $r(A) := \{\lambda \in \mathbb{C} \cup \{\infty\} : (A - \lambda)^{-1} \in L_{\mathrm{b}}(\operatorname{dom}(A))\}$ as the *points of regular type*,

where we set $(T-\infty)^{-1} := T$ and $\operatorname{dom}(T-\infty)^{-1} := \operatorname{dom} T$.

Definition 2.1.7. Let X be a vector space over $\mathbb{C}$ and $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$, then we define the mapping $\tau_M : X \times X \to X \times X$ by

$$\tau_M(x;y) := \begin{pmatrix} \delta I & \gamma I \\ \beta I & \alpha I \end{pmatrix} (x;y) := (\delta x + \gamma y; \beta x + \alpha y).$$

Facts 2.1.8. For $M, N \in \mathbb{C}^{2\times 2}$ we have $\tau_M \tau_N = \tau_{MN}$ and therefore, for invertible M also $\tau_{M^{-1}} = {\tau_M}^{-1}$.

Lemma 2.1.9. *Let A be a linear relation on a vector space X and $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$. If $\operatorname{mul} A = \{0\}$, then*

$$\tau_M(A) = (\alpha A + \beta I)(\gamma A + \delta I)^{-1}.$$

Proof. Let $(a;b) \in \tau_M(A)$. Then there exists a $(x;y) \in A$ such that $(a;b) = (\delta x + \gamma y; \beta x + \alpha y)$. By Definition of the addition and multiplication by a scalar for linear relations we have $(x; \alpha y + \beta x) \in (\alpha A + \beta I)$, $(x; \gamma y + \delta x) \in (\gamma A + I)$ and therefore $(\gamma y + \delta x; x) \in (\gamma A + I)^{-1}$. Consequently $(a;b) \in (\alpha A + \beta I)(\gamma A + \delta I)^{-1}$.

On the other hand let $(a;b) \in (\alpha A + \beta I)(\gamma A + \delta I)^{-1}$. Then there exists a $x \in \operatorname{dom} A$ such that $(a;x) \in (\gamma A + \delta I)^{-1}$ and $(x;b) \in (\alpha A + \beta I)$. Since $\operatorname{mul} A = \{0\}$, there exists a unique $y \in X$ such that $(x;y) \in A$. Hence, $a = \gamma y + \delta x$ and $b = \alpha y + \beta x$ and consequently $(a;b) \in \tau_M(A)$.

❑

Remark 2.1.10. For $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ with $\det M \neq 0$ we have the Möbius transformation

$$\phi_M(z) = \frac{\alpha z + \beta}{\gamma z + \delta} = (\alpha z + \beta)(\gamma z + \delta)^{-1}.$$

By Lemma 2.1.9, we can see that $\phi_M(A) := (\alpha A + \beta)(\gamma A + \delta)^{-1} = \tau_M(A)$ for any linear relation A with $\operatorname{mul} A = \{0\}$.

2.2 Linear Relations on Krein spaces

Definition 2.2.1. Let $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$ and $(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ be a Krein spaces and A a linear relation between them. Then the *adjoint linear relation* is defined by

$$A^+ := \{(x;y) \in \mathcal{K}_2 \times \mathcal{K}_1 : [x,v]_{\mathcal{K}_2} = [y,u]_{\mathcal{K}_1} \text{ for all } (u;v) \in A\}. \tag{2.1}$$

Remark 2.2.2. If $A \in L_{\mathrm{b}}(\mathcal{K}_1, \mathcal{K}_2)$ then the Krein space adjoint A^+ from Lemma 1.2.1 coincides with the adjoint linear relation of A. This justifies the same notation.

For the following Lemma we will extend the mapping τ_M for $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to $\mathcal{K}_1 \times \mathcal{K}_2 \cup \mathcal{K}_2 \times \mathcal{K}_1$ by

$$\tau_M(x; y) = (y; -x) \quad \text{for all} \quad (x, y) \in \mathcal{K}_1 \times \mathcal{K}_2 \cup \mathcal{K}_2 \times \mathcal{K}_1.$$

Lemma 2.2.3. *Let $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$,$(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ be Krein spaces, $A \leq \mathcal{K}_1 \times \mathcal{K}_2$ a linear relation between them and $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then we can write the adjoint of A by*

$$A^+ = \tau_M(A^{[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}}) = \tau_M(A)^{[\perp]_{\mathcal{K}_2 \times \mathcal{K}_1}},$$

where $[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}$ will denote orthogonal complement in $(\mathcal{K}_1 \times \mathcal{K}_2, [.,.]_{\mathcal{K}_1 \times \mathcal{K}_2})$ and $[\perp]_{\mathcal{K}_2 \times \mathcal{K}_1}$ the orthogonal complement in $(\mathcal{K}_2 \times \mathcal{K}_1, [.,.]_{\mathcal{K}_2 \times \mathcal{K}_1})$. Furthermore, A^+ is closed.

Proof. Let $(x; y) \in \mathcal{K}_2 \times \mathcal{K}_1$, $(u; v) \in \mathcal{K}_1 \times \mathcal{K}_2$. Then we have the following equivalences.

$$\begin{aligned} [x, v]_{\mathcal{K}_1} = [y, u]_{\mathcal{K}_2} &\Leftrightarrow [y, u]_{\mathcal{K}_1} - [x, v]_{\mathcal{K}_2} = 0 \Leftrightarrow [(y; -x), (u; v)]_{\mathcal{K}_1 \times \mathcal{K}_2} = 0 \\ &\Leftrightarrow [\tau_M(x; y), (u; v)]_{\mathcal{K}_1 \times \mathcal{K}_2} = 0 \Leftrightarrow \tau_M(x; y)[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}(u; v). \end{aligned}$$

On the other hand we have the equivalences

$$\begin{aligned} [x, v]_{\mathcal{K}_1} = [y, u]_{\mathcal{K}_2} &\Leftrightarrow [x, v]_{\mathcal{K}_2} + [y, -u]_{\mathcal{K}_1} = 0 \Leftrightarrow [(x; y), \tau_M(u; v)]_{\mathcal{K}_2 \times \mathcal{K}_1} = 0 \\ &\Leftrightarrow [(x; y), \tau_M(u; v)]_{\mathcal{K}_2 \times \mathcal{K}_1} = 0 \Leftrightarrow (x; y)[\perp]_{\mathcal{K}_2 \times \mathcal{K}_1} \tau_M(u; v). \end{aligned}$$

Hence, we conclude that the following sets coincides.

$$\begin{aligned} A^+ &= \{(x; y) \in \mathcal{K}_2 \times \mathcal{K}_1 : [x, v]_{\mathcal{K}_2} = [y, u]_{\mathcal{K}_1} \text{ for all } (u; v) \in A\} \\ &= \{(x; y) \in \mathcal{K}_2 \times \mathcal{K}_1 : \tau_M(x; y)[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}(u; v) \text{ for all } (u; v) \in A\} \\ &= \{(x; y) \in \mathcal{K}_2 \times \mathcal{K}_1 : (x; y)[\perp]_{\mathcal{K}_2 \times \mathcal{K}_1} \tau_M(u; v); \text{for all } (u; v) \in A\}. \end{aligned}$$

As a linear subspace of $\mathcal{K}_2 \times \mathcal{K}_1$ the set $A^{[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}}$ is a linear relation between $\mathcal{K}_2$ and $\mathcal{K}_1$. Since $\tau_M^{-1}(B) = \tau_M(B)$ holds true for every linear relation B, we conclude

$$A^+ = \tau_M(A^{[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}}) = \tau_M(A)^{[\perp]_{\mathcal{K}_2 \times \mathcal{K}_1}}.$$

The closedness of A^+ follows immediately.

❑

Lemma 2.2.4. *Let $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1})$, $(\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ and $(\mathcal{K}_3, [.,.]_{\mathcal{K}_3})$ Krein spaces and $A \leq \mathcal{K}_1 \times \mathcal{K}_2$ a linear relation between $\mathcal{K}_1$ and $\mathcal{K}_2$. Then*

(*i*) $\operatorname{mul} A^+ = (\operatorname{dom} A)^\perp$, $\ker A^+ = (\operatorname{ran} A)^\perp$,

(*ii*) $(BA)^+ \supseteq A^+B^+$ *for all linear relations* $B \leq \mathcal{K}_2 \times \mathcal{K}_3$,

(*iii*) $(BA)^+ = A^+B^+$ *for all operators* $B \in L_b(\mathcal{K}_2, \mathcal{K}_3)$,

(*iv*) $A^{++} = \overline{A}$

Proof.

(*i*) By the definition of A^+ (2.1), we have

$$\operatorname{mul} A^+ = \{y \in \mathcal{K}_1 : [0, v]_{\mathcal{K}_2} = [y, u]_{\mathcal{K}_1} \text{ for all } (u; v) \in A\} = (\operatorname{dom} A)^\perp,$$
$$\ker A^+ = \{x \in \mathcal{K}_1 : [x, v]_{\mathcal{K}_2} = [0, u]_{\mathcal{K}_1} \text{ for all } (u; v) \in A\} = (\operatorname{ran} A)^\perp.$$

(*ii*) If $(x; y) \in A^+B^+$, then there exist a $z \in \mathcal{K}_2$ such that $(x; z) \in B^+$ and $(z; y) \in A^+$. Moreover,

$$[x, w]_{\mathcal{K}_3} = [z, v]_{\mathcal{K}_2} \quad \text{for all} \quad (v; w) \in B,$$
$$[z, v]_{\mathcal{K}_2} = [y, u]_{\mathcal{K}_1} \quad \text{for all} \quad (u; v) \in A.$$

Hence, $[x, w]_{\mathcal{K}_3} = [y, u]_{\mathcal{K}_1}$ for all $(u; w) \in BA$ and consequently $(x; y) \in (BA)^+$.

(*iii*) Since B is an everywhere defined operator, we can write $BA = \{(u; Bv) : (u; v) \in A\}$. Therefore,

$$(BA)^+ = \{(x; y) \in \mathcal{K}_3 \times \mathcal{K}_1 : [x, Bv]_{\mathcal{K}_3} = [y, u]_{\mathcal{K}_1} \text{ for all } (u; v) \in A\}.$$

If $(x; y) \in (BA)^+$, then

$$[(x; Bv)]_{\mathcal{K}_3} = [B^+x, v]_{\mathcal{K}_2} = [y, u]_{\mathcal{K}_1} \quad \text{for all} \quad (u; v) \in A,$$

and in turn $(B^+x; y) \in A^+$. Clearly, we also have $(x; B^+x) \in B^+$. Hence $(x; y) \in A^+B^+$.

(*iv*) By Lemma 2.2.3 and Lemma 1.1.11 we have

$$A^{++} = \tau_M(\tau_M(A)^{[\perp]_{\mathcal{K}_2 \times \mathcal{K}_1}})^{[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}} = \tau_M(\tau_M(A))^{[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}}$$
$$= A^{[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}[\perp]_{\mathcal{K}_1 \times \mathcal{K}_2}} = \overline{A}.$$

❑

Definition 2.2.5. Let $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be a Krein space and A a linear relation on $\mathcal{K}$. We call A *symmetric*, if $A \subseteq A^+$ and *self-adjoint*, if $A = A^+$.

2.3 Diagonal Transform

Definition 2.3.1. Let $T : X \to Y$ be a linear operator between the vector spaces X and Y. We define the mapping

$$T \times T : \begin{cases} X \times X & \to & Y \times Y, \\ (a; b) & \mapsto & (Ta; Tb). \end{cases}$$

Facts 2.3.2. Let $T : X \to Y$ be a linear operator between the vector spaces X and Y, A a linear relation on Y, and B a linear relation on X. Then

(*i*) $T \times T$ is a linear mapping.

(*ii*) $(T \times T)(B) = \{(Tu; Tv) : (u; v) \in B\}$ is a linear relation.

(iii) $(T \times T)^{-1}(A) = \{(u;v) : (Tu;Tv) \in A\}$ is a linear relation. If T is additionally continuous and A is closed, then $(T \times T)^{-1}(A)$ is also closed.

Lemma 2.3.3. *Let $T : X \to Y$ a linear operator, B be a linear relation on X and A be a linear relation on Y. Then*

$$(T \times T)(B) = TBT^{-1} \quad \text{and} \quad (T \times T)^{-1}(A) = T^{-1}AT.$$

Proof. If $(a;b) \in (T \times T)(B)$, then there exists a pair $(x;y) \in B$ such that $(a;b) = (Tx;Ty)$. Since $(Tx;x) \in T^{-1}$ and $(y;Ty) \in T$ we have

$$\underbrace{(Tx;x)}_{\in T^{-1}}, \quad \underbrace{(x;y)}_{\in B}, \quad \underbrace{(y;Ty)}_{\in T}.$$

By the definition of the multiplication of linear relations we conclude that $(a;b) = (Tx;Ty) \in TBT^{-1}$.

On the other hand if $(a;b) \in TBT^{-1}$, then there are $x, y \in X$ such that $(a;x) \in T^{-1}$, $(x;y) \in B$ and $(y;b) \in T$. Since T is an operator we have that $a = Tx$ and $b = Ty$ and consequently $(a;b) = (Tx;Ty)$ for $(x;y) \in B$ which is the condition for $(a;b) \in (T \times T)(B)$.

Let $(x;y) \in (T \times T)^{-1}(A)$ then $(Tx;Ty) \in A$ and clearly $(x;Tx) \in T$ and $(Ty;y) \in T^{-1}$ which gives us

$$\underbrace{(x;Tx)}_{\in T}, \quad \underbrace{(Tx;Ty)}_{\in A}, \quad \underbrace{(Ty;y)}_{\in T^{-1}}.$$

By the definition of the multiplication of linear relations we conclude that $(x;y) \in T^{-1}AT$.

If $(x;y) \in T^{-1}AT$, then there are $a, b \in Y$ such that $(x;a) \in T$, $(a;b) \in A$ and $(b;y) \in T^{-1}$. Since T is an operator we have $a = Tx$ and $b = Ty$. Hence $(Tx;Ty) = (a;b) \in A$ which is the condition for $(x,y) \in (T \times T)^{-1}(A)$.

❑

Lemma 2.3.4. *Let $T : X \to Y$ be a linear operator between the vector spaces X and Y, A a linear relation on Y, and B a linear relation on X. Then the following statements are equivalent*

(i) $(T \times T)(B) \subseteq A$.

(ii) $B \subseteq (T \times T)^{-1}(A)$.

(iii) $TB \subseteq AT$.

If A and B are even everywhere defined operators, then all those statements are equivalent to $TB = AT$.

Proof. The statements *(i)* and *(ii)* are clearly equivalent. Let us assume *(ii)*: $B \subseteq (T \times T)^{-1}(A) = T^{-1}AT$. Because of $TT^{-1} \subseteq I$ this yields

$$TB \subseteq TT^{-1}AT \subseteq AT.$$

Conversely, $TB \subseteq AT$ implies $B \subseteq T^{-1}TB \subseteq T^{-1}AT$.

Let us assume statement (*iii*) for the following. If A and B are everywhere defined operators, then $\operatorname{dom} TB = \operatorname{dom} AT$. Therefore, if $(x; y) \in AT$, then there exists a $z \in Y$ such that $(x; z) \in TB$. Since $\operatorname{mul} AT = \{0\}$, we have that y and z must be equal. Hence, $(x; y)$ is also an element of TB and in consequence $AT = TB$.

❑

Lemma 2.3.5. *Let $T : X \to Y$ be a linear operator between to vector spaces X and Y, B a linear relation on X and A a linear relation on Y. For every $M \in \mathbb{C}^{2\times 2}$ we have*

$$\tau_M\big((T \times T)(B)\big) = (T \times T)(\tau_M(B)).$$

If M is additionally invertible, then we have

$$\tau_M\big((T \times T)^{-1}(A)\big) = (T \times T)^{-1}(\tau_M(A)).$$

Proof. Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Due to

$$\begin{aligned}\tau_M\big((T \times T)(B)\big) &= \{(\delta Tx + \gamma Ty; \beta Tx + \alpha Ty) : (x; y) \in B\} \\ &= \{(T(\delta x + \gamma y); T(\beta x + \alpha y)) : (x; y) \in B\} \\ &= (T \times T)(\tau_M(B)),\end{aligned}$$

we obtain the first equality.

If $(x; y) \in \tau_M\big((T \times T)^{-1}(A)\big)$, then there exists a $(a; b) \in X \times X$ such that $(Ta; Tb) \in A$ and $(x; y) = (\delta a + \gamma b; \beta a + \alpha b)$. This leads to

$$(Tx, Ty) = (\delta Ta + \gamma Tb; \beta Ta + \alpha Tb) = \tau_M((Ta; Tb)) \in \tau_M(A),$$

and furthermore to $(x; y) \in (T \times T)^{-1}(\tau_M(A))$. Hence,

$$\tau_M\big((T \times T)^{-1}(A)\big) \subseteq (T \times T)^{-1}(\tau_M(A)). \tag{2.2}$$

If M is invertible, we can substitute A with $\tau_M(A)$ and τ_M with $\tau_{M^{-1}}$ in (2.2). Therefore,

$$\tau_{M^{-1}}\big((T \times T)^{-1}(\tau_M(A))\big) \subseteq (T \times T)^{-1}(\tau_{M^{-1}}(\tau_M(A))).$$

Applying τ_M on both sides yields

$$(T \times T)^{-1}(\tau_M(A)) \subseteq \tau_M\big((T \times T)^{-1}(A)\big). \tag{2.3}$$

The combination of (2.2) and (2.3) completes the proof.

❑

Lemma 2.3.6. *Let $T : X \to Y$ be a linear operator between the vector spaces X and Y, A_1 and A_2 linear relations on Y, and $\lambda \in \mathbb{C} \setminus \{0\}$. Then we have*

$$\begin{aligned}(T \times T)^{-1}(\lambda A_1) &= \lambda (T \times T)^{-1}(A_1), \\ (T \times T)^{-1}(A_1 + A_2) &\supseteq (T \times T)^{-1}(A_1) + (T \times T)^{-1}(A_2), \\ (T \times T)^{-1}(A_1 A_2) &\supseteq (T \times T)^{-1}(A_1)(T \times T)^{-1}(A_2).\end{aligned}$$

Proof. Set $M = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then Lemma 2.3.5 yields the first equation.

If $(x;y) \in (T \times T)^{-1}(A_1) + (T \times T)^{-1}(A_2)$, then there exist $u, v \in X$ such that $(Tx;Tu) \in A_1$, $(Tx;Tv) \in A_2$ and $u + v = y$. Hence, $Tu + Tv = Ty$ and in turn $(Tx, Ty) \in A_1 + A_2$ which yields $(x;y) \in (T \times T)^{-1}(A_1 + A_2)$.

Since $TT^{-1} \subseteq I$, we have

$$\begin{aligned}(T \times T)^{-1}(A_1)(T \times T)^{-1}(A_2) &= T^{-1}A_1TT^{-1}A_2T \\ &\subseteq T^{-1}A_1A_2T = (T \times T)^{-1}(A_1A_2).\end{aligned}$$

❑

Lemma 2.3.7. *Let $(\mathcal{H}, [.,.]_{\mathcal{H}})$ and $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be Krein spaces. Then for a linear relation A on $\mathcal{K}$ and a linear mapping $T : \mathcal{H} \to \mathcal{K}$ we have*

$$\ker\big((T \times T)^{-1}(A) - \lambda\big) = T^{-1}\ker(T - \lambda) \quad \textit{for all} \quad \lambda \in \mathbb{C} \cup \{\infty\}.$$

In particular, $\sigma_p\big((T \times T)^{-1}(A)\big) \subseteq \sigma_p(A)$, if T is additionally injective.

Proof. First note that

$$y \in \operatorname{mul}\big((T \times T)^{-1}(A)\big) \Leftrightarrow (0;Ty) \in A \Leftrightarrow y \in T^{-1}(\operatorname{mul} A).$$

By definition, we have $\ker\big((T \times T)^{-1}(A) - \lambda\big) = T^{-1}\ker(T - \lambda)$ for $\lambda = \infty$. It is straightforward that every linear relation B satisfies $\ker B = \operatorname{mul} B^{-1}$. For $\lambda \in \mathbb{C}$ we set $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda \end{smallmatrix}\right)$. Since $\tau_M(B) = (B - \lambda)^{-1}$, we conclude

$$\ker(B - \lambda) = \operatorname{mul}(B - \lambda)^{-1} = \operatorname{mul} \tau_M(B).$$

Hence,

$$\begin{aligned}\ker\big((T \times T)^{-1}(A) - \lambda\big) &= \operatorname{mul} \tau_M\big((T \times T)^{-1}(A)\big) = \operatorname{mul}(T \times T)^{-1}(\tau_M(A)) \\ &= T^{-1}\operatorname{mul} \tau_M(A) = T^{-1}\ker(T - \lambda).\end{aligned}$$

If T is injective, then $T^{-1}\ker(A - \lambda) \neq \{0\}$ implies $\ker(A - \lambda) \neq \{0\}$. Therefore, $\sigma_p\big((T \times T)^{-1}(A)\big) \subseteq \sigma_p(A)$.

❑

Lemma 2.3.8. *Let $R : \mathcal{K}_1 \to \mathcal{K}_2$ be a bounded linear mapping between the Krein spaces $(\mathcal{K}_1, [.,.]_{\mathcal{K}_1}), (\mathcal{K}_2, [.,.]_{\mathcal{K}_2})$ and $L \subseteq \mathcal{K}_2$. Then we have*

$$R^+(L)^{[\perp]_{\mathcal{K}_1}} = R^{-1}(L^{[\perp]_{\mathcal{K}_2}}).$$

Proof. The varifaction of the stated equality follows from

$$\begin{aligned}R^+(L)^{[\perp]_{\mathcal{K}_1}} &= \{x \in \mathcal{K}_1 : [x, R^+l] = 0 \text{ for all } l \in L\} \\ &= \{x \in \mathcal{K}_1 : [Rx, l] = 0 \text{ for all } l \in L\} \\ &= \{x \in \mathcal{K}_1 : Rx \in L^{[\perp]_{\mathcal{K}_2}}\} \\ &= R^{-1}(L^{[\perp]_{\mathcal{K}_2}}).\end{aligned}$$

❑

Lemma 2.3.9. *Let $(\mathcal{H}, [.,.]_{\mathcal{H}})$, $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be Krein spaces and $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear mapping. For a linear relation A on $\mathcal{K}$ we have*

$$\big((T^+ \times T^+)(A)\big)^+ = (T \times T)^{-1}(A^+)$$

In particular $\big((T \times T)^{-1}(A^+)\big)^+$ is the closure of $(T^+ \times T^+)(A)$.

Proof. We regard $T \times T$ as a mapping from $\mathcal{H} \times \mathcal{H}$ to $\mathcal{K} \times \mathcal{K}$ where $\mathcal{K} \times \mathcal{K}$ is equipped with $[(x; y), (w; z)]_{\mathcal{K}\times\mathcal{K}} := [x, w]_{\mathcal{K}} + [y, z]_{\mathcal{K}}$ and $\mathcal{H} \times \mathcal{H}$ is equipped with the respective inner product. Hence, we can use Lemma 2.3.8 to obtain

$$\big((T^+ \times T^+)(A)\big)^{[\perp]} = (T \times T)^{-1}(A^{[\perp]}), \tag{2.4}$$

where $[\perp]$ denotes the orthogonal complement in $\mathcal{K} \times \mathcal{K}$ as well as in $\mathcal{H} \times \mathcal{H}$. By Lemma 2.2.3 we have

$$\begin{aligned}\big((T^+ \times T^+)(A)\big)^+ &= \tau_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\Big(\big((T^+ \times T^+)(A)\big)^{[\perp]}\Big) \overset{(2.4)}{=} \tau_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\Big((T \times T)^{-1}(A^{[\perp]})\Big) \\ &= (T \times T)^{-1}\left(\tau_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}(A^{[\perp]})\right) = (T \times T)^{-1}(A^+).\end{aligned}$$

By applying the adjoint $^+$ to both sides we obtain

$$\overline{(T^+ \times T^+)(A)} = \big((T \times T)^{-1}(A^+)\big)^+.$$

❑

Proposition 2.3.10. *Let $(\mathcal{H}, [.,.]_{\mathcal{H}})$, $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be a Krein spaces and $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear mapping between these spaces. If A is a closed linear relation on $\mathcal{K}$, which satisfies*

$$(TT^+ \times TT^+)(A^+) \subseteq A,$$

then the closure $(T \times T)^{-1}(A)^+$ of $(T^+ \times T^+)(A^+)$ is a symmetric linear relation on $\mathcal{H}$.

In the special case that T is injective, that $(\mathcal{H}, [.,.]_{\mathcal{H}})$ is a Hilbert space and that $\mathbb{C} \setminus \sigma_p(A)$ contains points from $\mathbb{C}^+$ and from $\mathbb{C}^-$, the linear relation $(T \times T)^{-1}(A)$ is self-adjoint.

Proof. The assumption $(T \times T)(T^+ \times T^+)(A^+) = (TT^+ \times TT^+)(A^+) \subseteq A$ implies $(T^+ \times T^+)(A^+) \subseteq (T \times T)^{-1}(A)$. By Lemma 2.3.9, $(T \times T)^{-1}(A)^+$ is the closure of $(T^+ \times T^+)(A^+)$. Since $(T \times T)^{-1}(A)$ is closed, we have

$$(T \times T)^{-1}(A)^+ = \overline{(T^+ \times T^+)(A^+)} \subseteq (T \times T)^{-1}(A) = (T \times T)^{-1}(A)^{++}.$$

Hence, $(T \times T)^{-1}(A)^+$ is symmetric.

If $(\mathcal{H}, [.,.]_{\mathcal{H}})$ is a Hilbert space, then $(T \times T)^{-1}(A)^+$ not being a self-adjoint relation on $\mathcal{H}$ implies, that its defect indices are not both equal to zero. This means

$$\operatorname{ran}\big((T \times T)^{-1}(A)^+ - \lambda\big)^\perp = \ker\big((T \times T)^{-1}(A) - \overline{\lambda}\big) \neq \{0\}$$

for all $\lambda \in \mathbb{C}^+$ or for all $\lambda \in \mathbb{C}^-$. Hence the point spectrum of $(T \times T)^{-1}(A)$ contains all points from the upper half-plane or all points from the lower half-plane. Due to Lemma 2.3.7 we have $\sigma_p\big((T \times T)^{-1}(A)\big) \subseteq \sigma_p(A)$ which leads to a contradiction to the assumption concerning $\mathbb{C} \setminus \sigma_p(A)$.

❑

The following Lemma is a consequence of Loewner's Theorem 2.2.6. However, in order to be more self-contained we will present a proof which uses the spectral calculus for self-adjoint operators on Hilbert spaces.

Lemma 2.3.11. *Let $(\mathcal{H}, [.,.]_{\mathcal{H}})$ be a Hilbert space and let $A, C \in L_{\mathrm{b}}(\mathcal{H})$ such that C and AC are self-adjoint and such that C is positive. Then we have $|[ACx, x]_{\mathcal{H}}| \leq \|A\| \, [Cx, x]_{\mathcal{H}}$ for all $x \in \mathcal{H}$.*

Proof. Since C is a positive operator we have $\sigma(C) \subseteq [0, +\infty)$. Consequently, $C + \epsilon$ is boundedly invertible for $\epsilon > 0$. The functional calculus for the self-adjoint operator C yields that $C(C + \epsilon)^{-1}$ has norm $\sup_{t\in\sigma(C)} \frac{t}{t+\epsilon} = \frac{\|C\|}{\|C\|+\epsilon}$.

Since for the spectral radius we have $\mathrm{spr}(FG) = \mathrm{spr}(GF)$ for all bounded operators F, G, we conclude

$$\mathrm{spr}((C + \epsilon)^{-\frac{1}{2}} AC(C + \epsilon)^{-\frac{1}{2}}) = \mathrm{spr}(AC(C + \epsilon)^{-1}) \leq \|A\| \, \frac{\|C\|}{\|C\| + \epsilon} \leq \|A\| \, .$$

For self-adjoint operators spectral radius and norm coincide. Hence, due to the Cauchy-Schwarz inequality,

$$\begin{aligned} |[ACx, x]_{\mathcal{H}}| &= \left| \left[(C + \epsilon)^{-\frac{1}{2}} AC(C + \epsilon)^{-\frac{1}{2}} (C + \epsilon)^{\frac{1}{2}} x, (C + \epsilon)^{\frac{1}{2}} x\right]_{\mathcal{H}} \right| \\ &\leq \left\|(C + \epsilon)^{-\frac{1}{2}} AC(C + \epsilon)^{-\frac{1}{2}}\right\| \left\|(C + \epsilon)^{\frac{1}{2}} x\right\| \\ &\leq \|A\| \, [(C + \epsilon)x, x]_{\mathcal{H}} \overset{\epsilon \searrow 0}{\longrightarrow} \|A\| \, [Cx, x]_{\mathcal{H}}. \end{aligned}$$

❑

Lemma 2.3.12. *Let $(\mathcal{H}, [.,.]_{\mathcal{H}})$ be a Hilbert space, $c \in [0, +\infty)$ and let B be a self-adjoint linear relation on $\mathcal{H}$ such that $\mathrm{mul}\, B = \{0\}$. If $|[y, x]_{\mathcal{H}}| \leq c[x, x]_{\mathcal{H}}$ for all $(x; y) \in B$, then B is a bounded linear operator on $\mathcal{H}$ such that $\|B\| \leq c$.*

Proof. By Remark 2.1.4, we regard B as a linear operator on $\mathrm{dom}\, B$. By Lemma 2.2.4, $\mathrm{dom}\, B$ is dense in $\mathcal{H}$ and $B = B^*$ is closed, because B is self-adjoint and $\mathrm{mul}\, B = \{0\}$. Therefore, we can apply the spectral theorem for unbounded self-adjoint operators on Hilbert spaces to obtain a spectral measure E on the Borel sets of $\mathbb{R}$; see [9, Theorem 13.30].

In the following we will use the following well-known result: An element $x \in \mathcal{H}$ belongs to the domain of $\int_{\mathbb{R}} \phi \, \mathrm{d}E$ if and only if $\int_{\mathbb{R}} |\phi|^2 \, \mathrm{d}E_{x,x} < +\infty$; see [9, Lemma 13.23, Theorem 13.24].

For every $n \in \mathbb{N}$ consider the interval $\Delta_n := [c + \frac{1}{n}, c + n]$ in $\mathbb{R}$. For $x \in \mathrm{ran}\, E(\Delta_n)$, we have

$$\int_{\mathbb{R}} |t|^2 \, \mathrm{d}E_{x,x}(t) = \int_{\Delta_n} |t|^2 \, \mathrm{d}E_{x,x}(t) \leq (c + n)^2 \, \|x\|^2 < +\infty,$$

which yields $x \in \operatorname{dom} B$. By our assumptions we have

$$\begin{aligned} c[x,x]_{\mathcal{H}} \geq |[Bx,x]_{\mathcal{H}}| = \Big| \int_{\Delta_n} t \, \mathrm{d}E_{x,x}(t) \Big| &\geq \big(c + \frac{1}{n}\big)[E(\Delta_n x, x)]_{\mathcal{H}} \\ &= \big(c + \frac{1}{n}\big)[x,x]_{\mathcal{H}}. \end{aligned}$$

Consequently x can only be 0 and therefore $E(\Delta_n) = 0$ for all $n \in \mathbb{N}$. By the σ-additivity we have that $E\big((c,+\infty)\big) = E\big(\bigcup_{n\in\mathbb{N}} \Delta_n\big) = 0$. Analogues, we can show $E\big((-\infty,-c)\big) = 0$, which yields $\operatorname{supp} E \subseteq [-c,c]$. We can write $B = \int_{[-c,c]} t \, \mathrm{d}E(t)$ which implies that B is a bounded linear operator on $\mathcal{H}$ with $\|B\| \leq \sup_{t\in[-c,c]} |t| = c$.

❑

Theorem 2.3.13. *Let $(\mathcal{H}, [.,.]_{\mathcal{H}})$ be a Hilbert space, $(\mathcal{K}, [.,.]_{\mathcal{K}})$ be a Krein space, $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear and injective mapping, and $A \in L_{\mathrm{b}}(\mathcal{K})$ such that $(TT^+ \times TT^+)(A^+) \subseteq A$. Then $(T\times T)^{-1}(A)$ is a bounded linear and self-adjoint operator on $\mathcal{H}$ with*

$$\left\|(T \times T)^{-1}(A)\right\| \leq \|A\| \,. \tag{2.5}$$

On the right-hand-side $\|.\|$ denotes the operator norm with respect to any fundamental symmetry J.

Proof. Since A is a bounded operator we have that $\sigma(A) \subseteq B_{\|A\|}$. In particular $\mathbb{C} \setminus \sigma_p(A)$ contains points from $\mathbb{C}^+$ and $\mathbb{C}^-$. Therefore by Proposition 2.3.10 $(T \times T)^{-1}(A)$ is self-adjoint and coincides with the closure of $(T^+ \times T^+)(A^+)$. By the injectivity of T, we have that $\operatorname{mul}(T \times T)^{-1}(A) = \operatorname{mul} T^{-1}AT = \{0\}$. Hence, $(T \times T)^{-1}(A)$ is a self-adjoint operator on its domain.

Due to Lemma 2.3.4, we have $TT^+A^+ = ATT^+$. Let J be any fundamental symmetry and let A^*, T^* denote the Hilbert space adjoint of A, T, when we endow $\mathcal{K}$ with $(.,.)_J$. Then $T^+ = T^*J$ and $A^+ = JA^*J$. Since $JJ = I$, we have

$$TT^*A^* = TT^*JJA^*JJ = TT^+A^+J = ATT^+J = ATT^*.$$

Consequently $(ATT^*)^* = TT^*A^* = ATT^*$ is self-adjoint on the Hilbert space $(\mathcal{K}, (.,.)_J)$. For $(x;y) \in (T^+ \times T^+)(A^+) \subseteq (T \times T)^{-1}(A)$ we have $(Tx;Ty) \in A$ and $x = T^+u$ for some $u \in \operatorname{dom} A^+$. Hence,

$$|[y,x]_{\mathcal{H}}| = |[y,T^+u]_{\mathcal{H}}| = |[Ty,u]_{\mathcal{K}}| = |[ATT^+u,u]_{\mathcal{K}}| = |(ATT^*Ju,Ju)_J|.$$

Lemma 2.3.11 yields

$$|[y,x]_{\mathcal{H}}| \leq \|A\| \, (TT^*Ju,Ju)_J = \|A\| \, [TT^+u,u]_{\mathcal{K}} = \|A\| \, [x,x]_{\mathcal{H}}.$$

Since $(T^+ \times T^+)(A^+)$ is dense in $(T \times T)^{-1}(A)$ we have $[y,x]_{\mathcal{H}} \leq \|A\| \, [x,x]_{\mathcal{H}}$ for all $(x;y) \in (T \times T)^{-1}(A)$. By Lemma 2.3.12, $(T \times T)^{-1}(A)$ is a linear operator on $\mathcal{H}$ bounded by $\|A\|$.

❑

Lemma 2.3.14. *Let $T : \mathcal{H} \to \mathcal{K}$ a bounded linear mapping. Then $(TT^+)'$ and $(T^+T)'$ are closed $*$-subalgebras.*

Proof. For $A, B \in (TT^+)'$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} &TT^+(A+\lambda B) = TT^+A + TT^+\lambda B = ATT^+ + \lambda BTT^+ = (A+\lambda B)TT^+, \\ &TT^+AB = ATT^+B = ABTT^+, \\ &TT^+A^+ = (ATT^+)^+ = (TT^+A)^+ = A^+TT^+. \end{aligned}$$

Consequently, $(TT^+)'$ is $*$-subalgebra. If $(A_n)_{n\in\mathbb{N}}$ is a sequence in $(TT^+)'$ that converges to $A \in L_b(\mathcal{K})$, then we have

$$TT^+A = \lim_{n\in\mathbb{N}} TT^+A_n = \lim_{n\in\mathbb{N}} A_nTT^+ = ATT^+.$$

Hence, $(TT^+)'$ is closed. Analogously, we can show that $(T^+T)'$ is also a closed $*$-subalgebra.

❑

Theorem 2.3.15. *Let $(\mathcal{K}, [.,.]_\mathcal{K})$ be a Krein space, $(\mathcal{H}, [.,.]_\mathcal{H})$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{K}$ be a bounded and injective linear mapping. Then*

$$\Theta : \begin{cases} (TT^+)' & \to \quad (T^+T)', \\ \quad C & \mapsto \quad (T\times T)^{-1}(C), \end{cases}$$

constitues a bounded $*$-homomorphism. Hereby, $\Theta(I) = I$, $\Theta(TT^+) = T^+T$, and

$$\ker\Theta = \{C \in (TT^+)' : \operatorname{ran} C \subseteq \ker T^+\}.$$

Moreover, $(T^+ \times T^+)(C)$ is densely contained in $\Theta(C)$ for all $C \in (TT^+)'$ and we have $T^+C = \Theta(C)T^+$.

Proof. Let $C \in (TT^+)'$ be a self-adjoint operator. Then we have by Lemma 2.3.4 that $(TT^+ \times TT^+)(C) \subseteq C$ and consequently

$$(TT^+ \times TT^+)(C^+) = (TT^+ \times TT^+)(C) \subseteq C.$$

Theorem 2.3.13 implies that $\Theta(C) = (T \times T)^{-1}(C)$ is a bounded linear and self-adjoint mapping on $\mathcal{H}$ containing $(T^+ \times T^+)(C)$ densely. Due to

$$(T^+T \times T^+T)((T\times T)^{-1}(C)) \subseteq (T^+ \times T^+)(C) \subseteq (T\times T)^{-1}(C)$$

and Lemma 2.3.4 we have $(T\times T)^{-1}(C) \in (T^+T)'$.

Clearly $\Theta(I) = (T\times T)^{-1}(I) = T^{-1}IT = I$ and $\Theta(TT^+) = (T\times T)^{-1}(TT^+) = T^{-1}TT^+T = T^+T$.

Let $C \in (TT^+)'$ be arbitrary. Since $(TT^+)'$ a $*$-algebra, we also have $C^+ \in (TT^+)'$. We set

$$\operatorname{Re} C = \frac{C + C^+}{2}, \quad \operatorname{Im} C = \frac{C - C^+}{2\mathrm{i}}.$$

Both are self-adjoint operators in $(TT^+)'$ and we have $C = \operatorname{Re} C + \mathrm{i}\operatorname{Im} C$, $C^+ = \operatorname{Re} C - \mathrm{i}\operatorname{Im} C$. By Lemma 2.3.6

$$\begin{aligned} (T\times T)^{-1}(\operatorname{Re} C + \mathrm{i}\operatorname{Im} C) &\supseteq (T\times T)^{-1}(\operatorname{Re} C) + \mathrm{i}(T\times T)^{-1}(\operatorname{Im} C), \\ (T\times T)^{-1}(\operatorname{Re} C - \mathrm{i}\operatorname{Im} C) &\supseteq (T\times T)^{-1}(\operatorname{Re} C) - \mathrm{i}(T\times T)^{-1}(\operatorname{Im} C). \end{aligned} \tag{2.6}$$

Since T is injective, the multi-value-part is $\{0\}$ on both sides of the inclusion. Moreover, by the already proven the right-hand-sides are everywhere defined operators. This yields that both sides must coincide and $(T\times T)^{-1}(C) \in (T^+T)'$. Furthermore we obtain from (2.6) that $(T\times T)^{-1}(C^+) = (T\times T)^{-1}(C)^*$. Hence, the mapping Θ is well-defined and satisfies $\Theta(C^+) = \Theta(C)^*$.

Again by employing Lemma 2.3.6 and using that the right-hand-side of the inclusion is a everywhere defined operator, we obtain that Θ is linear and mulitplicative.

Let J be a fundamental symmetry of $(\mathcal{K}, [.,.]_{\mathcal{K}})$. By

$$\|\Theta(C)\|^2 = \sup_{x\in\mathcal{H}, \|x\|=1} [\Theta(C)x, \Theta(C)x]_{\mathcal{H}} = \sup_{x\in\mathcal{H}, \|x\|=1} [\Theta(C^+C)x, x]_{\mathcal{H}}$$
$$\leq \left\|\Theta(C^+C)\right\| \overset{(2.5)}{\leq} \left\|C^+C\right\| = \|JC^*JC\| \leq \|J\|^2 \|C\|^2 \leq \|C\|^2,$$

we conclude that Θ is bounded. Lemma 2.3.9 yields

$$\big((T^+\times T^+)(C)\big)^* = (T\times T)^{-1}(C^+) = \big((T\times T)^{-1}(C)\big)^*.$$

This shows that $(T^+\times T^+)(C)$ is densely contained in $(T\times T)^{-1}(C)$. In particular, $(T\times T)^{-1}(C) = \Theta(C) = 0$ is equivalent to the fact that $(a;b) \in (T^+\times T^+)(C)$ always implies $b = 0$. Therefore, $T^+y = 0$ for all $(x;y) \in C$, which means $\operatorname{ran} C \subseteq \ker T^+$.

From $(T^+u; T^+Cu) \in (T^+\times T^+)(C) \subseteq \Theta(C)$ and $(T^+u, \Theta(C)T^+u) \in \Theta(C)$ we conlcude that $T^+Cu = \Theta(C)T^+u$ for every $u \in \mathcal{K}$.

❑

Lemma 2.3.16. *Let $T : \mathcal{H} \to \mathcal{K}$ be a bounded and injective linear mapping from the Hilbert space $(\mathcal{H}, [.,.]_{\mathcal{H}})$ into the Krein space $(\mathcal{K}, [.,.]_{\mathcal{K}})$. Then*

$$\Xi : \begin{cases} L_{\mathrm{b}}(\mathcal{H}) & \to & L_{\mathrm{b}}(\mathcal{K}), \\ D & \mapsto & TDT^+, \end{cases}$$

is bounded linear and injective. Moreover, Ξ maps $(T^+T)' \subseteq L_{\mathrm{b}}(\mathcal{H})$ into $(TT^+)' \subseteq L_{\mathrm{b}}(\mathcal{K})$ and satisfies for $C \in (TT^+)'$ and $D, D_1, D_2 \in (T^+T)'$

$$\Xi(D^*) = \Xi(D)^+, \quad \Xi(D\Theta(C)) = \Xi(D)C, \quad \Xi(\Theta(C)D) = C\Xi(D),$$
$$\Xi(D_1D_2T^+T) = \Xi(D_1)\Xi(D_2), \quad \Xi\circ\Theta(C) = TT^+C = CTT^+.$$

Moreover, $\Xi(D)$ commutes with all operators from $(TT^+)'$ if D commutes with all operators from $(T^+T)'$, i.e. $\Xi((T^+T)'') \subseteq (TT^+)''$.

Proof. The mapping $\Xi(D) = TDT^+$ is clearly linear and bounded by $\|T\|\,\|T^+\|$. Since T is injective and $\operatorname{ran} T^+$ is dense in $\mathcal{H}$, we obtain the injectivity of Ξ. It is easy to see that $\Xi(D)^+ = \Xi(D^*)$. Let $C \in (TT^+)'$ and $D \in (T^+T)'$. Then we have

$$\Xi(D)TT^+ = TDT^+TT^+ = TT^+TDT^+ = TT^+\Xi(D),$$

and in consequence $\Xi(D) \in (TT^+)'$. For $C \in (TT^+)'$, $D \in (T^+T)'$, due to $T^+C = \Theta(C)T^+$ we have $\Xi(D\Theta(C)) = TD\Theta(C)T^+ = TDT^+C = \Xi(D)C$. Applying this to C^+, D^* and taking adjoints yields $\Xi(\Theta(C)D) = C\Xi(D)$.

For $D_1, D_2 \in (T^+T)'$ we have

$$\Xi(D_1D_2T^+T) = TD_1D_2T^+T^+T = TD_1T^+TD_2T^+ = \Xi(D_1)\Xi(D_2).$$

Due to $T^+C = \Theta(C)T^+$ we conclude $\Xi \circ \Theta(C) = T\Theta(C)T^+ = TT^+C = CTT^+$.

Finally assume that D commutes with all operators from $(T^+T)'$. Since $\Theta(C) \in (T^+T)'$ for $C \in (TT^+)'$, we have

$$\Xi(D)C = \Xi(D\Theta(C)) = \Xi(\Theta(C)D) = C\Xi(D).$$

❑

3 Joint Spectral Theorem

3.1 Multiple embeddings

Assumptions 3.1.1. In the present section we fix a Krein space $(\mathcal{K}, [.,.]_\mathcal{K})$, a Hilbert space $(\mathcal{H}, [.,.]_\mathcal{H})$ and a number $n \in \mathbb{N}$. For every $i \in [1,n]_\mathbb{Z}$ let $(\mathcal{H}_i, [.,.]_{\mathcal{H}_i})$ be a further Hilbert space. Moreover we assume that bounded linear and injective mappings $T : \mathcal{H} \to \mathcal{K}$ and $T_i : \mathcal{H}_i \to \mathcal{K}$ for every $i \in [1,n]_\mathbb{Z}$ are given such that

$$TT^+ = \sum_{i=1}^n T_i T_i^+. \tag{3.1}$$

Lemma 3.1.2. *For every $i \in [1,n]_\mathbb{Z}$ there exists a injective contraction $R_i : \mathcal{H}_i \to \mathcal{H}$ such that $T_i = TR_i$ and*

$$\sum_{i=1}^n R_i R_i^* = I.$$

If $(T_i T_i^+)_{i=1}^n$ is a tuple of pairwise commuting operators, then for fixed $i \in [1,n]_\mathbb{Z}$ the operator $R_i R_i^*$ commutes with T^+T and $R_i^* R_i$ commutes with $T_i^+ T_i$.

Proof. For $x \in \mathcal{K}$ we have

$$\begin{aligned} \left\|T^+x\right\|_\mathcal{H}^2 = [T^+x, T^+x]_\mathcal{H} = [TT^+x, x]_\mathcal{K} &\overset{(3.1)}{=} \sum_{i=1}^n [T_iT_i^+x, x]_{\mathcal{K}_i} \\ = \sum_{i=1}^n [T_i^+x, T_i^+x]_{\mathcal{H}_i} = \sum_{i=1}^n \left\|T_i^+x\right\|_{\mathcal{H}_i}^2 &\geq \left\|T_k^+x\right\|_{\mathcal{H}_k}^2 \end{aligned} \tag{3.2}$$

for every $k \in [1,n]_\mathbb{Z}$. This inequality guarantees that

$$B_k : \begin{cases} \operatorname{ran} T^+ & \to & \operatorname{ran} T_k^+, \\ T^+x & \mapsto & T_k^+x \end{cases}$$

is a well-defined, linear and contractive mapping. Due to our assumptions T is injective and therefore $\{0\} = \ker T = (\operatorname{ran} T^+)^\perp$. This leads to $\operatorname{ran} T^+$ being dense in $\mathcal{H}$ the same counts for every T_k and the corresponding Hilbert space $\mathcal{H}_k$. This justifies that we can uniquely extend B_k by continuity to $\overline{B}_k : \mathcal{H} \to \mathcal{H}_k$. Clearly $\overline{B}_k$ is still a linear contractive map which has a dense range.

We define the desired mapping $R_i : \mathcal{H}_i \to \mathcal{H}$ by the adjoint of $\overline{B}_i$ i.e. $R_i = \overline{B}_i^*$. Since $\ker R_i = (\operatorname{ran} R_i^*)^\perp = \{0\}$ and $\|R_i\| = \|R_i^*\|$ we conclude that R_i is injective and contractive. By definition we have $R_i^* T^+ = \overline{B}_i T^+ = T_i^+$, which leads to $TR_i = T_i$.

The equation

$$T(I)T^+ = TT^+ \overset{(3.1)}{=} \sum_{i=1}^n \underbrace{TR_i}_{=T_i} \overbrace{R_i^* T^+}^{=T_i^+} = T\Big(\sum_{i=1}^n R_i R_i^*\Big)T^+$$

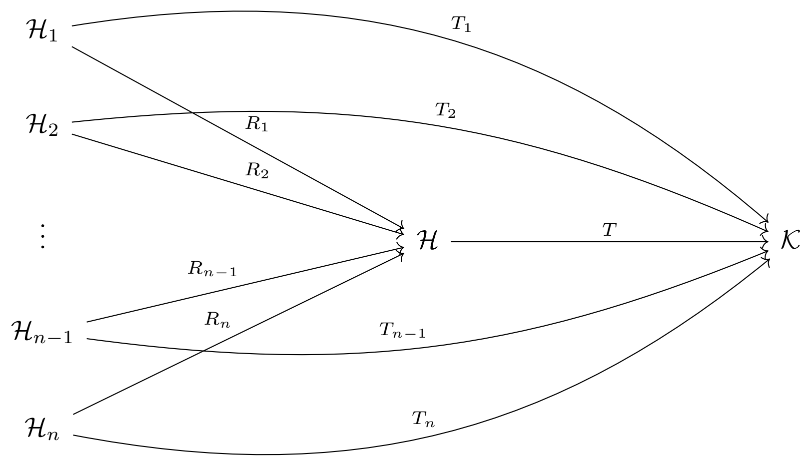

Figure 1: Setting of Lemma 3.1.2

together with the injectivity of T and the density of $\operatorname{ran} T^+$ yields $I = \sum_{i=1}^n R_i R_i^*$.

If $(T_i T_i^+)_{i=1}^n$ is a commuting tuple, then by (3.1) every $T_i T_i^+$ commutes with TT^+. From

$$T(T^+ \underbrace{TR_i}_{=T_i} \overbrace{R_i^*)T^+}^{=T_i^+} = TT^+ T_i T_i^+ = T_i T_i^+ TT^+ = \underbrace{T(R_i}_{=T_i} \overbrace{R_i^* T^+ T}^{=T_i^+})T^+.$$

and from $T's$ injectivity and the density of $\operatorname{ran} T^+$ we conclude that $R_i R_i^*$ and T^+T commute for every $i \in [1,n]_{\mathbb{Z}}$. Finally, we have

$$T_i^+ T_i R_i^* R_i = \underbrace{R_i^*(T^+}_{=T_i^+} \overbrace{TR_i}^{=T_i} R_i^*)R_i = R_i^*(R_i \underbrace{R_i^* T^+}_{=T_i^+} \overbrace{T)R_i}^{=T_i} = R_i^* R_i T_i^+ T_i.$$

❑

We want to recall the $*$-algebra homomorphisms from Theorem 2.3.15 corresponding to a injective mapping T. We will define such a $*$-algebra homomorphisms for each T_i and R_i for $i \in [1,n]_{\mathbb{Z}}$.

Definition 3.1.3. Let T, T_i for $i \in [1,n]_{\mathbb{Z}}$ be the mappings from Assumptions 3.1.1 and R_i the mappings from Lemma 3.1.2. Then we define $\Theta : (TT^+)' \to (T^+T)'$ and $\Theta_i : (T_i T_i^+)' \to (T_i^+ T_i)'$ by

$$\Theta(C) = (T \times T)^{-1}(C) = T^{-1}CT \quad \text{and} \quad \Theta_i(C) = (T_i \times T_i)^{-1}(C) = T_i^{-1}CT.$$

and $\Gamma_i : (R_i R_i^*)' \to (R_i^* R_i)'$ by

$$\Gamma_i(D) = (R_i \times R_i)^{-1}(D) = R_i^{-1} D R_i$$

for each $i \in [1, n]_{\mathbb{Z}}$.

Proposition 3.1.4. *With Assumptions 3.1.1 and Definition 3.1.3, we have* $\bigcap_{i=1}^{n} (T_i T_i^+)' \subseteq (TT^+)'$ *and* $\Theta\big(\bigcap_{i=1}^{n} (T_i T_i^+)'\big) \subseteq \bigcap_{i=1}^{n} (R_i R_i^*)' \cap (T^+T)'$, *where*

$$\Theta(C) R_i R_i^* = R_i \Theta_i(C) R_i^* = R_i R_i^* \Theta(C)$$

and

$$\Theta_i(C) = \Gamma_i \circ \Theta(C) \quad \textit{for all} \quad C \in \bigcap_{i=1}^{n} (T_i T_i^+)' \tag{3.3}$$

Proof. From (3.1) we easily conclude $\bigcap_{i=1}^{n} (T_i T_i^+)' \subseteq (TT^+)'$. According to Theorem 2.3.15 we have $\Theta(C)T^+ = T^+C$ and $\Theta_i(C)T_i^+ = T_i^+ C$ for $i \in [1, n]_{\mathbb{Z}}$. This leads to

$$T(R_i \Theta_i(C) R_i^*) T^+ = T_i \Theta_i(C) T_i^+ = T_i T_i^+ C = T R_i R_i^* T^+ C = T(R_i R_i^* \Theta(C)) T^+.$$

From the injectivity of T and the density of $\operatorname{ran} T^+$ we obtain $R_i \Theta_i(C) R_i^* = R_i R_i^* \Theta(C)$. Applying this equation to C^+ and taking adjoints yields

$$R_i \Theta_i(C^+)^* R_i^* = \big(R_i \Theta_i(C^+) R_i^*\big)^+ = \big(R_i R_i^* \Theta(C^+)\big)^+ = \Theta(C^+)^* R_i R_i^*.$$

Since Θ and Θ_i are $*$-homomorphisms we obtain $R_i \Theta_i(C) R_i^* = \Theta(C) R_i R_i^*$. Combining these two equations yields $\Theta(C) \in (R_i R_i^*)'$. This justifies the application of Γ_i to $\Theta(C)$.

$$\Gamma_i \circ \Theta(C) = R_i^{-1} T^{-1} C T R_i = T_i^{-1} C T_i = \Theta_i(C),$$

where $R_i^{-1} T^{-1} = (TR_i)^{-1}$ has to be understood in the sense of linear relations. ❑

Corollary 3.1.5. *Let us use Assumptions 3.1.1 and Definition 3.1.3, and let* $\boldsymbol{N} = (N_k)_{k=1}^{m}$ *be tuple of pairwise commuting, self-adjoint Operators in* $\bigcap_{i=1}^{n} (T_i T_i^+)'$.

Then $\Theta(\boldsymbol{N})$, $\Theta_i(\boldsymbol{N})$ *are also tuples of pairwise commuting, self-adjoint Operators in the Hilbert spaces* $(\mathcal{H}, [.,.]_{\mathcal{H}}), (\mathcal{H}_i, [.,.]_{\mathcal{H}_i})$, *repectively for* $i \in [1, n]_{\mathbb{Z}}$.

If E (E^i) *denotes the joint spectral measure for* $\Theta(\boldsymbol{N})$ ($\Theta_i(\boldsymbol{N})$), *then* $E(\Delta) \in \bigcap_{i=1}^{n} (R_i R_i^*)' \cap (T^+T)'$ *and*

$$\Gamma(E(\Delta)) = E^i(\Delta) \in (R_i^* R_i)' \cap (T_i^+ T_i)'$$

for all Borel subsets Δ *of* $\mathbb{R}^m$. *Moreover* $\int h \, \mathrm{d}E \in \bigcap_{i=1}^{n} (R_i R_i^*)' \cap (T^+T)'$ *and*

$$\Gamma_i\Big(\int h \, \mathrm{d}E\Big) = \int h \, \mathrm{d}E^i \in (R_i^* R_i)' \cap (T_i^+ T_i)' \tag{3.4}$$

for any bounded and measurable $h : \sigma(\Theta(\boldsymbol{N})) \to \mathbb{C}$.

Proof. Since Θ and Θ_i are $*$-homomorphisms, the images of commuting operators commute as well. From Proposition 3.1.4 we obtain $\Theta(N_k) \in \bigcap_{i=1}^n (R_iR_i^*)' \cap (T^+T)'$ for every $k \in [1,m]_{\mathbb{Z}}$. Therefore $E(\Delta) \in \bigcap_{i=1}^n (R_iR_i^*)' \cap (T^+T)'$ and, in turn, $\int h \, \mathrm{d}E \in \bigcap_{i=1}^n (R_iR_i^*)' \cap (T^+T)'$. This justifies the application of Γ_i to $E(\Delta)$ and $\int h \, \mathrm{d}E$. Theorem 2.3.15 tells us that $\Gamma_i(D)R_i^* = R_i^*D$ for $D \in (R_iR_i^*)'$. For $x \in \mathcal{H}$ and $y \in \mathcal{H}_i$ we get

$$[\Gamma_i(E(\Delta))R_i^*x, y]_{\mathcal{H}_i} = [R_i^*E(\Delta)x, y]_{\mathcal{H}_i} = [E(\Delta)x, R_iy]_{\mathcal{H}}$$

and in turn for and $s \in \mathbb{C}[z_1, \dots, z_m]$

$$\begin{aligned}\int\limits_{\mathbb{R}^m} s \, \mathrm{d}[\Gamma_i(E)R_i^*x, y]_{\mathcal{H}_i} &= \int\limits_{\mathbb{R}^m} s \, \mathrm{d}[E(\Delta)x, R_iy]_{\mathcal{H}} = \Big[s\left(\Theta(\boldsymbol{N})\right)x, R_iy\Big]_{\mathcal{H}} \\ &= \Big[R_i^* s\left(\Theta(\boldsymbol{N})\right)x, y\Big]_{\mathcal{H}_i} = \Big[\Gamma_i\left(s\left(\Theta(\boldsymbol{N})\right)\right)R_i^*x, y\Big]_{\mathcal{H}_i}\end{aligned}$$

Since Γ_i is a homomorphism, s is a polynom and $s\left(\Theta(\boldsymbol{N})\right)$ is in $\bigcap_{i=1}^n (T_iT_i^+)'$ we can use (3.3) to conclude $\Gamma_i\left(s\left(\Theta(\boldsymbol{N})\right)\right) = s\left(\Theta_i(\boldsymbol{N})\right)$. According to this equality we obtain

$$\int\limits_{\mathbb{R}^m} s \, \mathrm{d}[\Gamma_i(E)R_i^*x, y]_{\mathcal{H}_i} = \Big[s\left(\Theta_i(\boldsymbol{N})\right)R_i^*x, y\Big]_{\mathcal{H}_i} = \int\limits_{\mathbb{R}^m} s \, \mathrm{d}[E^iR_i^*x, y]_{\mathcal{H}_i}.$$

We can choose a compact $K \subseteq \mathbb{R}^m$ such that $E(\mathbb{R}^m \backslash K) = 0$ and $E^i(\mathbb{R}^m \backslash K) = 0$. Since $\mathbb{C}[z_1, \dots, z_m]$ is dense in $C(K)$, Riesz' Representation Theorem tells us that the measures must coincide:

$$[\Gamma_i(E(\Delta))R_i^*x, y]_{\mathcal{H}_i} = [E^i(\Delta)R_i^*x, y]_{\mathcal{H}_i} \quad \text{for all } x \in \mathcal{H},\ y \in \mathcal{H}_i$$

and all Borel subsets Δ of $\mathbb{R}^m$. The density of $\operatorname{ran} R_i^*$ gives us $[\Gamma_i(E(\Delta))z, y]_{\mathcal{H}_i} = [E^i(\Delta)z, y]_{\mathcal{H}_i}$ for all $y, z \in \mathcal{H}_i$. Consequently $\Gamma_i(E(\Delta)) = E^i(\Delta)$. The image of Γ_i is contained in $(R_i^*R_i)'$. Therefore, $E^i(\Delta)$ and $\int h \, \mathrm{d}E^i$ is also contained in $(R_i^*R_i{}')$ for every bounded and measurable h.

Since $\Gamma_i(E(\Delta)) = E^i(\Delta)$, we conclude $\operatorname{supp} E^i \subseteq \operatorname{supp} E$ and therefore $\sigma(\Theta_i(\boldsymbol{N})) \subseteq \sigma(\Theta(\boldsymbol{N}))$

Let $h : \sigma(\Theta(\boldsymbol{N})) \to \mathbb{C}$ be bounded and measurable. Clearly, also its restriction to $\sigma(\Theta_i(\boldsymbol{N}))$ is bounded and measurable. From the already shown fact that $E^i(\Delta)R_i^* = \Gamma_i(E(\Delta))R_i^* = R_i^*E(\Delta)$ we obtain

$$\begin{aligned}\Big[\Gamma_i\Big(\int h \, \mathrm{d}E\Big)R_i^*x, y\Big]_{\mathcal{H}_i} &= \Big[R_i^*\Big(\int h \, \mathrm{d}E\Big)x, y\Big]_{\mathcal{H}_i} = \Big[\Big(\int h \, \mathrm{d}E\Big)x, R_iy\Big]_{\mathcal{H}} \\ &= \int h \, \mathrm{d}[Ex, R_iy]_{\mathcal{H}} = \int h \, \mathrm{d}[E^iR_i^*x, y]_{\mathcal{H}_i} \\ &= \Big[\Big(\int h \, \mathrm{d}E^i\Big)R_i^*x, y\Big]_{\mathcal{H}_i}.\end{aligned}$$

Again the density of $\operatorname{ran} R_i^*$ yields the desired equation (3.4).

❑

We will use Lemma 2.3.16 to introduce the mappings Ξ and Ξ_i for each $i \in [1,n]_{\mathbb{Z}}$ referring to T and T_i:

$$\Xi : \begin{cases} (T^+T)' & \to & (TT^+)', \\ D_i & \mapsto & TDT^+, \end{cases} \qquad \Xi_i : \begin{cases} (T_i^+T_i)' & \to & (T_iT_i^+)', \\ D_i & \mapsto & T_iD_iT_i^+. \end{cases}$$

Again according to Lemma 2.3.16 we define

$$\Lambda_i : \begin{cases} (R_i^* R_i)' & \to & (R_i R_i^*)', \\ D_i & \mapsto & R_i D_i R_i^* \end{cases}$$

and we conclude that

$$\Xi_i(D_i) = T R_i D_i R_i^* T^+ = \Xi \circ \Lambda_i(D_i) \quad \text{for} \quad D_i \in (R_i^* R_i)' \cap (T_i^+ T_i)'. \tag{3.5}$$

According to Lemma 2.3.16 we have in our notation relating to R_i

$$\Lambda_i \circ \Gamma_i(D) = D R_i R_i^*. \tag{3.6}$$

Hence, using Corollary 3.1.5 and its notation we obtain

$$\begin{aligned} \Xi_i\left(\int h \, \mathrm{d}E^i\right) &\overset{C3.1.5}{=} \Xi_i \circ \Gamma_i\left(\int h \, \mathrm{d}E\right) \overset{(3.5)}{=} \Xi \circ \Lambda_i \circ \Gamma_i\left(\int h \, \mathrm{d}E\right) \\ &\overset{(3.6)}{=} \Xi\left(R_i R_i^* \int h \, \mathrm{d}E\right). \end{aligned} \tag{3.7}$$

Finally, $T^{-1} T_i T_i^+ T = T^{-1} T R_i R_i^* T^+ T = R_i R_i^* T^+ T$. If $(T_i T_i^+)_{i=1}^n$ is a tuple of pairwise commuting operators, then we have $T_i T_i^+ \in (T T^+)'$ and the later equality can be expressed as

$$\Theta(T_i T_i^+) = R_i R_i^* T^+ T \quad \text{for every} \quad i \in [1, n]_{\mathbb{Z}}. \tag{3.8}$$

3.2 Setting

Assumptions 3.2.1. Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be a tuple of pairwise commuting, selfadjoint and definitizable Operators in $L_{\mathrm{b}}(\mathcal{K})$. We denote a corresponding tuple of definitizing polynomials by $\boldsymbol{p} = (p_i)_{i=1}^n$, i.e. p_i is a definitizing polynomial for A_i. For convenience we will choose each p_i as a real polynomial; see Lemma 1.2.8.

According to Corollary 1.2.12 for each A_i there exists a Hilbert space $(\mathcal{H}_i, [.,.]_{\mathcal{H}_i})$ and an injective and bounded linear mapping

$$T_i : \mathcal{H}_i \to \mathcal{K} \quad \text{such that} \quad T_i T_i^+ = p_i(A_i). \tag{3.9}$$

Since $\sum_{i=1}^n p_i(A_i)$ is also a positiv Operator, we can apply Lemma 1.2.10 and obtain a Hilbert space $(\mathcal{H}, [.,.]_{\mathcal{H}})$ and an injective and bounded linear mapping $T : \mathcal{H} \to \mathcal{K}$ such that

$$T T^+ = \sum_{i=1}^n p_i(A_i) = \sum_{i=1}^n T_i T_i^+.$$

Hence, the mappings T and $(T_i)_{i=1}^n$ fulfill the Assumptions 3.1.1. By Lemma 3.1.2 there exists a tuple of injective contractions $\boldsymbol{R} = (R_i)_{i=1}^n$ such that $R_i : \mathcal{H}_i \to \mathcal{H}$ and $T_i = T R_i$.

Lemma 3.2.2. *Let T,T_i and R_i be as in Assumptions 3.2.1 and Θ the $*$-homomorphism according to T; see Definition 3.1.3. Then we have*

$$p_i(\Theta(A_i)) = R_i R_i^* \sum_{k=1}^n p_k(\Theta(A_k)),$$

where $R_i R_i^*$ commutes with $\sum_{k=1}^n p_k(\Theta(A_k))$ for all $i \in [1, n]_{\mathbb{Z}}$.

Proof. By the definition of Θ (Theorem 2.3.15), we have

$$T^+T = \Theta(TT^+) \overset{(3.1)}{=} \Theta\Big(\sum_{k=1}^n T_k T_k^+\Big) \overset{(3.9)}{=} \sum_{k=1}^n \Theta(p_k(A_k)) = \sum_{k=1}^n p_k(\Theta(A_k)).$$

Lemma 3.1.2 guarantees that $R_i R_i^*$ commutes with T^+T and hence it does with $\sum_{k=1}^n p_k(\Theta(A_k))$. We obtain

$$p_i(\Theta(A_i)) = \Theta(p_i(A_i)) = \Theta(T_i T_i^+) \overset{(3.8)}{=} R_i R_i^* T^+ T = R_i R_i^* \sum_{k=1}^n p_k(\Theta(A_k))$$

which completes the proof.

❑

Lemma 3.2.3. *Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be as in Assumptions 3.2.1. For $i \in [1,n]_{\mathbb{Z}}$ we then have*

$$\Big\{\boldsymbol{z} \in \mathbb{R}^n : |p_i(z_i)| > \|R_i R_i^*\| \cdot \Big|\sum_{k=1}^n p_k(z_k)\Big|\Big\} \subseteq \rho(\Theta(\boldsymbol{A})).$$

In particular, the zeros of $\sum_{k=1}^n p_k(z_k)$ are contained in

$$\rho(\Theta(\boldsymbol{A})) \cup \{\boldsymbol{z} \in \mathbb{R}^n : p_j(z_j) = 0 \text{ for all } j \in [1,n]_{\mathbb{Z}}\}.$$

Proof. Let E be the spectral measure of $\Theta(\boldsymbol{A})$ as in Theorem 1.5.1. For a fixed $i \in [1,n]_{\mathbb{Z}}$ and an arbitrary $m \in \mathbb{N}$ we introduce the set

$$\Delta_m := \Big\{\boldsymbol{z} \in \mathbb{R}^n : |p_i(z_i)|^2 > \frac{1}{m} + \|R_i R_i^*\|^2 \Big|\sum_{k=1}^n p_k(z_k)\Big|^2\Big\}.$$

For $x \in \operatorname{ran} E(\Delta_m)$ we have

$$\begin{aligned}
\|p_i(\Theta(A_i))x\|^2 = \|p_i(\Theta(A_i))E(\Delta_m)x\|^2 &= \int_{\Delta_m} |p_i(z_i)|^2 \,\mathrm{d}[E(\boldsymbol{z})x,x] \\
&\geq \int_{\Delta_m} \frac{1}{m}\,\mathrm{d}[E(\boldsymbol{z})x,x] + \|R_i R_i^*\|^2 \int_{\Delta_m} \Big|\sum_{k=1}^n p_k(z_k)\Big|^2 \mathrm{d}[E(\boldsymbol{z})x,x] \\
&\geq \frac{1}{m}\|x\|^2 + \Big\|\underbrace{R_i R_i^* \sum_{k=1}^n p_k(\Theta(A_k))}_{=p_i(\Theta(A_i))}\, x\Big\|^2.
\end{aligned}$$

This inequality can only hold true for $x = 0$. Hence, $E(\Delta_m) = 0$. The fact that Δ_m is open implies that $\Delta_m \subseteq (\operatorname{supp} E)^{\mathsf{c}} = \sigma(\boldsymbol{A})^{\mathsf{c}} = \rho(\boldsymbol{A})$. Since $m \in \mathbb{N}$ was arbitrary, we finally obtain

$$\rho(\boldsymbol{A}) \supseteq \bigcup_{m \in \mathbb{N}} \Delta_m = \Big\{\boldsymbol{z} \in \mathbb{R}^n : |p_i(z_i)| > \|R_i R_i^*\| \cdot \Big|\sum_{k=1}^n p_k(z_k)\Big|\Big\}.$$

If $\sum_{k=1}^{n} p_k(z_k) = 0$ and $\boldsymbol{z} \notin \{\boldsymbol{w} \in \mathbb{R}^n : p_i(w_i) = 0 \text{ for all } i \in [1,n]_{\mathbb{Z}}\}$ then there exists a $j \in [1,n]_{\mathbb{Z}}$ such that $|p_j(z_j)| > 0 = \|R_j R_j^*\| \left|\sum_{k=1}^{n} p_k(z_k)\right|$. From the already shown we conclude that $\boldsymbol{z} \in \rho(\boldsymbol{A})$.

❑

In order to be more self contained we will proof the following Lemma, which will be needed for the next Corollary.

Lemma 3.2.4. *Let* $(\mathcal{H}, [.,.])$ *be a Hilbert space and* $N : \mathcal{H} \to \mathcal{H}$ *be a normal Operator then* $\ker N = (\operatorname{ran} N)^{\perp}$.

Proof. Since N is normal, we have

$$\|Nx\|^2 = [Nx, Nx] = [N^*Nx, x] = [NN^*x, x] = [N^*x, N^*x] = \|N^*x\|^2 .$$

This leads to $\ker N = \ker N^*$. From the well-known result $\ker N^* = (\operatorname{ran} N)^{\perp}$ we conlcude the statement.

❑

Corollary 3.2.5. *With the notation and assumptions from Lemma 3.2.3 and* $\Delta := \{\boldsymbol{z} \in \mathbb{R}^n : p_k(z_k) \neq 0 \text{ for some } k \in [1,n]_{\mathbb{Z}}\}$ *we have*

$$R_i R_i^* E(\Delta) = \int_{\Delta} \frac{p_i(z_i)}{\sum_{k=1}^{n} p_k(z_k)} \, \mathrm{d}E(\boldsymbol{z})$$

for every $i \in [1,n]_{\mathbb{Z}}$

Proof. By Lemma 3.2.3 we have $|p_i(z_i)| \leq \|R_i R_i^*\| \left|\sum_{k=1}^{n} p_k(z_k)\right|$ for every $\boldsymbol{z} \in \operatorname{supp} E$. Hence, the integrand is bounded on $\operatorname{supp} E$ and consequently the integral on right-hand-side exists.

Clearly, both sides vanish on the range of $E(\Delta^{\mathsf{c}})$. For

$$\mathcal{U} := \operatorname{ran} E(\Delta) = \left(\operatorname{ran} E(\Delta^{\mathsf{c}})\right)^{\perp}$$

we have that $\mathcal{U}^{\perp} = \operatorname{ran} E(\Delta^{\mathsf{c}})$ is contained in the kernel of the operator

$$\int \sum_{k=1}^{n} p_k(z_k) \, \mathrm{d}E(\boldsymbol{z}) = \sum_{k=1}^{n} p_k(\Theta(A_k)).$$

By Lemma 3.2.3 all zeros of $\boldsymbol{z} \mapsto \sum_{k=1}^{n} p_k(z_k)$ which are also contained in $\operatorname{supp} E$ can only be found in Δ^{c}. For $x \in \mathcal{U}$, $x \neq 0$ we have

$$\begin{aligned} \left\| \int \sum_{k=1}^{n} p_k(z_k) \, \mathrm{d}E(\boldsymbol{z}) x \right\|^2 &= \left\| \int \sum_{k=1}^{n} p_k(z_k) \, \mathrm{d}E(\boldsymbol{z}) E(\Delta) x \right\|^2 \\ &= \int_{\Delta} \underbrace{\left| \sum_{k=1}^{n} p_k(z_k) \right|^2}_{>0 \text{ on } \Delta} \mathrm{d}[E(\boldsymbol{z})x, x] > 0. \end{aligned}$$

Therefore, $\ker \int \sum_{k=1}^{n} p_k(z_k) \, \mathrm{d}E(\boldsymbol{z}) = \mathcal{U}^{\perp}$. Since $\sum_{k=1}^{n} p_k(\Theta(A_k))$ is normal, we obtain from Lemma 3.2.4 that its range is dense in $\mathcal{U}$. Let x be in this dense subspace. Then we can write $x = \sum_{k=1}^{n} p_k(\Theta(A_k))y$ for some $y \in \mathcal{U}$ and obtain

$$\int_\Delta \frac{p_i(z_i)}{\sum_{k=1}^n p_k(z_k)} \,\mathrm{d}E(\boldsymbol{z})x = \int_\Delta p_i(z_i)\,\mathrm{d}E(\boldsymbol{z})y = p_i(\Theta(A_i))y$$
$$= R_i R_i^* \sum_{k=1}^{n} p_k(\Theta(A_k))y = R_i R_i^* x.$$

By density every $x \in \mathcal{U}$ fulfills this equation.

❑

3.3 Function class

Definition 3.3.1. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ we define the multi-index sets

$$\hat{I}_\alpha := \{\beta \in \mathbb{N}_0^n : \beta_i < \alpha_i \text{ for all } i \in [1,n]_\mathbb{Z}\}$$
$$I_\alpha := \hat{I}_\alpha \cup \{\alpha_i e_i : i \in [1,n]_\mathbb{Z}\},$$

where $e_i = (\delta_{i,j})_{j=1}^n$ and $\delta_{i,j}$ is the Kronecker delta. Furthermore we denote by $\mathfrak{A}_\alpha$ the set of all

$$a = (a_\beta)_{\beta \in I_\alpha} \quad \text{such that} \quad a_\beta \in \mathbb{C},$$

and by $\mathfrak{B}_\alpha$ we denote the set of all $a = (a_\beta)_{\beta \in \hat{I}_\alpha}$ such that $a_\beta \in \mathbb{C}$. There exists a canonical addition, scalar multiplication and conjugate linear involution on $\mathfrak{A}_\alpha$:

$$\begin{aligned} a + b &:= (a_\beta + b_\beta)_{\beta \in I_\alpha} && \text{for } a, b \in \mathfrak{A}_\alpha \\ \lambda a &:= (\lambda a_\beta)_{\beta \in I_\alpha} && \text{for } \lambda \in \mathbb{C} \text{ and } a \in \mathfrak{A}_\alpha \\ \overline{a} &:= (\overline{a}_\beta)_{\beta \in I_\alpha} && \text{for } a \in \mathfrak{A}_\alpha. \end{aligned}$$

Analogously, we can define these operations on $\mathfrak{B}_\alpha$. Additionally we can define a multiplication on these sets by

$$a \cdot b := \Big(\sum_{\gamma+\delta=\beta} a_\gamma b_\delta \Big)_{\beta \in I_\alpha} \quad \text{and} \quad a \cdot b := \Big(\sum_{\gamma+\delta=\beta} a_\gamma b_\delta \Big)_{\beta \in \hat{I}_\alpha} \quad \text{respectively.}$$

Finally, we want to introduce the projection

$$\pi_\alpha : \begin{cases} \mathfrak{A}_\alpha \cup \mathfrak{B}_\alpha & \to \quad \mathfrak{B}_\alpha, \\ \qquad a & \mapsto \quad (a_\beta)_{\beta \in \hat{I}_\alpha}. \end{cases}$$

Remark 3.3.2. For $a \in \mathfrak{B}_\alpha$ the projection π_α maps a on itself. For $a \in \mathfrak{A}_\alpha$ the projection π_α forgets all indices $\{\alpha_i e_i : i \in [1,n]_\mathbb{Z}\}$.

Example 3.3.3. For $\alpha = (n,m)$ we have $I_\alpha = [0, n-1]_\mathbb{Z} \times [0, m-1]_\mathbb{Z} \cup \{(n,0),(m,0)\}$

Remark 3.3.4. The sets $\mathfrak{A}_\alpha$ and $\mathfrak{B}_\alpha$ endowed with the operations that are presented in Definition 3.3.1 yield commutative unital $*$-algebras. The unit

$e = (e_\beta)_{\beta\in I_\alpha}$ in $\mathfrak{A}_\alpha$ is given by $e_0 = 1$ and $e_\beta = 0$ if $\beta \neq 0$. Analogously, $e = (e_\beta)_{\beta\in \hat{I}_\alpha}$ is the unit in $\mathfrak{B}_\alpha$.

Moreover it is easy to check that an element a of $\mathfrak{A}_\alpha$ ($\mathfrak{B}_\alpha$) has a multiplicative inverse in $\mathfrak{A}_\alpha$ ($\mathfrak{B}_\alpha$) if and only if $a_0 \neq 0$.

Definition 3.3.5. We define for every polynomial $q \in \mathbb{C}[z]$ the function

$$\mathfrak{d}_q : \begin{cases} \mathbb{C} & \to \quad \mathbb{N}_0, \\ z & \mapsto \quad \min\{j \in \mathbb{N}_0 : q^{(j)}(z) \neq 0\} \end{cases}$$

For a tuple of polynomials $\boldsymbol{q} = (q_i)_{i=1}^n$ where $q_i \in \mathbb{C}[z]$ and a vector $\boldsymbol{z} \in \mathbb{C}^n$ we employ the following notation

$$\mathfrak{d}_{\boldsymbol{q}}(\boldsymbol{z}) := \big(\mathfrak{d}_{q_i}(z_i)\big)_{i=1}^n \in \mathbb{N}_0^n.$$

Definition 3.3.6. Let p be polynomial in $\mathbb{C}[z]$ then we want to define the set of all zeros of q and the set of all real zeros of q by

$$Z_q := q^{-1}\{0\} \quad \text{and} \quad Z_q^{\mathbb{R}} := Z_q \cap \mathbb{R}$$

For a tuple of polynomials $\boldsymbol{q} = (q_i)_{i=1}^n$ where $q_i \in \mathbb{C}[z]$ we define the set of joint zeros, the set of joint real zeros and the set of joint complex zeros

$$Z_{\boldsymbol{q}} := \prod_{i=1}^n Z_{q_i}, \quad Z_{\boldsymbol{q}}^{\mathbb{R}} := Z_{\boldsymbol{q}} \cap \mathbb{R}^n \quad \text{and} \quad Z_{\boldsymbol{q}}^{\mathrm{i}} := Z_{\boldsymbol{q}} \backslash \mathbb{R}^n$$

as subsets of $\mathbb{C}^n$.

Furthermore let $\boldsymbol{p} = (p_i)_{i=1}^n$ be a tuple of real definitizing polynomials corresponding to the tuple of operators $\boldsymbol{A} = (A_i)_{i=1}^n$.

(*i*) Then we denote the space of all functions ϕ with domain

$$\Big(\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}^{\mathbb{R}}\Big) \,\dot\cup\, Z_{\boldsymbol{p}}^{\mathrm{i}} \subseteq \mathbb{C}^n$$

such that $\phi(\boldsymbol{z}) \in \mathfrak{C}(\boldsymbol{z})$, where

$$\mathfrak{C}(\boldsymbol{z}) := \begin{cases} \mathbb{C}, & \text{if } \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \mathfrak{A}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}, & \text{if } \boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \mathfrak{B}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}, & \text{if } \boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}, \end{cases}$$

by $\mathcal{M}_{\boldsymbol{A}}$. If $\boldsymbol{A}$ contains only one element A, we will write $\mathcal{M}_A$ instead.

(*ii*) We endow $\mathcal{M}_{\boldsymbol{A}}$ with pointwise scalar multiplication, addition and multiplication, where the operations on $\mathfrak{A}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}$ or $\mathfrak{B}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}$ are as in Definition 3.3.1. We also define a conjugate linear involution $(.)^{\#}$ on $\mathcal{M}_{\boldsymbol{A}}$ by

$$\phi^{\#}(\boldsymbol{z}) = \overline{\phi(\overline{\boldsymbol{z}})} \quad \text{for} \quad \boldsymbol{z} \in \Big(\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}^{\mathbb{R}}\Big) \,\dot\cup\, Z_{\boldsymbol{p}}^{\mathrm{i}}$$

This is well-defined, since $\boldsymbol{p}$ contains only real polynomials, which implies $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$ is equivalent to $\overline{\boldsymbol{z}} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$ and $\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z}) = \mathfrak{d}_{\boldsymbol{p}}(\overline{\boldsymbol{z}})$.

(*iii*) By $\mathcal{R}_{\boldsymbol{A}}$ we denote the set of all elements $\phi \in \mathcal{M}_{\boldsymbol{A}}$ such that $\pi_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}(\phi(\boldsymbol{w})) = 0$ for all $\boldsymbol{w} \in Z_{\boldsymbol{p}}$.

Remark 3.3.7. The function space $\mathcal{M}_{\boldsymbol{A}}$ is a commutative unital $*$-algebra with the operations defined in Defintion 3.3.6. Moreover $\mathcal{R}_{\boldsymbol{A}}$ is an ideal of $\mathcal{M}_{\boldsymbol{A}}$.

Definition 3.3.8. For $\boldsymbol{x} = (x_i)_{i=1}^n \in \mathbb{C}^n$ and $\beta \in \mathbb{N}_0^n$ we set

$$\boldsymbol{x}^\beta := \prod_{i=1}^n x_i^{\beta_i}, \quad \beta! := \prod_{i=1}^n \beta_i! \quad \text{and} \quad |\beta| = \sum_{i=1}^n \beta_i.$$

Definition 3.3.9. Let $f : \operatorname{dom} f \to \mathbb{C}$ be a function with

$$\left(\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}^{\mathbb{R}}\right) \dot\cup Z_{\boldsymbol{p}}^{\mathrm{i}} \subseteq \operatorname{dom} f \subseteq \mathbb{C}^n,$$

such that f is sufficiently smooth – more exactly, at least $\max_{\boldsymbol{w}\in Z_{\boldsymbol{p}}^{\mathbb{R}}} |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1$ times continuously differentiable – on an open neighborhood of $Z_{\boldsymbol{p}}^{\mathbb{R}}$ as subset of $\mathbb{R}^n$, and such that f is holomorphic on an open neighborhood of $Z_{\boldsymbol{p}}^{\mathrm{i}}$ as subset of $\mathbb{C}^n$.

Then f can be considered as an element $f_{\boldsymbol{A}}$ of $\mathcal{M}_{\boldsymbol{A}}$ by setting

$$f_{\boldsymbol{A}}(\boldsymbol{z}) := \begin{cases} f(\boldsymbol{z}), & \text{if } \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \left(\frac{1}{\beta!} D^\beta f(\boldsymbol{z})\right)_{\beta \in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}, & \text{if } \boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \left(\frac{1}{\beta!} D^\beta f(\boldsymbol{z})\right)_{\beta \in \hat I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}, & \text{if } \boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}. \end{cases}$$

For $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ the derivative should be understood in the sense of real derivation and for $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$ it is a complex derivative.

Remark 3.3.10. Let f, g be functions which satisfy the conditions of Definition 3.3.9. For $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ and $\beta \in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}$ the Leibniz rule yields

$$\begin{aligned} (fg)_{\boldsymbol{A}}(\boldsymbol{z}) &= \frac{1}{\beta!} D^\beta (fg)(\boldsymbol{z}) = \frac{1}{\beta!} \sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^\gamma f(\boldsymbol{z}) D^\delta g(\boldsymbol{z}) \\ &= \sum_{\gamma+\delta=\beta} \underbrace{\frac{1}{\gamma!} D^\gamma f(\boldsymbol{z})}_{=\left(f_{\boldsymbol{A}}(\boldsymbol{z})\right)_\gamma} \underbrace{\frac{1}{\delta!} D^\delta g(\boldsymbol{z})}_{=\left(g_{\boldsymbol{A}}(\boldsymbol{z})\right)_\delta} = \left(f_{\boldsymbol{A}}(\boldsymbol{z}) \cdot g_{\boldsymbol{A}}(\boldsymbol{z})\right)_\beta. \end{aligned}$$

Therefore, $(fg)_{\boldsymbol{A}}(\boldsymbol{z}) = f_{\boldsymbol{A}}(\boldsymbol{z}) \cdot g_{\boldsymbol{A}}(\boldsymbol{z})$. Analogously, we can show that this equation holds for $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$. Consequently,

$$(fg)_{\boldsymbol{A}} = f_{\boldsymbol{A}} \cdot g_{\boldsymbol{A}}.$$

Moreover, it is easy to check that for $\lambda, \mu \in \mathbb{C}$

$$(\lambda f + \mu g)_{\boldsymbol{A}} = \lambda f_{\boldsymbol{A}} + \mu g_{\boldsymbol{A}}.$$

Furthermore, we define the function $f^{\#}$ by $f^{\#}(\boldsymbol{z}) = \overline{f(\overline{\boldsymbol{z}})}$ for $\boldsymbol{z} \in \operatorname{dom} f$. Then

$$(f^{\#})_{\boldsymbol{A}} = (f_{\boldsymbol{A}})^{\#}.$$

Example 3.3.11. Let $i \in [1,n]_{\mathbb{Z}}$ be fixed and p_i be a real definitizing polynomial of A_i. Then we can regard p_i also as an element of $\mathbb{C}[z_1,\dots,z_n]$ just by setting $p_i(\boldsymbol{z}) = p_i(z_i)$. Clearly, $p_i : \mathbb{C}^n \to \mathbb{C}$ satisfies all conditions of Definition 3.3.9 and we can build $p_{i\boldsymbol{A}}$. Since $p_i(\boldsymbol{z})$ is constant in every direction z_k for $k \neq i$, every derivative in these directions vanishes. Moreover, for $\boldsymbol{z} \in Z_{\boldsymbol{p}}$

$$p_i^{(l)}(z_i) = 0 \quad \text{if} \quad l < \mathfrak{d}_{p_i}(z_i).$$

Thus, we can easily conclude that

- for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ we have $p_{i\boldsymbol{A}}(\boldsymbol{z}) = p_i(z_i)$,
- for $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$ we have $p_{i\boldsymbol{A}}(\boldsymbol{z}) = 0 \in \mathfrak{B}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}$ and
- for $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ we have $p_{i\boldsymbol{A}}(\boldsymbol{z}) = (p_{i\boldsymbol{A}}(\boldsymbol{z})_\beta)_{\beta \in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}$, where

$$\big(p_{i\boldsymbol{A}}(\boldsymbol{z})\big)_\beta = \begin{cases} 0, & \text{if } \beta \neq \mathfrak{d}_{p_i}(z_i)e_i, \\ \frac{1}{\mathfrak{d}_{p_i}(z_i)!} p^{\mathfrak{d}_{p_i}(z_i)}(z_i), & \text{if } \beta = \mathfrak{d}_{p_i}(z_i)e_i. \end{cases}$$

Furthermore, if we have a sufficiently smooth function f, then we can evaluate $(p_i f)_{\boldsymbol{A}}$ at $\boldsymbol{z} \in Z_{\boldsymbol{p}}$

$$\Big((p_i f)_{\boldsymbol{A}}(\boldsymbol{z})\Big)_\beta = \frac{1}{\beta!}(D^\beta p_i f)(\boldsymbol{z}) = \begin{cases} 0, & \text{if } \beta \neq \mathfrak{d}_{p_i}(z_i)e_i, \\ \frac{1}{\mathfrak{d}_{p_i}(z_i)!} p^{\mathfrak{d}_{p_i}(z_i)}(z_i) f(\boldsymbol{z}), & \text{if } \beta = \mathfrak{d}_{p_i}(z_i)e_i. \end{cases}$$

For $\sum_{k=1}^n p_k f$ we obtain

$$\Big(\Big(\sum_{k=1}^n p_k f\Big)_{\boldsymbol{A}}(\boldsymbol{z})\Big)_\beta = \begin{cases} 0, & \text{if } \forall i \in [1,n]_{\mathbb{Z}} : \beta \neq \mathfrak{d}_{p_i}(z_i)e_i, \\ \frac{1}{\mathfrak{d}_{p_i}(z_i)!} p^{\mathfrak{d}_{p_i}(z_i)}(z_i) f(\boldsymbol{z}), & \text{if } \exists i \in [1,n]_{\mathbb{Z}} : \beta = \mathfrak{d}_{p_i}(z_i)e_i. \end{cases}$$

Definition 3.3.12. Let $\boldsymbol{q} = (q_i)_{i=1}^n$ be a tuple of polynomials $q_i \in \mathbb{C}[z] \setminus \{0\}$ of positive degree $\deg q_i$. We will denote the space of all polynomials from $\mathbb{C}[z_1,\dots,z_n]$ with z_i-degree less than $\deg q_i$ for all $i \in [1,n]_{\mathbb{Z}}$ by $\mathcal{P}_{\boldsymbol{q}}$.

Lemma 3.3.13. *Let $\boldsymbol{q} = (q_i)_{i=1}^n$ be a tuple of polynomials $q_i \in \mathbb{C}[z] \setminus \{0\}$ of positive degree m_i for every $i \in [1,n]_{\mathbb{Z}}$, and set $m = \prod_{i=1}^n m_i$. By $Z_{\boldsymbol{q}}$ we denote the set of all joint zeros of $\boldsymbol{q}$ in $\mathbb{C}^n$; see Definition 3.3.6. Then any $s \in \mathbb{C}[z_1,\dots,z_n]$ can be written as*

$$s(\boldsymbol{z}) = \sum_{i=1}^n q_i(z_i) u_i(\boldsymbol{z}) + r(\boldsymbol{z})$$

with $u_i, r \in \mathbb{C}[z_1,\dots,z_n]$ for all $i \in [1,n]_{\mathbb{Z}}$ such that $r \in \mathcal{P}_{\boldsymbol{q}}$. Here u_i, r can be found in $\mathbb{R}[z_1,\dots,z_n]$ if $q_i \in \mathbb{R}[z]$ and $s \in \mathbb{R}[z_1,\dots,z_n]$.

Furthermore, for

$$\varpi : \begin{cases} \mathbb{C}[z_1, \dots, z_n] & \to & \mathbb{C}^m, \\ s & \mapsto & \left(\left(\frac{1}{\beta!} D^\beta s(\boldsymbol{z}) \right)_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{q}}(\boldsymbol{z})}} \right)_{\boldsymbol{z} \in Z_{\boldsymbol{q}}} \end{cases}$$

we have $s \in \ker \varpi$ *if and only if* $s(\boldsymbol{z}) = \sum_{i=1}^n q_i(z_i) u_i(\boldsymbol{z})$ *for some* $u_i \in \mathbb{C}[z_1, \dots, z_n]$ *for* $i \in [1, n]_{\mathbb{Z}}$. *Moreover,* ϖ *restricted to* $\mathcal{P}_{\boldsymbol{q}}$ *is bijective.*

Proof. Applying the Euclidean algorithm to $s \in \mathbb{C}[z_1, \dots, z_n]$ and q_1 we obtain $s(\boldsymbol{z}) = q_1(z_1)u_1(\boldsymbol{z}) + r_1(\boldsymbol{z})$ where $u_1, r_1 \in \mathbb{C}[z_1, \dots, z_n]$ such that the z_1-degree of r_1 is less than m_1. Let r_k be the polynomial we obtain when we apply the Euclidean algorithm to r_{k-1} and q_k. Then we get $r_{k-1}(\boldsymbol{z}) = q_k(z_k)u_k(\boldsymbol{z}) + r_k(\boldsymbol{z})$, where $u_k, r_k \in \mathbb{C}[z_1, \dots, z_n]$ such that for all $i \in [1, k-1]_{\mathbb{Z}}$ the z_i-degree of r_k is less than the z_i-degree of r_{k-1} and the z_k-degree is less than m_k.

By induction $r := r_n$ fulfills the desired properties and

$$s(\boldsymbol{z}) = \sum_{i=1}^n q_i(z_i) u_i(\boldsymbol{z}) + r(\boldsymbol{z})$$

The resulting polynomials $(u_i)_{i=1}^n, (r_i)_{i=1}^n$ belong to $\mathbb{R}[z_1, \dots, z_n]$ if $q_i \in \mathbb{R}[z]$ and $s \in \mathbb{R}[z_1, \dots, z_n]$.

The Leibniz rule ensures that $\varpi(q_i u_i) = 0$ for all $i \in [1, n]_{\mathbb{Z}}$. Hence, $\varpi(s) = \varpi(r)$. Consequently, $s \in \ker \varpi$ if $r = 0$. On the other hand, if $0 = \varpi(s) = \varpi(r)$ then we will show that r must be 0 by induction. At first we define the projection

$$\pi_l^k : \begin{cases} \mathbb{C}^n & \to & \mathbb{C}^{k-l+1}, \\ (z_i)_{i=1}^n & \mapsto & (z_i)_{i=l}^k \end{cases}$$

and the set $\hat{I}_\alpha^k := \{\beta \in \hat{I}_\alpha : \beta_i = 0 \; \forall i \in [1, k]_{\mathbb{Z}}\}$.

Induction hypothesis: For $k \in \mathbb{N}_0$, $k \leq n$, for all $(w_i)_{i=k+1}^n \in \pi_{k+1}^n(Z_{\boldsymbol{q}})$, all $\beta \in \hat{I}_\alpha^k$ and all $(x_i)_{i=1}^k \in \mathbb{C}^k$ we have

$$D^\beta r(x_1, \dots, x_k, w_{k+1}, \dots, w_n) = 0.$$

Induction start: For $k = 0$ the induction hypothesis is nothing else than $\varpi(r) = 0$.

Induction step: Assuming that the induction hypothesis is satisfied by k for arbitrary $(w_i)_{i=k+1}^n \in \pi_{k+1}^n(Z_{\boldsymbol{q}})$, $\beta \in \hat{I}_\alpha^{k+1}$ and $(x_i)_{i=1}^k \in \mathbb{C}^k$ the mapping

$$x \mapsto D^\beta r(x_1, \dots, x_k, x, w_{k+2}, \dots, w_m)$$

has zeros at $x \in Z_{q_{k+1}}$ with multiplicity at least $\mathfrak{d}_{q_{k+1}}(x)$. Since this mapping is a polynomial of degree less than $m_{k+1} = \deg q_{k+1} = \sum_{x \in Z_{q_{k+1}}} \mathfrak{d}_{q_{k+1}}(x)$, it must be identically equal to zero. Hence $k+1$ fulfills the induction hypothesis.

This proves that $r = 0$.

Our discription of $\ker \varpi$ shows in particular that ϖ restricted to $\mathcal{P}_{\boldsymbol{q}}$ is one-to-one. Comparing dimensions shows that this restriction of ϖ is also onto.

❑

Corollary 3.3.14. *For every* $\phi \in \mathcal{M}_{\boldsymbol{A}}$ *there exists an* $s \in \mathbb{C}[z_1, \dots, z_n]$ *such that* $\phi - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$

Proof. The mapping ϖ from Lemma 3.3.13 is bijective. Hence there exists an $s \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\varpi(s)_{\boldsymbol{w}} = \pi_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}(\phi(\boldsymbol{w}))$ for every $\boldsymbol{w} \in Z_{\boldsymbol{p}}$. As a consequence we obtain $\phi - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$.

❏

Example 3.3.15. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and assume that $Z_{\boldsymbol{p}}^{\mathbb{R}} = \{\boldsymbol{w}\}$. Then we can write

$$\begin{aligned} f(\boldsymbol{z}) &= \sum_{\beta \in \mathbb{N}_0^n} \frac{1}{\beta!} D^\beta f(\boldsymbol{w})(\boldsymbol{z} - \boldsymbol{w})^\beta \\ &= \underbrace{\sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w})(\boldsymbol{z} - \boldsymbol{w})^\beta}_{=:s(\boldsymbol{z})} + \sum_{\beta \in \hat{I}^{\mathfrak{c}}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w})(\boldsymbol{z} - \boldsymbol{w})^\beta \end{aligned}$$

It is easy to see that $f_{\boldsymbol{A}} - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$. We can rewrite this equation as

$$f(\boldsymbol{z}) = s(\boldsymbol{z}) + \sum_{i=1}^n p_i(z_i) \underbrace{\frac{\sum_{\beta \in \hat{I}^{\mathfrak{c}}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w})(\boldsymbol{z} - \boldsymbol{w})^\beta}{\sum_{i=1}^n p_i(z_i)}}_{=:g(\boldsymbol{z})}$$

for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus \{\boldsymbol{w}\}$. This representation is well-defined, since denominator of $g(\boldsymbol{z})$ can only be zero for $\boldsymbol{z} = \boldsymbol{w}$; see Lemma 3.2.3. If we could extent g to $\{\boldsymbol{w}\}$, we would have a useful decomposition of f. Unfortunately, in general this is not possible, since $\lim_{\boldsymbol{z} \to \boldsymbol{w}} g(\boldsymbol{z})$ may not exist. For example by L'Hôpital's rule we have

$$\lim_{t \to 0} g(\boldsymbol{w} + te_i) = \frac{D^{\mathfrak{d}_{p_i}(w_i)e_i} f(\boldsymbol{w})}{p^{(\mathfrak{d}_{p_i}(w_i))}(w_i)} = \frac{\mathfrak{d}_{p_i}(w_i)! f_{\boldsymbol{A}}(\boldsymbol{w})_{\mathfrak{d}_{p_i}(w_i)e_i}}{p^{(\mathfrak{d}_{p_i}(w_i))}(w_i)}$$

which does not coincide for every $i \in [1, n]_{\mathbb{Z}}$ in general. If $g(\boldsymbol{w})$ would exist, then we could compute $(f_{\boldsymbol{A}} - s_{\boldsymbol{A}})(\boldsymbol{w})_\beta$, according to Example 3.3.11, in the following way

$$\begin{aligned} (f_{\boldsymbol{A}} - s_{\boldsymbol{A}})(\boldsymbol{w})_\beta &= \frac{1}{\beta} D^\beta \Big(\sum_{i=1}^n p_i g \Big)(\boldsymbol{w}) \\ &= \begin{cases} 0, & \text{if } \beta \neq \mathfrak{d}_{p_i}(w_i)e_i, \\ \dfrac{p^{\mathfrak{d}_{p_i}(w_i)}(w_i)}{\mathfrak{d}_{p_i}(w_i)!} g(\boldsymbol{w}), & \text{if } \exists i \in [1, n]_{\mathbb{Z}} : \beta = \mathfrak{d}_{p_i}(w_i)e_i. \end{cases} \end{aligned}$$

This would lead us to the equations

$$\frac{1}{\mathfrak{d}_{p_i}(w_i)!} D^{\mathfrak{d}_{p_i}(w_i)e_i} f(\boldsymbol{w}) = \frac{p^{\mathfrak{d}_{p_i}(w_i)}(w_i)}{\mathfrak{d}_{p_i}(w_i)!} g(\boldsymbol{w}) \quad \text{for all} \quad i \in [1, n]_{\mathbb{Z}}.$$

This motivates the following Remark

Remark 3.3.16. Recall from Lemma 3.2.3 that $\sum_{i=1}^n p_i(z_i) = 0$ with $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))$ implies $p_i(z_i) = 0$ for all $i \in [1, n]_{\mathbb{Z}}$, i.e. $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$.

If $\phi \in \mathcal{R}_{\boldsymbol{A}}$, then we find a function g on $\sigma(\Theta(\boldsymbol{A}))$ with

$$g(\boldsymbol{z}) \in \begin{cases} \mathbb{C}, & \text{if } \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \mathbb{C}^n, & \text{if } \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}. \end{cases}$$

such that $\phi(\boldsymbol{z}) = \sum_{i=1}^n p_{i\boldsymbol{A}}(z_i) \cdot g(\boldsymbol{z})$ for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))$, where the multiplication is defined as the multiplication in $\mathbb{C}$ in the case that $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$, and as

$$\Big(\sum_{i=1}^n p_{i\boldsymbol{A}}(z_i) \cdot g(\boldsymbol{z})\Big)_\beta := \begin{cases} 0, & \text{if } \beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}, \\ \big(p_{j\boldsymbol{A}}(z_j)\big)_{\mathfrak{d}_{p_j}(z_j)e_j} g(\boldsymbol{z})_j, & \text{if } \beta = \mathfrak{d}_{p_j}(z_j)e_j, \end{cases}$$

otherwise. The desired function is defined by $g(\boldsymbol{z}) := \frac{\phi(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)}$ for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ and

$$g(\boldsymbol{z})_i := \frac{\mathfrak{d}_{p_i}(z_i)!\,\phi(\boldsymbol{z})_{\mathfrak{d}_{p_i}(z_i)e_i}}{p_i^{(\mathfrak{d}_{p_i}(z_i))}(z_i)} \quad \text{for} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$$

for every $i \in [1,n]_{\mathbb{Z}}$.

Remark 3.3.17. If the tuple $\boldsymbol{A}$ contains only one single operator A (i.e. $n = 1$), then Example 3.3.15 would work and Remark 3.3.16 would give a $\mathbb{C}$-valued function g.

Definition 3.3.18. With the notation from Definition 3.3.6 we denote by $\mathcal{F}_{\boldsymbol{A}}$ the set of all $\phi \in \mathcal{M}_{\boldsymbol{A}}$ such that $\boldsymbol{z} \mapsto \phi(\boldsymbol{z})$ is Borel measurable and bounded on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$, and such that for each $\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$, which is not isolated in $\sigma(\Theta(\boldsymbol{A}))$

$$\frac{\phi(\boldsymbol{z}) - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta}{\max_{k \in [1,n]_{\mathbb{Z}}} |z_k - w_k|^{\mathfrak{d}_{p_k}(w_k)}} \tag{3.10}$$

is bounded for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap B_r(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$, where $r > 0$ is sufficiently small.

Example 3.3.19. Let $\boldsymbol{w} \in Z_{\boldsymbol{p}}$ be an isolated point of $\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}$, $a \in \mathcal{M}_{\boldsymbol{A}}$ and $\delta_{\boldsymbol{w}} : \sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}} \to \mathbb{C}$ defined by

$$\delta_{\boldsymbol{w}}(\boldsymbol{z}) := \begin{cases} 1, & \text{if } \boldsymbol{z} = \boldsymbol{w}, \\ 0, & \text{else.} \end{cases}$$

Then $\delta_{\boldsymbol{w}} a$ defined by $\delta_{\boldsymbol{w}} a(\boldsymbol{z}) := \delta_{\boldsymbol{w}}(\boldsymbol{z}) a(\boldsymbol{z})$ is an element of $\mathcal{F}_{\boldsymbol{A}}$. Cleary, every element of $Z_{\boldsymbol{p}}^{\mathrm{i}}$ is isolated in $\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}$.

Example 3.3.20. Let h be defined on an open subset D of $\mathbb{R}^n$ with values in $\mathbb{C}$ and let $\boldsymbol{w} \in D$. Moreover assume that for $\alpha \in \mathbb{N}^n$ the function h is $|\alpha| - n + 1$ times continuously differentiable. The Taylor Approximation Theorem from multidimensional calculus yields [4, 10.2.10 and 10.2.13]

$$h(\boldsymbol{z}) = \sum_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \le |\alpha| - n}} \frac{1}{\beta!} D^\beta h(\boldsymbol{w})(\boldsymbol{z}-\boldsymbol{w})^\beta + O\big(\, \|\boldsymbol{z}-\boldsymbol{w}\|_\infty^{|\alpha|-n+1} \big)$$

for $\boldsymbol{z} \to \boldsymbol{w}$. Since $\alpha_k \geq 1$ for all $k \in [1,n]_{\mathbb{Z}}$, we conclude that $|\alpha| - n + 1 \geq \alpha_i$ for every $i \in [1,n]_{\mathbb{Z}}$ which leads to

$$\|\boldsymbol{z} - \boldsymbol{w}\|_\infty^{|\alpha|-n+1} = \max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{|\alpha|-n+1} = O\Big(\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\alpha_i}\Big)$$

If $\|\boldsymbol{z} - \boldsymbol{w}\|_\infty \leq 1$ and if there exists a $k \in [1,n]_{\mathbb{Z}}$ such that $\beta_k \geq \alpha_k$, then

$$\big|(\boldsymbol{z} - \boldsymbol{w})^\beta\big| \leq |z_k - w_k|^{\beta_k} \leq |z_k - w_k|^{\alpha_k} \leq \max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\alpha_i}.$$

Hence, $(\boldsymbol{z} - \boldsymbol{w})^\beta$ is also an $O(\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\alpha_i})$ if there exists an $k \in [1,n]_{\mathbb{Z}}$ such that $\beta_k \geq \alpha_k$. This yields

$$h(\boldsymbol{z}) = \sum_{\beta\in\hat{I}_\alpha} \frac{1}{\beta!} D^\beta h(\boldsymbol{w})(\boldsymbol{z} - \boldsymbol{w})^\beta + O\Big(\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\alpha_i}\Big).$$

Lemma 3.3.21. *Let $f : \operatorname{dom} f \to \mathbb{C}$ be a function with the properties mentioned in Definition 3.3.9. Then $f_{\boldsymbol{A}}$ belongs to $\mathcal{F}_{\boldsymbol{A}}$.*

Proof. For a fixed $\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$ which is non-isolated and an arbitrary $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ by Example 3.3.20 the expression

$$f_{\boldsymbol{A}}(\boldsymbol{z}) - \sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} f_{\boldsymbol{A}}(\boldsymbol{w})_\beta(\boldsymbol{z} - \boldsymbol{w})^\beta = f(\boldsymbol{z}) - \sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w})(\boldsymbol{z} - \boldsymbol{w})^\beta$$

is an $O(\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\alpha_i})$ for $\boldsymbol{z} \to \boldsymbol{w}$. Therefore, $f_{\boldsymbol{A}} \in \mathcal{F}_{\boldsymbol{A}}$.

❑

Lemma 3.3.22. *If $\phi \in \mathcal{F}_{\boldsymbol{A}}$ is such that $\phi(\boldsymbol{z})$ is invertible in $\mathfrak{C}(\boldsymbol{z})$ for all $\boldsymbol{z} \in (\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}^{\mathbb{R}}) \dot\cup Z_{\boldsymbol{p}}^{\mathrm{i}}$ and such that 0 does not belong to the closure of $\phi(\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}})$, then $\phi^{-1} : \boldsymbol{z} \mapsto \phi(\boldsymbol{z})^{-1}$ also belongs to $\mathcal{F}_{\boldsymbol{A}}$.*

Proof. Since 0 is not in $\overline{\phi(\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}})}$ the mapping $\boldsymbol{z} \mapsto \frac{1}{\phi(\boldsymbol{z})}$ is bounded on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$. By the first assumption ϕ^{-1} is a well-defined object belonging to $\mathcal{M}_{\boldsymbol{A}}$. Since ϕ is measurable on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ also $\boldsymbol{z} \mapsto \frac{1}{\phi(\boldsymbol{z})}$ is measurable on this set.

It remains to verify the boundedness of (3.10) on a certain neighborhood of $\boldsymbol{w}$ for each $\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$ for ϕ^{-1}, when $\boldsymbol{w}$ is non-isolated in $\sigma(\Theta(\boldsymbol{A}))$. For $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ we calculate

$$\begin{aligned}
\phi^{-1}(\boldsymbol{z}) &- \sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi^{-1}(\boldsymbol{w})\big)_\beta(\boldsymbol{z} - \boldsymbol{w})^\beta \\
&= \frac{1}{\phi(\boldsymbol{z})} - \frac{1}{\sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi(\boldsymbol{w})\big)_\beta(\boldsymbol{z} - \boldsymbol{w})^\beta} && (3.11)\\
&\quad + \frac{1}{\sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi(\boldsymbol{w})\big)_\beta(\boldsymbol{z} - \boldsymbol{w})^\beta} - \sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi^{-1}(\boldsymbol{w})\big)_\beta(\boldsymbol{z} - \boldsymbol{w})^\beta. && (3.12)
\end{aligned}$$

The term (3.11) can be written as

$$-\frac{1}{\phi(\boldsymbol{z})} \cdot \frac{1}{\sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta} \cdot \Big(\phi(\boldsymbol{z}) - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta \Big).$$

By assumption $\frac{1}{\phi(\boldsymbol{z})}$ is bounded and $\phi(\boldsymbol{z}) - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta$ is an $O(\max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)})$. The invertibility of $\phi(\boldsymbol{w})$ guarantees $\big(\phi(\boldsymbol{w})^{-1}\big)_0 \neq 0$, which yields

$$\frac{1}{\sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta} = O(1)$$

for $\boldsymbol{z} \to \boldsymbol{w}$. Thus, (3.11) is an $O(\max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)})$.

Factoring out $\frac{1}{\sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} (\phi(\boldsymbol{w}))_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta}$ from (3.12) results in

$$\underbrace{\frac{1}{\sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta}}_{=O(1)} \Bigg(1 - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} \underbrace{\sum_{\gamma_1+\gamma_2=\beta} \big(\phi(\boldsymbol{w})\big)_{\gamma_1} \big(\phi(\boldsymbol{w})^{-1}\big)_{\gamma_2}}_{=e_\beta} (\boldsymbol{z}-\boldsymbol{w})^\beta$$
$$\underbrace{- \sum_{\beta \in J} \sum_{\gamma_1+\gamma_2=\beta} \big(\phi(\boldsymbol{w})\big)_{\gamma_1} \big(\phi(\boldsymbol{w})^{-1}\big)_{\gamma_2} (\boldsymbol{z}-\boldsymbol{w})^\beta \Bigg)}_{=O(\max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)})}$$

where $J := \{\gamma_1 + \gamma_2 \in \mathbb{N}_0^n : \gamma_1, \gamma_2 \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}} \text{ and } \gamma_1 + \gamma_2 \notin \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}\}$ and e is the multiplicative unit of $\mathfrak{B}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}$. Since $\sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}(\boldsymbol{w})}}} e_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta = 1$, we see that (3.12) is an $O(\max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)})$. Consequently, $\phi^{-1} \in \mathcal{F}_{\boldsymbol{A}}$.

❑

3.4 The Spectral Theorem

Lemma 3.4.1. *For every $\phi \in \mathcal{F}_{\boldsymbol{A}}$ there exists a polynomial $s \in \mathbb{C}[z_1, \dots, z_n]$ and a function g on $\sigma(\Theta(\boldsymbol{A}))$ with values in $\mathbb{C}$ on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ and values in $\mathbb{C}^n$ on $\sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$ such that $\phi - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$, g is bounded and measurable on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$, and*

$$\phi(\boldsymbol{z}) = s_{\boldsymbol{A}}(\boldsymbol{z}) + \sum_{i=1}^n p_{i\boldsymbol{A}}(z_i) \cdot g(\boldsymbol{z}) \quad \text{for} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))\,, \tag{3.13}$$

where the multiplication has to be understood in the sense of Remark 3.3.16. We will call such a pair s, g a decomposition of ϕ.

Proof. According to Corollary 3.3.14 there exists an $s \in \mathbb{C}[z_1, \dots, z_n]$ such that $\phi - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$, and by Remark 3.3.16 we then find a function g such that (3.13) holds true. The measurability of

$$g(\boldsymbol{z}) = \frac{\phi(\boldsymbol{z}) - s(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)} \quad \text{on} \quad \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$$

follows from the assumption $\phi \in \mathcal{F}_{\boldsymbol{A}}$; in particular from the measurability of ϕ itself.

In order to show g's boundedness, first recall from Lemma 3.2.3 that

$$\max_{i\in[1,n]_{\mathbb{Z}}} |p_i(z_i)| \le \max_{i\in[1,n]_{\mathbb{Z}}} \|R_i R_i^*\| \left|\sum_{i=1}^{n} p_i(z_i)\right| \quad \text{for} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))\,.$$

Hence, for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ we have

$$\frac{\max_{i\in[1,n]_{\mathbb{Z}}} |p_i(z_i)|}{\left|\sum_{i=1}^{n} p_i(z_i)\right|} \le \max_{i\in[1,n]_{\mathbb{Z}}} \|R_i R_i^*\|\,.$$

As $\phi \in \mathcal{F}_{\boldsymbol{A}}$ for each $\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$ which non-isolated in $\sigma(\Theta(\boldsymbol{A}))$ we find an open neighborhood $B_{r_{\boldsymbol{w}}}(\boldsymbol{w})$ of $\boldsymbol{w}$ such that (3.10) is bounded for $\boldsymbol{z} \in B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$. Clearly, we can choose $r_{\boldsymbol{w}}$ even smaller such that the family of neighborhoods is pairwise disjoint. For $\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$ and for each $i \in [1,n]_{\mathbb{Z}}$ the number w_i is real and a zero of p_i with multiplicity $\mathfrak{d}_{p_i}(w_i)$. Therefore

$$|p_i(z_i)| = \left|a_{\mathfrak{d}_{p_i}(w_i)}(z_i - w_i)^{\mathfrak{d}_{p_i}(w_i)} + O\Big((z_i - w_i)^{\mathfrak{d}_{p_i}(w_i)+1}\Big)\right| \ge c_i |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}$$

for $c_i > 0$ and $\boldsymbol{z} \in B_{r_{\boldsymbol{w}}}(\boldsymbol{w})$. Hence,

$$\frac{\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}}{\max_{i\in[1,n]_{\mathbb{Z}}} |p_i(z_i)|} \le C_{\boldsymbol{w}}$$

on $\sigma(\Theta(\boldsymbol{A})) \cap B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$ for some $C_{\boldsymbol{w}} > 0$. Since s is holomorphic as a polynomial and $\phi - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$ implies $\phi(\boldsymbol{w})_\beta = \frac{1}{\beta!} D^\beta s(\boldsymbol{w})$ for $\boldsymbol{w} \in Z_{\boldsymbol{p}}$ and $\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}$, we have

$$s(\boldsymbol{z}) = \sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \phi(\boldsymbol{w})_\beta (\boldsymbol{z} - \boldsymbol{w})^\beta + O(\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)})$$

and in consequence of the choice of $B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$ (see (3.10))

$$\frac{|\phi(\boldsymbol{z}) - s(\boldsymbol{z})|}{\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}} \le D_{\boldsymbol{w}}$$

for some $D_{\boldsymbol{w}} > 0$ and $\boldsymbol{z} \in B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$. Altogether

$$|g(\boldsymbol{z})| = \underbrace{\frac{\max_{i\in[1,n]_{\mathbb{Z}}} |p_i(z_i)|}{\left|\sum_{i=1}^{n} p_i(z_i)\right|}}_{\le \max_{i\in[1,n]_{\mathbb{Z}}} \|R_i R_i^*\|} \underbrace{\frac{\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}}{\max_{i\in[1,n]_{\mathbb{Z}}} |p_i(z_i)|}}_{\le C_{\boldsymbol{w}}} \underbrace{\frac{|\phi(\boldsymbol{z}) - s(\boldsymbol{z})|}{\max_{i\in[1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}}}_{\le D_{\boldsymbol{w}}}\,.$$

This leads us to the boundedness of g on $\sigma(\Theta(\boldsymbol{A})) \cap \bigcup_{\boldsymbol{w}\in Z_{\boldsymbol{p}}^{\mathbb{R}}} B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$. On $\sigma(\Theta(\boldsymbol{A})) \setminus \bigcup_{\boldsymbol{w}\in Z_{\boldsymbol{p}}^{\mathbb{R}}} B_{r_{\boldsymbol{w}}}(\boldsymbol{w})$ the boundedness is clear. Hence g is bounded on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$.

❑

Definition 3.4.2. For every $\phi \in \mathcal{F}_{\boldsymbol{A}}$ we define

$$\phi(\boldsymbol{A}) := s(\boldsymbol{A}) + \Xi\Big(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \, \mathrm{d}E \Big)$$

where s, g is a decomposition of ϕ in the sense of Lemma 3.4.1, and where

$$\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \, \mathrm{d}E := \int_{\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} g \, \mathrm{d}E \;\; + \sum_{w \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}} \sum_{i=1}^{n} g(w)_i R_i R_i^* E\{w\}$$

Remark 3.4.3. For a one-tuple $\boldsymbol{A} = (A)$ the corresponding mapping R fulfills $RR^* = I$. Moreover the function g of the decomposition has only $\mathbb{C}$ as range. Hence, we can write

$$\phi(A) = s(A) + \int_{\sigma(\Theta(A))} g \, \mathrm{d}E.$$

At first we have to guarantee that $\phi(\boldsymbol{A})$ is well-defined.

Theorem 3.4.4. *Let $\phi \in \mathcal{F}_{\boldsymbol{A}}$, s, g and $\tilde{s}, \tilde{g}$ be decompositions of ϕ in the sense of Lemma 3.4.1. Then*

$$s(\boldsymbol{A}) + \Xi\Big(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \, \mathrm{d}E \Big) = \tilde{s}(\boldsymbol{A}) + \Xi\Big(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} \tilde{g} \, \mathrm{d}E \Big)$$

Proof. By assumption we have $\phi - s_{\boldsymbol{A}}, \phi - \tilde{s}_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$. Subtracting these functions yields $\tilde{s}_{\boldsymbol{A}} - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$ and consequently $\varpi(\tilde{s}_{\boldsymbol{A}} - s_{\boldsymbol{A}}) = 0$ for ϖ as in Lemma 3.3.13. Since $\tilde{s}_{\boldsymbol{A}} - s_{\boldsymbol{A}} \in \ker \varpi$, this Lemma implies

$$s(\boldsymbol{z}) - \tilde{s}(\boldsymbol{z}) = \sum_{i=1}^{n} p_i(z_i) u_i(\boldsymbol{z}) \tag{3.14}$$

for some $(u_i)_{i=1}^n$ where $u_i \in \mathbb{C}[z_1, \dots, z_n]$.

By Lemma 2.3.16 and $T_i T_i^+ = p_i(A_i)$ we have

$$\Xi_i(u_i(\Theta_i(\boldsymbol{A}))) = \Xi_i(\Theta_i(u_i(\boldsymbol{A}))) = p_i(A_i) u_i(\boldsymbol{A}) \tag{3.15}$$

for every $i \in [1, n]_{\mathbb{Z}}$. Recall the notation from Corollary 3.1.5 for the operator tuple $\boldsymbol{A}$. Since $u(\Theta_i(\boldsymbol{A})) = \int u_i \, \mathrm{d}E^i$, we obtain

$$\Xi_i(u_i(\Theta_i(\boldsymbol{A}))) = \Xi_i\Big(\int u_i \, \mathrm{d}E^i \Big) \overset{(3.7)}{=} \Xi\Big(R_i R_i^* \int u_i \, \mathrm{d}E \Big). \tag{3.16}$$

for all $i \in [1, n]_{\mathbb{Z}}$. This leads to

$$\tilde{s}(\boldsymbol{A}) - s(\boldsymbol{A}) = \sum_{i=1}^{n} p_i(A_i) u_i(\boldsymbol{A}) \overset{(3.15)}{=} \sum_{i=1}^{n} \Xi_i(u_i(\boldsymbol{A})) \overset{(3.16)}{=} \Xi\Big(\sum_{i=1}^{n} R_i R_i^* \int u_i \, \mathrm{d}E \Big)$$

By Corollary 3.2.5, we have

$$\tilde{s}(\boldsymbol{A}) - s(\boldsymbol{A}) = \tag{3.17}$$
$$\Xi\left(\int\limits_{\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} \frac{\sum_{i=1}^n p_i(z_i)u_i(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)}\,\mathrm{d}E(\boldsymbol{z}) + \sum_{\boldsymbol{w}\in\sigma(\Theta(\boldsymbol{A}))\cap Z_{\boldsymbol{p}}^{\mathbb{R}}} \sum_{i=1}^n R_iR_i^* u_i(\boldsymbol{w})E\{\boldsymbol{w}\}\right).$$

On the other hand, since both s, g and $\tilde{s}, \tilde{g}$ are decompositions of ϕ in sense of Lemma 3.4.1 we have

$$(\tilde{s}_{\boldsymbol{A}} - s_{\boldsymbol{A}})(\boldsymbol{z}) = \sum_{i=1}^n p_{i\boldsymbol{A}}(z_i)\cdot(g(\boldsymbol{z}) - \tilde{g}(\boldsymbol{z})) \quad \text{for} \quad \boldsymbol{z}\in\sigma(\Theta(\boldsymbol{A})) \tag{3.18}$$

In particular, for $\boldsymbol{z}\in\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$

$$\sum_{i=1}^n p_i(z_i)u_i(\boldsymbol{z}) \overset{(3.14)}{=} \tilde{s}(\boldsymbol{z}) - s(\boldsymbol{z}) = \sum_{i=1}^n p_i(z_i)(g(\boldsymbol{z}) - \tilde{g}(\boldsymbol{z}))$$

and in turn

$$(g(\boldsymbol{z}) - \tilde{g}(\boldsymbol{z})) = \frac{\sum_{i=1}^n p_i(z_i)u_i(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)}.$$

Considering the entries with index $\mathfrak{d}_{p_i}(z_i)e_i$ of (3.18) and (3.14) multiplied by $\mathfrak{d}_{p_i}(z_i)!$ for $\boldsymbol{z}\in\sigma(\Theta(\boldsymbol{A}))\cap Z_{\boldsymbol{p}}^{\mathbb{R}}$, we obtain

$$p_i^{(\mathfrak{d}_{p_i}(z_i))}(z_i)u_i(\boldsymbol{z}) = \frac{\partial^{\mathfrak{d}_{p_i}(z_i)}}{\partial z^{\mathfrak{d}_{p_i}(z_i)}}\big(\tilde{s}(\boldsymbol{z}) - s(\boldsymbol{z})\big) = p_i^{(\mathfrak{d}_{p_i}(z_i))}(z_i)\big(g(\boldsymbol{z})_i - \tilde{g}(\boldsymbol{z})_i\big),$$

where we used the general Leibniz rule for derivatives and the fact that z_i is a zero of p_i with multiplicity $\mathfrak{d}_{p_i}(z_i)$ for the left-hand-side. Since $p_i^{(\mathfrak{d}_{p_i}(z_i))}(z_i)$ does not vanish, we conlcude $u_i(\boldsymbol{z}) = g(\boldsymbol{z})_i - \tilde{g}(\boldsymbol{z})_i$ for $i\in[1,n]_{\mathbb{Z}}$. Therefore, we can write (3.17) as

$$\tilde{s}(\boldsymbol{A}) - s(\boldsymbol{A}) = \Xi\left(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} (g - \tilde{g})\,\mathrm{d}E\right)$$

and showing the asserted equality.

❏

Lemma 3.4.5. *Let $\phi_1, \phi_2 \in \mathcal{F}_{\boldsymbol{A}}$, s_1, g_1 a decomposition of ϕ_1 and s_2, g_2 a decomposition of ϕ_2 in the sense of Lemma 3.4.1. Then*

$$s(\boldsymbol{z}) = s_1(\boldsymbol{z})s_2(\boldsymbol{z}),$$
$$g(\boldsymbol{z}) = s_1(\boldsymbol{z})g_2(\boldsymbol{z}) + s_2(\boldsymbol{z})g_1(\boldsymbol{z}) + \sum_{i=1}^n p_i(z_i)g_1(\boldsymbol{z})g_2(\boldsymbol{z})$$

for $\boldsymbol{z}\in\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ and

$$g(\boldsymbol{z})_i = g_1(\boldsymbol{z})_i s_2(\boldsymbol{z}) + g_2(\boldsymbol{z})_i s_1(\boldsymbol{z}) \quad \text{for all} \quad i\in[1,n]_{\mathbb{Z}}$$

for $\boldsymbol{z}\in\sigma(\Theta(\boldsymbol{A}))\cap Z_{\boldsymbol{p}}^{\mathbb{R}}$, is a decomposition of $\phi_1\cdot\phi_2$.

Proof. Clearly, g is bounded and measurable for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}$ because g_1 and g_2 have these properties. Since $\mathcal{R}_{\boldsymbol{A}}$ is an ideal we obtain

$$\phi_1\phi_2 - s_{1\boldsymbol{A}}s_{2\boldsymbol{A}} = (\phi_1 - s_{1\boldsymbol{A}})\phi_2 + (\phi_2 - s_{2\boldsymbol{A}})s_{1\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$$

Since for $k = 1, 2$ the pair s_k, g_k is a decomposition of ϕ_k, we have

$$g_k(\boldsymbol{z}) = \frac{\phi_k(\boldsymbol{z}) - s_k(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)} \quad \text{for all} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}.$$

Therefore, we can rewrite $g(\boldsymbol{z})$ for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}$ as

$$\frac{s_1(\boldsymbol{z})(\phi_2(\boldsymbol{z})-s_2(\boldsymbol{z}))}{\sum_{i=1}^n p_i(z_i)} + \frac{s_2(\boldsymbol{z})(\phi_1(\boldsymbol{z})-s_1(\boldsymbol{z}))}{\sum_{i=1}^n p_i(z_i)} + \frac{(\phi_1(\boldsymbol{z})-s_1(\boldsymbol{z}))(\phi_2(\boldsymbol{z})-s_2(\boldsymbol{z}))}{\sum_{i=1}^n p_i(z_i)}.$$

After expanding the terms, this simplifies to

$$g(\boldsymbol{z}) = \frac{(\phi_1\phi_2)(\boldsymbol{z}) - (s_1s_2)(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)}.$$

For $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap Z^{\mathbb{R}}_{\boldsymbol{p}}$ we have

$$g_k(\boldsymbol{z})_i = \frac{\mathfrak{d}_{p_i}(z_i)!(\phi_k(\boldsymbol{z}) - s_{k\boldsymbol{A}}(\boldsymbol{z}))_{\mathfrak{d}_{p_i}(z_i)e_i}}{p_i^{(\mathfrak{d}_{p_i}(z_i))}(z_i)}.$$

Let $r = \mathfrak{d}_{p_i}(z_i)$ and $\beta = re_i$. Then we have

$$\begin{aligned} g(\boldsymbol{z})_i &= \frac{r!}{p_i^{(r)}(z_i)}\Big((\phi_1(\boldsymbol{z}) - s_{1\boldsymbol{A}}(\boldsymbol{z}))_\beta s_2(\boldsymbol{z}) + (\phi_2(\boldsymbol{z}) - s_{2\boldsymbol{A}}(\boldsymbol{z}))_\beta s_1(\boldsymbol{z})\Big) \\ &= \frac{r!}{p_i^{(r)}(z_i)}\Big(\phi_1(\boldsymbol{z})_\beta s_2(\boldsymbol{z}) - s_{1\boldsymbol{A}}(\boldsymbol{z})_\beta s_2(\boldsymbol{z}) + \phi_2(\boldsymbol{z})_\beta s_1(\boldsymbol{z}) - s_{2\boldsymbol{A}}(\boldsymbol{z})_\beta s_1(\boldsymbol{z})\Big). \end{aligned}$$

Note that $\phi_k(\boldsymbol{z})_0 = s_k(\boldsymbol{z}) = s_{k\boldsymbol{A}}(\boldsymbol{z})_0$ for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap Z^{\mathbb{R}}_{\boldsymbol{p}}$. Hence,

$$\begin{aligned} g(\boldsymbol{z})_i = \frac{r!}{p_i^{(r)}(z_i)}\Big(&\phi_1(\boldsymbol{z})_\beta \phi_2(\boldsymbol{z})_0 + \phi_2(\boldsymbol{z})_\beta \phi_1(\boldsymbol{z})_0 \\ &- s_{1\boldsymbol{A}}(\boldsymbol{z})_\beta s_{2\boldsymbol{A}}(\boldsymbol{z})_0 - s_{2\boldsymbol{A}}(\boldsymbol{z})_\beta s_{1\boldsymbol{A}}(\boldsymbol{z})_0\Big). \end{aligned}$$

Recall the definition of multiplication in $\mathfrak{A}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}$.

$$\begin{aligned} g(\boldsymbol{z})_i &= \frac{r!}{p_i^{(r)}(z_i)}\Big(\big(\phi_1(\boldsymbol{z}) \cdot \phi_2(\boldsymbol{z})\big)_\beta - \big(s_{1\boldsymbol{A}}(\boldsymbol{z}) \cdot s_{2\boldsymbol{A}}(\boldsymbol{z})\big)_\beta\Big) \\ &= \frac{r!}{p_i^{(r)}(z_i)}\Big((\phi_1 \cdot \phi_2)(\boldsymbol{z}) - s_{\boldsymbol{A}}(\boldsymbol{z})\Big)_\beta. \end{aligned}$$

This justifies that s, g is a decomposition of $\phi_1 \cdot \phi_2$ in the sense of Lemma 3.4.1. ❑

Theorem 3.4.6. *The mapping $\phi \mapsto \phi(\boldsymbol{A})$ defined in Definition 3.4.2 constitutes a $*$-homomorphism from $\mathcal{F}_{\boldsymbol{A}}$ into $\boldsymbol{A}'' \subseteq L_{\mathrm{b}}(\mathcal{K})$ such that $s_{\boldsymbol{A}}(\boldsymbol{A}) = s(\boldsymbol{A})$ for every polynomial $s \in \mathbb{C}[z_1, \ldots, z_n]$.*

Proof. As $s_{\boldsymbol{A}} = s_{\boldsymbol{A}} + \sum_{i=1}^n p_{i\boldsymbol{A}} \cdot 0$ Theorem 3.4.4 yields $s_{\boldsymbol{A}}(\boldsymbol{A}) = s(\boldsymbol{A})$ for all $s \in \mathbb{C}[z_1, \ldots, z_n]$.

Let $\phi_1, \phi_2 \in \mathcal{F}_{\boldsymbol{A}}$. According to Lemma 3.4.1 we find $s_1, s_2 \in \mathbb{C}[z_1, \ldots, z_n]$ and g_1, g_2 such that $\phi_k - s_{k\boldsymbol{A}} \in \mathcal{R}$, g_k is bounded and measurable on $\sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$, and

$$\phi_k(\boldsymbol{z}) = s_{k\boldsymbol{A}}(\boldsymbol{z}) + \sum_{i=1}^n p_{i\boldsymbol{A}}(z_i) \cdot g_k(\boldsymbol{z}) \quad \text{for} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \quad \text{and} \quad k = 1, 2.$$

For $\lambda, \mu \in \mathbb{C}$ Remark 3.3.10 guarantees $(\lambda s_1 + \mu s_2)_{\boldsymbol{A}} = \lambda s_{1\boldsymbol{A}} + \mu s_{2\boldsymbol{A}}$ and therefore

$$(\lambda\phi_1 + \mu\phi_2)(\boldsymbol{z}) = (\lambda s_1 + \mu s_2)_{\boldsymbol{A}}(\boldsymbol{z}) + \sum_{i=1}^n p_{i\boldsymbol{A}}(z_i) \cdot (\lambda g_1 + \mu g_2)(\boldsymbol{z})$$

for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))$. It is easy to verify that $\lambda s_1 + \mu s_2, \lambda g_1 + \mu g_2$ is a decomposition of $\lambda\phi_1 + \mu\phi_2$ in the sense of Lemma 3.4.1. Since the definition of $\phi(\boldsymbol{A})$ in Definition 3.4.2 depends linearly on s and g, we conclude from Theorem 3.4.4 that

$$(\lambda\phi_1 + \mu\phi_2)(\boldsymbol{A}) = \lambda\phi_1(\boldsymbol{A}) + \mu\phi_2(\boldsymbol{A}).$$

As $\sigma(\Theta(\boldsymbol{A})) \subseteq \mathbb{R}^n$ and since we chose $p_i \in \mathbb{R}[z]$, we obtain $\phi^\#(\boldsymbol{z}) = s_1{}^\#_{\boldsymbol{A}}(\boldsymbol{z}) + \sum_{i=1}^n p_{i\boldsymbol{A}}(z_i) \cdot \overline{g}_1(\boldsymbol{z})$ for all $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))$. $\phi_1^\# - (s_1^\#)_{\boldsymbol{A}} = (\phi - s_{1\boldsymbol{A}})^\# \in \mathcal{R}$ holds true due to the fact that $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}} \Leftrightarrow \overline{\boldsymbol{z}} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$ which is a consequence of $p_i \in \mathbb{R}[z]$ for all $i \in [1, n]_{\mathbb{Z}}$. Hence, $s_1^\#, \overline{g}_1$ is a decomposition of $\phi_1^\#$ in the sense of Lemma 3.4.1. On the hand we have

$$\begin{aligned}\phi_1(\boldsymbol{A})^+ &= s_1(\boldsymbol{A})^+ + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_1 \,\mathrm{d}E\bigg)^+ = s_1^\#(\boldsymbol{A}) + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} \overline{g}_1 \,\mathrm{d}E\bigg)\\ &= \phi_1^\#(\boldsymbol{A})\end{aligned}$$

where the last equality is derived from Theorem 3.4.4.

Let g be defined as in Lemma 3.4.5. By Theorem 3.4.4 we have

$$(\phi_1 \cdot \phi_2)(\boldsymbol{A}) = (s_1 s_2)(\boldsymbol{A}) + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \,\mathrm{d}E\bigg)$$

On the other hand we obtain

$$\begin{aligned}\phi_1(\boldsymbol{A})\phi_2(\boldsymbol{A}) &= \bigg[s_1(\boldsymbol{A}) + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_1 \,\mathrm{d}E\bigg)\bigg]\bigg[s_2(\boldsymbol{A}) + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_2 \,\mathrm{d}E\bigg)\bigg]\\ &= s_1(\boldsymbol{A})s_2(\boldsymbol{A}) + \underbrace{s_1(\boldsymbol{A})\Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_2 \,\mathrm{d}E\bigg) + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_1 \,\mathrm{d}E\bigg)s_2(\boldsymbol{A})}_{=:U}\\ &\quad + \underbrace{\Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_1 \,\mathrm{d}E\bigg)\Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g_2 \,\mathrm{d}E\bigg)}_{=:V}\end{aligned}$$

The identities $C\Xi(D) = \Xi(\Theta(C)D)$ and $\Xi(D)C = \Xi(D\Theta(C))$ from Lemma 2.3.16 can be used to expand the multiplication to

$$U = \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} \underbrace{(s_1 g_2 + s_2 g_1)}_{=g-\sum_{i=1}^n p_i g_1 g_2} \,\mathrm{d}E + \sum_{\boldsymbol{w}\in\sigma(\Theta(\boldsymbol{A}))\cap Z_{\boldsymbol{p}}^{\mathbb{R}}} \sum_{i=1}^n \underbrace{(s_1(\boldsymbol{w})g_2(\boldsymbol{w})_i + s_2(\boldsymbol{w})g_1(\boldsymbol{w})_i)}_{=g(\boldsymbol{w})_i} R_i R_i^* E\{\boldsymbol{w}\}\bigg).$$

From $\Xi(D_1)\Xi(D_2) = \Xi(D_1 D_2 T T^+)$ and Lemma 2.3.16 we derive

$$V = \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} \sum_{i=1}^n p_i g_1 g_2 \,\mathrm{d}E \bigg).$$

By linearity of Ξ and Definition 3.4.2 we can sum up the above terms and obtain

$$\phi_1(\boldsymbol{A})\phi_2(\boldsymbol{A}) = (s_1 s_2)(\boldsymbol{A}) + \Xi\bigg(\int_{\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}}^{\boldsymbol{R}} g \,\mathrm{d}E \bigg) = (\phi_1 \cdot \phi_2)(\boldsymbol{A}),$$

which showes that the mapping $\phi \mapsto \phi(\boldsymbol{A})$ is compatible with multiplications.

Finally, we shall show that $\phi(\boldsymbol{A}) \in \boldsymbol{A}''$. Clearly, $s(\boldsymbol{A}) \in \boldsymbol{A}''$ for $s \in \mathbb{C}[z_1, \dots, z_n]$. If $C \in \boldsymbol{A}' \subseteq \bigcap_{i=1}^n (T_i T_i^+)'$, then $\Theta(C) \in \Theta(\boldsymbol{A})'$ because Θ is a homomorphism. By the spectral theorem in Hilbert spaces $\Theta(C)$ commutes with $E(\Delta)$ for all Borel sets Δ and by Proposition 3.1.4 $\Theta(C)$ commutes with all $R_i R_i^*$ for $i \in [1,n]_{\mathbb{Z}}$. Consequently, it commutes with

$$D := \int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \,\mathrm{d}E.$$

According to Lemma 2.3.16 we then obtain

$$\Xi(D)C = \Xi(D\Theta(C)) = \Xi(\Theta(C)D) = C\Xi(D).$$

Hence, $\Xi(D) \in \boldsymbol{A}''$ and altogether $\phi(\boldsymbol{A}) \in \boldsymbol{A}''$.

❑

Definition 3.4.7. Let $B(\boldsymbol{w})$ for $\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ be pairwise disjoint balls in $\mathbb{R}^n \subseteq \mathbb{C}^n$. We endow the vector space $\mathcal{F}_{\boldsymbol{A}}$ with the norm

$$\|\phi\|_{\mathcal{F}_{\boldsymbol{A}}} := \sup_{\boldsymbol{z}\in\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} |\phi(\boldsymbol{z})| + \sum_{\boldsymbol{w}\in Z_{\boldsymbol{p}}^{\mathbb{R}}} \max_{\alpha\in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} |\phi(\boldsymbol{w})_\alpha| + \sum_{\boldsymbol{w}\in Z_{\boldsymbol{p}}^{\mathrm{i}}} \max_{\alpha\in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} |\phi(\boldsymbol{w})_\alpha| + \sum_{\substack{\boldsymbol{w}\in Z_{\boldsymbol{p}}^{\mathbb{R}} \\ \boldsymbol{w}\text{ non isolated}}} \sup_{\boldsymbol{z}\in B(\boldsymbol{w})} \left| \frac{\phi(\boldsymbol{z}) - \sum_{\beta\in\hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z}-\boldsymbol{w})^\beta}{\max_{k\in[1,n]_{\mathbb{Z}}} |z_k - w_k|^{\mathfrak{d}_{p_k}(w_k)}} \right|$$

Remark 3.4.8. If we choose a different family of balls in Definition 3.4.7, we would obtain an equivalent norm.

Lemma 3.4.9. *Let* $\epsilon > 0$, $L := B_\epsilon(\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}^{\mathbb{R}})$ *and* $m := \max_{\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}} |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1$. *Furthermore let* f *be a sufficiently smooth function as in Definition 3.3.9 such that*

$$\|f\| := \max_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \le m}} \sup_{\boldsymbol{z} \in L} |D^\beta f(\boldsymbol{z})|$$

is bounded. Then the mapping $f \mapsto f_{\boldsymbol{A}}$ *is continuous.*

Proof. Let $\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$, $B(\boldsymbol{w})$ the corresponding ball as in Definition 3.4.7 and $\boldsymbol{z} \in B(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$. Then we have

$$\begin{aligned}
\left| f_{\boldsymbol{A}}(\boldsymbol{z}) - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(f_{\boldsymbol{A}}(\boldsymbol{w})\big)_\beta (\boldsymbol{z} - \boldsymbol{w})^\beta \right| &= \left| f(\boldsymbol{z}) - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w}) (\boldsymbol{z} - \boldsymbol{w})^\beta \right| \\
&= \left| f(\boldsymbol{z}) - \sum_{\substack{\beta \in \mathbb{N}_0^n \\ |\beta| \le |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w}) (\boldsymbol{z} - \boldsymbol{w})^\beta + \sum_{\substack{\beta \notin \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})} \\ |\beta| \le |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w}) (\boldsymbol{z} - \boldsymbol{w})^\beta \right| \\
&\le |R_{|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}(\boldsymbol{z})| + \left| \sum_{\substack{\beta \notin \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})} \\ |\beta| \le |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w}) (\boldsymbol{z} - \boldsymbol{w})^\beta \right|
\end{aligned}$$

where $R_{|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}(\boldsymbol{z})$ is the remainder of the Taylor approximation. For $\boldsymbol{z} \in B(\boldsymbol{w}) \setminus \{w\}$ we can bound the remainder by

$$\begin{aligned}
|R_{|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}(\boldsymbol{z})| &\le \sup_{\substack{\boldsymbol{u} \in B(\boldsymbol{w}) \\ |\beta| = |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1}} |D^\beta f(\boldsymbol{u})| \frac{n^{|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1}}{(|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1)!} \|\boldsymbol{z} - \boldsymbol{w}\|_\infty^{|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1} \\
&\le \|f\| \frac{n^{|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1}}{(|\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n + 1)!} c_1 \max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)},
\end{aligned}$$

for some $c_1 > 0$, which is independent of f. For the second summand we will use that $|(\boldsymbol{z} - \boldsymbol{w})^\beta|$ is an $O(\max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)})$ for $\beta \notin \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}$ like we already did in Example 3.3.20:

$$\left| \sum_{\substack{\beta \notin \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})} \\ |\beta| \le |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}} \frac{1}{\beta!} D^\beta f(\boldsymbol{w}) (\boldsymbol{z} - \boldsymbol{w})^\beta \right| \le \max_{\substack{\beta \notin \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})} \\ |\beta| \le |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - n}} |D^\beta f(\boldsymbol{w})| c_2 \max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}$$

for some $c_2 > 0$, which does not depend on f.

Altogether, for some $C_{\boldsymbol{w}} > 0$ we have

$$\left| \frac{f_{\boldsymbol{A}}(\boldsymbol{z}) - \sum_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(f_{\boldsymbol{A}}(\boldsymbol{w})\big)_\beta (\boldsymbol{z} - \boldsymbol{w})^\beta}{\max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}} \right| \le C_{\boldsymbol{w}} \|f\| .$$

Consequently, for $C := \sum_{\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}} C_{\boldsymbol{w}}$ we have $\|f_{\boldsymbol{A}}\|_{\mathcal{F}_{\boldsymbol{A}}} \le (1 + |Z_{\boldsymbol{p}}| + C) \|f\|$. ❑

Theorem 3.4.10. *The functional calculus* $\phi \mapsto \phi(\boldsymbol{A})$ *defined in Definition 3.4.2 from* $(\mathcal{F}_{\boldsymbol{A}}, \|.\|_{\mathcal{F}_{\boldsymbol{A}}})$ *into* $(L_{\mathrm{b}}(\mathcal{K}), \|.\|_{L_{\mathrm{b}}(\mathcal{K})})$ *is continuous.*

Proof. Since Theorem 3.4.4 states that the concrete decomposition does not affect the functional calculus, we will use a distinct decomposition in the following.

As a first step we define a mapping which provides us with a polynomial s of a decomposition of ϕ. Consider,

$$\pi_{\boldsymbol{p}} : \begin{cases} \mathcal{F}_{\boldsymbol{A}} & \to \quad \mathbb{C}^m, \\ \quad \phi & \mapsto \quad \left(\left(\phi(\boldsymbol{w})\right)_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \right)_{\boldsymbol{w} \in Z_{\boldsymbol{p}}}, \end{cases}$$

where $m = \sum_{\boldsymbol{w} \in Z_{\boldsymbol{p}}} \prod_{i=1}^{n} \mathfrak{d}_{p_i}(w_i)$. Recall the mapping $\varpi : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}^m$ from Lemma 3.3.13 according to $\boldsymbol{p}$. The lemma also states that the restriction of ϖ to $\mathcal{P}_{\boldsymbol{p}}$ is bijective. Hence, we can compose

$$\varpi\big|_{\mathcal{P}_{\boldsymbol{p}}}^{-1} \circ \pi_{\boldsymbol{p}} : \begin{cases} \mathcal{F}_{\boldsymbol{A}} & \to \quad \mathcal{P}_{\boldsymbol{p}}, \\ \quad \phi & \mapsto \quad s. \end{cases}$$

It can be easily seen that $\|\pi_{\boldsymbol{p}}(\phi)\|_{\infty, \mathbb{C}^m} \leq \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}$. Hence, $\pi_{\boldsymbol{p}}$ is continuous as a linear mapping. Since every norm on $\mathbb{C}^m$ is equivalent, the continuity of $\pi_{\boldsymbol{p}}$ is independent of the chosen norm. The linearity and the finite dimensional domain of $\varpi\big|_{\mathcal{P}_{\boldsymbol{p}}}^{-1}$ implies its continuity for every norm on $\mathcal{P}_{\boldsymbol{p}}$. Consequently, the composition $\varpi\big|_{\mathcal{P}_{\boldsymbol{p}}}^{-1} \circ \pi_{\boldsymbol{p}}$ is continuous.

We want to endow $\mathcal{P}_{\boldsymbol{p}}$ with the norm from Lemma 3.4.9, and denote it by $\|.\|_{\mathcal{P}_{\boldsymbol{p}}}$. Then we have

$$\|s\|_{\mathcal{P}_{\boldsymbol{p}}} = \left\| \varpi\big|_{\mathcal{P}_{\boldsymbol{p}}}^{-1} \circ \pi_{\boldsymbol{p}}(\phi) \right\|_{\mathcal{P}_{\boldsymbol{p}}} \leq \tilde{C} \, \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}$$

for some $\tilde{C} > 0$.

Since $\phi - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$, Remark 3.3.16 and Lemma 3.4.1 provide a g such that s, g is a decomposition of ϕ. In order to show that $\phi \mapsto g$ is continuous, we introduce a norm on the space of all such g:

$$\|g\| := \max \left\{ \sup_{\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} |g(\boldsymbol{z})| \right\} \cup \left\{ \, \|g(\boldsymbol{w})\|_{\infty, \mathbb{C}^n} : \boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}} \right\}.$$

We distinguish between three cases:

- g on $\sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$

$$\begin{aligned} \|g(\boldsymbol{w})\|_{\infty} &= \max_{i \in [1,n]_{\mathbb{Z}}} |g(\boldsymbol{w})_i| = \max_{i \in [1,n]_{\mathbb{Z}}} \left| \frac{\mathfrak{d}_{p_i}(w_i)! (\phi - s_{\boldsymbol{A}})(\boldsymbol{w})_{\mathfrak{d}_{p_i}(w_i) e_i}}{p_i^{(\mathfrak{d}_{p_i}(w_i))}(w_i)} \right| \\ &= \max_{i \in [1,n]_{\mathbb{Z}}} \left| \frac{\mathfrak{d}_{p_i}(w_i)! \phi(\boldsymbol{w})_{\mathfrak{d}_{p_i}(w_i) e_i} - D^{\mathfrak{d}_{p_i}(w_i) e_i} s(\boldsymbol{w})}{p_i^{(\mathfrak{d}_{p_i}(w_i))}(w_i)} \right| \\ &\leq \max_{i \in [1,n]_{\mathbb{Z}}} \left| \frac{\mathfrak{d}_{p_i}(w_i)!}{p_i^{(\mathfrak{d}_{p_i}(w_i))}(w_i)} \right| \left(\|\phi(\boldsymbol{w})\|_{\infty} + \|s\|_{\mathcal{P}_{\boldsymbol{p}}} \right) \leq C_{\boldsymbol{w}} \, \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}} \end{aligned}$$

  for some $C_{\boldsymbol{w}} > 0$. For $C_1 := \max_{\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}} C_{\boldsymbol{w}}$ we obtain

$$\max_{\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}} \|g(\boldsymbol{w})\| \leq C_1 \, \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}.$$

- g on a neighborhood of $\sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$. According to Lemma 3.2.3 for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))$ the inequality $\|R_i R_i^*\| \, |\sum_{k=1}^n p_k(z_k)| \geq |p_i(z_i)|$ holds true. Consequently,

$$\max_{i \in [1,n]_{\mathbb{Z}}} \|R_i R_i^*\| \left| \sum_{k=1}^n p_k(z_k) \right| \geq \max_{i \in [1,n]_{\mathbb{Z}}} |p_i(z_i)|.$$

  Furthermore, there exists a $r_{\boldsymbol{w}} > 0$ such that for $\boldsymbol{z} \in B_{r_{\boldsymbol{w}}}(\boldsymbol{w})$ we have $|p_i(z_i)| \geq c_i |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}$ for some $c_i > 0$ for every $i \in [1,n]_{\mathbb{Z}}$. This leads to

$$\left| \sum_{k=1}^n p_k(z_k) \right| \geq D_{\boldsymbol{w}} \max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}$$

  for a certain $D_{\boldsymbol{w}} > 0$ and $\boldsymbol{z} \in B_{r_{\boldsymbol{w}}}(\boldsymbol{w})$. Therefore,

$$\begin{aligned} |g(\boldsymbol{z})| &= \left| \frac{\phi(\boldsymbol{z}) - s(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)} \right| \leq \left| \frac{\phi(\boldsymbol{z}) - s(\boldsymbol{z})}{D_{\boldsymbol{w}} \max_{i \in [1,n]_{\mathbb{Z}}} |z_i - w_i|^{\mathfrak{d}_{p_i}(w_i)}} \right| \\ &\leq \left| \frac{\phi(\boldsymbol{z}) - \sum\limits_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z} - \boldsymbol{w})^\beta}{D_{\boldsymbol{w}} \max\limits_{k \in [1,n]_{\mathbb{Z}}} |z_k - w_k|^{\mathfrak{d}_{p_k}(w_k)}} \right| + \left| \frac{s(\boldsymbol{z}) - \sum\limits_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})}} \big(\phi(\boldsymbol{w})\big)_\beta (\boldsymbol{z} - \boldsymbol{w})^\beta}{D_{\boldsymbol{w}} \max\limits_{k \in [1,n]_{\mathbb{Z}}} |z_k - w_k|^{\mathfrak{d}_{p_k}(w_k)}} \right| \\ &\leq \frac{1}{D_{\boldsymbol{w}}} \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}} + \frac{1}{D_{\boldsymbol{w}}} \|s_{\boldsymbol{A}}\|_{\mathcal{F}_{\boldsymbol{A}}}. \end{aligned}$$

  By Lemma 3.4.9, we have $\|s_{\boldsymbol{A}}\|_{\mathcal{F}_{\boldsymbol{A}}} \leq \hat{C} \|s\|_{\mathcal{P}_{\boldsymbol{p}}} \leq \hat{C}\tilde{C} \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}$. This yields

$$|g(\boldsymbol{z})| \leq C_{\boldsymbol{w},2} \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}.$$

  Since $C_{\boldsymbol{w},2}$ is independent of $\boldsymbol{z} \in B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}$, the inequality holds true for all these $\boldsymbol{z}$. Taking the maximum C_2 of all $C_{\boldsymbol{w},2}$ for $\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ yields

$$|g(\boldsymbol{z})| \leq C_2 \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}} \quad \text{for all} \quad \boldsymbol{z} \in \bigcup_{\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}} B_{r_{\boldsymbol{w}}}(\boldsymbol{w}) \setminus \{\boldsymbol{w}\}.$$

- g on $\sigma(\Theta(\boldsymbol{A})) \setminus \bigcup_{\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}} B_{r_{\boldsymbol{w}}}(\boldsymbol{w})$. Since zeros of $\sum_{i=1}^n p_i(z_i)$ can only be in $Z_{\boldsymbol{p}}^{\mathbb{R}}$, we have $|\sum_{i=1}^n p_i(z_i)| > d$ for a $d > 0$. Hence,

$$|g(\boldsymbol{z})| = \left| \frac{\phi(\boldsymbol{z}) - s(\boldsymbol{z})}{\sum_{i=1}^n p_i(z_i)} \right| \leq \frac{1}{d} (|\phi(\boldsymbol{z})| + |s(\boldsymbol{z})|) \leq C_3 \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}.$$

Taking these three inequalities into account yields

$$\|g\| \leq \max\{C_1, C_2, C_3\} \|\phi\|_{\mathcal{F}_{\boldsymbol{A}}}.$$

Therefore, we proved the continuity of $\phi \mapsto g$ and the continuity of $\phi \mapsto (s, g)$.

It is left to show that

$$(s, g) \mapsto s(\boldsymbol{A}) + \int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \, \mathrm{d}E$$

is continuous. The continuity of $s \mapsto s(\boldsymbol{A})$ for $s \in \mathcal{P}_{\boldsymbol{p}}$ follows from $\dim \mathcal{P}_{\boldsymbol{p}} < \infty$. By the spectral theorem in Hilbert spaces we know that $g \mapsto \int_{\sigma(\Theta(\boldsymbol{A}))\setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} g \,\mathrm{d}E$ is continuous. Since the remaining part of $\int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \,\mathrm{d}E$ is a finite sum we can find a $C > 0$ such that

$$\left\| \sum_{w \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}} \sum_{i=1}^{n} g(w)_i R_i R_i^* E\{w\} \right\| \leq C \|g\| .$$

Hence $(s, g) \mapsto s(\boldsymbol{A}) + \int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} g \,\mathrm{d}E$ is continuous and consequently $\phi \mapsto \phi(\boldsymbol{A})$ is also continuous as a composition of continuous mappings.

❑

3.5 Compatibility of the Spectral Theorem

In this section we want to regard the spectral calculus of a tuple $\boldsymbol{A} = (A_i)_{i=1}^n$ compared to the spectral calculus of a fixed entry A_i of $\boldsymbol{A}$. More precisely, we want to check, if

$$\phi(A_i) = (\phi \circ \pi_i)(\boldsymbol{A}),$$

where on the left-hand-side we use the functional calculus of A_i and on the right-hand-side we use the functional calculus of $\boldsymbol{A}$.

At first we have to define what we exactly mean by $\phi \circ \pi_i$.

Example 3.5.1. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $\pi_i : \mathbb{C}^n \to \mathbb{C}$ be the projection on the i-th coordinate. Then we want to take a look at $(f \circ \pi_i)_{\boldsymbol{A}}$:

$$\big((f \circ \pi_i)_{\boldsymbol{A}}(\boldsymbol{z})\big)_\beta = \frac{1}{\beta!} D^\beta (f \circ \pi_i)(\boldsymbol{z}).$$

Since the entries z_j for $j \neq i$ do not affect the function $f \circ \pi_i$, the derivative in these directions vanish. If $\beta = \beta_i e_i$ where $e_i = (\delta_{i,j})_{j=1}^n$, then we have

$$\frac{1}{\beta!} D^\beta (f \circ \pi_i)(\boldsymbol{z}) = \frac{1}{\beta_i!} f^{(\beta_i)}(z_i) = (f_{A_i}(z_i))_{\beta_i}.$$

Therefore,

$$\big((f \circ \pi_i)_{\boldsymbol{A}}(\boldsymbol{z})\big)_\beta = \begin{cases} 0, & \text{if } \exists j \neq i \ : \beta_j \neq 0, \\ (f_{A_i}(z_i))_{\beta_i}, & \text{if } \beta = \beta_i e_i. \end{cases}$$

In view of Example 3.5.1 we want define an adequate function composition.

Definition 3.5.2. Let $\phi \in \mathcal{F}_{A_i}$ and $\pi_i : \mathbb{C}^n \to \mathbb{C}$ be the projection on the i-th coordinate. We set $\phi \circ \pi_i(\boldsymbol{z}) = \phi(z_i)$ for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}$ and

$$\big((\phi \circ \pi_i)(\boldsymbol{z})\big)_\beta = \begin{cases} 0, & \text{if } \exists j \neq i \ : \beta_j \neq 0, \\ (\phi(z_i))_{\beta_i}, & \text{if } \beta = \beta_i e_i. \end{cases}$$

for $\boldsymbol{z} \in \big(\sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}\big) \,\dot{\cup}\, Z_{\boldsymbol{p}}^{\mathrm{i}}$ and $\mathrm{dom}(\phi \circ \pi_i) := \pi^{-1}(\mathrm{dom}\,\phi)$.

Remark 3.5.3. For a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ we obtain from Example 3.5.1 and Definition 3.5.2

$$(f \circ \pi_i)_{\boldsymbol{A}} = f_{A_i} \circ \pi_i.$$

Furthermore, the composition defined in Definition 3.5.2 is distributive, i.e. for $\phi_1, \phi_2 \in \mathcal{F}_{A_i}$ we have

$$\begin{aligned}(\phi_1 + \phi_2) \circ \pi_i &= (\phi_1 \circ \pi_i) + (\phi_2 \circ \pi_i),\\ (\phi_1 \cdot \phi_2) \circ \pi_i &= (\phi_1 \circ \pi_i) \cdot (\phi_2 \circ \pi_i).\end{aligned}$$

Lemma 3.5.4. *Fix $i \in [1, n]_{\mathbb{Z}}$. If $\phi \in \mathcal{F}_{A_i}$ then $\phi \circ \pi_i \in \mathcal{F}_{\boldsymbol{A}}$. For every $s \in \mathbb{C}[z]$ such that $\phi - s_{A_i} \in \mathcal{R}_{A_i}$ we have $\phi \circ \pi_i - (s \circ \pi_i)_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$. Moreover, if $\phi = s_{A_i} + p_{i A_i} \cdot g$ is a decomposition for $\phi \in \mathcal{F}_{A_i}$ in the sense of Lemma 3.4.1 then $\phi \circ \pi_i = (s \circ \pi_i)_{\boldsymbol{A}} + \sum_{k=1}^n p_{k\boldsymbol{A}} \cdot \hat{g}$ is a decomposition for $\phi \circ \pi_i \in \mathcal{F}_{\boldsymbol{A}}$, where*

$$\hat{g}(\boldsymbol{z}) = \frac{p_i(z_i)}{\sum_{k=1}^n p_k(z_k)} g(z_i) \quad \textit{for} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}},$$

and

$$\hat{g}(\boldsymbol{z})_k = \begin{cases} g(z_i), & \textit{if } k = i, \\ 0, & \textit{else}, \end{cases} \quad \textit{for} \quad \boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}.$$

Proof. Recall that $\phi \circ \pi_i \in \mathcal{F}_{\boldsymbol{A}}$ means nothing else but the fact that for every $\boldsymbol{\omega} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ the term

$$\left| \frac{\phi \circ \pi_i(\boldsymbol{x}) - \sum\limits_{\beta \in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{\omega})}} \big((\phi \circ \pi_i)(\boldsymbol{\omega})\big)_\beta (\boldsymbol{x} - \boldsymbol{\omega})^\beta}{\max\limits_{k \in [1,n]_{\mathbb{Z}}} |x_k - \omega_k|^{\mathfrak{d}_{p_k}(\omega_k)}} \right|$$

is bounded for $\boldsymbol{x} \in B_r(\boldsymbol{\omega}) \setminus \{\boldsymbol{\omega}\} \cap \sigma(\Theta(\boldsymbol{A}))$ for a sufficiently small $r > 0$. By Definition 3.5.2, $\big((\phi \circ \pi_i)(\boldsymbol{x})\big)_\beta(\boldsymbol{x}) = 0$ if $\beta \neq \beta_i e_i$. Hence, the sum can be reduced to

$$\left| \frac{\phi(x_i) - \sum\limits_{k=0}^{\mathfrak{d}_{p_i}(\omega_i)} \big((\phi)(\omega_i)\big)_k (x_k - \omega_k)^k}{\max\limits_{k \in [1,n]_{\mathbb{Z}}} |x_k - \omega_k|^{\mathfrak{d}_{p_k}(\omega_k)}} \right| \leq \left| \frac{\phi(x_i) - \sum\limits_{k=0}^{\mathfrak{d}_{p_i}(\omega_i)} \big((\phi)(\omega_i)\big)_k (x_k - \omega_k)^k}{|x_i - \omega_i|^{\mathfrak{d}_{p_i}(\omega_i)}} \right|.$$

Due to our assumption $\phi \in \mathcal{F}_{A_i}$ there exists a $r_0 > 0$ such that the right-hand-side is bounded for $x_i \in B_{r_0}(\omega_i) \setminus \{\omega_i\} \cap \sigma(\Theta_i(A_i))$. Consequently, the left-hand-side is also bounded for $\boldsymbol{x} \in B_{r_0}(\boldsymbol{\omega}) \setminus \{\boldsymbol{\omega}\} \cap \sigma(\Theta(\boldsymbol{A}))$. Hence, $\phi \circ \pi_i \in \mathcal{F}_{\boldsymbol{A}}$.

Let $s \in \mathbb{C}[z]$ be such that $\phi - s_{A_i} \in \mathcal{R}_{A_i}$. By definition

$$(\phi \circ \pi_i(\boldsymbol{z}) - (s \circ \pi_i)_{\boldsymbol{A}}(\boldsymbol{z}))_\beta = \begin{cases} 0, & \text{if } \exists j \neq i \ : \beta_j \neq 0, \\ (\phi(z_i))_{\beta_i} - (s_{A_i}(z_i))_{\beta_i}, & \text{if } \beta = \beta_i e_i. \end{cases}$$

and consequently $\phi \circ \pi_i - s_{\boldsymbol{A}} \in \mathcal{R}_{\boldsymbol{A}}$.

Since s, g is a decomposition of ϕ, we have $g(z_i) = \frac{\phi(z_i) - s(z_i)}{p_i(z_i)}$ for $z_i \in \sigma(\Theta_i(A_i)) \setminus Z^{\mathbb{R}}_{p_i} \supseteq \pi_i\big(\sigma(\Theta(\boldsymbol{A})) \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}\big)$. Lemma 3.2.3 guarantees that if $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A}))$ and $p_i(z_i) = 0$, then $\boldsymbol{z} \in Z^{\mathbb{R}}_{\boldsymbol{p}}$ which justifies the definition

$$\hat{g}(\boldsymbol{z}) = \frac{p_i(z_i)}{\sum_{k=1}^n p_k(z_k)} g(z_i) = \frac{p_i(z_i)}{\sum_{k=1}^n p_k(z_k)} \frac{\phi(z_i) - s(z_i)}{p_i(z_i)} = \frac{\phi \circ \pi_i(\boldsymbol{z}) - (s \circ \pi_i)(\boldsymbol{z})}{\sum_{k=1}^n p_k(z_k)}$$

for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}$. Additionally we obtain from this equation that $\phi \circ \pi_i(\boldsymbol{z}) = s_{\boldsymbol{A}}(\boldsymbol{z}) + \sum_{k=1}^n p_{k\boldsymbol{A}}(z_k) \cdot \hat{g}(\boldsymbol{z})$ holds true for $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}$.

For $\boldsymbol{z} \in \sigma(\Theta(\boldsymbol{A})) \cap Z^{\mathbb{R}}_{\boldsymbol{p}}$ it is left to show

$$(\phi \circ \pi_i - s_{\boldsymbol{A}})(\boldsymbol{z})_{\mathfrak{d}_{p_k}(z_k) e_k} = \frac{p^{(\mathfrak{d}_{p_k}(z_k))}(z_k)}{\mathfrak{d}_{p_k}(z_k)!} \hat{g}(\boldsymbol{z})_k.$$

By definition for $k \neq i$ both sides are equal to zero. For $k = i$

$$\begin{aligned} \hat{g}(\boldsymbol{z})_i = g(z_i) &= \frac{\mathfrak{d}_{p_i}(z_i)!\big((\phi(z_i))_{\mathfrak{d}_{p_i}(z_i)e_i} - (s_{A_i}(z_i))_{\mathfrak{d}_{p_i}(z_i)e_i}\big)}{p^{(\mathfrak{d}_{p_i}(z_i))}(z_i)} \\ &= \frac{\mathfrak{d}_{p_i}(z_i)!}{p^{(\mathfrak{d}_{p_i}(z_i))}(z_i)} (\phi \circ \pi_i - (s \circ \pi_i)_{\boldsymbol{A}})(\boldsymbol{z})_{\mathfrak{d}_{p_i}(z_i)e_i}, \end{aligned}$$

which completes the proof.

❑

Theorem 3.5.5. *Let $\boldsymbol{A} = (A_i)_{i=1}^n$ be a tuple of operators satisfying Assumptions 3.2.1, $i \in [1, n]_{\mathbb{Z}}$ and $\phi \in \mathcal{F}_{A_i}$. Then*

$$\phi(A_i) = (\phi \circ \pi_i)(\boldsymbol{A}),$$

where both sides have to be understood in the sense of Definition 3.4.2 according to the respective function class $\mathcal{F}_{A_i}$ and $\mathcal{F}_{\boldsymbol{A}}$, and $\phi \circ \pi_i$ is defined as in Definition 3.5.2.

Proof. Let s, g be a decomposition of ϕ in the sense of Lemma 3.4.1. By Lemma 3.5.4 we have $s \circ \pi_i, \hat{g}$ as a decomposition for $\phi \circ \pi_i$.

We will extend g to $\mathbb{R}$ by setting $g(z) = 0$ for all $z \in \mathbb{R} \setminus \sigma(\Theta_i(A_i))$. By Remark 3.4.3, we obtain

$$\phi(A_i) = s(A_i) + \Xi_i\Big(\int_{\mathbb{R}} g \,\mathrm{d}E^i_i\Big) \overset{(1.6)}{=} s(A_i) + \Xi_i\Big(\int_{\mathbb{R}^n} g \circ \pi_i \,\mathrm{d}E^i\Big).$$

Applying the identity (3.7) yields

$$\phi(A_i) = s(A_i) + \Xi\Big(R_i R_i^* \int_{\mathbb{R}^n} g \circ \pi_i \,\mathrm{d}E\Big).$$

We can split up $\mathbb{R}^n$ in $Z^{\mathbb{R}}_{\boldsymbol{p}} \,\dot{\cup}\, \big(\mathbb{R}^n \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}\big)$ and use the fact $\int_\Delta f \,\mathrm{d}E = \int_\Delta \mathbb{1}_\Delta f \,\mathrm{d}E = E(\Delta) \int_\Delta f \,\mathrm{d}E$ in order to obtain

$$\phi(A_i) = s(A_i) + \Xi\Bigg(R_i R_i^* E\big(\mathbb{R}^n \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}\big) \int\limits_{\mathbb{R}^n \setminus Z^{\mathbb{R}}_{\boldsymbol{p}}} g \circ \pi_i \,\mathrm{d}E + R_i R_i^* \int\limits_{Z^{\mathbb{R}}_{\boldsymbol{p}}} g \circ \pi_i \,\mathrm{d}E\Bigg).$$

By Corollary 3.2.5, we have $R_i R_i^* E\big(\mathbb{R}^n \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}\big) = \int_{\mathbb{R}^n \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} \frac{p_i}{\sum_{k=1}^n p_k} \,\mathrm{d}E$. Hence,

$$\phi(A_i) = s(A_i) + \Xi\left(\int\limits_{\mathbb{R}^n \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} \frac{p_i}{\sum_{k=1}^n p_k} \,\mathrm{d}E \int\limits_{\mathbb{R}^n \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} g \circ \pi_i \,\mathrm{d}E + \sum_{\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}} R_i R_i^* E(\{\boldsymbol{w}\}) g(\boldsymbol{w})_i \right).$$

Using the compatibility with multiplications of the integral and the definition of $\hat{g}$ we obtain

$$\phi(A_i) = s(A_i) + \Xi\bigg(\int_{\mathbb{R}^n \setminus Z_{\boldsymbol{p}}^{\mathbb{R}}} \hat{g} \,\mathrm{d}E + \sum_{\boldsymbol{w} \in \sigma(\Theta(\boldsymbol{A})) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}} \sum_{k=1}^n \hat{g}(\boldsymbol{w})_k R_k R_k^* E(\{\boldsymbol{w}\}) \bigg),$$

which is by defintion nothing else but

$$\phi(A_i) = s \circ \pi_i(\boldsymbol{A}) + \int_{\sigma(\Theta(\boldsymbol{A}))}^{\boldsymbol{R}} \hat{g} \,\mathrm{d}E = (\phi \circ \pi_i)(\boldsymbol{A}).$$

❑

3.6 Spectrum

In this section we will show that only the values of $\phi \in \mathcal{F}_{\boldsymbol{A}}$ on $\sigma(\boldsymbol{A})$ are essential for our functional calculus. This means that if $\phi_1, \phi_2 \in \mathcal{F}_{\boldsymbol{A}}$ differ only on $\big(\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}\big) \setminus \sigma(\boldsymbol{A})$, then $\phi_1(\boldsymbol{A}) = \phi_2(\boldsymbol{A})$.

Remark 3.6.1. Let $\boldsymbol{w} \in Z_{\boldsymbol{p}}$ be an isolated point of $\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}$ and let $e = 1_{\boldsymbol{A}}$ the multiplicative neutral element of $\mathcal{F}_{\boldsymbol{A}}$. Then by Example 3.3.19, $\delta_{\boldsymbol{w}} e$ belongs to $\mathcal{F}_{\boldsymbol{A}}$. Since $\delta_{\boldsymbol{w}} e \cdot \delta_{\boldsymbol{w}} e = \delta_{\boldsymbol{w}} e$ the corresponding operator $\delta_{\boldsymbol{w}} e(\boldsymbol{A})$ is a projection.

Furthermore let $\boldsymbol{\lambda} \in \mathbb{C}^n \setminus \{\boldsymbol{w}\}$ and $\boldsymbol{s}(\boldsymbol{z}) := \boldsymbol{z} - \boldsymbol{\lambda}$ and $s_i(\boldsymbol{z}) := z_i - \lambda_i$ for all $i \in [1, n]_{\mathbb{Z}}$. Then there exists an $i \in [1, n]_{\mathbb{Z}}$ such that $s_i(\boldsymbol{w}) \neq 0$. For this $i \in [1, n]_{\mathbb{Z}}$ we have $(s_{i\boldsymbol{A}} \delta_{\boldsymbol{w}} e)(.) = \delta_{\boldsymbol{w}}(.) s_{i\boldsymbol{A}}(\boldsymbol{w})$ where $s_{i\boldsymbol{A}}(\boldsymbol{w})$ is invertible in $\mathfrak{C}(\boldsymbol{w})$ because of $s_{i\boldsymbol{A}}(\boldsymbol{w})_0 \neq 0$. Let b denote its inverse. Then we have

$$s_{i\boldsymbol{A}} \delta_{\boldsymbol{w}} e \cdot \delta_{\boldsymbol{w}} b = \delta_{\boldsymbol{w}} e.$$

We see that $A_i\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})} - \lambda_i$ has $\delta_{\boldsymbol{w}} b(\boldsymbol{A})\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})}$ as its inverse operator. By Remark 1.3.18 also $\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})} - \boldsymbol{\lambda}$ is invertible, where $\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})} := \big(A_i\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})}\big)_{i=1}^n$. Since $\boldsymbol{\lambda}$ was arbitrary in $\mathbb{C}^n \setminus \{\boldsymbol{w}\}$, we conclude that the spectrum $\sigma\Big(\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})}\Big)$ can only contain $\boldsymbol{w}$ or in other words $\sigma\Big(\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})}\Big) \subseteq \{\boldsymbol{w}\}$.

Lemma 3.6.2. *Let $\phi \in \mathcal{F}_{\boldsymbol{A}}$. If $\phi(\boldsymbol{z}) = 0$ for all $\boldsymbol{z} \in \sigma(\boldsymbol{A})$, then $\phi(\boldsymbol{A}) = 0$.*

Proof. As $\sigma(\Theta(\boldsymbol{A})) \subseteq \sigma(\boldsymbol{A})$ every $\boldsymbol{w} \in Z_{\boldsymbol{p}} \setminus \sigma(\boldsymbol{A})$ is an isolated point of $\sigma(\Theta(\boldsymbol{A})) \cup Z_{\boldsymbol{p}}$. We can apply Remark 3.6.1. By assumption the operator tuple $\boldsymbol{A} - \boldsymbol{w}$ is invertible. This implies the invertibility of $\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})} - \boldsymbol{w}$. By Remark 3.6.1 $\boldsymbol{w}$ was the only possible candidate for a spectral point of $\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})}$. Hence, we obtain $\sigma\Big(\boldsymbol{A}\big|_{\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A})}\Big) = \emptyset$. By Corollary 1.4.5, this is only possible if $\operatorname{ran} \delta_{\boldsymbol{w}} e(\boldsymbol{A}) = \{0\}$. Thus, $\delta_{\boldsymbol{w}} e(\boldsymbol{A}) = 0$.

By our assumptions ϕ can be written as $\sum_{\boldsymbol{w}\in Z_{\boldsymbol{p}}\setminus\sigma(\boldsymbol{A})}\delta_{\boldsymbol{w}}\phi(\boldsymbol{w})$ which implies

$$\phi(\boldsymbol{A}) = \sum_{\boldsymbol{w}\in Z_{\boldsymbol{p}}\setminus\sigma(\boldsymbol{A})} \delta_{\boldsymbol{w}}\phi(\boldsymbol{w})(\boldsymbol{A}) = \sum_{\boldsymbol{w}\in Z_{\boldsymbol{p}}\setminus\sigma(\boldsymbol{A})} \phi(\boldsymbol{w})\delta_{\boldsymbol{w}}e(\boldsymbol{A}) = 0$$

❑

Since Lemma 3.6.2 tells us that $\phi(\boldsymbol{A})$ depends only on ϕ's values on $\sigma(\boldsymbol{A})$ we can redefine the domain of the functions in $\mathcal{F}_{\boldsymbol{A}}$.

Definition 3.6.3. We will redefine the set $\mathcal{F}_{\boldsymbol{A}}$. In fact, let $\mathcal{F}_{\boldsymbol{A}}$ contain all functions ϕ with domain $\sigma(\boldsymbol{A})$ such that $\phi(\boldsymbol{z}) \in \mathfrak{C}(\boldsymbol{z})$ – see Definition 3.3.6 –, such that $\boldsymbol{z} \mapsto \phi(\boldsymbol{z})$ is measurable and bounded on $\sigma(\boldsymbol{A}) \setminus Z_{\boldsymbol{p}}$ and such that (3.10) is locally bounded at $\boldsymbol{w}$ for all $\boldsymbol{w} \in \sigma(\boldsymbol{A}) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}$, which are non-isolated.

We will also redefine $f_{\boldsymbol{A}}$. We reduce the conditions of Definition 3.3.9 to $\sigma(\boldsymbol{A}) \subseteq \operatorname{dom} f$ and the requested differentiability (holomorphy) is only necessary for points of $Z_{\boldsymbol{p}}^{\mathbb{R}}$ ($Z_{\boldsymbol{p}}^{\mathrm{i}}$) which also belong to $\sigma(\boldsymbol{A})$. Hence, we define

$$f_{\boldsymbol{A}}(\boldsymbol{z}) := \begin{cases} f(\boldsymbol{z}), & \text{if } \boldsymbol{z} \in \sigma(\boldsymbol{A}) \setminus Z_{\boldsymbol{p}}, \\ \left(\frac{1}{\beta!}D^{\beta}f(\boldsymbol{z})\right)_{\beta\in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}, & \text{if } \boldsymbol{z} \in \sigma(\boldsymbol{A}) \cap Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \left(\frac{1}{\beta!}D^{\beta}f(\boldsymbol{z})\right)_{\beta\in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}, & \text{if } \boldsymbol{z} \in \sigma(\boldsymbol{A}) \cap Z_{\boldsymbol{p}}^{\mathrm{i}}. \end{cases}$$

Remark 3.6.4. In fact, the redefined $\mathcal{F}_{\boldsymbol{A}}$ contains all functions ϕ such that $\hat{\phi}$ defined by

$$\hat{\phi}(\boldsymbol{z}) := \begin{cases} \phi(\boldsymbol{z}), & \text{if } \boldsymbol{z} \in \sigma(\boldsymbol{A}), \\ e, & \text{else}, \end{cases}$$

is an element of the previous definition of $\mathcal{F}_{\boldsymbol{A}}$ – see Definition 3.3.18 – where e is the neutral element of $\mathfrak{C}(\boldsymbol{z})$.

Definition 3.6.5. For convenience we define $\phi(\boldsymbol{A})$ as $\hat{\phi}(\boldsymbol{A})$, where $\hat{\phi}$ is the mapping in Remark 3.6.4 and $\phi \in \mathcal{F}_{\boldsymbol{A}}$ – Definition 3.6.3.

Remark 3.6.6. It is easy to check that the mapping $\phi \mapsto \hat{\phi} - \hat{0}$ from the new to the old definition of $\mathcal{F}_{\boldsymbol{A}}$ is a $*$-homomorphism. By Lemma 3.6.2 the zero mapping 0 satisfies $0(\boldsymbol{A}) := \hat{0}(\boldsymbol{A}) = 0$. This yields $(\hat{\phi} - \hat{0})(\boldsymbol{A}) = \hat{\phi}(\boldsymbol{A})$ and $\phi \mapsto \phi(\boldsymbol{A})$ is the composition of the $*$-homomorphisms $\phi \mapsto \hat{\phi} - \hat{0}$ and $\hat{\phi} \mapsto \hat{\phi}(\boldsymbol{A})$. Hence, the functional calculus $\phi \mapsto \phi(\boldsymbol{A})$ is also a $*$-homomorphism.

Lemma 3.6.7. *If ϕ is an element of the redefined set $\mathcal{F}_{\boldsymbol{A}}$ – Definition 3.6.3 – such that $\phi(\boldsymbol{z})$ is invertible in $\mathfrak{C}(\boldsymbol{z})$ for all $\boldsymbol{z} \in \sigma(\boldsymbol{A})$ and such that 0 does not belong to the closure of $\phi(\sigma(\boldsymbol{A}) \setminus Z_{\boldsymbol{p}}^{\mathbb{R}})$, then $\phi(\boldsymbol{A})$ is invertible.*

Proof. Let $\hat{\phi}$ be defined as in Remark 3.6.4. Then $\hat{\phi}$ satisfies all conditions of Lemma 3.3.22 and therefore $\phi^{-1} = (\hat{\phi})^{-1}\big|_{\sigma(\boldsymbol{A})} \in \mathcal{F}_{\boldsymbol{A}}$. The functional calculus yields

$$\phi(\boldsymbol{A})\phi^{-1}(\boldsymbol{A}) = \phi\phi^{-1}(\boldsymbol{A}) = 1_{\boldsymbol{A}}(\boldsymbol{A}) = I.$$

❑

4 Spectral Theorem for Normal Operators

In this section we will use the Spectral Calculus for families of definitizable self-adjoint operators presented in Section 3.4 to introduce a Spectral Theorem for definitizable normal operators.

4.1 Spectral Theorem

Definition 4.1.1. Let $\mathcal{K}$ be a Krein space. A normal operator $N \in L_{\mathrm{b}}(\mathcal{K})$ is called *definitizable* if the self-adjoint operators $A_1 := \frac{N+N^+}{2}$ and $A_2 := \frac{N-N^+}{2\mathrm{i}}$ are both definitizable.

Assumptions 4.1.2. Let N be a normal definitizable operator. We will define $\boldsymbol{A} = (A_1, A_2) := \left(\frac{N+N^+}{2}, \frac{N-N^+}{2\mathrm{i}}\right)$ and $\boldsymbol{p} = (p_1, p_2)$ where p_i is a definitizing polynomial of A_i. Furthermore, we define the mapping $\iota : \mathbb{C}^2 \to \mathbb{C}$, $\boldsymbol{z} \mapsto z_1 + \mathrm{i}z_2$.

Theorem 4.1.3. *Let N be normal and definitizable operator in a Krein space $\mathcal{K}$ and A_1, A_2 the corresponding real and imaginary part of N. Then we have*

$$\sigma(N) = \iota(\sigma(\boldsymbol{A})).$$

Proof. If $\lambda \notin \sigma(N)$, then $T := (N - \lambda)^{-1}$ exists. For every $\boldsymbol{\lambda} \in \mathbb{C}^2$ which fulfills $\iota(\boldsymbol{\lambda}) = \lambda$ we have

$$(A_1 + \mathrm{i}A_2 - \iota(\boldsymbol{\lambda}))T = I.$$

Defining $\boldsymbol{B} := (T, \mathrm{i}T)$ we get

$$\begin{aligned}(\boldsymbol{A} - \boldsymbol{\lambda}) \cdot \boldsymbol{B} &= (A_1 - \lambda_1)T + (A_2 - \lambda_2)\mathrm{i}T = (A_1 + \mathrm{i}A_2 - \underbrace{(\lambda_1 + \mathrm{i}\lambda_2)}_{=\iota(\boldsymbol{\lambda})})T \\ &= (A_1 + \mathrm{i}A_2 - \lambda)T = I.\end{aligned}$$

Similarly, $\boldsymbol{B} \cdot (\boldsymbol{A} - \boldsymbol{\lambda}) = I$. Thus, $(\boldsymbol{A} - \boldsymbol{\lambda})$ is invertible. Therefore, we conclude $\lambda \notin \iota(\sigma(\boldsymbol{A}))$.

On the other hand let $\lambda \notin \iota(\sigma(\boldsymbol{A}))$. Then $f(\boldsymbol{z}) := \iota(\boldsymbol{z}) - \lambda \neq 0$ for $\boldsymbol{z} \in \sigma(\boldsymbol{A})$ and $f_{\boldsymbol{A}}$ belongs to $\mathcal{F}_{\boldsymbol{A}}$. Therefore, $f_{\boldsymbol{A}}$ has a multiplicative inverse $(f_{\boldsymbol{A}})^{-1} \in \mathcal{F}_{\boldsymbol{A}}$. Since $f_{\boldsymbol{A}}(N) = N - \lambda$, we have

$$(f_{\boldsymbol{A}})^{-1}(N) = (N - \lambda)^{-1}$$

and consequently $\lambda \notin \sigma(N)$.

❑

Definition 4.1.4. Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a function such that $\sigma(N) \subseteq D$ and such that D contains an open neighborhood of $\iota(Z_{\boldsymbol{p}})$. Furthermore let f be $\max_{\boldsymbol{w} \in Z_{\boldsymbol{p}}^{\mathbb{R}}} |\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{w})| - 1$ times continuously real differentiable in an open neighborhood of $\iota(Z_{\boldsymbol{p}}^{\mathbb{R}})$ and holomorphic in an open neighborhood of $\iota(Z_{\boldsymbol{p}}^{\mathrm{i}})$. Then f can be considered as an element of f_N of $\mathcal{M}_{\boldsymbol{A}}$

$$f_N(\boldsymbol{z}) := \begin{cases} f \circ \iota(\boldsymbol{z}), & \text{if } \boldsymbol{z} \in \sigma(\boldsymbol{A}) \setminus Z_{\boldsymbol{p}}, \\ \left(\frac{1}{\beta!} D^{\beta} f \circ \iota(\boldsymbol{z})\right)_{\beta \in I_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}, & \text{if } \boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}, \\ \left(\frac{1}{\beta!} D^{\beta} f \circ \iota(\boldsymbol{z})\right)_{\beta \in \hat{I}_{\mathfrak{d}_{\boldsymbol{p}}(\boldsymbol{z})}}, & \text{if } \boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}, \end{cases}$$

For $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathbb{R}}$ the derivative should be understood in the sense of real derivation and for $\boldsymbol{z} \in Z_{\boldsymbol{p}}^{\mathrm{i}}$ it is a complex derivative.

Lemma 4.1.5. *If f satisfy all conditions of Definition 4.1.4, then $f_N \in \mathcal{F}_{\boldsymbol{A}}$.*

Proof. By definition $f_N = \left(f \circ \iota\big|_{\iota^{-1}(\operatorname{dom} f)}\right)_{\boldsymbol{A}}$ and $\left(f \circ \iota\big|_{\iota^{-1}(\operatorname{dom} f)}\right)$ satisfies all conditions of Lemma 3.3.21 which implies that $f_N = \left(f \circ \iota\big|_{\iota^{-1}(\operatorname{dom} f)}\right)_{\boldsymbol{A}} \in \mathcal{F}_{\boldsymbol{A}}$. ❑

Definition 4.1.6. Let N be normal definitizable operator, which fulfills Assumptions 4.1.2, and $\phi \in \mathcal{F}_{\boldsymbol{A}}$. We define

$$\phi(N) := \phi(\boldsymbol{A}).$$

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