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Applications of Vector Measures with Tensor Products of Banach Spaces

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Ao. Univ. Prof. Dipl.-Ing. Dr.techn. Michael Kaltenbäck

durch

Philip Ludwig Scheberan, BSc

Verfasser

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Contents

Introduction

The present thesis is a continuation of the work I have done in my Bachelor Thesis [S], the contents of which are fully contained in Chapters 3, 4 and 5. There the basic theory of tensor products of Banach spaces is discussed. In particular the projective and injective tensor products and their relation to nuclear and integral operators. Section 5.2 provides a close examination of integral operators. It shows that every nuclear operator is integral. Immediately the question arises, under which circumstances those classes of operators coincide.

We will also show that Pietsch integral operators, which are a special case of integral operators, are the link that connects nuclear and integral operators. The case when integral operators and Pietsch integral operators coincide has already been discussed in [S]. Chapter 6 is concerned with examining nuclear and Pietsch integral operators more closely. Under which conditions do they coincide? The fundamental key to understand the connections between these operator classes will be the notion of vector measures, in particular the representing measure, as can be found in [BDS]. The basic theory of vector measures is introduced in Section 2.1.

The results which are presented in the first chapters will lead towards to groundbreaking work of Alexander Grothendieck in [Gr1] and [Gr2]. Combining the statements of the previous chapters, tying up loose ends and showing, how these different topics work together, produces astounding results. In the final Chapter, the metric approximation property is studied. On the first glance not inherently connected to this topic, the theory of tensor products of Banach spaces will prove essential for its understanding. Together with the Radon-Nikodým Property, studied in Section 2.3, it turns out that reflexive Banach spaces and separable dual spaces satisfy the metric approximation property.

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Notation and Terminology

In this thesis we assume that the reader is familiar with the basic results of functional analysis, as can be found in [BKW] as well as with basic results from measure theory, in particular complex measures; see [G], [Ku] and [Ru]. Hence, we will forgo any references.

We are going to summarize all symbols and notational conventions which are found in this thesis. In general, we assume vector spaces and Banach spaces to be spaces over the scalar field $\mathbb{C}$. Given a Banach space X, by X^* we will denote the algebraic dual space, by X' the topological dual space. Usually elements of X, X' and X^* will be denoted by x, x' and x^*, respectively. On rare occasion elements of X' will be denoted by f, g, h, if we want to stress their role as functionals on X rather than their role as elements of another Banach space. Furthermore, we employ the following notation:

- $U_r^X(x)$ denotes the open ball in X with radius r around $x \in X$ and $K_r^X(x)$ the closed ball in X with radius r around x.
- By $\iota_X : X \to X''$ we will denote the natural embedding of X into its bidual X''.
- Given a subset S of X, the relative topology on S is the subspace topology induced by X, i.e. the coarsest topology, such that the natural inclusion map from S into X is continuous.
- $\sigma(X, X')$ denotes the weak topology on X, $\sigma(X', X)$ denotes the weak*-topology on X'.
- Given another Banach space Y, $L(X, Y)$ denotes the vector space of all linear functions on X into Y. $L_b(X, Y)$ denotes the Banach space of all bounded linear functions on X into Y, which is provided with the operator norm.
- We write $X \cong Y$ if X is isometrically isomorphic to Y.
- Given $T \in L_b(X, Y)$, by $T' \in L_b(Y', X')$ we will denote is adjoint operator.
- Given topologies τ_1 on X and τ_2 on Y, a function $f : X \to Y$ is called τ_1-to-τ_2 continuous, if it is continuous with respect to these topologies.
- Given a measure space $(\Omega, \mathcal{A}, \mu)$, by $L_1(\mu)$ we will denote the usual Banach space of equivalence classes of complex valued, measurable functions, whose absolute value is integrable. The corresponding integral is the norm on $L_1(\mu)$. By $L_\infty(\mu)$ we will denote the space of equivalence classes of complex valued, measurable functions, which are essentially bounded, with the essential supremum of the absolute value of the functions as its norm.
- Given $A \subseteq \Omega$, $\mathbb{1}_A$ stands for the characteristic function of A on Ω.
- $\mathbb{N}$ denotes the positive integers, starting with 1, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$.

- ℓ_1 stands for the usual Banach space of absolutely summing sequences. c_0 denotes the Banach space of all null-sequences and ℓ_∞ the Banach space of all bounded sequences.

Chapter 1

Preliminary Results

In this chapter we will introduce definitions and bring results, not inherently related to the upcoming topics. They will occur in the proofs throughout the upcoming sections. Although often cited, the presented results are scarcely proven in their entirety.

1.1 Weakly Compact Operators

We will start of by some of the basic facts about weakly compact operators, first found in [G] and [N]. The proofs in this section, are taken from Section 5.3 of [M], where one can also find other closely related results.

Lemma 1.1.1. Given a Banach space X, the following statements hold true.

(i) The natural embedding ι_X from X into X'' is weak-to-weak* continuous.

(ii) $\iota_X : X \to \iota_X(X)$ is a weak-to-relative weak* homeomorphism.

(iii) The relative weak- and relative weak*-topologies coincide on $\iota_X(X)$. Consequently $\iota_X : X \to \iota_X(X)$ is a weak-to-relative weak homeomorphism.

Proof. Given $x \in X$ and a net $(x_i)_{i \in I}$ for some index set I we have $x_i \xrightarrow{w} x \in X$ if and only if $\langle x_i, x' \rangle \to \langle x, x' \rangle$ for each $x' \in X'$. This is equivalent to $\iota_X(x_i) \xrightarrow{w^*} \iota_X(x)$. Thus ι_X is weak-to-weak* continuous and a weak-to-relative weak* homeomorphism onto $\iota_X(X)$. Since $\iota_X(X)$ is a subspace of X'', we attain from the Hahn-Banach Theorem that the relative-weak topology of $\iota(X)$ as a subspace of X'' coincides with the weak-topology of the space $\iota(X)$, seen as an independent topological space. As $\iota(X)$ is an isomorphic copy of X, the latter is the coarsest topology, such that all elements of $\iota_{X'}(X')$ are continuous. Thus the weak topology on $\iota(X)$ coincides with the relative weak*-topology. □

Lemma 1.1.2. Let X, Y be normed spaces and $T \in L_b(X, Y)$. Then $T'' \circ \iota_X(X) \subseteq \iota_Y(Y)$ and $T = \iota_Y^{-1} \circ T'' \circ \iota_X$.

Proof. Given $x \in X$ we have

$$\langle y', T'' \circ \iota_X(x) \rangle = \langle T'y', \iota_X(x) \rangle = \langle x, T'y' \rangle = \langle Tx, y' \rangle = \langle y', \iota_Y(Tx) \rangle$$

for all $y' \in Y'$. Hence, $T'' \circ \iota_X(x) = \iota_Y \circ T(x) \in \iota_Y(Y)$ implying also $T = \iota_Y^{-1} \circ T'' \circ \iota_X$. □

Definition 1.1.3. Given Banach spaces X, Y, an operator $T : X \to Y$ is called *weakly compact*, if $T(K_1^X(0))$ is relatively compact in $(Y, \sigma(Y, Y'))$.

Lemma 1.1.4. A bounded operator $T : X \to Y$ is weakly compact if and only if $T'' : X'' \to Y''$ has its values in $\iota_Y(Y)$.

Proof. If T is weakly compact, then $K := \overline{T(K_1^X(0))}^{\sigma(Y,Y')}$ is $\sigma(Y, Y')$-compact. Therefore $\iota_Y(K) \subseteq \iota_Y(Y)$ is $\sigma(Y'', Y')$-compact. Since $\iota_X(K_1^X(0))$ is $\sigma(X'', X')$-dense in $K_1^{X''}(0)$, we obtain from the weak*-to-weak* continuity of T''

$$\begin{aligned} T''(K_1^{X''}(0)) &= T''(\overline{\iota_X(K_1^X(0))}^{\sigma(X'',X')}) \subseteq \overline{T''(\iota_X(K_1^X(0)))}^{\sigma(Y'',Y')} \\ &= \overline{\iota_Y(T(K_1^X(0))}^{\sigma(Y'',Y')} \subseteq \overline{\iota_Y(K)}^{\sigma(Y'',Y')} = \iota_Y(K) \subseteq \iota_Y(Y). \end{aligned}$$

For the converse suppose $T''(X'') \subseteq \iota_Y(Y)$. By Lemma 1.1.1 $\iota_Y^{-1} : \iota_Y(Y) \to Y$ is relative weak*-to weak continuous. As T'' is weak*-to-weak* continuous, $\iota_Y^{-1} \circ T''$ is weak*-to-weak continuous. From the weak*-compactness of $K_1^{X''}(0)$ we deduce the weak-compactness of $\iota_Y^{-1} \circ T''(K_1^{X''}(0))$. Thus, its subset $T(K_1^X(0)) = \iota_Y^{-1} \circ T'' \circ \iota_X(K_1^X(0))$ is relatively weakly compact. □

Lemma 1.1.5. Let X, Y be Banach spaces. An operator $T \in L_b(X, Y)$ is weakly compact if and only if its adjoint operator $T' \in L_b(Y', X')$ is weakly compact.

Proof. Suppose T is weakly compact. Given a net $(x_i'')_{i\in I} \in X''$ which is weak*-convergent to $x'' \in X''$, we have $T''x_i'' \xrightarrow{w*} T''x''$. By (iii) from Lemma 1.1.1 in combination with Lemma 1.1.4 this is equivalent to $T''x_i'' \xrightarrow{w} T''x''$. Given $y''' \in Y'''$ we have

$$\langle x_i'', T'''y''' \rangle = \langle T''x_i'', y''' \rangle \longrightarrow \langle T''x'', y''' \rangle = \langle x'', T'''y''' \rangle.$$

Hence $T'''y'''$ is weak*-continuous on X'' and in turn $T'''y''' \in \iota_{X'}(X')$. By Lemma 1.1.4 T' is weakly compact.
For the converse suppose T' is weakly compact. By the preceding paragraph T'' must be weakly compact. Hence, $K := \overline{T''(K_1^{X''}(0))}^{w}$ and, in turn, $K \cap \iota_Y(Y)$ is weakly compact in Y''. By Lemma 1.1.1 ι_Y^{-1} is relative weak-to-weak continuous showing that $\iota_Y^{-1}(K \cap \iota_Y(Y)) = \iota_Y^{-1}(K)$ is weakly-compact in Y. From $T = \iota_Y^{-1} \circ T'' \circ \iota_X$ we conclude that $T(K_1^X(0))$ is a subset of the weakly-compact set $\iota_Y^{-1}(K)$. Consequently, its weak closure is weakly compact. □

1.2 Banach Space Theory

This section is devoted to general results in Banach space theory. The statements and proofs in this section have been assembled from various parts and comments in [DF], [DJT], [DU], [FHHMZ] and [Ry].

Definition 1.2.1. A Banach space Y is called *injective*, if for every Banach space X, every subspace $U \subseteq X$ and every $T \in L_b(U, Y)$ there exists an extension $\tilde{T} \in L_b(X, Y)$ of T, such that $\tilde{T}|_U = T$ and $\|\tilde{T}\| = \|T\|$.

Example 1.2.2. For any probability measure μ the space $L_\infty(\mu)$ is an injective Banach space; see Theorem 4.14 in [DJT].

Lemma 1.2.3. For complex $z_1, \dots, z_n \in \mathbb{C}$, $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} |z_i| = \sup\{ \, |\sum_{i=1}^{n} \lambda_i z_i| : \; \lambda_i \in \mathbb{C}, \; |\lambda_i| = 1\}.$$

The statement holds true even if the λ_i are chosen from a dense subset of $K_1^{\mathbb{C}}(0)$.

Proof. On the one hand,

$$\sup\{ \, |\sum_{i=1}^{n} \lambda_i z_i| : \; \lambda_i \in K_1^{\mathbb{C}}(0)\} \leq \sup\{ \, \big|\sum_{i=1}^{n} |\lambda_i z_i|\big| : \; \lambda_i \in K_1^{\mathbb{C}}(0)\} = \sum_{i=1}^{n} |z_i|.$$

On the other hand, choosing $\tilde{\lambda}_i \in \mathbb{C}$, $|\tilde{\lambda}_i| = 1$, such that $\tilde{\lambda}_i z_i = |z_i|$ for all $i = 1, \dots, n$ gives

$$\sum_{i=1}^{n} |z_i| = \big|\sum_{i=1}^{n} \tilde{\lambda}_i z_i\big| \leq \sup\{ \, |\sum_{i=1}^{n} \lambda_i z_i| : \; \lambda_i \in \mathbb{C}, \; |\lambda_i| = 1\}.$$

For a dense subset D of $K_1^{\mathbb{C}}(0)$ and $\epsilon > 0$ first choosing $\lambda_i \in K_1^{\mathbb{C}}(0)$ such that $\sum_{i=1}^{n} |z_i| \leq |\sum_{i=1}^{n} \lambda_i z_i| + \epsilon/2$, then choosing $\tilde{\lambda}_i \in D$ such that $|\tilde{\lambda}_i - \lambda_i| \leq \epsilon/(2n \cdot \max_j\{|z_j|, 1\})$ yields

$$\begin{aligned}\sum_{i=1}^{n} |z_i| - \epsilon &\leq |\sum_{i=1}^{n} \lambda_i z_i| - \epsilon/2 \leq |\sum_{i=1}^{n} \tilde{\lambda}_i z_i| + \sum_{i=1}^{n} |\tilde{\lambda}_i - \lambda_i| \cdot |z_i| - \epsilon/2 \\ &\leq |\sum_{i=1}^{n} \tilde{\lambda}_i z_i| \leq \sum_{i=1}^{n} |z_i|.\end{aligned}$$

□

Lemma 1.2.4. Let X be a Banach space and K be a compact subset of X. Then for every $\epsilon > 0$ there exist subsets F and L of X, such that

- F is finite and $F \subseteq 2K$;
- L is compact and $L \subseteq K_\epsilon^X(0)$;
- $K \subseteq co(F \cup L)$.

Proof. K being compact we find $x_1, \dots, x_n \in K$, such that $K \subseteq \cup_{i=1}^{n} U_{\epsilon/2}(x_i)$. $F := \{2x_1, \dots, 2x_n\}$ is finite and satisfies $F \subseteq 2K$.
For each $i = 1, \dots, n$ we set $L(x_i) := 2(K - x_i) \cap K_\epsilon^X(0)$ and define $L = \cup_{i=1}^{n} L(x_i)$. Then L is compact as a finite union of compact sets and satisfies $\sup_{x \in L} \|x\| \leq \epsilon$.
In order to show that $K \subseteq co(F \cup L)$ we fix $x \in K$. Then $\|x - x_i\| < \epsilon/2$ for some $i \in \{1, \dots, n\}$. From $\|2(x - x_i)\| < \epsilon$ we derive $2(x - x_i) \in L(x_i)$. $2x_i \in F$ finally yields

$$x = \frac{1}{2}(2x_i + 2x - 2x_i) = \frac{1}{2}(2x_i + 2(x - x_i)) \in co(F \cup L).$$

□

Lemma 1.2.5. Let X be a Banach space and K be a compact subset of X. Then there exists a sequence in X converging in norm to zero, such that K is contained in the closed convex hull of this sequence.

Proof. We will inductively generate two sequences of sets as follows. Applying the previous lemma for the compact set $L_0 := K$ and $\epsilon = 2^{-1}$, we get a finite set $F_1 \subseteq 2K$ and a compact set L_1, such that $\|x\| \leq 2^{-1}$ for all $x \in L_1$ and $K \subseteq co(F_1 \cup L_1)$. We repeat the process for the newly generated compact set L_1 and so on:
Assume that F_i is finite and L_i is compact, $i = 1, \ldots, n$, such that $\|x\| \leq 2^{-i}$ for all $x \in L_i$, $F_i \subseteq 2L_{i-1}$ and $K \subseteq co(\cup_{i=1}^n F_i \cup L_n)$. Applying the previous lemma again to the compact set L_n and $\epsilon = 2^{-(n+1)}$ yields a finite set $F_{n+1} \subseteq 2L_n$ and a compact set L_{n+1}, such that $\|x\| \leq 2^{-(n+1)}$ for all $x \in L_{n+1}$ as well as $L_n \subseteq co(F_{n+1} \cup L_{n+1})$. In consequence,

$$K \subseteq co(\cup_{i=1}^n F_i \cup L_n) \subseteq co(\cup_{i=1}^{n+1} F_i \cup L_{n+1}).$$

By induction we end up with a sequence of finite sets $(F_n)_{n\in\mathbb{N}}$ and a sequence of compact sets $(L_n)_{n\in\mathbb{N}}$, such that $\sup_{x\in L_n}\|x\| \leq 2^{-n}$ as well as $F_n \subseteq 2L_{n-1}$. Therefore, $\cup_{i=1}^\infty F_n$ forms a countable set and, if appropriately listed in a sequence, this sequence tends to zero in norm.
Given $x \in K$ and $\epsilon > 0$, choose $n \in \mathbb{N}$, such that $2^{-n} < \epsilon$. Then we can choose $\lambda_1, \ldots, \lambda_n, \lambda \geq 0$, such that $\lambda + \sum_{i=1}^n \lambda_i = 1$ and $x = \sum_{i=1}^n \lambda_i f_i + \lambda l$ for certain $f_1, \ldots, f_n \in \cup_{i=1}^n F_i$ and $l \in L_n$. From

$$\Big\|x - \sum_{i=1}^n \lambda_i f_i\Big\| = \|\lambda l\| \leq 2^{-n} < \epsilon$$

we conclude that K is contained in the closed convex hull of that sequence. □

The next results involve Minkowski functionals. We assume the reader to be familiar with their basic properties as can be found in Section 5.1 of [BKW].

Lemma 1.2.6. Let K be a closed, bounded, convex and circular subset of a Banach space X and set $Y := \mathrm{span}(K)$. Then the following statements hold true.

(i) The Minkowski functional $\mu_K(y) = \inf\{t > 0 : \; y \in tY\}$ defines a norm on Y such that the embedding $\iota : Y \to X$ is bounded.

(ii) Y is complete under this norm. Moreover, if $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y then it is a Cauchy sequence in X. The corresponding limits coincide.

(iii) If $A \subseteq Y$ is closed in X, it is closed in Y.

(iv) K coincides with the closed unit ball of Y.

Proof. (i): As K is convex and circular we have $Y = \cup_{t>0} tK$. Hence, the Minkowski functional μ_K is well defined and constitutes a seminorm on Y. Given $y \in Y$ satisfying $\mu_K(y) = 0$, there exists a null-sequence $(t_n)_{n\in\mathbb{N}} \in (0, +\infty)$, such that $(1/t_n)y \in K$ for all $n \in \mathbb{N}$. If we had $\|y\|_X > 0$, then $\|(1/t_n)y\|_X \overset{n\to\infty}{\longrightarrow} +\infty$ would imply that K is unbounded.

Hence $\|y\|_X = 0$, which identifies μ_K as a norm on Y. As $\{y \in Y : \mu_K(y) < 1\} \subseteq K$, the embedding $\iota : Y \to X$ satisfies

$$\|\iota\| = \sup\{\|y\|_X : \mu_K(y) < 1\} \leq \sup\{\|y\|_X : y \in K\} < +\infty.$$

(ii): Suppose that with respect to μ_K there exists a Cauchy sequence $(y_n)_{n\in\mathbb{N}} \in Y$ which does not converge. We choose positive constants L, $M > 0$, such that $\mu_K(y_n) < L$ for all $n \in \mathbb{N}$, as well as $K \subseteq K^X_M(0)$. The former inequality implies the existence of $t_n < L$, such that $(1/t_n)y_n \in K$ for all $n \in \mathbb{N}$. As K is assumed to be circular we obtain

$$\frac{1}{L} \cdot y_n = \frac{t_n}{L}\big(\frac{1}{t_n} \cdot y_n\big) \in K \quad \text{for all } n \in \mathbb{N}.$$

Hence, by replacing $(y_n)_{n\in\mathbb{N}}$ with the sequence $((1/L)y_n)_{n\in\mathbb{N}}$ we can assume without loss of generality that $y_n \in K$ for all $n \in \mathbb{N}$.
ι being bounded implies that $(y_n)_{n\in\mathbb{N}}$ is Cauchy in X and thus converges to $y \in K$. We define $x_n := y_n - y$, $n \in \mathbb{N}$. By assumption $(x_n)_{n\in\mathbb{N}}$ is Cauchy in Y, but does not converge there. In particular, it does not converge towards zero. Consequently, there exists a constant $C > 0$ and a subsequence $(x_{n(k)})_{k\in\mathbb{N}}$, such that $\mu_K(x_{n(k)}) > C$ for all $k \in \mathbb{N}$. On the other hand, as $(x_n)_{n\in\mathbb{N}}$ is Cauchy in Y, there exists $N \in \mathbb{N}$, such that $\mu_K(x_n - x_m) < C$ for all $n, m \geq N$. This means $x_n - x_m \in C \cdot K$ for all $n, m \geq N$. Taking the limit $m \to \infty$ in X we attain $x_n \in C \cdot K$. Thus, $\mu_K(x_n) \leq C$ for all $n \geq N$, which contradicts $\mu_K(x_{n(k)}) > C$ for all $k \in \mathbb{N}$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is a null-sequence in Y and, as a consequence, $(y_n)_{n\in\mathbb{N}}$ converges to y in Y.
(iii) is a simple consequence of $\iota : Y \to X$ being continuous. As K is convex we have $\{y \in Y : \mu_K(y) < 1\} \subseteq K \subseteq \{y \in Y : \mu_K(y) \leq 1\}$ and (iv) follows immediately from (iii). □

Corollary 1.2.7. Let X be a Banach space and K be a compact subset of X. Then there exists a subspace Y of X, such that $(Y, \|\cdot\|_Y)$ is a Banach space, K is compact in Y and the embedding mapping $\iota : Y \to X$ is compact.

Proof. By Lemma 1.2.5 there exists a null-sequence $(x_n)_{n\in\mathbb{N}}$, such that K is contained in its closed convex hull. Choose a sequence $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}$ converging to $+\infty$, such that $(\lambda_n x_n)_{n\in\mathbb{N}}$ still is a null-sequence. The closed, convex and circular hull L of $(\lambda_n x_n)_{n\in\mathbb{N}}$ is compact. From the previous lemma we attain a Banach space $Y \subseteq X$, such that L coincides with the closed unit ball of Y. Here Y is the linear span of L and the norm on Y is the Minkowski functional of L. In consequence, $\iota : Y \to X$ is compact. Since for all $n \in \mathbb{N}$ we have

$$\|x_n\|_Y = \mu_L(x_n) \leq \lambda_n^{-1},$$

the closure $\overline{V}^Y$ of $V := co(\{x_n : n \in \mathbb{N}\})$ is compact in Y. Given $y \in K$ there exists a sequence $(y_n)_{n\in\mathbb{N}} \in V$ converging to y in X. As the closure of V is compact in Y we obtain a subsequence $(y_{n(k)})_{k\in\mathbb{N}}$ with limit $\tilde{y} \in \overline{V}^Y$. Applying (ii) of the previous lemma we obtain $\tilde{y} = y$. Thus $K \subset \overline{V}^Y$, and by (iii) from the previous lemma K is closed and thus compact even in Y. □

Definition 1.2.8. A series $\sum_{i=1}^{\infty} x_i$ in a Banach space X is subseries convergent, if $\sum_{k=1}^{\infty} x_{i(k)}$ is convergent for any subsequence $(x_{i(k)})_{k\in\mathbb{N}}$ of $(x_i)_{i\in\mathbb{N}}$.

For a proof of the following result see theorems 1.5 and 1.8 in [DJT].

Theorem 1.2.9 (Orlicz-Pettis)**.** Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in a Banach space X. Then $\sum_{i\in\mathbb{N}} x_i$ is unconditionally convergent if and only if $\sum_{i=1}^{\infty} x_i$ is subseries convergent with respect to the weak topology on X.

1.3 Measure Theory

Throughout this thesis we will make use of complex regular measures. We shall give a quick review on the important facts about such measures needed in the upcoming chapters. Given a complex measure μ on a measurable space $(\Omega, \mathcal{A})$ its variation $|\mu| : \mathcal{A} \to [0, +\infty]$ is defined by

$$|\mu|(A) := \sup\{\sum_{i=1}^{\infty} |\mu(A_i)| : \ A_k \in \mathcal{A}, \ k \in \mathbb{N}, \ \sum_{i=1}^{\infty} A_i = A\}$$

and constitutes a positive finite measure on $(\Omega, \mathcal{A})$. Moreover, for each such μ there exists a complex integrable function ϕ, $|\phi| = 1$ almost everywhere, such that

$$\int_{\Omega} f \ d\mu = \int_{\Omega} f\phi \ d|\mu|, \tag{1.1}$$

see Sections 3.1 und 3.2 in [Ka3]. This justifies $L_p(\mu) := L_p(|\mu|)$. Given a $\sigma-$finite measure ν, we can identify the dual space of $L_1(\nu)$ with $L_\infty(\nu)$; see Satz 13.40 in [Ku]. Hence, for a complex measure μ we have $L_1(|\mu|)' \cong L_\infty(|\mu|)$ and $L_\infty(\mu) \ni g \mapsto (f \mapsto \int fg \ d|\mu|) \in L_1(\mu)'$ constitutes an isometric isomorphism.

Lemma 1.3.1. Given a complex measure μ we have $L_1(\mu)' \cong L_\infty(\mu)$, $L_\infty(\mu) \ni g \mapsto (f \mapsto \int fg \ d\mu) \in L_1(\mu)'$ constituting an isometric isomorphism.

Proof. If ϕ is as in (1.1), $g \mapsto g\phi$ constitutes an isometric isomorphism from $L_\infty(\mu)$ into $L_\infty(\mu)$. Consequently, $g \mapsto (f \mapsto \int fg\phi \ d|\mu|) = (f \mapsto \int fg \ d\mu)$ constitutes an isometric isomorphism as well. □

Lemma 1.3.2. If $(\Omega, \mathcal{C}, \mu)$ is a finite measure space and $\mathcal{B}$ an algebra which generates $\mathcal{C}$, then for every $C \in \mathcal{C}$ and $\epsilon > 0$ there exists $B \in \mathcal{B}$, such that $\mu(C\Delta B) < \epsilon$.

Proof. We define

$$\mathcal{A} := \{C \in \mathcal{C} : \ \forall\epsilon > 0 \exists B \in \mathcal{B} \text{ with } \mu(C\Delta B) < \epsilon\}.$$

As $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{C}$ it suffices to show that $\mathcal{A}$ constitutes a σ-algebra.
(i): $\mathcal{B}$ being an algebra implies $\Omega \in \mathcal{B}$ and in turn $\Omega \in \mathcal{C}$. (ii): Given $A \in \mathcal{A}$ and $\epsilon > 0$ choose $B \in \mathcal{B}$, such that $\mu(A\Delta B) < \epsilon$. Then $\mu(A^c\Delta B^c) = \mu(A\Delta B) < \epsilon$ and $A^c \in \mathcal{A}$ since $\epsilon > 0$ was arbitrary.

(iii): Given A_1, $A_2 \in \mathcal{A}$ and $\epsilon > 0$ choose B_1, $B_2 \in \mathcal{B}$, such that $\mu(A_1 \Delta B_1)$, $\mu(A_2 \Delta B_2) < \epsilon/2$. Because of $(A_1 \cup A_2)\Delta(B_1 \cup B_2) \subseteq (A_1 \Delta B_1) \cup (A_2 \Delta B_2)$ we have

$$\mu((A_1 \cup A_2)\Delta(B_1 \cup B_2)) \leq \mu(A_1 \Delta B_1) + \mu(A_2 \Delta B_2) < \epsilon.$$

Hence, $A_1 \cup A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = (A^c \cup B^c)^c \in \mathcal{A}$.
(iv): Finally, let $(A_i)_{i \in \mathbb{N}} \in \mathcal{A}$ be pairwise disjoint and $\epsilon > 0$. Choose $n \in \mathbb{N}$, such that $\mu(\cup_{i=n+1}^{\infty} A_i) < \epsilon/2$. For $i = 1, \ldots n$ there exist $B_i \in \mathcal{B}$, such that $\mu(A_i \Delta B_i) < \epsilon/(2n)$. Because of

$$\begin{aligned}
\mu\Big((\bigcup_{i=1}^{\infty} A_i)\Delta(\bigcup_{i=1}^{n} B_i)\Big) &\leq \mu\Big(\big[(\bigcup_{i=1}^{n} A_i)\Delta(\bigcup_{i=1}^{n} B_i)\big] \cup (\bigcup_{i=n+1}^{\infty} A_i)\Big) \\
&\leq \mu(\bigcup_{i=1}^{n} A_i \Delta B_i) + \mu(\bigcup_{i=n+1}^{\infty} A_i) \\
&\leq \sum_{i=1}^{n} \mu(A_i \Delta B_i) + \epsilon/2 < \epsilon.
\end{aligned}$$

$\cup_{i=1}^{\infty} A_i$ belongs to $\mathcal{A}$. $\square$

Chapter 2

Vector Measures and the Bochner Integral

In this chapter we introduce the principal definitions and properties of vector measures. We will establish a theory with notions and results very similar to the already known results of standard measure theory. We will study countably additive vector measures and their variations and how a basic integral can be defined and introduce the representing measure. The third section of this chapter examines how we can define an integral of a Banach space valued function with respect to a finite measure μ. This leads to the notion of the Bochner integral. These results will be combined in the last section where we study the Radon-Nikodým Property for Banach spaces.

2.1 Introduction to Vector Measures

The first section is based upon the results in [BDS], the structure and definitions are orientated on [Ry]. The proofs can be found in Chapter 5 of [Ry] and Chapter III of [DS]. For a more general view on vector measures a look into [DU] is recommended.

Definition 2.1.1. Let $(\Omega, \mathcal{A})$ be a measurable space. A function m on $\mathcal{A}$ with values in a Banach space X is called a *finitely additive vector measure*, if $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ holds true for all disjoint $A_1, A_2 \in \mathcal{A}$.
m is called a *countably additive vector measure*, or simply a *vector measure*, if $m(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i=1}^{\infty} m(A_i)$ for every sequence $(A_i)_{i \in \mathbb{N}}$ of pairwise disjoint sets $A_i \in \mathcal{A}$. Here this sequence converges with respect to the norm on X.
We denote the set of all vector measures from $\mathcal{A}$ to X by $\mathcal{M}(\mathcal{A}, X)$. In the case $X = \mathbb{C}$ we simply write $\mathcal{M}(\mathcal{A})$.

Definition 2.1.2. Let $(\Omega, \mathcal{A})$ be a measurable space and m a be (finitely additive) vector measure on $\mathcal{A}$ with values in a Banach space X. The mapping $|m| : \mathcal{A} \to [0, +\infty]$ defined by

$$|m|(A) = \sup\{\sum_{i=1}^{n} \|m(A_i)\| : \text{ for pairwise disjoint } A_i \in \Sigma, \bigcup_{i=1}^{n} A_i = A\},$$

is called the *variation of* m and $\|m\|_1 := |m|(\Omega)$ is called the *variation norm* of m.
If $\|m\|_1 < +\infty$, we say that m is a *measure of bounded variation*.

Remark 2.1.3. If m is a vector measure of bounded variation, then the variation $|m|$ defines a finite, positive measure on $\mathcal{A}$; see Proposition I.1.9 in [DU].

Proposition 2.1.4. For a measurable space $(\Omega, \mathcal{A})$ the set of all vector measures of bounded variation on $\mathcal{A}$ with values in a Banach space X endowed with the variation norm constitutes a Banach space. We denote this Banach space by $\mathcal{M}_1(\mathcal{A}, X)$.

Proof. It is easy to check, that the set of all vector measures of bounded variation with the usual addition and scalar multiplication forms a vector space, and that the variation norm is in fact a norm. It remains to show that this space is complete.
Assume $(m_k)_{k\in\mathbb{N}}$ to be a Cauchy-sequence. Clearly, there exists an $M \in (0, +\infty)$, such that $|m_k|(A) \leq \|m_k\|_1 < M$ for all $A \in \mathcal{A}$ and $k \in \mathbb{N}$.
For $\epsilon > 0$ choose $N \in \mathbb{N}$, such that $\|m_k - m_l\|_1 < \epsilon$ for all $k, l \geq N$. Given $A \in \mathcal{A}$ it follows that

$$\|m_k(A) - m_l(A)\|_X = \|(m_k - m_l)(A)\|_X \leq |m_k - m_l|(A) \leq \|m_k - m_l\|_1 < \epsilon, \quad k, l \geq N.$$

Consequently, $(m_k(A))_{k\in\mathbb{N}}$ being a Cauchy-sequence in X we can define

$$m(A) := \lim_{k\to\infty} m_k(A), \;\; A \in \mathcal{A}.$$

It remains to show that m is indeed a finitely additive vector measure, has bounded variation, $\lim_{k\to\infty}\|m - m_k\|_1 = 0$ and that m is countably additive.
The first assertion immediately follows from the fact that m is X-valued and that limits are compatible with addition.
In order to show bounded variation, let $A_1, A_2, \ldots, A_n \in \mathcal{A}$ be finitely many disjoint sets with $\cup_{i=1}^n A_i = \Omega$. Choosing $k \in \mathbb{N}$, such that $\sum_{i=1}^n \|m(A_i) - m_k(A_i)\|_X < 1$, we obtain

$$\begin{aligned}\sum_{i=1}^n \|m(A_i)\|_X &\leq \sum_{i=1}^n \Big(\|m_k(A_i)\|_X + \|m(A_i) - m_k(A_i)\|_X\Big) \\ &\leq |m_k|(\Omega) + 1 < M + 1 < +\infty.\end{aligned}$$

Taking the supremum over over all appropriate $A_1, \ldots, A_n \in \mathcal{A}$ yields $|m|(\Omega) = \|m\|_1 < +\infty$.
For $\lim_{k\to\infty}\|m_k - m\|_1 = 0$ let $\epsilon > 0$ and choose $N \in \mathbb{N}$, such that $\|m_k - m_l\|_1 < \epsilon$, $k, l \geq N$. If $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint with $\Omega = \cup_{i=1}^n A_i$, then for $k, l \geq N$ we have

$$\sum_{i=1}^n \|m_k(A_i) - m_l(A_i)\|_X \leq |m_k - m_l|(\Omega) = \|m_k - m_l\|_1 < \epsilon.$$

Taking the limit $l \to \infty$ we obtain

$$\sum_{i=1}^n \|m_k(A_i) - m(A_i)\|_X \leq \epsilon.$$

and then taking the supremum over all appropriate sequences $A_1, \ldots, A_n \in \mathcal{A}$ yields $\|m - m_k\|_1 \leq \epsilon$ for all $k \geq N$.

Lastly, let $(A_i)_{i\in\mathbb{N}}$ be a pairwise disjoint sequence of sets from $\mathcal{A}$ and $A := \cup_{i\in\mathbb{N}} A_i$. Given $\epsilon > 0$ choose $k \in \mathbb{N}$, such that $\|m_k - m\|_1 < \epsilon/2$. We infer

$$\|m_k(A) - m(A)\|_X \leq |m_k - m|(A) \leq \|m_k - m\|_1 < \epsilon/2,$$

as well as

$$\Big\| \sum_{i=1}^{n} \Big(m_k(A_i) - m(A_i)\Big) \Big\|_X \leq \sum_{i=1}^{n} \|m_k(A_i) - m(A_i)\|_X \leq |m_k - m|(A) < \epsilon/2$$

for any $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}
\Big\|m(A) - \sum_{i=1}^{n} m(A_i)\Big\|_X &\leq \Big\|m(A) - m_k(A) + m_k(A) - \sum_{i=1}^{n}(m(A_i) - m_k(A_i) + m_k(A_i))\Big\|_X \\
&\leq \|m(A) - m_k(A)\| \\
&\qquad + \Big\|m_k(A) - \sum_{i=1}^{n} m_k(A_i)\Big\|_X + \Big\|\sum_{i=1}^{n}(m_k(A_i) - m(A_i))\Big\|_X \\
&\leq \Big\|m_k(A) - \sum_{i=1}^{n} m_k(A_i)\Big\|_X + \epsilon.
\end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \Big\|m(A) - \sum_{i=1}^{n} m(A_i)\Big\|_X \leq \epsilon.$$

As $\epsilon > 0$ was chosen arbitrarily, we obtain $\sum_{i=1}^{\infty} m(A_i) = \mu(A)$, i.e. μ is countably additive. □

Definition 2.1.5. Let $(\Omega, \mathcal{A})$ be a measurable space and m be a vector measure on $\mathcal{A}$ with values in a Banach space X. We call $|m|_\infty : \mathcal{A} \to [0, +\infty]$ defined by

$$|m|_\infty(A) = \sup\{|x'm|(A) : \; x' \in X', \; \|x'\| \leq 1\},$$

the *semivariation* of m, where $|x'm|$ denotes the variation of the complex measure $x'm$. We call $\|m\|_\infty := |m|_\infty(\Omega)$ the semivariation norm of m.

Remark 2.1.6. If $A = \cup_{i=1}^{n} A_i$ for pairwise disjoint $A_1, \ldots, A_n$ we have

$$|x'm|(A) = \sum_{i=1}^{n} |x'm|(A_i) \leq \sum_{i=1}^{n} \|m(A_i)\|.$$

Taking the supremum over all $x' \in X'$ and then over all pairwise disjoint $A_1, \ldots, A_n$ as above we get $|m|_\infty(A) \leq |m|(A)$.

In Definition 2.1.5 and Remark 2.1.6 we implicitly assumed that $x'm$ constitutes a complex measure and the variation $|x'm|$ a finite positive measure. The simple verification of this fact is found in the proof of the following lemma.

Lemma 2.1.7. Let $(\Omega, \mathcal{A})$ be a measurable space and m be a vector measure on $\mathcal{A}$ with values in a Banach space X. Then $x'm$ is a complex measure for every $x' \in X'$ and

$$T_m : \begin{cases} X' & \to & \mathcal{M}(\mathcal{A}), \\ x' & \mapsto & x'm. \end{cases}$$

is a bounded, linear operator. Moreover, $|m|_\infty(\Omega) = \|T_m\|$.

Proof. As m is countably additive, we derive

$$x'm(\bigcup_{i=1}^{\infty} A_i) = x'(m(\bigcup_{i=1}^{\infty} A_i)) = x'(\sum_{i=1}^{\infty} m(A_i)) = \sum_{i=1}^{\infty} x'(m(A_i)) = \sum_{i=1}^{\infty} x'm(A_i),$$

for any $x' \in X'$ and any sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$, i.e. $x'm$ is a complex measure for all $x' \in X'$. It follows that T_m is a well defined linear mapping. Consider a sequence $(x'_n)_{n\in\mathbb{N}} \in X'$ converging to $x' \in X'$ and suppose $\lim_{n\to\infty} T(x'_n) =: \mu$ exists in $\mathcal{M}(\mathcal{A})$. For $A \in \mathcal{A}$ we have

$$T_m(x')(A) = x'(m(A)) = \lim_{n\to\infty} x'_n(m(A)) = \lim_{n\to\infty} T_m(x'_n)(A) = \mu(A).$$

Hence, T_m is a closed operator. By the Closed Graph Theorem it is bounded with $\|T_m\| = \sup\{|x'm|(\Omega) : \|x'\| \leq 1\} = |m|_\infty(\Omega)$. □

Corollary 2.1.8. The semivariation of a vector measure m takes its values in a bounded subset of $\mathbb{R}$.

Remark 2.1.9. The result of Corollary 2.1.8 is remarkable. All vector measures are naturally of bounded semivariation. The fact that vector measures are functions on a σ-Algebra guarantees that the objects $x'm$ are elements of a Banach space and the Closed Graph Theorem is applicable.
We might also take a more general approach as seen in [DU], in which a vector measure is a function on a set $\mathcal{F}$ of subsets of a set Ω, where $\mathcal{F}$ does not necessarily have to be a σ-Algebra. The function F on the fields of subsets of $\mathbb{N}$, that are finite or have finite complement, defined by

$$F(A) := \begin{cases} |A| & \text{if } A \text{ is finite,} \\ |A^c| & \text{if } \mathbb{N} \setminus A \text{ is finite,} \end{cases}$$

is a real valued vector measure in the sense of [DU] with unbounded semivariation.

The next statements are fundamental results about the properties of vector measures. Due to their lenghts we omit their proofs, that can be found for example in [BDS], Theorem 1.3 in combination with Lemma 2.3 and Theorem 2.9.

Lemma 2.1.10. Let $(\Omega, \mathcal{A})$ be a measurable space and X be a Banach space. Given a vector measure $m : \mathcal{A} \to X$, the set $\{x'm : x' \in K_1^{x'}(0)\}$ of complex measures is a relatively weakly compact subset of $\mathcal{M}(\mathcal{A})$. In particular, the operator T_m from Lemma 2.1.7 is weakly compact.

Theorem 2.1.11. Let $(\Omega, \mathcal{A})$ be a measurable space and X be a Banach space. Given a vector measure $m : \mathcal{A} \to X$, the range of m is a weakly compact subset of X.

Definition 2.1.12. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X be a Banach space. A vector measure $m : \mathcal{A} \to X$ is called *absolutely continuous* with respect to μ, or *μ-continuous*, if $\mu(A) = 0$ implies $m(A) = 0$ for all $A \in \mathcal{A}$.

We will finish this section about vector measures by introducing an integral for such a measure. We will start of with some considerations.

Remark 2.1.13. Let m be a vector measure on a measurable space $(\Omega, \mathcal{A})$ with values in a Banach space X. Given a simple function $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, where $a_i \in \mathbb{C}$ and the sets $A_i \in \mathcal{A}$ are pairwise disjoint, we define

$$S(f) := \sum_{i=1}^{n} a_i m(A_i).$$

It is easy to verify that this definition is independent from the explicit representation $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$ and that S constitutes a linear operator on the set of all simple functions. Because of

$$\begin{aligned} \|Sf\| &= \Big\| \sum_{i=1}^{n} a_i m(A_i) \Big\| \leq \|f\|_\infty \cdot \Big\| \Big(\sum_{i=1}^{n} \frac{a_i}{\|f\|_\infty} m(A_i) \Big) \Big\| \\ &= \|f\|_\infty \cdot \sup_{x' \in K_1^{X'}(0)} \Big| \sum_{i=1}^{n} \frac{a_i}{\|f\|_\infty} x' m(A_i) \Big| \\ &\leq \|f\|_\infty \cdot \sup_{x' \in K_1^{X'}(0)} \sum_{i=1}^{n} |x' m(A_i)| \qquad (2.1) \\ &\leq \|f\|_\infty \cdot |m|_\infty \Big(\bigcup_{i=1}^{n} A_i \Big) \leq \|f\|_\infty \cdot \|m\|_\infty, \end{aligned}$$

S is bounded with respect to the supremum norm on the space of all simple functions with $\|S\| \leq \|m\|_\infty$. Consider the space $B(\mathcal{A})$ of all bounded complex valued functions on Ω which are measurable with respect to $\mathcal{A}$ provided with the supremum norm. Since the simple functions are dense in $B(\mathcal{A})$, S extends to a continuous operator from $B(\mathcal{A})$ into X, also denoted by S, with $\|S\| \leq \|m\|_\infty$.

Definition 2.1.14. Let $(\Omega, \mathcal{A})$ be a measurable space, X be a Banach space and $m : \mathcal{A} \to X$ be a vector measure. Let $S : B(\mathcal{A}) \to X$ be the operator from the previous remark. We define the *integral* of $f \in B(\mathcal{A})$ over $A \in \mathcal{A}$ by

$$\int_A f \; dm := S(\mathbb{1}_A f) \in X.$$

Proposition 2.1.15. Given a measurable space $(\Omega, \mathcal{A})$, a Banach space X, a vector measure $m : \mathcal{A} \to X$ and $f \in B(\mathcal{A})$ the following statements hold true:

(i) If Y is another Banach space and $B \in L_b(X, Y)$ then $B \circ m : \mathcal{A} \to Y$ constitutes a vector measure satisfying $B(\int_\Omega f \; dm) = \int_\Omega f \; d(B \circ m)$, $|B \circ m|(A) \leq \|B\| \cdot |m|(A)$ and $|B \circ m|_\infty \leq \|B\| \cdot |m|_\infty(A)$ for all $A \in \mathcal{A}$.

(ii) $x'(\int_\Omega f \; dm) = \int_\Omega f \; d(x'm)$ for all $x' \in X'$;

(iii) $\big\| \int_A f \; dm \big\|_X \le \|f\|_\infty |m|_\infty(A)$ for all $A \in \mathcal{A}$.

(iv) $\big\| \int_\Omega f \; dm \big\|_X \le \int_\Omega |f| \; d|m|$ in the case that m is of bounded variation

Proof. (i): If $\cup_{i=1}^\infty A_i = A \in \mathcal{A}$ for pairwise disjoint $(A_i)_{i\in\mathcal{A}} \in \mathcal{A}$ then

$$(B \circ m)(A) = B(m(A)) = B(\sum_{i=1}^\infty m(A_i)) = \sum_{i=1}^\infty B(m(A_i)) = \sum_{i=1}^\infty (B \circ m)(A_i),$$

thus $B \circ m$ constitutes a vector measure. For a simple function $g = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$ with $a_1, \dots, a_n \in \mathbb{C}$, $A_1, \dots, A_n \in \mathcal{A}$, we have

$$B\big(\int_\Omega g \; dm\big) = B\big(\sum_{i=1}^n a_i \cdot m(A_i)\big) = \sum_{i=1}^n a_i \cdot (B \circ m)(A_i) = \int_\Omega g \; d(B \circ m).$$

The general statement for $f \in B(\mathcal{A})$ follows from the continuity of the operator S from Remark 2.1.13 and the density of all simple functions in $B(\mathcal{A})$.
Given $y' \in Y'$ $y' \circ B$ is an element of X' with $\|y' \circ B\| \le \|B\|$. Hence, for $A = \cup_{i=1}^n A_i \in \mathcal{A}$ with pairwise disjoint $A_1, \dots, A_n \in \mathcal{A}$

$$\begin{aligned}\sum_{i=1}^n |(y' \circ B \circ m)(A_i)| &\le \sup\big\{\sum_{i=1}^n |(x' \circ m)(A_i)| : \; x' \in X', \; \|x'\| \le B\big\}\\ &= \|B\| \cdot \sup\big\{\sum_{i=1}^n |x' m(A_i)| : \; x' \in X', \; \|x'\| \le 1\big\}\\ &\le \|B\| \cdot \sup\{|x'm|(A) : \; x' \in X', \; \|x'\| \le 1\big\} = \|B\| \cdot |m|_\infty(A).\end{aligned}$$

Taking the supremum over all such $A_1, \dots A_n$, $n \in \mathbb{N}$, then over all $y' \in K_1^{Y'}(0)$ yields $|B \circ m|_\infty(A) \le \|B\| \cdot |m|_\infty(A)$. Similarly,

$$\sum_{i=1}^n \|(B \circ m)(A_i)\| \le \|B\| \cdot \sum_{i=1}^n \|m(A_i)\| \le \|B\| \cdot |m|(A)$$

shows $|B \circ m|(A) \le \|B\| \cdot |m|(A)$, $A \in \mathcal{A}$.
(ii) is a special case of (i).
(iii) is a consequence of (2.1) and the density of simple functions in $B(\mathcal{A})$.
(iv): As m is of bounded variation, $|m|$ constitutes a positive measure, and $\int_\Omega |f| \; d|m|$ is well-defined. We first prove the statement for simple functions. Given $a_1, \dots, a_n \in \mathbb{C}$ and pairwise disjoint $A_1, \dots, A_n \in \mathcal{A}$ for $g = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$ we have

$$\begin{aligned}\big\| \int_\Omega g \; dm \big\|_X = \big\| \sum_{i=1}^n a_i m(A_i) \big\|_X &\le \sum_{i=1}^n \big(|a_i| \cdot \|m(A_i)\|_X\big)\\ &\le \sum_{i=1}^n \big(|a_i| \cdot |m|(A_i)\big) = \int_\Omega \big(\sum_{i=1}^n |a_i| \mathbb{1}_{A_i}\big) \; d|m|\\ &= \int_\Omega |g| \; d|m|.\end{aligned}$$

The general statement again follows from the continuity of the integral. □

2.2 The representing measure

The present section will introduce a first application of vector measures and is based on Section VI.2 in [DU]. Here, the representing measure will be examined, which later turns out to be the link between the operator classes introduced in chapter 5, whose relation will be studied in detail in chapter 6.

We start this sections off with some considerations. Let K be a compact Hausdorff space and X be a Banach space. According to the Riesz Representation Theorem for linear functionals on $C(K)$ each element of $C(K)'$ can be identified with a unique complex regular measure μ on the σ-Algebra $\mathfrak{B}(K)$ of Borel subsets of K. Further, for $A \in \mathfrak{B}(K)$ the function $\phi_A(\mu) = \mu(A)$ clearly constitutes a bounded linear functional on the set of all such measures and thus can be seen as an element of $C(K)''$.
For a bounded linear operator $T : C(K) \to X$ its adjoint T' acts on X' and has values in $C(K)'$. Hence, $T'(x')$ can be understood as a complex regular Borel measure on K and

$$T'x'(A) = \langle T'(x'), \phi_A \rangle = \langle x', T''(\phi_A) \rangle \tag{2.2}$$

for all $A \in \mathfrak{B}(K)$

Theorem 2.2.1. Let K be a compact Hausdorff space and X be a Banach space and denote with $\mathfrak{B}(K)$ the Borel subsets of K. Given $T \in L_b(C(K), X)$ the set function $m_T : \mathfrak{B}(K) \to X''$, $m_T(A) = T''(\phi_A)$, where $\phi_A(\mu) = \mu(A)$, has the following properties.

(i) $\iota_{X'}(x')m_T = [A \mapsto \langle x', m_T(A) \rangle]$ is a complex regular Borel measure on K for all $x' \in X'$, i.e. m_T is weak*-regular, where $\iota_{X'} : X' \to X'''$ denotes the canonical embedding.

(ii) $\langle T(f), x' \rangle = \int_K f \; d(\iota_{X'}(x')m_T)$, for all $f \in C(K)$, $x' \in X'$

(iii) m_T is finitely additive.

(iv) If m_T is a vector measure, then $\iota_X \circ T(f) = \int f \; dm_T$ for all $f \in C(K)$, where $\iota_X : X \to X''$ denotes the canonical embedding and the integral has to be understood in the sense of Definition 2.1.14.

We call m_T the *representing measure* of the operator T.

Proof. (2.2) can be written as

$$\iota_{X'}(x')m_T(A) = \langle x', m_T(A) \rangle = T'x'(A).$$

As noted in the preluding paragraph, $T'x'$ is a complex regular Borel measure proving the first point. The second point follows from

$$\langle Tf, x' \rangle = \langle f, T'x' \rangle = \int_K f \; d(T'x') = \int_K f \; d(\iota_{X'}(x')m_T)$$

and from the linearity of T'' we immediately obtain (iii).

(iv). With (ii) from Proposition 2.1.15 we get

$$\begin{aligned}\langle x', \iota_X \circ T(f)\rangle = \langle T(f), x'\rangle &= \int_K f \ d(\iota_X(x')m_T) \\ &= \iota_{X'}(x')(\int_K f \ dm_T) = \langle x', \int_K f \ dm_T\rangle\end{aligned}$$

for all $x' \in X'$ and $f \in C(K)$. □

Lemma 2.2.2. Let K be a compact Hausdorff space, X be a Banach space and $T \in L_b(C(K), X)$. If the representing measure $m_T : \mathfrak{B}(K) \to X''$ is of bounded variation, then it constitutes a vector measure.

Proof. Given a sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint Borel sets, we have

$$\sum_{i=1}^{\infty} \|m_T(A_i)\|_{X''} \leq |m_T|(\bigcup_{i=1}^{\infty} A_i) \leq \|m_T\| < +\infty.$$

Thus, $\sum_{i=1}^{\infty} m_T(A_i)$ is absolutely convergent and in consequence unconditionally convergent. Given $x' \in X'$, $\iota'_X(x')m_T$ constitutes a complex regular Borel measure. From

$$\langle x', \sum_{i=1}^{\infty} m_T(A_i)\rangle = \sum_{i=1}^{\infty} \iota_{X'}(x')m_T(A_i) = \iota_{X'}(x')m_T(\bigcup_{i\in\mathbb{N}} A_i) = \langle x', m_T(\bigcup_{i\in\mathbb{N}} A_i)\rangle$$

for all $x' \in X'$ we conclude $\sum_{i=1}^{\infty} m_T(A_i) = m_T(\cup_{i\in\mathbb{N}} A_i)$. Hence, m_T is countably additive and this was the definition of m_T being a vector measure. □

Theorem 2.2.3. Let K be a compact Hausdorff space, X be a Banach space and $T \in L_b(C(K), X)$ be weakly compact. Then the representing measure m_T of T has the following properties:

(i) m_T takes its values in $\iota(X)$, where $\iota : X \to X''$ denotes the canonical embedding. Moreover, it constitutes a vector measure.

(ii) $Tf = \int_K f \ d(\iota_X^{-1} \circ m_T)$ for all $f \in C(K)$; in the sense of Definition 2.1.14.

(iii) $\|T\| = \|m_T\|_\infty$.

Conversely, given a vector measure m on the Borel subsets of K with values in X, $Tf := \int_K f \ dm$ defines a weakly compact operator from $C(K)$ into X satisfying $m_T = \iota_X \circ m$; see Definition 2.1.14.

Proof. (i): If T is a weakly compact operator then by Lemma 1.1.4 T'' takes its values in $\iota_X(X)$. As $m_T(A) = T''(\phi_A)$ for all $A \in \mathfrak{B}(K)$, m_T takes all its values in $\iota_X(X)$. By Theorem 2.2.1 m_T is weak*-countably additive. Hence, $\iota_X^{-1} \circ m_T$ is weakly-countably additive; see Lemma 1.1.1. Thus, by the Theorem of Orlicz-Pettis, Theorem 1.2.9, $\iota_X^{-1} \circ m_T$ and, in turn, m_T is countably additive and constitutes a vector measure.
(ii): As m_T constitutes a vector measure, the integral over bounded and measurable complex valued functions as in Definition 2.1.14 together with (2.2) implies

$$\int_K \mathbb{1}_A \ d(T'x') = \langle x', \int_K \mathbb{1}_A \ dm_T\rangle = \langle \int_K \mathbb{1}_A \ d(\iota_X^{-1} \circ m_T), x'\rangle, \quad A \in \mathfrak{B}(K).$$

As this holds true for all indicator functions the density of the simple functions in the Banach space of bounded measurable functions yields

$$\langle Tf, x' \rangle = \langle f, T'x' \rangle = \int_K f \, d(T'x') = \langle \int_K f \, d(\iota_X^{-1} \circ m_T), x' \rangle$$

for $f \in C(K)$ and $x' \in X'$. Hence, $Tf = \int_K f \, d(\iota_X^{-1} \circ m_T)$.
(iii): For every $f \in C(K)$ we have

$$\|Tf\| = \Big\| \int_K f \, d(\iota_X^{-1} \circ m_T) \Big\| \overset{2.1.15}{\leq} \|f\|_\infty \|m_T\|_\infty,$$

as well as

$$\|m_T\|_\infty = \sup_{x' \in K_1^{X'}(0)} \|\iota_{X'}(x') m_T\| \overset{(2.2)}{=} \sup_{x' \in K_1^{X'}(0)} \|T'x'\| \leq \|T'\| = \|T\|.$$

Regarding the last statement, given a vector measure m and defining $Tf := \int_K f dm$,

$$\langle Tf, x' \rangle = x'\big(\int_K f \, dm \big) \overset{2.1.15}{=} \int_K f \, d(x'm) = \langle f, T_m x' \rangle$$

shows that the operator T_m from Lemma 2.1.7 is the adjoint of T. By Lemma 2.1.10 T_m is weakly compact and by Lemma 1.1.5 so must be T. Finally, by

$$\langle x', m_T(A) \rangle = \langle T'x', \phi_A \rangle = \langle T_m x', \phi_A \rangle = x'm(A) = \langle m(A), x' \rangle$$

for $x' \in X'$ we have $\iota_X \circ m = m_T$. □

Corollary 2.2.4. Let K be a compact Hausdorff space, X be a Banach space. If an operator $T \in L_b(C(K), X)$ has representing measure m_T of bounded variation, then T is weakly compact.

Proof. According to Lemma 2.2.2 m_T constitutes a vector measure and by Theorem 2.2.3 the operator $S : C(K) \to X''$, $S(f) = \int_K f \, dm_T$, is weakly compact. Because of

$$\begin{aligned} \langle x', \iota_X T(f) \rangle = \langle T(f), x' \rangle &= \int_K f \, d(\iota_{X'}(x') m_T) \overset{2.1.15}{=} \iota_{X'}(x') \big(\int_K f \, dm_T \big) \\ &= \langle S(f), \iota_{X'}(x') \rangle = \langle x', S(f) \rangle, \end{aligned}$$

for all $x' \in X'$ we obtain $\iota_X T = S$. Since S is weakly compact, $S(K_1^{C(K)}(0))$ is relatively weakly compact in $\iota_X(X)$ and with Lemma 1.1.1 we see that $T = \iota_X^{-1} \circ S$ is weakly compact as well. □

2.3 Integration Theory

In the present section we introduce the Bochner integral of functions with values in a Banach space over a finite measure space $(\Omega, \mathcal{A}, \mu)$. Throughout this section we will assume such measure spaces to be complete. This assumption is commonly found in literature upon this topic and will make the proofs in this chapter slightly easier. The proofs and structure of this section are loosely orientated upon Section 3 of [DU], as well Chapter III of [DS] and Sections 2.2 and 2.3 in [Ry].

Definition 2.3.1. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X be a Banach space.

(i) A function $f : \Omega \to X$ is called *simple*, if there exist $x_1, \ldots, x_n \in X$ and a pairwise disjoint $A_1, \ldots, A_n \in \mathcal{A}$, such that $f = \sum_{k=1}^n \mathbb{1}_{A_k} x_k$.

(ii) A function $f : \Omega \to X$ is called *μ-strongly measurable*, if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions, such that $\lim_{n\to\infty}\|f_n(\omega) - f(\omega)\| = 0$ for μ-almost all $\omega \in \Omega$.

(iii) A function $f : \Omega \to X$ is called *weakly measurable*, if $x'f$ is measurable for every $x' \in X'$.

(iv) A function $f : \Omega \to X$ is called *weakly μ-integrable*, if $x'f$ is integrable with respect to μ for every $x' \in X'$.

Remark 2.3.2. Given a sequence of simple functions $(g_n)_{n\in\mathbb{N}}$, such that $g_n(x) \overset{n\to\infty}{\longrightarrow} g(x)$ for almost all x, $(\|g_n\|)_{n\in\mathbb{N}}$ defines a sequence of real-valued simple functions, which are thus measurable. As

$$\lim_{n\to\infty} \big|\|g\| - \|g_n\|\big| \leq \lim_{n\to\infty} \|g - g_n\| = 0, \ \mu\text{-almost everywhere},$$

$\|g\|$ must be measurable as a μ-almost everywhere limit of measurable functions, as we assume the completeness of μ.

Theorem 2.3.3 (Egorov's Theorem)**.** Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $f : \Omega \to X$ a μ-strongly measurable function. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f almost uniformly, i.e. for every $\epsilon > 0$ exists a set $A_\epsilon \in \mathcal{A}$ with $\mu(A_\epsilon) < \epsilon$, such that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on A_ϵ^c .

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be as in Definition 2.3.1 (ii). As $f_n - f$ is μ-strongly measurable, by the previous remark $\|f_n - f\|$ is measurable. Hence, given $k \in \mathbb{N}$ the set $\{\omega \in \Omega : \ \|f_n(\omega) - f(\omega)\| \geq 1/k\}$ belongs to $\mathcal{A}$. For $m \in \mathbb{N}$ we define

$$A_{m,k} := \bigcup_{n\geq m} \{\omega \in \Omega : \ \|f_n(\omega) - f(\omega)\| \geq 1/k\} \in \mathcal{A}.$$

As $(f_n)_{n\in\mathbb{N}}$ converges to f μ-almost everywhere, we have

$$\lim_{m\to\infty} \mu(A_{m,k}) = \mu(\bigcap_{m\in\mathbb{N}} A_{m,k}) \leq \mu\big(\Omega \setminus \{\omega : \lim_{n\to\infty} f_n(\omega) = f(\omega)\}\big) = 0.$$

Given $\epsilon > 0$, for each $k \in \mathbb{N}$ we choose $m(k) \in \mathbb{N}$, such that $\mu(A_{m(k),k}) < 2^{-k}\epsilon$. Defining

$$A_\epsilon := \bigcup_{k\in\mathbb{N}} A_{m(k),k},$$

we have uniform convergence on $\Omega \setminus A_\epsilon$ and

$$\mu(A_\epsilon) \leq \sum_{k\in\mathbb{N}} \mu(A_{m(k),k}) < \sum_{k\in\mathbb{N}} 2^{-k}\epsilon = \epsilon.$$

□

Proposition 2.3.4 (Pettis Measurability Theorem). Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, X be a Banach space and $f : \Omega \to X$. Then the following statements are equivalent.

(i) f is μ-strongly measurable.

(ii) f is weakly measurable and μ-essentially separably valued, i.e. there exists a set $A \in \mathcal{A}$ with $\mu(A) = 0$, such that $f(\Omega \setminus A)$ is a separable subset of X.

Proof. (i) $\Rightarrow$ (ii). Suppose, $(f_n)_{n\in\mathbb{N}}$ is a sequence of simple functions converging μ-almost everywhere to f. Given $x' \in X'$, $x'f_n$ is a simple function with values in $\mathbb{C}$ and $x'f$ is the pointwise limit of $(x'f_n)_{n\in\mathbb{N}}$ μ-almost everywhere. Thus, $x'f$ is measurable, i.e. f is weakly measurable. The closed subspace generated by $\cup_{n\in\mathbb{N}} \operatorname{ran}(f_n)$ is separable and, as f is the μ-almost everywhere limit of the sequence $(f_n)_{n\in\mathbb{N}}$, it contains $f(w)$ for μ-almost all $w \in \Omega$.
(ii) $\Rightarrow$ (i). Let $A \in \mathcal{A}$ be such that $\mu(A) = 0$, $f(\Omega \setminus A)$ is separable. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of $f(\Omega \setminus A)$ and choose $x'_n \in K_1^{X'}(0)$, such that $x'_n(x_n) = \|x_n\|$. We claim that $\|f(\omega)\| = \sup_{n\in\mathbb{N}} |\langle f(\omega), x'_n\rangle|$ for $\omega \in \Omega \setminus A$. Given $x \in f(\Omega \setminus A)$ and $\epsilon > 0$ choose $n \in \mathbb{N}$, such that $\|x_n - x\| < \epsilon$. Then

$$\begin{aligned} \big|\langle x, x'_n\rangle\big| &= \big|\langle x_n, x'_n\rangle + \langle x - x_n, x'_n\rangle\big| \geq |\langle x_n, x'_n\rangle| - |\langle x - x_n, x'_n\rangle| \\ &\geq |\langle x_n, x'_n\rangle| - \epsilon = \|x_n\| - \epsilon \geq \|x\| - 2\epsilon. \end{aligned}$$

Consequently, $\|x\| - 2\epsilon \leq \sup_{n\in\mathbb{N}}\langle x, x'_n\rangle \leq \|x\|$. As $\epsilon > 0$ was arbitrary the claim follows. $\omega \mapsto \langle f(\omega), x'\rangle$ being measurable for all $x' \in X'$ implies that $\|f(\cdot)\| = \sup_{n\in\mathbb{N}} |\langle f(\cdot), x'_n\rangle|$ constitutes a measurable function. Exactly the same line of arguments shows that $g_n := \|f(\cdot) - x_n\|$ is measurable for all $n \in \mathbb{N}$.
In order to show that f is μ-strongly measurable, we will construct a sequence of simple functions converging μ-almost everywhere to f. Given $m, n \in \mathbb{N}$ consider $A_{n,m} := \{\omega \in \Omega : g_n(\omega) < 1/m\}$. As g_n is measurable the $A_{n,m}$ lie in $\mathcal{A}$. Setting $B_{n,m} := A_{n,m} \setminus \cup_{k<n} A_{k,m}$, we define $h_m : \Omega \to X$ for all $m \in \mathbb{N}$ by

$$h_m(\omega) = \begin{cases} x_n & \text{if } \omega \in B_{n,m}, \\ 0 & \text{otherwise.} \end{cases}$$

Given $m \in \mathbb{N}$ for $\omega \in B_{k,m}$ with $k \in \mathbb{N}$, we have $\|h_m(\omega) - f(\omega)\| = \|x_k - f(\omega)\| = g_k(\omega) < 1/m$ because of $B_{k,m} \subseteq A_{k,m}$. Thus, $\|h_m - f\| < 1/m$ on $\cup_{n\in\mathbb{N}} B_{n,m}$. For arbitrary $\omega \in \Omega \setminus A$ there exists $n \in \mathbb{N}$, such that $g_n(\omega) = \|f(\omega) - x_n\| < 1/m$. Hence, $\omega \in A_{n,m}$, consequently $\Omega \setminus A \subseteq \cup_{n\in\mathbb{N}} B_{n,m}$ and $\|h_m - f\| < 1/m$ μ-almost everywhere.
We can write these functions as $h_m = \sum_{k=1}^{\infty} \mathbb{1}_{B_{k,m}} x_k$ and want to clip them off at a suitable point to obtain our final sequence of simple functions. Given $l \in \mathbb{N}$ choose $n(l) \in \mathbb{N}$, such that

$$\mu\Big(\bigcup_{k=n(l)+1}^{\infty} B_{k,l}\Big) \leq 2^{-l}.$$

Defining $C_l := \cup_{k=n(l)+1}^{\infty} B_{k,l}$ and $f_l := \sum_{k=1}^{n(l)} \mathbb{1}_{B_{k,l}} x_k$ we obtain $\|f_l - f\| < 1/l$ for $w \in \Omega \setminus C_l$. For $C := \cap_{r=1}^{\infty} \cup_{l=r}^{\infty} C_l$ we have $\mu(C) = 0$ as

$$\mu(C) \leq \mu\Big(\bigcup_{l=m}^{\infty} C_l\Big) \leq \sum_{l=m}^{\infty} \mu(C_l) \leq \sum_{l=m}^{\infty} \frac{1}{2^l} = \frac{1}{2^{m-1}}, \quad m \in \mathbb{N}.$$

Given $\in \Omega \setminus C$, there exists $r \in \mathbb{N}$, such that $\omega \in \Omega \setminus C_l$ for all $l \geq r$. Consequently $\|f_l(\omega) - f(\omega)\| < 1/l$ for all $l \geq r$, showing that f_l converges pointwise to f outside the μ-null set C. □

Examining the proof of the theorem above and the construction of the functions h_m in (ii)⇒(i) yields the following result.

Corollary 2.3.5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and X be a Banach space. A function $f : \Omega \to X$ is μ-strongly measurable if and only if f is μ-almost everywhere uniform limit of a sequence $(f_k)_{k\in\mathbb{N}}$ of countably valued, μ-strongly measurable functions. These functions can be written as $f_k = \sum_{j=1}^{\infty} \mathbb{1}_{A_{k,j}} x_{k,j}$ for pairwise disjoint $A_{k,j} \in \mathcal{A}$ and $x_{k,j} \in X$.

Definition 2.3.6. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let X be a Banach space.

(i) Let $f = \sum_{k=1}^{n} \mathbb{1}_{A_k} x_k$ for a pairwise disjoint partition $A_1, \ldots, A_n \in \mathcal{A}$, i.e. f is a simple function. For a measurable subset A of Ω, we define the *integral of f over A* as

$$\int_A f \, d\mu := \sum_{k=1}^{n} \mu(A_k \cap A) x_i.$$

It is straightforward to verify that this definition does not depend on the concrete representation $\sum_{k=1}^{n} \mathbb{1}_{A_k} x_k$ of f.

(ii) If $f : \Omega \to X$ is a μ-strongly measurable function, then f is called *Bochner integrable*, if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} f_n = f \;\mu \text{ almost everywhere and}$$
$$\lim_{n\to\infty} \int_\Omega \|f - f_n\| d\mu = 0.$$

(iii) If $f : \Omega \to X$ is Bochner integrable, $(f_n)_{n\in\mathbb{N}}$ is as in (ii) and $A \in \mathcal{A}$, the *Bochner integral of f over A* is defined by

$$\int_A f \, d\mu := \lim_{n\to\infty} \int_A f_n \, d\mu$$

Proposition 2.3.7. The previous definition of the Bochner integral is valid, i.e. the limit in (iii) exists and does not depend upon the specific sequence of simple functions.

Proof. Choose a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f, such that point (ii) in the previous definition holds true. By

$$\begin{aligned}\Big\| \int_A f_n \, d\mu - \int_A f_m \, d\mu \Big\|_X = \Big\| \int_A f_n - f_m \, d\mu \Big\|_X &\leq \int_A \|f_n - f_m\| \, d\mu \\ &\leq \int_\Omega \|f_n - f\| \, d\mu + \int_\Omega \|f_m - f\| d\mu\end{aligned}$$

$(\int_A f_n \; d\mu)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and hence convergent. If $(g_n)_{n\in\mathbb{N}}$ is another sequence of simple functions with the same properties, then

$$\begin{aligned}\| \lim_{n\to\infty} \int_A f_n \; d\mu - \lim_{n\to\infty} \int_A g_n \; d\mu \|_X &= \lim_{n\to\infty} \| \int_A f_n \; d\mu - \int_A g_n \; d\mu \|_X \\ &\leq \limsup_{n\to\infty} \Big(\int_A \|f_n - f\| \; d\mu + \int_A \|g_n - f\| \; d\mu \Big) = 0.\end{aligned}$$

□

Proposition 2.3.8. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X be a Banach space. A μ-strongly measurable function $f : \Omega \to X$ is Bochner integrable if and only if the scalar function $\|f\|$ is integrable.

Proof. If f is Bochner integrable, then there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \int_\Omega \|f - f_n\| d\mu = 0$. Consequently,

$$\int_\Omega \|f\| \; d\mu \leq \int_\Omega \|f - f_n\| \; d\mu + \int_\Omega \|f_n\| \; d\mu < +\infty$$

if n is sufficiently large.
For the converse suppose that f is μ-strongly measurable and $\int_\Omega \|f\| \; d\mu < +\infty$. The former implies the existence of a sequence $(f_n)_{n\in\mathbb{N}}$ of X-valued simple functions converging μ-almost everywhere to f. We fix $\delta > 0$ and define

$$g_n(x) := \begin{cases} f_n(x) & \text{if } \|f_n(x)\| \leq (1+\delta)\|f(x)\|, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{n\to\infty} f_n(\omega) = f(\omega) \neq 0$ implies $\|f_n(\omega)\| \leq (1+\delta)\|f(\omega)\|$ for sufficiently large $n \in \mathbb{N}$, g_n constitutes a sequence of simple functions, which converges to f μ-almost everywhere. In addition the scalar function $\|f - g_n\|$ is dominated by the integrable function $(2+\delta)\|f\|$. By the Dominated Convergence Theorem $\int_\Omega \|f - g_n\| d\mu \to 0$ for $n \to \infty$, i.e. f is Bochner integrable. □

Lemma 2.3.9. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X a Banach space. For a Bochner integrable function $f : \Omega \to X$ the following statements hold true.

(i) Let Y be a Banach space and $T \in L_b(X, Y)$, then Tf is Bochner integrable and

$$T\Big(\int_\Omega f \; d\mu \Big) = \int_\Omega Tf \; d\mu.$$

(ii) For every $x' \in X'$ we have $\langle f, x' \rangle := (\omega \mapsto \langle f(\omega), x' \rangle) \in L_1(\mu)$ and

$$\langle \int_\Omega f \; d\mu, x' \rangle = \int_\Omega \langle f, x' \rangle \; d\mu.$$

(iii) $\|\int_\Omega f \; d\mu\| \leq \int_\Omega \|f\| \; d\mu$.

Proof. Regarding (i), suppose $(f_n)_{n\in\mathbb{N}}$ are as in Definition 2.3.6 (ii)-(iii). Then the Tf_n constitute simple functions. Since T is continuous, we have $Tf = \lim_{n\to\infty} Tf_n$ μ-almost everywhere. Because of

$$\int_\Omega \|Tf - Tf_n\|\ d\mu \le \|T\| \int_\Omega \|f - f_n\|\ d\mu \overset{n\to\infty}{\longrightarrow} 0$$

the sequence $(Tf_n)_{n\in\mathbb{N}}$ satisfies (ii)-(iii) from Definition 2.3.6, i.e. Tf is Bochner integrable. Since the integral of a simple function is interchangeable with linear mappings, we obtain

$$\int_\Omega Tf\ d\mu = \lim_{n\to\infty} \int_\Omega Tf_n\ d\mu = \lim_{n\to\infty} T\Big(\int_\Omega f_n\ d\mu\Big) = T\Big(\lim_{n\to\infty} \int_\Omega f_n\ d\mu\Big) = T\Big(\int_\Omega f\ d\mu\Big).$$

(ii) is a special case of (i). The last point follows from

$$\big|\langle \int_\Omega f\ d\mu, x'\rangle\big| = \big| \int_\Omega \langle f, x'\rangle\ d\mu\big| \le \int_\Omega |\langle f, x'\rangle|\ d\mu \le \int_\Omega \|f\|\ d\mu,$$

if we take the supremum over all $x' \in X'$ with $\|x'\| \le 1$ on the left hand side. □

Remark 2.3.10. Given Bochner integrable functions $f,\ g : \Omega \to X$ and corresponding sequences of simple functions $(f_n)_{n\in\mathbb{N}},\ (g_n)_{n\in\mathbb{N}}$ as in Definition 2.3.6 (ii)-(iii), for $a \in \mathbb{C}$ we have

$$\int_\Omega \|f + ag - (f_n + ag_n)\|\ d\mu \le \int_\Omega \|f - f_n\|\ d\mu + |a| \int_\Omega \|g - g_n\|\ d\mu \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that the set of all Bochner integrable functions constitutes a vector space. Moreover, it is straight forward to verify that $f \sim_1 g :\Leftrightarrow \int_\Omega \|f - g\|\ d\mu = 0$ defines an equivalence relation on this vector space. Furthermore, $\|[f]_{\sim_1}\|_1 := \int_\Omega \|f\|\ d\mu$ defines a norm on the set of all equivalence classes with respect to $\sim_1$. Therefore, the following definition makes sense.

Definition 2.3.11. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X be a Banach space. We define $L_1(\mu, X)$ as the space of all equivalence classes of Bochner integrable functions on Ω with values in X with respect to $\sim_1$ equipped with the norm $\|\cdot\|_1$ as defined in the preluding remark. We will use the symbol f for the equivalence class containing f.

Given a μ-strongly measurable function g, we have seen in Remark 2.3.2 that $\|g\|$ constitutes a measurable function. Consequently, the essential supremum of $\|g\|$ is well defined.

Definition 2.3.12. Given a finite measure space $(\Omega, \mathcal{A}, \mu)$ and a Banach space X, for a Bochner integrable functions g we set

$$\|g\|_\infty = \text{ess sup}\|g\| \quad (\in [0, +\infty]).$$

g is called μ-essentially bounded, if ess sup$\|g\| < +\infty$.

Remark 2.3.13. Given essentially bounded g_1, g_2, it is easy to verify that $g_1 \sim_\infty g_2 :\Leftrightarrow \|g_1 - g_2\|_\infty = 0$ constitutes an equivalence relation on the vector space of Bochner integrable and essentially bounded functions. Additionally, $\|[g]_{\sim_\infty}\|_\infty := \|g\|_\infty$ defines a norm on all equivalence classes with respect to $\sim_\infty$.

Definition 2.3.14. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X be a Banach space. By $L_\infty(\mu, X)$ we will denote the space of equivalence classes with respect to $\sim_\infty$ of μ-essentially bounded, Bochner integrable functions equipped with the norm $\|\cdot\|_\infty$ as defined in the previous remark. We use the symbol g for the equivalent class containing g.

Remark 2.3.15. In the previous two definitions, if $X = \mathbb{C}$ then $L_1(\mu, X) = L_1(\mu)$ and $L_\infty(\mu, X) = L_\infty(\mu)$. This follows from the fact that μ-strongly measurability is equivalent to measurability for a finite dimensional X such as $X = \mathbb{C}$. The Lebesgue integral and the Bochner integral coincide.

Lemma 2.3.16. For a finite measure space $(\Omega, \mathcal{A}, \mu)$ and a Banach space X, the space $L_\infty(\mu, X)$ is a Banach space.

Proof. Clearly, $\|\cdot\|_\infty$ constitutes a norm. It remains to prove completeness. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_\infty(\mu, X)$ and choose $M \in (0, +\infty)$, such that $\|g_n\|_\infty < M$ for all $n \in \mathbb{N}$. Given $\epsilon > 0$ there exists a μ-null set A_ϵ and $N(\epsilon) \in \mathbb{N}$, such that $\|g_n(\omega) - g_m(\omega)\| < \epsilon$ and $\|g_n(\omega)\| < M$ for all $n, m > N(\epsilon)$ and $\omega \in \Omega \setminus A_\epsilon$. Then

$$A := \bigcup_{l\in\mathbb{N}} A_{1/l}.$$

is a μ-null set and for $\omega \in \Omega \setminus A$ we have

$$\|g_n(\omega) - g_m(\omega)\| \leq 1/l,$$

if only $n, m > N(1/l)$. In consequence, $g_n(\omega)$ is a Cauchy sequence for all $\omega \in \Omega \setminus A$. We define

$$g(\omega) := \begin{cases} \lim_{n\to\infty} g_n(\omega) & \text{if } \omega \in \Omega \setminus A, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\epsilon > 0$ choose $l \in \mathbb{N}$, such that $1/l < \epsilon$. For $\omega \in \Omega \setminus A$ and $n, m > N(1/l)$ we have

$$\|g_n(\omega) - g_m(\omega)\| < 1/l < \epsilon,$$

Taking the limit $m \to \infty$ yields

$$\|g_n(\omega) - g(\omega)\| < \epsilon$$

for all $n > N(1/l)$ and $\omega \in \Omega \setminus A$. Hence, $\|g_n - g\|_\infty \leq \epsilon$, i.e. $\lim_{n\to\infty} g_n = g$ with respect to $\|\cdot\|_\infty$. Since g is the μ-almost everywhere limit of the sequence $(g_n)_{n\in\mathbb{N}}$ where every g_n is μ-strongly measurable, g turns out to be μ-strongly measurable. It remains to show that g is Bochner integrable. Because of

$$\|g(\omega)\| \leq \|g_n(\omega)\| + \|g(\omega) - g_n(\omega)\| \leq M + 1$$

for $\omega \in \Omega \setminus A$ and suitably large n $\|g\|$ is μ essentially bounded and hence integrable. By Proposition 2.3.8 g is Bochner integrable. □

Remark 2.3.17. If $g \in L_\infty(\mu, X)$ has μ-essentially relatively norm compact range, then there exists a sequence of simple functions converging to g in $L_\infty(\mu, X)$. To see this, assume that $g(\Omega \setminus A) \subseteq K$ for some compact $K \subseteq X$ and some μ-null set $A \in \mathcal{A}$. Given $\epsilon > 0$ there exist $x_1, \ldots, x_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $K \subseteq \cup_{i=1}^n U_\epsilon(x_i)$. Since g is μ-strongly measurable, so is $\omega \mapsto g(\omega) - x_i$ and $\omega \mapsto \|g(\omega) - x_i\|$ is measurable for all $i = 1, \ldots, n$; see Remark 2.3.2. We define $A_i := \{\omega \in \Omega : \|g(\omega) - x_i\| < \epsilon\} \in \mathcal{A}$, $i = 1, \ldots n$, and set $B_1 := A_1$, $B_k := A_k \setminus (\cup_{i=1}^{k-1} A_i)$ for $k = 2, \ldots, n$. Defining $g_\epsilon = \sum_{i=1}^n \mathbb{1}_{B_i} x_i$ we have $\|g_\epsilon - g\|_\infty < \epsilon$ μ-almost everywhere.

Theorem 2.3.18 (Dominated Convergence Theorem)**.** Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, X a Banach space and $f_n : \Omega \to X$, $n \in \mathbb{N}$, be a sequence of Bochner integrable functions converging pointwise to $f : \Omega \to X$ μ-almost everywhere. If there exists $h \in L_1(\mu)$, such that $\|f_n\| \leq h$ μ-almost everywhere, then $\lim_{n\to\infty} \int_\Omega \|f_n - f\| \, d\mu = 0$. Moreover, f is Bochner integrable and

$$\int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega f_n \, d\mu. \tag{2.3}$$

Proof. Let A_0 be a μ-null set, such that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f on $\Omega \setminus A_0$. By the Pettis Measurability Theorem, Proposition 2.3.4, f_n is weakly measurable and there exist μ-null sets A_n, such that $f_n(\Omega \setminus A_n)$ is separable. $A := \cup_{n\in\mathbb{N}\cup\{0\}} A_n$ constitutes a μ-null set and $f(\Omega \setminus A)$ is a separable subset of X. Given $x' \in X'$, $x' f_n$ is a sequence of measurable functions converging pointwise to $x' f$ μ-almost everywhere, i.e. f is weakly measurable. Proposition 2.3.4 shows that f is μ-strongly measurable.
Since $\|f\| \leq h$ μ-almost everywhere f is Bochner integrable. Moreover, applying the Dominated Convergence Theorem for the scalar function $\|f - f_n\|$ which is dominated by $2h$ yields $\lim_{n\to\infty} \int_\Omega \|f_n - f\| \, d\mu = 0$. Together with (iii) from Lemma 2.3.9 we get

$$\lim_{n\to\infty} \left\| \int_\Omega (f - f_n) \, d\mu \right\| \leq \lim_{n\to\infty} \int_\Omega \|f - f_n\| \, d\mu = 0.$$

□

The last result of this section is a technical lemma, which will be of use much later in the proof of Proposition 6.2.5.

Lemma 2.3.19. If $(\Omega, \mathcal{A}, \mu)$ is a finite measure space, X a Banach space space and $f : \Omega \to X$ a Bochner integrable function, then for every $\epsilon > 0$ there exist sequences $(x_j)_{j=1}^\infty \in X$ and $(A_j)_{j=1}^\infty \in \mathcal{A}$, not necessarily disjoint, such that

(i) $\sum_{j=1}^\infty \mathbb{1}_{A_j} x_j$ converges to f μ-almost everywhere and

(ii) $\int_\Omega \|f\| \, d\mu \leq \sum_{j=1}^\infty \|x_j\| \mu(A_j) \leq \int_\Omega \|f\| \, d\mu + \epsilon$.

Proof. By Corollary 2.3.5 there exists a sequence $(f_k)_{k=1}^\infty$ of functions with values in X, each of the functions being countably valued, such that $\lim_{k\to\infty} f_k = f$ uniformly on Ω outside of a certain μ-null set A. We set $f_0 = 0$ and $n(0) = 0$. Choose $n(1) \in \mathbb{N}$, such that

$$\left\| f(\omega) - f_{n(1)}(\omega) \right\| \leq \epsilon \cdot \frac{1}{2 \cdot \mu(\Omega)}, \quad \omega \in \Omega \setminus A. \tag{2.4}$$

For $k \geq 2$ choose $n(k) \in \mathbb{N}$, such that

$$\left\|f_{n(k)(\omega)} - f_{(k-1)}(\omega)\right\| \leq \left\|f_{n(k)}(\omega) - f(\omega)\right\| + \left\|f(\omega) - f_{n(k-1)}(\omega)\right\| \leq \epsilon \cdot \frac{1}{2^k \cdot \mu(\Omega)} \quad (2.5)$$

if $\omega \in \Omega \setminus A$.

Fix $k \in \mathbb{N}$. We can write $f_{n(k)} = \sum_{i=1}^{\infty} \mathbb{1}_{B_i} x_i$ and $f_{n(k-1)} = \sum_{j=1}^{\infty} \mathbb{1}_{C_j} y_j$, where $(B_i)_{i\in\mathbb{N}}$ and $(C_j)_{j\in\mathbb{N}}$ are both sequences of pairwise disjoint sets in $\mathcal{A}$; see Corollary 2.3.5. The sets $B_i \setminus (\cup_{j\in\mathbb{N}} C_j)$, $i \in \mathbb{N}$, $C_j \setminus (\cup_{i\in\mathbb{N}} B_i)$, $j \in \mathbb{N}$ and $B_i \cap C_j$, $i, j \in \mathbb{N}$ are a countable amount of pairwise disjoint sets in $\mathcal{A}$. $f_{n(k)} - f_{n(k-1)}$ takes the values x_i, $(-y_j)$, $(x_i - y_j)$ on these sets respectively and is zero otherwise. Set $B_0 := \Omega \setminus (\cup_{i\in\mathbb{N}} B_i)$, $C_0 := \Omega \setminus (\cup_{j\in\mathbb{N}} C_j)$, $x_0 := 0$, $y_0 := 0$ and $m(n) := n(n+1)/2 - 1$, $n \geq 1$. For $l \in \mathbb{N}$ and $n \in \mathbb{N}$ with $m(n) < l \leq m(n+1)$ define

$$\begin{aligned} A_{k,l} &= B_{n-(l-m(n)-1)} \cap C_{l-m(n)-1}, \\ x_{k,l} &= x_{n-(l-m(n)-1)} - y_{l-m(n)-1}. \end{aligned}$$

The sequence $(A_{k,l})_{l\in\mathbb{N}}$ is pairwise disjoint since $A_{k,m(n)+1} = B_n \cap C_0 = B_n \setminus (\cup_{j\in\mathbb{N}} C_j)$, $A_{k,m(n)+2} = B_{n-1} \cap C_1$, $A_{k,m(n)+3} = B_{n-2} \cap C_2$, ..., $A_{k,m(n+1)-1} = B_1 \cap C_{n-1}$, $A_{k,m(n+1)} = B_0 \cap C_n = C_n \setminus (\cup_{i\in\mathbb{N}} B_i)$. Given $\omega \in \Omega$ with $f_{n(k)}(\omega) - f_{n(k-1)}(\omega) \neq 0$ we consider three cases. If $\omega \notin \cup_{j\in\mathbb{N}} C_j$, then there exists $m \in \mathbb{N}$, such that $\omega \in B_m \setminus (\cup_{j\in\mathbb{N}} C_j) = B_m \cap C_0$ and

$$\sum_{l=1}^{\infty} \mathbb{1}_{A_{k,l}}(\omega) x_{k,l} = \mathbb{1}_{B_m \cap C_0}(\omega) x_{k,m} = \mathbb{1}_{B_m}(\omega) x_m = f_{n(k)}(\omega) = f_{n(k)}(\omega) - f_{n(k-1)}(\omega).$$

If $\omega \notin \cup_{i\in\mathbb{N}} B_i$, then there exists $p \in \mathbb{N}$, such that $\omega \in C_p \setminus (\cup_{i\in\mathbb{N}} B_i) = C_p \cap B_0$ and

$$\sum_{l=1}^{\infty} \mathbb{1}_{A_{k,l}}(\omega) x_{k,l} = \mathbb{1}_{C_p \cap B_0}(\omega) x_{k,m} = \mathbb{1}_{C_p}(\omega)(-y_m) = /f_{n(k-1)}(\omega) = f_{n(k)}(\omega) - f_{n(k-1)}(\omega).$$

For $\omega \in B_m \cap C_p$ with $m, p \in \mathbb{N}$ we have

$$\sum_{l=1}^{\infty} \mathbb{1}_{A_{k,l}}(\omega) x_{k,l} = \mathbb{1}_{C_m \cap B_p}(\omega) x_{k,m} = \mathbb{1}_{C_m}(\omega) x_m - \mathbb{1}_{C_p}(\omega) y_p = f_{n(k)}(\omega) - f_{n(k-1)}(\omega).$$

Thus, $f_{n(k)} - f_{n(k-1)} = \sum_{l=1}^{\infty} \mathbb{1}_{A_{k,l}} x_{k,l}$ and furthermore

$$f = \sum_{k=1}^{\infty} \left(f_{n(k)} - f_{n(k-1)}\right) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \mathbb{1}_{A_{k,l}} x_{k,l}$$

pointwise on $\Omega \setminus A$. Together with (2.5) we calculate

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mathbb{1}_{A_{k,l}}(\omega) &= \sum_{k=1}^{\infty} \left\|f_{n(k)}(\omega) - f_{n(k-1)}(\omega)\right\| \\ &= \left\|f_{n(1)}(\omega)\right\| + \sum_{k=2}^{\infty} \left\|f_{n(k)}(\omega) - f_{n(k-1)}(\omega)\right\| \\ &\leq \left\|f_{n(1)}(\omega)\right\| + \sum_{k=2}^{\infty} \epsilon \cdot \frac{1}{2^k \cdot \mu(\Omega)} \\ &\leq \left\|f_{n(1)}(\omega)\right\| + \frac{\epsilon}{2 \cdot \mu(\Omega)} \end{aligned} \quad (2.6)$$

for all $\omega \in \Omega \setminus A$. Having in mind that we can disregard μ-null sets when integrating, doing so for the first and last term of the previous inequality yields

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mu(A_{k,l}) \leq \int_{\Omega} \left\|f_{n(1)}\right\| \, d\mu + \frac{\epsilon}{2}$$
$$\overset{(2.4)}{\leq} \int_{\Omega} \big(\|f\| + \frac{\epsilon}{2 \cdot \mu(\Omega)}\big) \, d\mu + \frac{\epsilon}{2} = \int_{\Omega} \|f\| \, d\mu + \epsilon.$$

Finally we have

$$\int_{\Omega} \|f\| \, d\mu \leq \int_{\Omega} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mathbb{1}_{A_{k,l}} \, d\mu = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mu(A_{k,l}).$$

Let $\mathbb{N} \ni j \mapsto (k(j), l(j)) =: \beta(j) \in \mathbb{N} \times \mathbb{N}$ be a bijection. Applying Fubini and the monotone convergence Theorem for the counting measure by (2.6)

$$\sum_{j=1}^{\infty} \|x_{\beta(j)}\| \cdot \mathbb{1}_{A_{\beta(j)}}(\omega) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mathbb{1}_{A_{k,l}}(\omega) < +\infty$$

for $\omega \in \Omega \setminus A$. Furthermore we have

$$\sum_{j=1}^{\infty} \|x_{\beta(j)}\| \cdot \mu(A_{\beta(j)}) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mu(A_{k,l}) < +\infty.$$

Thus, $\sum_{j=1}^{\infty} \mathbb{1}_{A_{\beta(j)}}(\omega) x_{\beta(j)}$ converges absolutely in X. We fix $\omega \in \Omega \setminus A$. Given $\epsilon > 0$ choose $N_1 \in \mathbb{N}$, such that

$$\sum_{k=N+1}^{\infty} \sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mathbb{1}_{A_{k,l}}(\omega) < \epsilon$$

and

$$\|f(\omega) - \sum_{k=1}^{N_1}\sum_{l=1}^{\infty} \|x_{k,l}\| \cdot \mathbb{1}_{A_{k,l}}\| < \epsilon.$$

Since for fixed k the sets $(A_{k,l})_{l \in \mathbb{N}}$ are pairwise disjoint there exists $N_2 \in \mathbb{N}$, such that $\sum_{k=1}^{N_1} \sum_{l=N_2+1}^{\infty} \|x_{k,l}\| \cdot \mathbb{1}_{A_{k,l}}(\omega) = 0$. If $J_0 \in \mathbb{N}$ is chosen, such that $\{1, \ldots, N_1\} \times \{1, \ldots, N_2\} \subseteq \beta(\{1, \ldots, J_0\})$, then for $J \geq J_0$ we have

$$\|f(\omega) - \sum_{j=1}^{J} \mathbb{1}_{A_{\beta(j)}}(\omega) \cdot x_{\beta(j)}\| \leq \|f(\omega) - \sum_{k=1}^{N_1}\sum_{l=1}^{N_2} \mathbb{1}_{A_{k,l}}(\omega) \cdot x_{k,l}\| + \epsilon$$
$$= \|f(\omega) - \sum_{k=1}^{N_1}\sum_{l=1}^{\infty} \mathbb{1}_{A_{k,l}}(\omega) \cdot x_{k,l}\| + \epsilon < 2\epsilon.$$

Hence, $A_j := A_{\beta(j)}$ and $x_j := x_{\beta(j)}$ constitute the desired sequences. □

2.4 The Radon-Nikodým Property

The present section is based upon [DP] and [Ph], in which the results of this Section have been first discussed. The structure of this sections is based upon the ideas of Section 5 in [Ri]. The proofs below can be found in Chapter III of [DU] and Section VI.8 in [DS].

Two of the most remarkable results of measure theory are the Radon-Nikodým Theorem and the Riesz Representation Theorem. The question arises immediately whether similar theorems hold true for vector measures and Bochner integrals. In detail, let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Given a vector measure $m : \mathcal{A} \to X$, which is μ-continuous, as defined in Definition 2.1.12, does there exist a Bochner integrable functions $g \in L_1(\mu, X)$, such that

$$m(A) = \int_A g \, d\mu, \quad A \in \mathcal{A}. \tag{2.7}$$

Given $T \in L_b(L_1(\mu), X)$, does there exist a function $g \in L_\infty(\mu, X)$, such that

$$Tf = \int_\Omega fg \, d\mu, \quad f \in L_1(\mu), \tag{2.8}$$

where the right side denotes the Bochner integral. In general, the desired generalizations are not valid.

Definition 2.4.1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and X a Banach space.

- A vector measure $m : \mathcal{A} \to X$ of bounded variation is said to have the *Radon-Nikodým Property (RNP)* if (2.7) is satisfied for some $g \in L_1(\mu, X)$.
- X is said to have the *Radon-Nikodým Property with respect to* μ, if any μ-continuous vector measure $m : \mathcal{A} \to X$ of bounded variation has RNP.
- X is said to have the *Radon-Nikodým Property (RNP)*, if X has the Radon Nikodým Property with respect to every finite measure.
- An operator $T \in L_b(L_1(\mu), X)$ is called *representable* if (2.8) is satisfied for some $g \in L_\infty(\mu, X)$.

Proposition 2.4.2. A Banach space X satisfies RNP if and only if for every finite measure space $(\Omega, \mathcal{A}, \mu)$ every operator $T \in L_b(L_1(\mu), X)$ is representable. In this case for $g \in L_\infty(\mu, X)$ with

$$Tf = \int_\Omega fg \, d\mu, \quad f \in L_1(\mu),$$

we have $\|T\| = \|g\|_\infty$.

Proof. Suppose X satisfies RNP and let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. For $T \in L_b(L_1(\mu), X)$ we define $m(A) := T(\mathbb{1}_A)$, $A \in \mathcal{A}$ and observe that m is μ-continuous because of

$$\|m(A)\| = \|T(\mathbb{1}_A)\| \le \|T\| \cdot \|\mathbb{1}_A\|_{L_1(\mu)} = \|T\| \cdot \mu(A) \tag{2.9}$$

for every $A \in \mathcal{A}$. Given $A \in \mathcal{A}$ let $(A_i)_{i\in\mathbb{N}} \in \mathcal{A}$ be a pairwise disjoint, such that $A = \cup_{i\in\mathbb{N}} A_i$. Obviously, m is finitely additive by (2.9) and

$$\sum_{i=1}^{n} \|m(A_i)\| \le \|m(\bigcup_{i=n+1}^{\infty} A_i)\| + \sum_{i=1}^{n} \|m(A_i)\|$$
$$\le \|T\| \cdot \Big(\mu(\bigcup_{i=n+1}^{\infty} A_i) + \sum_{i=1}^{n} \mu(A_i)\Big) = \|T\| \cdot \mu(A). \tag{2.10}$$

Taking the supremum over $n \in \mathbb{N}$ yields $\sum_{i=1}^{\infty} \|m(A_i)\| \le \|T\| \cdot \mu(A)$. Thus, $\sum_{i=1}^{\infty} m(A_i)$ is absolutely convergent. Countable additivity of m follows from

$$\big\|m(A) - \sum_{i=1}^{n} m(A_i)\big\| = \big\|m(\bigcup_{i=n+1}^{\infty} A_i)\big\| \le \|T\| \cdot \mu(\bigcup_{i=n+1}^{\infty} A_i) \overset{n\to\infty}{\longrightarrow} 0.$$

For pairwise disjoint $B_1, \ldots, B_n \in \mathcal{A}$ with $\cup_{i=1}^{n} B_i = A$ we have

$$\sum_{i=1}^{n} \|m(B_i)\| \le \|T\| \cdot \mu(A).$$

Taking the supremum over all such partitions of A we obtain $|m|(A) \le \|T\| \cdot \mu(A)$ for all $A \in \mathcal{A}$. In particular, $\|m\| \le \|T\| \cdot \mu(\Omega)$. In total we have shown that m is a μ-continuous vector measure of bounded variation. As X satisfies RNP there exists a Bochner integrable $g \in L_1(\mu, X)$, such that $m(A) = \int_A g \, d\mu$ for each $A \in \mathcal{A}$. We claim that g is μ-essentially bounded. Assume for the moment, that we have shown this. Since $|f|$ is integrable for $f \in L_1(\mu)$ and $\|g\| \in L_\infty(\mu)$ the function $\|fg\|$ is integrable. Invoking Proposition 2.3.8 we obtain that fg is Bochner integrable. The operator $S : L_1(\mu) \to X$,

$$S(f) := \int_\Omega fg \, d\mu, \quad f \in L_1(\mu),$$

by Lemma 2.3.9, (iii), satisfies $\|Sf\| \le \int_\Omega \|fg\| \, d\mu \le \|g\|_\infty \|f\|_{L_1(\mu)}$, i.e. S is bounded with $\|S\| \le \|g\|_\infty$. However, this operator coincides with T for simple functions, which are dense in $L_1(\mu)$. Therefore,

$$T(f) = \int_\Omega fg \, d\mu, \quad f \in L_1(\mu).$$

It remains to verify our claim on the boundedness of g. By Lemma 2.3.9, (iii), for $A \in \mathcal{A}$ and pairwise disjoint $A_1, \ldots, A_n \in \mathcal{A}$ with $A = \cup_{i=1}^{n} A_i$ we have

$$\sum_{i=1}^{n} \|m(A_i)\| = \sum_{i=1}^{n} \Big\| \int_{A_i} g \, d\mu \Big\| \le \sum_{i=1}^{n} \int_{A_i} \|g\| \, d\mu = \int_A \|g\| \, d\mu.$$

Taking the supremum over all such partitions of A we obtain $|m|(A) \le \int_A \|g\| \, d\mu$. We choose a sequence of simple functions $(g_n)_{n\in\mathbb{N}} \in L_1(\mu, X)$, such that $g = \lim_{n\to\infty} g_n$ μ-almost everywhere and $\lim_{n\to\infty} \int_\Omega \|g_n - g\| \, d\mu = 0$. Then

$$\Big| \int_\Omega \|g_n\| \, d\mu - \int_\Omega \|g\| \, d\mu \Big| \le \int_\Omega \big| \|g_n\| - \|g\| \big| \, d\mu \le \int_\Omega \|g_n - g\| \, d\mu \overset{n\to\infty}{\longrightarrow} 0$$

shows $\int_\Omega \|g\| \ d\mu = \lim_{n\to\infty} \int_\Omega \|g_n\| \ d\mu$. Given $\epsilon > 0$ choose $N \in \mathbb{N}$, such that

$$\int_\Omega \|g_n - g\| d\mu \leq \epsilon, \quad n > N.$$

If g_n is given by $g_n = \sum_{i=1}^{k(n)} \mathbb{1}_{A_{i,n}} x_{i,n}$ for $x_{i,n} \in X$ and pairwise disjoint $A_{i,n}$, $n = 1, \ldots, k(n)$, then

$$\begin{aligned}
\int_A \|g_n\| \ d\mu &= \sum_{i=1}^{k(n)} \|x_{i,n}\| \mu(A_{i,n} \cap A) = \sum_{i=1}^{k(n)} \Big\| \int_{A_{i,n}\cap A} g_n \ d\mu \Big\| \\
&\leq \sum_{i=1}^{n} \Big\| \int_{A_{i,n}\cap A} (g_n - g) \ d\mu \Big\| + \sum_{i=1}^{n} \Big\| \int_{A_{i,n}\cap A} g \ d\mu \Big\| \\
&\leq \sum_{i=1}^{n} \int_{A_{i,n}\cap A} \|g_n - g\| \ d\mu + \sum_{i=1}^{n} \|m(A_{i,n} \cap A)\| \\
&\leq \epsilon + |m|(A),
\end{aligned}$$

if only $n > N$. Thus, taking the limit $n \to \infty$ over the left hand side of the above equation we get $\int_A \|g\| \ d\mu \leq |m|(A) + \epsilon$. As $\epsilon > 0$ was arbitrary, we conclude $\int_A \|g\| \ d\mu \leq |m|(A)$. Together with the previously shown $|m|(A) \leq \int_A \|g\| \ d\mu$ we get $\int_A \|g\| \ d\mu = |m|(A) \leq \|T\|\mu(A)$, which implies $\|g\|_\infty \leq \|T\|$.
To proof the converse let $m : \mathcal{A} \to X$ be a vector measure of bounded variation and note, that the variation $|m|$ constitutes a finite, positive measure on $\mathcal{A}$; see Remark 2.1.3. We will extend our integral definition over vector measures as in Definition 2.1.14 to the space $L_1(|m|)$. For a simple function $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$ with pairwise disjoint $A_i \in \mathcal{A}$ we define $Tf \in X$ by Sf as defined in Remark 2.1.13, i.e.

$$Tf = \sum_{i=1}^{n} a_i m(A_i).$$

Because of

$$\|Tf\| \leq \sum_{i=1}^{n} |a_i| \cdot \|m(A_i)\| \leq \sum_{i=1}^{n} |a_i| \int_{A_i} d|m| = \int_{\cup_{i=1}^n A_i} |f| \ d|m|) = \|f\|_{L_1(|m|)}$$

we have $\|Tf\| \leq \|f\|_{L_1(|m|)}$. Consequently, T extends to a contractive operator on all of $L_1(|m|)$, also called T. Hence, by assumption that every such T is representable, there exists $g \in L_\infty(|m|, X)$, such that

$$Tf = \int_\Omega fg \ d|m|, \quad f \in L_1(|m|),$$

and in turn $m(A) = T(\mathbb{1}_A) = \int_A g \ d|m|$ for all $A \in \mathcal{A}$. Suppose, m is absolutely continuous with respect to some finite positive measure μ. For $A \in \mathcal{A}$ let $A_1, \ldots, A_n \in \mathcal{A}$ be a pairwise disjoint partition with $A = \cup_{i=1}^n A_i$. If $\mu(A) = 0$, then also $\mu(A_i) = 0$ and in consequence $m(A_i) = 0$. We obtain $\sum_{i=1}^n \|m(A_i)\| = 0$. Taking the supremum over all such pairwise disjoint $A_1, \ldots, A_n$ yields $|m|(A) = 0$. Thus, $|m|$ is absolutely continuous with respect to μ and by the Theorem of Radon-Nikodým we obtain a positive $\phi \in L_1(\mu)$ with $|m|(A) = \int_A \phi \; d\mu$. Let $(\phi_n)_{n\in\mathbb{N}}$ be a positive, monotonously increasing sequence of simple functions with $\lim_{n\to\infty} \phi_n = \phi$ μ-almost everywhere. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of X-valued simple functions with $\lim_{n\to\infty} g_n = g$ μ-almost everywhere and $\lim_{n\to\infty} \int_\Omega \|g - g_n\| \; d\mu = 0$. As $\int_\Omega \phi \cdot \|g_n\| \; d\mu = \int_\Omega \|g_n\| \; d|m| < +\infty$, by Proposition 2.3.8 the functions $\phi \cdot g_n$ and, with a similar reasoning, $\phi \cdot g$ are Bochner integrable with respect to μ. Because of

$$\int_\Omega \|\phi g - \phi g_n\| \; d\mu = \int_\Omega \phi \cdot \|g - g_n\| \; d\mu = \int_\Omega \|g - g_n\| \; d|m| \overset{n\to\infty}{\longrightarrow} 0,$$

$\int_\Omega \phi \cdot g \; d\mu = \lim_{n\to\infty} \int_\Omega \phi \cdot g_n \; d\mu$. By the Dominated Convergence Theorem 2.3.18 the Bochner integrals $\int_\Omega \phi_n \cdot x \; d\mu$ converge to $\int_\Omega \phi \cdot x \; d\mu$ and it is easy to see that $\int_\Omega \phi_n \cdot x \; d\nu = (\int_\Omega \phi_n \; d\nu) \cdot x$ for $x \in X$ and every finite measure ν on $(\Omega, \mathcal{A})$. In consequence

$$\begin{aligned} \int_A \phi \cdot x \; d\mu &= \lim_{n\to\infty} \int_A \phi_n \cdot x \; d\mu = \lim_{n\to\infty} \Big(\int_A \phi_n \; d\mu \Big) \cdot x \\ &= \Big(\int_A \phi \; d\mu \Big) \cdot x = \Big(\int_A \mathbb{1}_\Omega \; d|m| \Big) \cdot x = \int_A x \; d|m|, \quad A \in \mathcal{A}. \end{aligned}$$

Linearity of the Bochner integral yields $\int_\Omega \phi \cdot g_n \; d\mu = \int_\Omega g_n \; d|m|$, $n \in \mathbb{N}$. We calculate

$$m(A) = \int_A g \; d|m| = \lim_{n\to\infty} \int_\Omega g_n \; d|m| = \lim_{n\to\infty} \int_\Omega \phi \cdot g_n \; d\mu = \int_\Omega \phi \cdot g \; d\mu$$

and in turn (2.7) holds true. □

Remark 2.4.3. Again examine the first half of the previous proof. Given $T \in L_b(L_1(\mu), X)$, we have used the Radon-Nikodým Property of X only to propose the existence of $g \in L_\infty(\mu)$, such that T is representable by

$$Tf = \int_\Omega fg \; d\mu, \quad f \in L_1(\mu).$$

For Banach space X, not necessarily satisfying *RNP*, if $T \in L_b(L_1(\mu), X)$ is assumed to be representable, the existence of such g follows by definition. Moreover, all the calculations can be done in the exact same fashion. In particular, $Tf = \int_\Omega fg \; d\mu$ implies $\|g\|_\infty \leq \|T\|$. Clearly, $\|T\| \leq \|g\|_\infty$, showing $\|g\|_\infty = \|T\|$.

Lemma 2.4.4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X be a Banach space. For every finite $\pi \subseteq \mathcal{P}(\Omega)$ consisting of pairwise disjoint sets from $\mathcal{A}$, the function (here we set $0/0 = 0$)

$$E_\pi : f \mapsto \sum_{A\in\pi} \mathbb{1}_A \frac{\int_A f \; d\mu}{\mu(A)}, \quad f \in L_1(\mu, X),$$

defines a contraction on $L_1(\mu, X)$. Moreover, E_π has the following properties.

(i) Consider the set Π of all finite, pairwise disjoint partitions of Ω with elements in $\mathcal{A}$ directed by refinement, i.e. $\pi_1 \preceq \pi_2$ if for every $A_2 \in \pi_2$ there exists $A_1 \in \pi_1$, such that $A_2 \subseteq A_1$. Then

$$\lim_{\pi\in\Pi} \|E_\pi(f) - f\|_1 = 0, \quad f \in L_1(\mu, X).$$

(ii) For $g \in L_\infty(\mu, X)$ we have $\|E_\pi g\|_\infty \leq \|g\|_\infty$. Furthermore, if g has μ-essentially relatively norm compact range, i.e. there exists a compact set $K \subseteq X$ and a μ-null set A, such that $g(\Omega \setminus A) \subseteq K$, then

$$\lim_{\pi\in\Pi} \|E_\pi(g) - g\|_\infty = 0.$$

(iii) In the case $X = \mathbb{C}$ for $f \in L_1(\mu)$, $g \in L_\infty(\mu)$ we have

$$\langle E_\pi(f), g\rangle = \langle f, E_\pi(g)\rangle.$$

Proof. Because of

$$\begin{aligned}\|E_\pi(f)\|_1 &= \|\sum_{A\in\pi} \mathbb{1}_A \frac{\int_A f\, d\mu}{\mu(A)}\|_1 = \sum_{A\in\pi} \|\int_A f\, d\mu\| \\ &\leq \int_\Omega \|f\|\, d\mu = \|f\|_1, \quad f \in L_1(\mu, X),\end{aligned}$$

E_π is a contraction. Consider the net $(E_\pi(f))_{\pi\in\Pi}$ and begin with the case where f is a simple function with representation $f = \sum_{i=1}^n \mathbb{1}_{A_i} x_i$ for pairwise disjoint $A_1, \dots, A_n \in \mathcal{A}$. We set $\pi_0 := \{A_1, \dots, A_n, \Omega \setminus (\cup_{i=1}^n A_i)\} \in \Pi$. Suppose $\pi \in \Pi$ with $\pi \succeq \pi_0$. Then we can write π as

$$\pi = \bigcup_{i=1}^n \{B_{i,1}, \dots, B_{i,k(i)}\} \cup \{B_1, \dots, B_l\}$$

where $\cup_{j=1}^{k(i)} B_{i,j} = A_i$ for $i = 1, \dots, n$ and $\cup_{j=1}^l B_j = \Omega \setminus (\cup_{i=1}^n A_i)$. Because of

$$\begin{aligned}E_\pi(f) &= \sum_{i=1}^n \sum_{j=1}^{k(i)} \mathbb{1}_{B_{i,j}} \frac{\int_{B_{i,j}} f\, d\mu}{\mu(B_{i,j})} = \sum_{i=1}^n \sum_{j=1}^{k(i)} \mathbb{1}_{B_{i,j}} \frac{\int_{B_{i,j}} x_i\, d\mu}{\mu(B_{i,j})} \\ &= \sum_{i=1}^n \sum_{j=1}^{k(i)} \mathbb{1}_{B_{i,j}} x_i = \sum_{i=1}^n \mathbb{1}_{A_i} x_i. \end{aligned} \tag{2.11}$$

$E_\pi(f) = f$, if only $\pi \succeq \pi_0$. Thus, $E_\pi(f)$ becomes constant for simple functions f and in turn converges to f. For the general case, given $f \in L_1(\mu, X)$ and $\epsilon > 0$, choose a simple function f_ϵ, such that $\|f - f_\epsilon\|_1 < \epsilon$ and $\pi_0 \in \Pi$ in dependence on f_ϵ in the same way as we did above. We get $E_\pi(f_\epsilon) = f_\epsilon$ and hence

$$\begin{aligned}\|E_\pi(f) - f\|_1 &= \|E_\pi(f - f_\epsilon) + f_\epsilon - f\|_1 \\ &\leq \|f - f_\epsilon\|_1 + \|f_\epsilon - f\|_1 = 2\epsilon,\end{aligned} \tag{2.12}$$

if only $\pi \succeq \pi_0$, which finishes the proof for the case $L_1(\mu, X)$.
Given $g \in L_\infty(\mu, X)$ we have

$$\begin{aligned}\|E_\pi(g)\|_\infty &= \max\{\frac{\|\int_A g\ d\mu\|}{\mu(A)}:\ A \in \pi\}\\ &\le \max\{\frac{\|g\|_\infty \mu(A)}{\mu(A)}:\ A \in \pi\} = \|g\|_\infty.\end{aligned}$$

If g has essentially relatively norm compact range, by Remark 2.3.17 g is limit of a sequence of simple functions in $L_\infty(\mu, X)$ and the rest follows exactly as for the case $L_1(\mu, X)$; see (2.11) and (2.12). The last point follows from

$$\int_\Omega E_\pi(f) g\ d\mu = \sum_{A \in \Pi} \frac{\int_A f\ d\mu \int_A g\ d\mu}{\mu(A)} = \int_\Omega f E_\pi(g)\ d\mu.$$

□

Theorem 2.4.5. For a finite measure space $(\Omega, \mathcal{A}, \mu)$ and a Banach space X every compact operator $T \in L_b(L_1(\mu), X)$ is representable.

Proof. Let Π and E_π be as in Lemma 2.4.4 for $X = \mathbb{C}$. In this case we have seen that $\langle E_\pi(f), g\rangle = \langle f, E_\pi(g)\rangle$, $f \in L_1(\mu)$, $g \in L_\infty(\mu)$. We obtain $(TE_\pi)' = E_\pi T'$. As every $g \in L_\infty(\mu)$ has essentially compact range, $\lim_{\pi\in\Pi} E_\pi(g) = g$ for every $g \in L_\infty(\mu)$. Given a compact subset $K \subseteq L_\infty(\mu)$ and $\epsilon > 0$, choose $g_1, \dots, g_n \in L_\infty(\mu)$, $n \in \mathbb{N}$, such that $K \subseteq \cup_{i=1}^n U_\epsilon(g_i)$. For each $i = 1, \dots, n$ choose $\pi_i \in \Pi$, such that $\|E_\pi(g_i) - g_i\|_\infty < \epsilon$ if only $\pi_i \preceq \pi$. Given an arbitrary $g \in K$ there exists $j \in \{1, \dots, n\}$, such that $\|g_j - g\|_\infty < \epsilon$. Because of $\|E_\pi(g - g_j)\|_\infty \le \|g - g_j\|_\infty$

$$\|E_\pi(g) - g\|_\infty \le \|E_\pi(g - g_j)\|_\infty + \|E_\pi(g_j) - g_j\|_\infty + \|g_j - g\|_\infty \le 3\epsilon,$$

if only $\pi_i \preceq \pi$ for all $i = 1, \dots, n$. Since T is compact, T' is compact and from the above we conclude that E_π converges uniformly on the compact set $\overline{T'(K_1^{X'}(0))} \subseteq L_\infty(\mu)$. In consequence $E_\pi T'$ converges to T' with respect to the operator norm, from which we obtain that even TE_π converges to T in the operator norm. Defining

$$g_\pi := \sum_{A \in \Pi} \mathbb{1}_A \frac{T(\mathbb{1}_A)}{\mu(A)},\ \pi \in \Pi,$$

we easily see that $TE_\pi(f) = \int_\Omega f g_\pi\ d\mu$ and in turn

$$(TE_{\pi_1} - TE_{\pi_2})(f) = \int_\Omega f(g_{\pi_1} - g_{\pi_2})\ d\mu,$$

for all $\pi_1, \pi_2 \in \Pi$. By Remark 2.4.3 $\|g_{\pi_1} - g_{\pi_2}\|_\infty = \|TE_{\pi_1} - TE_{\pi_2}\|$. Since the latter is Cauchy, $(g_\pi)_{\pi\in\Pi}$ is Cauchy in $L_\infty(\mu, X)$. By Lemma 2.3.16 this space is a Banach space. We obtain the existence of some $g \in L_\infty(\mu, X)$, such that $\lim_{\pi\in\Pi}\|g_\pi - g\|_\infty = 0$. Since $\|fg_\pi\| \le \|g_\pi\|_\infty \cdot |f|$, $\|fg\| \le \|g\|_\infty \cdot |f|$, the functions fg_π, fg are Bochner integrable by Proposition 2.3.8. Because of

$$\|\int_\Omega (fg - fg_\pi)\ d\mu\| \le \int_\Omega |f| \cdot \|g - g_\pi\|\ d\mu \le \|f\|_{L_1(\mu)} \cdot \|g - g_\pi\|_\infty \overset{\pi\in\Pi}{\longrightarrow} 0,$$

$\lim_{\pi\in\Pi}\int_\Omega fg_\pi\ d\mu = \int_\Omega fg\ d\mu$ and

$$T(f) = \lim_{\pi\in\Pi} TE_\pi(f) = \lim_{\pi\in\Pi}\int_\Omega fg_\pi\ d\mu = \int_\Omega fg\ d\mu.$$

Hence T is representable. □

Lemma 2.4.6. Let X be a separable Banach space and $K \subseteq X$ be weakly compact. Then there exists a norm $|||\cdot|||$ on X, such that $|||x||| \le \|x\|$ for all $x \in X$, K is compact in the normed space $(X, |||\cdot|||)$ and the weak topologies with respect to $\|\cdot\|$ and $|||\cdot|||$ coincide on K.

Proof. Let $\{x_n \in X :\ n \in \mathbb{N}\}$ be a dense subset of $K_1^X(0)$. For all $n \in \mathbb{N}$ choose $x'_n \in X'$, such that $x'_n(x_n) = \|x_n\|$ and $\|x'_n\| = 1$. We claim that $\{x'_n :\ n \in \mathbb{N}\}$ is a norming subset of X', i.e. $\|x\| = \sup_{n\in\mathbb{N}} |\langle x, x'_n\rangle|$ for all $x \in X$. To see this, first consider $x \in X$ with $\|x\| = 1$ and let $0 < \epsilon < 1/3$. Choose $n \in \mathbb{N}$, such that $\|x_n - x\| \le \epsilon$; note that $\|x\| - \epsilon > 0$ and $\|x_n\| - \epsilon > 0$. Then

$$\begin{aligned}1 = \|x\| \ge |\langle x, x'_n\rangle| = |\langle x_n, x'_n\rangle + \langle x - x_n, x'_n\rangle| &\ge |\langle x_n, x'_n\rangle| - |\langle x_n - x, x'_n\rangle| \\ &\ge \|x_n\| - \epsilon = \|x_n\| - \epsilon \ge \|x\| - 2\epsilon.\end{aligned}$$

Taking the supremum over $n \in \mathbb{N}$ and then letting $\epsilon \to 0$ yields $\|x\| \ge \sup_{n\in\mathbb{N}} |x'_n(x)| \ge \|x\|$. The general case for arbitrary x follows by applying this with $x/\|x\|$. We define

$$|||x||| = \sum_{n=1}^{\infty} 2^{-n} |\langle x, x'_n\rangle|.$$

This clearly defines a norm and from $|\langle x, x'_n\rangle| \le \|x\|$ for all $x \in X$ we get $|||\cdot||| \le \|\cdot\|$. As a weakly compact set K is bounded with respect to the original norm by a constant $C > 0$. Next will we show that every sequence in K has a convergent subsequence with respect to $|||\cdot|||$, which in turn proves that K is compact with respect $|||\cdot|||$. Given a sequence $(y_k)_{k\in\mathbb{N}}$, since K is weakly compact, by Eberlein-Szmulian, Theorem 3.109 in [FHHMZ], there exists a subsequence $(y_{k(l)})_{l\in\mathbb{N}}$ converging weakly to y with respect to the original norm. Given $\epsilon > 0$ choose $N \in \mathbb{N}$, such that $\sum_{n=N+1}^{\infty} 2^{-n} < \epsilon/(4C)$. Choose $l_0 \in \mathbb{N}$, such that $|\langle y - y_{k(l)}, x'_n\rangle| < \epsilon/2$ for $l \ge l_0$ and $n \in \{1, \dots, N\}$. Note that $|\langle y, x'_n\rangle| \le \|y\| \le C$ and $|\langle y_{k(l)}, x'_n\rangle| \le \left\|y_{k(l)}\right\| \le C$ for $l \in \mathbb{N}$. We calculate for $l \ge l_0$

$$\begin{aligned}|||y - y_{k(l)}||| &= \sum_{n=1}^{N} 2^{-n} |\langle y - y_{k(l)}, x'_n\rangle| + \sum_{n=N+1}^{\infty} 2^{-n} |\langle y - y_{k(l)}, x'_n\rangle| \\ &\le \frac{\epsilon}{2} \cdot \sum_{n=1}^{N} 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n} \cdot (|\langle y, x'_n\rangle| + |\langle y_{k(l)}, x'_n\rangle|) \\ &\le \frac{\epsilon}{2} + 2C \cdot \sum_{n=N+1}^{\infty} 2^{-n} < \frac{\epsilon}{2} + 2C \cdot \frac{\epsilon}{4C} = \epsilon.\end{aligned}$$

Since $\|\cdot\| \ge |||.|||$, the identity mapping $x \mapsto x$ from $(X, \|\cdot\|)$ into $(X, |||\cdot|||)$ is continuous. In consequence, the weak-$\|\cdot\|$ topology is finer than the weak-$|||\cdot|||$ topology. Moreover, K is compact in the former and Hausdorff in the latter, thus both topologies must coincide on K. □

Theorem 2.4.7. Given a finite measure space $(\Omega, \mathcal{A}, \mu)$ and a separable Banach space X, every weakly compact operator $T \in L_b(L_1(\mu), X)$ is representable.

Proof. We call K the weak closure of $T(K_1^{L_1(\mu)}(0))$ in X. Let $|||.|||$ be the norm on X satisfying $||| \cdot ||| \leq \|\cdot\|_X$ that we obtain by applying the previous lemma, and denote by $(Y, ||| \cdot |||)$ the closure of X under $||| \cdot |||$. Let D be a countable dense subset of X and $y \in Y$. Given $\epsilon > 0$ choose $x \in X$, such that $|||y - x||| < \epsilon/2$ and then $d \in D$, such that $\|x - d\|_X < \epsilon/2$. We get

$$|||y - d||| \leq |||y - x||| + |||x - d||| \leq |||y - x||| + \|x - d\|_X < \epsilon.$$

Thus, D is dense even in Y, i.e. Y is separable. If $\iota : X \to Y$ denotes the natural embedding $x \mapsto x$ from X into Y, then $\iota T : L_1(\mu) \to Y$ is a compact operator and, according to Theorem 2.4.5, representable. We obtain $g \in L_\infty(\mu, Y)$, such that

$$(\iota T)(f) = \int_\Omega fg \, d\mu, \quad f \in L_1(\mu).$$

We will show that the value of this integral lies in X and that g can be considered an element of $L_\infty(\mu, X)$, which completes the proof. For $A \in \mathcal{A}$ with $\mu(A) > 0$ the function $\mu(A)^{-1}\mathbb{1}_A$ is an element of the closed unit ball in $L_1(\mu)$ and

$$\mu(A)^{-1} \int_A g \, d\mu = (\iota T)(\mu(A)^{-1}\mathbb{1}_A) \in K. \tag{2.13}$$

By Remark 2.3.2

$$\begin{aligned} g^{-1}(U_\delta(y)) &= \{\omega \in \Omega : \; \|g(\omega) - y\| < \delta\} \\ &= (\|g(\cdot) - y\|)^{-1}\big((-\delta, \delta)\big) \in \mathcal{A}, \quad \delta > 0, \; y \in Y. \end{aligned}$$

Since Y is a separable metric space we have

$$V := Y \setminus K = \bigcup_{y \in D, \; U^Y_{1/n}(y) \subseteq Y \setminus K, \; n \in \mathbb{N}} U_{1/n}(y);$$

see the proof of Proposition 12.13.7 in [Ka]. Suppose that g does not have almost all its values in K. Since V is the countable union of sets of the form $U_{1/n}(y)$, $g^{-1}(V)$ not being a μ-null set is equivalent to the existence of $n \in \mathbb{N}$ and $y \in D$, such that $U^Y_{1/n}(y) \subseteq V$ and $\mu(g^{-1}(U^Y_{1/n}(y))) \neq 0$. From $g^{-1}(U^Y_{1/n}(y)) = \cup_{0<r<1/n} g^{-1}(U^Y_r(y))$ we obtain the existence of $r > 0$ and $y_0 \in D$, such that $\mu(g^{-1}(U^Y_r(y_0))) \neq 0$ and $K \cap K^Y_r(y_0) = \emptyset$, which means $|||y_0 - k||| > r$ for all $k \in K$. Setting $U := U^Y_r(y_0)$, $\mu(g^{-1}(U))^{-1} \cdot \int_U g \, d\mu$ is an element of K by (2.13) and

$$\begin{aligned} |||y_0 - \frac{1}{\mu(g^{-1}(U))} \int_{g^{-1}(U)} g \, d\mu||| &= \frac{1}{\mu(g^{-1}(U))} \cdot ||| \int_{g^{-1}(U)} (y_0 - g) \, d\mu||| \\ &\leq \frac{1}{\mu(g^{-1}(U))} \int_{g^{-1}(U)} |||y_0 - g||| \, d\mu \leq r, \end{aligned}$$

which contradicts $|||y_0 - k||| > r$ for all $k \in K$. In consequence g takes almost all its values in K. Disregarding the values of g on μ-null sets, we can assume that g takes all its values in X. In turn, g is μ-essentially bounded as an X-valued function since K is bounded as a weakly compact set. As $g : \Omega \to Y$ is μ-strongly measurable, by the Pettis Measurability Theorem 2.3.4 it is weakly measurable in Y. Since the weak topology on K coincides for $||| \cdot |||$ and $\|\cdot\|_X$, g is weakly measurable if seen as a function into X. Furthermore, g is separably valued as a functions with values in a separable space. Applying the Pettis Measurability Theorem again g turns out to be μ-strongly measurable as a function from Ω into X. Since g is essentially bounded, the function $|f| \cdot \|g\|_\infty$ is integrable. Hence, the Bochner integral $\int_\Omega fg \; d\mu$ exists in X and coincides with Tf for every $f \in L_1(\mu)$. □

Remark 2.4.8. Consider a finite measure μ and a Banach space X with a separable dual space. By the Banach-Alaoglu Theorem the closed unit ball in X' is weak*-compact. Thus, any bounded operator from $L_1(\mu)$ into X' is compact with respect to the weak*-topology. In this setting, we can invoke nearly the same line of arguments as in the previous two statements, in order to prove that such operator must be representable:
First note that the separability of X' means that X is separable. Therefore we obtain a countable dense subset $\{x_n : n \in \mathbb{N}\}$ of $K_1^X(0)$. We define a norm $|||.|||$ on X' by $|||x'||| = \sum_{n\in\mathbb{N}} 2^{-n}\langle x_n, x'\rangle$. Lemma 2.4.6 as previously formulated for weakly compact sets now applies for weak*-compact sets K as follows. The first claim, that $|||.|||$ is a norm and $|||.||| \leq \|\cdot\|$ is proven in the exact same fashion. Regarding the other assertions, the problem arises that we cannot apply the Eberlein-Szmulian Theorem, which we have used to show that $(K, |||.|||)$ is compact. However, as a weak*-compact subset of a separable dual space there exists a metric d on K, such that the topology τ_d induced by d coincides with the weak*-topology. Then every sequence $(y_n)_{n\in\mathbb{N}}$ in (K, τ_d) has a convergent subsequence and the compactness of $(K, |||.|||)$ follows in the same way as in the Lemma. Furthermore, from $\|\cdot\| \geq |||.|||$ we get that the τ_d-topology on K, which coincides with the weak*-topology, is finer than the weak*-topology with respect to $|||.|||$. Since K is compact with respect to the former and Hausdorff with respect to the latter, both topologies must coincide. Theorem 2.4.7 then follows with exactly the same line of arguments. Summarizing these thoughts, we conclude the following.

Corollary 2.4.9. Separable dual spaces satisfy RNP.

Definition 2.4.10. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. A subset of K of $L_1(\mu)$ is called *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta > 0$, such that

$$\int_A |f| \; d\mu < \epsilon,$$

if only $\mu(A) < \delta$.

Lemma 2.4.11. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let X be a Banach space. If $T \in L_b(L_1(\mu), X)$ is representable, then T maps any bounded and uniformly integrable set into a norm compact set.

Proof. Let $g \in L_\infty(\mu, X)$, such that $Tf = \int_\Omega fg \; d\mu$, $f \in L_1(\mu)$, $K \subseteq L_1(\mu)$ be bounded and uniformly integrable and let $\epsilon > 0$. Choose $\delta > 0$ in dependence of $\epsilon/(\|T\| + 1)$ as in Definition 2.4.10 and a sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions converging to g μ-almost everywhere. Using Egorov's Theorem 2.3.3 there exists a set $A \in \mathcal{A}$ with $\mu(\Omega \setminus A) < \delta$, such that $(g_n)_{n\in\mathbb{N}}$ converges uniformly on A. By our choice of δ we have

$$\int_{\Omega\setminus A} |f| \, d\mu < \frac{\epsilon}{\|T\| + 1}.$$

Without loss of generality we assume $\|(g_n(\omega) - g(\omega)) \cdot \mathbb{1}_A(\omega)\| < 1/n$ for all $\omega \in \Omega$. We claim that

$$\operatorname{ran}(g \cdot \mathbb{1}_A) \cup \bigcup_{n=1}^{\infty} \operatorname{ran}(g_n \cdot \mathbb{1}_A)$$

is relatively norm compact. It suffices to show that this set has a $2/n$-cover for every $n \in \mathbb{N}$. Let $\operatorname{ran}(g_n \cdot \mathbb{1}_A) = \{x_1, \dots, x_k\}$ and consider $V = U_{2/n}(x_1) \cup \cdots U_{2/n}(x_k)$. $\|(g_n(\omega) - g(\omega))\mathbb{1}_A\omega)\| < 1/n$ implies $\operatorname{ran}(g \cdot \mathbb{1}_A) \subseteq V$. Moreover,

$$\begin{aligned} \|(g_n(\omega) - g_m(\omega))\mathbb{1}_A(\omega)\| &\le \|(g_n(\omega) - g(\omega))\mathbb{1}_A(\omega)\| + \|(g(\omega) - g_m(\omega))\mathbb{1}_A(\omega)\| \\ &< 1/n + 1/m < 2/n \end{aligned}$$

for $m \ge n$ implies $\operatorname{ran}(g_m \cdot \mathbb{1}_A) \subseteq V$. Since $L := \cup_{i=1}^{n-1} \operatorname{ran}(g_i)$ is finite, the union of V with all open balls of radius $2/n$ around each point in L is a suitable cover. Thus, we have found a compact C, such that $g_n(\omega), g(\omega) \in C$ for all $\omega \in A$, $n \in \mathbb{N}$ and set $\tilde{C} := \overline{\text{co}}\{2C \cup 2iC \cup -2C \cup -2iC \cup \{0\}\}$. Consider $f \in L_1(\mu)$ and choose a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, converging to f in $L_1(\mu)$. The f_n can be chosen such that $|f_n| \le |f|$, thus $\|f_n\|_{L_1(\mu)} \le \|f\|_{L_1(\mu)}$; see the proof of Lemma 16.6.2 in [Ka2]. For $n \in \mathbb{N}$ we have

$$\begin{aligned} \Big\| \int_A (f_n g_n - fg) \, d\mu \Big\| &\le \int_A |f_n - f| \cdot \|g_n\| \, d\mu + \int_A |f| \cdot \|g_n - g\| \, d\mu \\ &\le \|f_n - f\|_{L_1(\mu)} \cdot \|g_n \mathbb{1}_A\|_\infty + \|f\|_{L_1(\mu)} \cdot \|(g_n - g)\mathbb{1}_A\|_\infty. \end{aligned}$$

Since we have uniform convergence of $(g_n)_{n\in\mathbb{N}}$ on A, the right side gets arbitrarily small if n is chosen suitably large showing that the Bochner integrals $\int_A f_n g_n \, d\mu$ converge to $\int_A fg \, d\mu$.
Additionally, suppose that $\|f\|_{L_1(\mu)} \le 1$. Without loss of generality we can assume $f_n = \sum_{i=1}^m a_i \mathbb{1}_{A_i}$, $g_n = \sum_{i=1}^m \mathbb{1}_{A_i} x_i$ with pairwise disjoint A_i and (not necessarily pairwise distinct) $a_i \in \mathbb{C} \setminus \{0\}$, $x_i \in X$ for $i = 1, \dots, m$. We have

$$1 \ge \|f_n\|_{L_1(\mu)} = \int_\Omega |f_n| \, d\mu = \sum_{i=1}^m |a_i| \mu(A_i).$$

Furthermore, since for the next calculation we only integrate over the set A, we will assume $A_i \subseteq A$ and get

$$\int_A f_n g_n \, d\mu = \sum_{i=1}^n a_i x_i \mu(A_i) = \sum_{i=1}^n |a_i| \mu(A_i) \big(\frac{a_i}{|a_i|} x_i \big).$$

Writing $a_i/(|a_i|) = r + is$ for $r, s \in \mathbb{R}$ with $r^2 + s^2 = 1$ we have $(1/2)(|r| + |s|) \leq 1$. By

$$\frac{|r|}{2} + \frac{|s|}{2} + \big(1 - \frac{|r| + |s|}{2}\big) = 1$$

the vector

$$\frac{a_i}{|a_i|} x_i = \frac{r}{2} \cdot 2x_i + \frac{is}{2} \cdot 2x_i + \big(1 - \frac{|r| + |s|}{2}\big) \cdot 0$$

belongs to $\tilde{C}$. In consequence, $\int_A f_n g_n \, d\mu \in \tilde{C}$ as a convex combination of elements in $\tilde{C}$ and further $\int_A fg \, d\mu = \lim_{n\to\infty} \int_A f_n g_n \, d\mu \in \tilde{C}$. In total, we have shown that the linear operator $f \mapsto \int_A fg \, d\mu$ maps the closed unit ball of $L_1(\mu)$ into the compact set $\tilde{C}$, i.e. $f \mapsto \int_A fg \, d\mu$ is a compact operator. In turn, K being bounded implies that $V := \{\int_A fg \, d\mu : \; f \in K\}$ is relatively norm compact, i.e. it possesses an ϵ-cover $V \subseteq U_\epsilon(x_1) \cup \ldots \cup U_\epsilon(x_n)$ for certain $x_1, \ldots x_n \in V$. By Remark 2.4.3 $\|g\|_\infty = \|T\|$, hence

$$\Big\| \int_{\Omega\setminus A} fg \, d\mu \Big\| \leq \|g\|_\infty \cdot \int_{\Omega\setminus A} |f| \, d\mu \leq \frac{\|T\|\epsilon}{\|T\| + 1} < \epsilon, \quad f \in K. \tag{2.14}$$

Because of

$$\begin{aligned}
\{T(f) : \; f \in K\} &= \{\int_A fg \, d\mu + \int_{\Omega\setminus A} fg \, d\mu : f \in K\} \\
&\subseteq \{\int_A fg \, d\mu : \; f \in K\} + \{\int_{\Omega\setminus A} fg \, d\mu : \; f \in K\} \\
&\subseteq \big[U_\epsilon(x_1) \cup \ldots \cup U_\epsilon(x_n)\big] + U_\epsilon(0) \subseteq U_{2\epsilon}(x_1) \cup \ldots \cup U_{2\epsilon}(x_n),
\end{aligned}$$

$T(K)$ possesses a 2ϵ-cover for arbitrary $\epsilon > 0$, and therefore turns out to be relatively compact. □

Theorem 2.4.12. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and X a Banach space. Every weakly compact operator $T \in L_b(L_1(\mu), X)$ has norm separable range. As a consequence, every weakly compact operator on $L_1(\mu)$ is representable.

Proof. Since the characteristic functions span the simple functions, which are dense in $L_1(\mu)$, it suffices to show that the range of $T(\{\mathbb{1}_A : \; A \in \mathcal{A}\})$ is separable. For this, we will show that $\{T(\mathbb{1}_A) : A \in \mathcal{A}\}$ is relatively compact with respect to $\|\cdot\|_X$. This finalizes the proof since compact subsets of a metric space are clearly separable.
Given a sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}$ let $\mathcal{B}$ be the algebra and $\mathcal{C}$ be the σ-algebra generated by $\{A_n : \; n \in \mathbb{N}\}$. Define $\nu := \mu|_\mathcal{C}$. $\mathcal{B}$ is countable since it is generated by countably many sets. Given $c \in \mathbb{C}$, $C \in \mathcal{C}$ and $0 < \epsilon \leq 1$ choose $b \in \{q + ip : \; q, p \in \mathbb{Q}\} =: D$ such that $|c - b| \leq \epsilon/\mu(C \cap B)$. By Lemma 1.3.2 we choose $B \in \mathcal{B}$, such that $\mu(C \Delta B) \leq \epsilon/(2 \cdot |c| + 1)$, and get

$$\begin{aligned}
\|c \cdot \mathbb{1}_C - b \cdot \mathbb{1}_B\|_{L_1(\nu)} &= \int_\Omega |c \cdot \mathbb{1}_C - b \cdot \mathbb{1}_B| \, d\mu \\
&= \int_{C\setminus B} |c| \, d\mu + \int_{C\cap B} |c - b| \, d\mu + \int_{B\setminus C} |b| \, d\mu \\
&\leq |c - b| \cdot \mu(C \cap B) + (|c| + |b|) \cdot \mu(C \Delta B) < 2\epsilon.
\end{aligned}$$

In consequence, every simple function $\sum_{i=1}^{n} c_i \cdot \mathbb{1}_{C_i}$, $c_i \in \mathbb{C}$, $C_i \in \mathcal{C}$ can be approximated by a function in $\{\sum_{j=1}^{n} d_j \cdot \mathbb{1}_{B_j} :\ d_j \in D,\ B_j \in \mathcal{B}\}$ arbitrarily well. Since the latter set is countable and the simple functions are dense in $L_1(\nu)$, $L_1(\nu)$ is separable and a closed linear subspace of $L_1(\mu)$. As a restriction of a weakly compact operator, $\hat{T} := T|_{L_1(\nu)}$ is also weakly compact and, since $L_1(\nu)$ is separable, has separable range. Hence, $\hat{T}$ is representable by Theorem 2.4.7. The set $\{\mathbb{1}_{A_n} :\ n \in \mathbb{N}\}$ clearly is an uniformly integrable subset of $L_1(\nu)$ and Lemma 2.4.11 shows that $\{\hat{T}(\mathbb{1}_{A_n}) :\ n \in \mathbb{N}\} = \{T(\mathbb{1}_{A_n}) :\ n \in \mathbb{N}\}$ is relatively compact. In total we have shown that every sequence $(T(\mathbb{1}_{A_n}))_{n\in\mathbb{N}}$ possesses a convergent subsequence and in turn $\{T(\mathbb{1}_A) :\ A \in \mathcal{A}\}$ is relatively norm compact. □

Corollary 2.4.13. Reflexive Banach spaces satisfy RNP.

Proof. By the Banach-Alaoglu Theorem the closed unit ball in X'' is compact with respect to the weak*-topology. As weak and weak*-topology coincide on the bidual for reflexive Banach spaces, the closed unit ball of X'' is compact in the weak topology. Thus, the closed unit ball in X is compact in the weak topology. Hence, every linear bounded operator into X is weakly compact and the statement follows from the previous theorem. □

Chapter 3

The Algebraic Tensor Product

In the present chapter we will discuss the algebraic construction and essential properties of tensor products . The presented results constitute a transcription from German of the first chapter of [S].

3.1 The Algebraic Tensor Product

Definition 3.1.1. Let X, Y and Z be vector spaces. A mapping $A : X \times Y \to Z$ is called *bilinear*, if

$$A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y)$$

and

$$A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2)$$

for all $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. By $B(X \times Y, Z)$ we will denote the set of all bilinear mapping from $X \times Y$ into Z. In the case that $Z = \mathbb{C}$ we simply write $B(X \times Y)$. Its elements are called *bilinear forms*.

Definition 3.1.2. Let X and Y be vector spaces. For $x \in X$ and $x \in Y$ we define $x \otimes y$ as the point evaluation functional on $B(X \times Y)$ in the point $(x, y) \in X \times Y$, i.e.

$$x \otimes y : \begin{cases} B(X \times Y) & \to & \mathbb{C}, \\ A & \mapsto & A(x, y). \end{cases}$$

for every $A \in B(X \times Y)$.

Definition 3.1.3. Let X and Y be vector spaces. By $X \otimes Y$ we will denote the subspace of $B(X \times Y)^*$, which is spanned by all elements of the form $x \otimes y$ as defined above. We call this space *the (algebraic) tensor product* of X and Y. Its elements are called *tensors*.

Remark 3.1.4. By definition a tensor $v \in X \otimes Y$ has a representation

$$v = \sum_{i=1}^{n} \lambda_i (x_i \otimes y_i),$$

where $n \in \mathbb{N}$, $x_i \in X$, $y_i \in Y$ and $\lambda_i \in \mathbb{C}$, $i \in \mathbb{N}$.
As objects of the form $x \otimes y$ operate on bilinear mappings, the mapping

$$\tau : \begin{cases} X \times Y & \to & X \otimes Y \\ (x,y) & \mapsto & x \otimes y \end{cases}$$

as well constitutes a bilinear mapping. We formulate this fact as

Corollary 3.1.5. Given vector spaces X and Y, for $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$ and $\lambda \in \mathbb{C}$ we have

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,

(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$,

(iii) $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$,

(iv) $0 \otimes y = x \otimes 0 = 0$.

In particular, every element of $X \otimes Y$ can be written as $\sum_{i=1}^n x_i \otimes y_i$ for certain $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, $y_1, \dots, y_n \in Y$.

In general, the specific representation of a tensor is not unique. The next results will deal with the question when two tensors represent the same element of $B(X,Y)^*$.

Proposition 3.1.6. Let X, Y be vector spaces, $M \subseteq X^*$ be a separating set for X and $N \subseteq Y^*$ be a separating set for Y, i.e. $x^*(x) = 0$ for all $x^* \in M$ implies $x = 0$, $y^*(y) = 0$ for all $y^* \in N$ implies $y = 0$. Given $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ the following statements are equivalent.

(i) $v = 0$;

(ii) $\sum_{i=1}^n x^*(x_i) y^*(y_i) = 0$ for all $x^* \in M$, $y^* \in N$;

(iii) $\sum_{i=1}^n x^*(x_i) y_i = 0$ for all $x^* \in M$;

(iv) $\sum_{i=1}^n y^*(y_i) x_i = 0$ for all $y^* \in N$.

Proof. (i) $\Rightarrow$ (ii): Given $x^* \in M$ and $y^* \in N$ the bilinear form B on $X \times Y$ defined by $B(x,y) = x^*(x)y^*(y)$, $x \in X$, $y \in Y$ satisfies $0 = v(B) = \sum_{i=1}^n x^*(x_i)y^*(y_i)$.
(ii) $\Rightarrow$ (iii): Since N is a separating set for Y,

$$0 = \sum_{i=1}^n x^*(x_i) y^*(y_i) = y^*\big(\sum_{i=1}^n x^*(x_i) y_i\big)$$

for all y^* yields $\sum_{i=1}^n x^*(x_i) y_i = 0$.
(iii) $\Rightarrow$ (i): Suppose $v = \sum_{i=1}^n x_i \otimes y_i \neq 0$. If $y_n = \sum_{j=1}^{n-1} \lambda_j y_j$ with $\lambda_1, \dots \lambda_n \in \mathbb{C}$ then

$$\begin{aligned} v &= \sum_{i=1}^{n-1} x_i \otimes y_i + x_n \otimes \big(\sum_{j=1}^{n-1} \lambda_j y_j\big) \\ &= \sum_{i=1}^{n-1} (x_i + \lambda_i x_n) \otimes y_i. \end{aligned}$$

By repeating this argument we can assume that $v = \sum_{i=1}^n x_i \otimes y_i$ with linearly independent $y_1, \ldots, y_n$. In view of $0 \otimes y = 0$ we can additionally assume $x_1 \neq 0$. As M is a separating set for X there exists $x^* \in M$, such that $x^*(x_1) \neq 0$. Linear independence of $y_1, \ldots, y_n$ implies $\sum_{i=1}^n x^*(x_i) y_i \neq 0$.
In the same fashion as in the verification of (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) we get (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i). □

Remark 3.1.7. For $v \in X \otimes Y$ the sum in point (ii) of Proposition 3.1.6 coincides with $v(B_{x^*,y^*})$, where $B_{x^*,y^*} \in B(X,Y)$ is defined by $B_{x^*,y^*}(x,y) = x^*(x) y^*(y)$, .

Proposition 3.1.8. Let X, Y and Z be vector spaces. The function $\Psi : B(X \times Y, Z) \to L(X \otimes Y, Z)$ defined by

$$\Psi(B) : \begin{cases} \quad X \otimes Y & \longrightarrow \quad Z \\ \sum_{i=1}^n x_i \otimes y_i & \longmapsto \quad \sum_{i=1}^n B(x_i, y_i), \end{cases}$$

constitutes a linear bijection. $\Psi(B)$ is the unique element in $L(X \otimes Y, Z)$, such that $x \otimes y$ is mapped to $B(x,y)$ for all $(x,y) \in X \times Y$. $\Psi(B)$ is called the *linearization* of $B \in B(X \times Y, Z)$.

Proof. Given a bilinear mapping $B : X \times Y \to Z$ we define a mapping $\tilde{B} : X \otimes Y \to Z$ by

$$\tilde{B}\big(\sum_{i=1}^n x_i \otimes y_i\big) = \sum_{i=1}^n B(x_i, y_i).$$

For this expression to be well defined we have to show that $v = \sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n \tilde{x}_i \otimes \tilde{y}_i$ implies $\sum_{i=1}^n B(x_i, y_i) = \sum_{i=1}^n B(\tilde{x}_i, \tilde{y}_i)$. Equivalently, if $\sum_{i=1}^n x_i \otimes y_i = 0$ then $\sum_{i=1}^n B(x_i, y_i) = 0$. Given $z^* \in Z^*$, the composition $z^* \circ B$ constitutes a bilinear functional. From $\sum_{i=1}^n x_i \otimes y_i = 0$ we conclude

$$z^*\big(\sum_{i=1}^n B(x_i, y_i)\big) = \sum_{i=1}^n z^* \circ B(x_i, y_i) = \big\langle z^* \circ B, \sum_{i=1}^n x_i \otimes y_i \big\rangle = 0.$$

Since this holds true for all $z \in Z^*$, $\sum_{i=1}^n B(x_i, y_i) = 0$. Obviously, $\Psi(B) := \tilde{B}$ constitutes a linear mapping on $X \otimes Y$ into Z and $\Psi : B(X \times Y, Z) \to L(X \otimes Y, Z)$ is linear. In order to show the bijectivity of Ψ we will explicitly construct its inverse mapping. Given $S \in L(X \otimes Y, Z)$, recall that by Remark 3.1.4 the mapping $\tau : X \times Y \to X \otimes Y$, $(x,y) \mapsto x \otimes y$ is bilinear. In consequence $S \circ \tau$ is an element of $B(X \times Y, Z)$. Because of

$$\Psi(S \circ \tau)(x \otimes y) = (S \circ \tau)(x,y) = S(x \otimes y), \quad x \in X, \; y \in Y,$$

we have $\Psi(S \circ \tau) = S$. Together with

$$(\Psi(B) \circ \tau)(x,y) = \Psi(B)(x \otimes y) = B(x,y), \quad B \in B(X \times Y, Z),$$

this identifies Ψ as a bijection with inverse mapping $\Psi^{-1}(S) = S \circ \tau$. Every function in $L(X \otimes Y, Z)$ is uniquely defined by its values on all elements of the form $x \otimes y$, $x \in X$, $y \in Y$, since $X \otimes Y$ is exactly the linear span of these elements. Hence, $\Psi(B)$ is uniquely defined by $\Psi(B)(x \otimes y) = B(x,y)$. □

Corollary 3.1.9. For vector spaces X and Y we have

$$B(X \times Y) \simeq (X \otimes Y)^*$$

Corollary 3.1.10. Let X, Y, Z, W be vector spaces and $S \in L(X, Z)$, $T \in L(Y, W)$. There exists a unique linear mapping $S \otimes T : X \otimes Y \to Z \otimes W$, which maps $x \otimes y$ to $(Sx) \otimes (Ty)$ for all $x \in X$ and $y \in Y$.

Proof. The mapping $(x, y) \mapsto (Sx) \otimes (Sy) \in Z \otimes W$ is well defined and bilinear for all $x \in X$ and $y \in Y$. Applying Ψ from Proposition 3.1.8 yields a unique mapping with the desired properties. $\square$

3.2 Tensors as Linear Functions

So far our understanding of tensors is one of linear functionals on $B(X \times Y)$. However, tensors can be identified in many different ways.

Proposition 3.2.1. Let X, Y be vector spaces. For $x \in X$ and $y \in Y$ we define $B_{x,y} \in B(X^* \times Y^*)$ by $B_{x,y}(x^*, y^*) = x^*(x)y^*(y)$. Then

$$\iota : \begin{cases} X \otimes Y & \to & B(X^* \times Y^*), \\ x \otimes y & \mapsto & B_{x,y}, \end{cases}$$

constitutes a one-to-one linear mapping.

Proof. Clearly, $(x, y) \mapsto B_{x,y} \in B(X^* \times Y^*)$ is bilinear. By Proposition 3.1.8 there exists a unique, linear mapping $\iota : X \otimes Y \to B(X^* \times Y^*)$, such that $\iota(x \otimes y) = B_{x,y}$. If $\sum_{i=1}^n B_{x_i,y_i} = 0$, then by Proposition 3.1.6 applied with $M = X^*$ and $N = Y^*$ we conclude $\sum_{i=1}^n x_i \otimes y_i = 0$. Hence, ι is one-to-one. $\square$

Proposition 3.2.2. Let X, Y be vector spaces and $L_{x,y} \in L(X^*, Y)$, $R_{x,y} \in L(Y^*, X)$ be defined by $L_{x,y} : x^* \mapsto x^*(x)y$ and $R_{x,y} : y^* \mapsto y^*(y)x$, respectively, for all $x \in X$ and $y \in Y$. Then

$$\iota_L : \begin{cases} X \otimes Y & \to & L(X^*, Y), \\ (x \otimes y) & \mapsto & L_{x,y}, \end{cases} \quad \text{and} \quad \iota_R : \begin{cases} X \otimes Y & \to & L(Y^*, X), \\ (x \otimes y) & \mapsto & R_{x,y}, \end{cases}$$

constitute one-to-one linear mappings.

Proof. Clearly, $(x, y) \mapsto L_{x,y}$ and $(x, y) \mapsto R_{x,y}$ are bilinear. Hence, appealing to Proposition 3.1.8, there exist unique linear mappings $\iota_L : X \otimes Y \to L(X^*, Y)$ and $\iota_R : X \otimes Y \to L(Y^*, X)$, such that $\iota_L(x \otimes y) = L_{x,y}$ and $\iota_R(x \otimes y) = R_{x,y}$.
Given $v = \sum_{i=1}^n x_i \otimes y_i$ we have

$$\iota_L(v)(x^*) = \sum_{i=1}^n x^*(x_i)y_i \quad \text{and} \quad \iota_R(v)(y^*) = \sum_{i=1}^n y^*(y_i)x_i.$$

Suppose $\sum_{i=1}^n x^*(x_i)y_i = 0$. By Proposition 3.1.6 applied with $M = X^*$ we get $v = 0$. Hence, ι_L is one-to-one. Similarly, ι_R is one-to-one. $\square$

Proposition 3.2.3. Given vector spaces X and Y let $L_{x^*,y} \in L(X,Y)$ be defined by $L_{x^*,y} : x \mapsto x^*(x)y$ for $x^* \in X^*$ and $y \in Y$ and let $R_{x,y^*} \in L(Y,X)$ be defined by $R_{x,y^*} : y \mapsto y^*(y)x$ for $x \in X$ and $y^* \in Y^*$. Then

$$\iota_L : \begin{cases} X^* \otimes Y & \to & L(X,Y) \\ (x^* \otimes y) & \mapsto & L_{x^*,y} \end{cases}, \qquad \iota_R : \begin{cases} X \otimes Y^* & \to & L(Y,X) \\ (x \otimes y^*) & \mapsto & R_{x,y^*} \end{cases}$$

constitute one-to-one linear mappings.

Proof. Clearly, $(x^*, y) \mapsto L_{x^*,y}$ is bilinear. By Proposition 3.1.8 there exists a unique linear mapping $\iota_L : X^* \otimes Y \to L(X,Y)$, such that $\iota_L(x^* \otimes y) = L_{x^*,y}$.
Suppose $0 \neq v = \sum_{i=1}^n x_i^* \otimes y_i \in X^* \otimes Y$. If ι_X denotes the canonical embedding from X into X^{**}, $\iota_X(X)$ is a separating set for X^*. By Proposition 3.1.6 there exists $x \in X$, such that $0 \neq \sum_{i=1}^n \iota_X(x)(x_i^*)y_i = \sum_{i=1}^n x_i^*(x)y_i = \iota_L(v)x$. Hence, ι_L is one-to-one. Analogously the statement is proven for ι_R. □

Remark 3.2.4. The embeddings mentioned in the previous results will be called *canonical embeddings* from the respective tensor products into the respective spaces of bilinear and linear functions.

Chapter 4

Tensor Products of Banach spaces

Given two Banach spaces the question arises whether their tensor product can be equipped with a suitable norm, such that after a possibly necessary completion a new Banach space emerges. In general, there exist a multitude of useful norms, called *reasonable cross norms*. This chapter is solely concerned with the projective and injective norms, which come up in a very natural way.

4.1 The Projective Tensor Product

Given Banach spaces X and Y, $x \in X$ and $y \in Y$, it is natural to postulate

$$\|x \otimes y\| \leq \|x\| \|y\|$$

for a norm on $X \otimes Y$. This requirement encourages the following definition.

Theorem 4.1.1. Let X and Y be Banach spaces. Then

$$\pi(v) := \inf\{\sum_{i=1}^{n} \|x_i\| \|y_i\| : \ v = \sum_{i=1}^{n} x_i \otimes y_i\}$$

constitutes a norm on $X \otimes Y$, for which $\pi(x \otimes y) = \|x\| \|y\|$ holds true for all $x \in X$, $y \in Y$.

Proof. If $\pi(v) = 0$ for a tensor $v \in X \otimes Y$, then given $\epsilon > 0$ there exists a representation $v = \sum_{i=1}^{n} x_i \otimes y_i$, such that $\sum_{i=1}^{n} \|x_i\| \|y_i\| \leq \epsilon$. Given $x' \in X'$ ($\subseteq X^*$) and $y' \in Y'$ ($\subseteq Y^*$) we apply v to the bilinear form $B_{x',y'}$ as defined in Proposition 3.2.1 and get

$$|v(B_{x',y'})| = \big| \sum_{i=1}^{n} x'(x_i) y'(y_i) \big| \leq \epsilon \|x'\| \|y'\|.$$

Since $\epsilon > 0$ was arbitrary, it follows $v(B_{x',y'}) = 0$ and by Proposition 3.1.6, applied with $M = X'$ and $N = Y'$, we obtain $v = 0$.
Next we show $\pi(\lambda v) = |\lambda| \pi(v)$. The case $\lambda = 0$ immediately follows from the beginning

of the present proof. For $\lambda \neq 0$ and an arbitrary representation $v = \sum_{i=1}^n x_i \otimes y_i$ of a tensor v we derive from $\lambda v = \sum_{i=1}^n (\lambda x_i) \otimes y_i$

$$\pi(\lambda v) \leq \sum_{i=1}^n \|\lambda x_i\| \|y_i\| = |\lambda| \sum_{i=1}^n \|x_i\| \|y_i\|.$$

As the specific representation of v was chosen arbitrary, taking the infimum over all such representations yields $\pi(\lambda v) \leq |\lambda| \pi(v)$. Applying this to λ^{-1} yields

$$\pi(v) = \pi(\lambda^{-1} \lambda v) \leq |\lambda^{-1}| \pi(\lambda v).$$

Therefore, $|\lambda| \pi(v) \leq \pi(\lambda v)$ and in total $\pi(\lambda v) = |\lambda| \pi(v)$.
For the triangle inequality, let $u, v \in X \otimes Y$ and $\epsilon > 0$. We choose representations $u = \sum_{i=1}^n x_i \otimes y_i$ and $v = \sum_{i=1}^n w_i \otimes z_i$, such that

$$\sum_{i=1}^n \|x_i\| \|y_i\| \leq \pi(u) + \frac{\epsilon}{2} \quad \text{and} \quad \sum_{i=1}^n \|w_i\| \|z_i\| \leq \pi(v) + \frac{\epsilon}{2}.$$

Then $\sum_{i=1}^n x_i \otimes y_i + \sum_{i=1}^n w_i \otimes z_i$ is a representation of $u + v$, which satisfies

$$\pi(u + v) \leq \sum_{i=1}^n \|x_i\| \|y_i\| + \sum_{i=1}^n \|w_i\| \|z_i\| \leq \pi(u) + \pi(v) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude $\pi(u + v) \leq \pi(u) + \pi(v)$.
Finally we show $\pi(x \otimes y) = \|x\| \|y\|$. By the Hahn-Banach Theorem there exist $x' \in K_1^{X'}(0)$ and $y' \in K_1^{Y'}(0)$, such that $x'(x) = \|x\|$ und $y'(y) = \|y\|$. We define the bilinear form B on $X \times Y$ by $B(z, w) = x'(z) y'(w)$. Its linearization $\tilde{B} = \Psi(B)$ according to Proposition 3.1.8 with $Z = \mathbb{C}$ satisfies

$$\big|\tilde{B}\big(\sum_{i=1}^n x_i \otimes y_i\big)\big| \leq \sum_{i=1}^n |\tilde{B}(x_i \otimes y_i)| = \sum_{i=1}^n |x'(x_i) y'(y_i)| \leq \sum_{i=1}^n \|x_i\| \|y_i\|,$$

implying $|\tilde{B}(v)| \leq \pi(v)$ for all $v \in X \otimes Y$. Hence, $\tilde{B}$ is a bounded linear functional on $(X \otimes Y, \pi)$ with $\|\tilde{B}\| \leq 1$. In particular, we have

$$\|x\| \|y\| = B(x, y) = \tilde{B}(x \otimes y) \leq \pi(x \otimes y).$$

The reverse inequality $\pi(x \otimes y) \leq \|x\| \|y\|$ follows immediately from the definition of π. □

Definition 4.1.2. Let X and Y be Banach spaces and π be as in the previous Theorem.

- We call the π the *projective norm*. In case we have to specify, which spaces this norm emerges from, we denote it by $\pi_{X,Y}$.
- We denote by $X \otimes_\pi Y$ the tensor product $X \otimes Y$ endowed with the projective norm and by $X \hat{\otimes}_\pi Y$ its completion. The Banach space $X \hat{\otimes}_\pi Y$ is called the *projective tensor product* of X and Y.

Remark 4.1.3. Clearly, $x \otimes y \mapsto y \otimes x$ defines an isometric isomorphism from $X \otimes_\pi Y$ onto $Y \otimes_\pi X$. In consequence, $X \hat{\otimes}_\pi Y \cong Y \hat{\otimes}_\pi X$.

Definition 4.1.4. Given Banach spaces X, Y and Z, a bilinear function $B : X \times Y \to Z$ is called *bounded*, if there exists a constant $C > 0$, such that

$$\|B(x, y)\|_Z \leq C\|x\|_X \|y\|_Y$$

for all $x \in X$ and $y \in Y$. By $\mathcal{B}(X \times Y, Z)$ we will denote the set of all bounded, bilinear functions on $X \times Y$ with values in Z.

Theorem 4.1.5. For Banach spaces X, Y und Z the set $\mathcal{B}(X \times Y, Z)$ of bounded, linear functions on $X \times Y$ into Z is a vector space. Furthermore,

$$\|B\| = \sup\{\|B(x, y)\|_Z : \ x \in X, \ y \in Y, \ \|x\|_X \leq 1, \ \|y\|_Y \leq 1\} \tag{4.1}$$

constitutes a norm on this vector space. $\mathcal{B}(X \times Y, Z)$ is complete under this norm.

Proof. It is easily seen that $\mathcal{B}(X \times Y, Z)$ is indeed a vector space and that (4.1) defines a norm. We are going to show completeness.
Let $(B_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{B}(X \times Y, Z)$. For a certain $M > 0$ we have $\|B_n\| \leq M$ for all $n \in \mathbb{N}$. Because of

$$\|B_n(x, y) - B_m(x, y)\| \leq \|B_n - B_m\| \|x\| \|y\|$$

$(B_n(x, y))_{n=1}^\infty$ is a Cauchy sequence in Z and in consequence convergent for all $x \in X$, $y \in Y$. We define $B \in B(X \times Y, Z)$ by

$$B(x, y) := \lim_{n \to \infty} B_n(x, y).$$

The bilinearity of the functions $B_n(x, y)$ immediately implies the bilinearity of B.
Let $x \in X, y \in Y$ and $\epsilon > 0$ be given. If $N \in \mathbb{N}$ is chosen such that $\|B_n(x, y) - B(x, y)\| < \epsilon$ for all $n > N$, then

$$\begin{aligned} \|B(x, y)\| &\leq \|B_n(x, y)\| + \|B_n(x, y) - B(x, y)\| \\ &\leq \epsilon + M\|x\| \|y\| \end{aligned}$$

for all $n > N$. Since $\epsilon > 0$ was arbitrary, we derive $\|B(x, y)\| \leq M\|x\| \|y\|$ for all $x \in X$ und $y \in Y$, i.e. B is bounded.
It remains to show that the Cauchy sequence $(B_n)_{n=1}^\infty$ converges to B in $\mathcal{B}(X \times Y, Z)$. Given $x \in K_1^X(0)$, $y \in K_1^Y(0)$ and fixed $\epsilon > 0$ choose $N \in \mathbb{N}$, such that $\|B_n - B_m\| \leq \epsilon$ for all $n, m > N$. It follows that

$$\|B_n(x, y) - B_m(x, y)\| \leq \|B_n - B_m\| \|x\| \|y\| \leq \epsilon \|x\| \|y\|.$$

Taking the limit $m \longrightarrow \infty$ yields

$$\|B_n(x, y) - B(x, y)\| \leq \epsilon \|x\| \|y\| \quad \text{for } n > N.$$

As this inequality holds true for all $x \in K_1^X(0)$, $y \in K_1^Y(0)$, taking the supremum over such elements we obtain $\|B_n - B\| \leq \epsilon$ for $n > N$. □

Theorem 4.1.6. Given Banach spaces X, Y and Z, a bilinear functions $B \in B(X \times Y, Z)$ is bounded if and only if it is continuous with respect to the maximum norm on $X \times Y$.

Proof. Let $B \in \mathcal{B}(X \times Y, Z)$ be bounded by constant $C > 0$. For $(x_n, y_n)_{n\in\mathbb{N}} \in X \times Y$ converging to (x, y) we have

$$\begin{aligned}\|B(x,y) - B(x_n,y_n)\| &= \|B(x,y) - B(x_n,y) + B(x_n,y) - B(x_n,y_n)\| \\ &\leq \|B(x - x_n, y)\| + \|B(x_n, y - y_n)\| \\ &\leq C\|x - x_n\|\|y\| + C\|x_n\|\|y - y_n\| \overset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

proving that B is continuous.
Regarding the reverse statement, let $B \in B(X \times Y, Z)$ be continuous. The continuity at zero implies the existence of $\delta > 0$, such that $\|B(x,y)\|_Z \leq 1$ if only $\|x\|_X, \|y\|_Y \leq \delta$. Given arbitrary non-zero $x \in X$, $y \in Y$ we obtain

$$\begin{aligned}\|B(x,y)\|_Z &= \|B(\frac{\|x\|_X}{\delta}\frac{\delta x}{\|x\|_X}, \frac{\|y\|_Y}{\delta}\frac{\delta y}{\|y\|_Y})\|_Z \\ &= \delta^{-2}\|x\|_X\|y\|_Y\|B(\frac{\delta x}{\|x\|}, \frac{\delta y}{\|y\|})\|_Z \leq \delta^{-2}\|x\|_X\|y\|_Y.\end{aligned}$$

□

Remark 4.1.7. Given Banach spaces X and Y, a bounded bilinear form $B \in \mathcal{B}(X \times Y)$ and $x \in X$ define $T_B(x) : Y \to \mathbb{C}$ by $T_B(y) = B(x, y)$. Since B is continuous, so is $T_B(x)$. In consequence $T_B(x)$ constitutes a uniquely determined element of Y'. Thus, $x \mapsto T_B(x)$ defines a linear operator T_B from X into Y'. The boundedness of B immediately implies the boundedness of T_B.

Remark 4.1.8. If $(X, \|.\|)$ is a normed space, $(Y, \|.\|)$ is a Banach space and $(\hat{X}, \|.\|)$ is the completion of X, then

$$\begin{array}{rcl} L_b(\hat{X}, Y) & \to & L_b(X, Y), \\ T & \mapsto & T|_X, \end{array} \tag{4.2}$$

constitutes an isometric isomorphism. This follows from the fact that every bounded linear function from X into Y has an extension on $\hat{X}$.

The following statement extends Proposition 3.1.8 to the tensor product of Banach spaces and identifies the dual space of the projective tensor product.

Lemma 4.1.9. Let X, Y, Z be Banach spaces, $\mathcal{B}(X \times Y, Z)$ be the Banach space of all bounded bilinear functions on $X \times Y$ with values in Z. If $\Psi : B(X \times Y, Z) \to L(X \otimes Y, Z)$ denotes the mapping as in Proposition 3.1.8, then $\Psi|_{\mathcal{B}(X\times Y,Z)} : \mathcal{B}(X \times Y, Z) \to L_b(X \otimes Y, Z)$ constitutes an isometric isomorphism. Here $X \otimes Y$ is endowed with the projective norm π.

Proof. Let $B : X \times Y \to Z$ be bounded and bilinear and consider $\tilde{B} = \Psi(B) \in L(X \otimes Y, Z)$. Given $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we have

$$\left\|\tilde{B}(v)\right\|_Z = \left\|\sum_{i=1}^n B(x_i, y_i)\right\|_Z \leq \|B\| \sum_{i=1}^n \|x_i\|_X\|y_i\|_Y.$$

Taking the infimum over all such representations of v yields $\|\tilde{B}(v)\| \leq \|B\|\pi(v)$. Hence, $\tilde{B}$ is bounded with norm $\|\tilde{B}\| \leq \|B\|$. On the other hand $\|B\| \leq \|\tilde{B}\|$, since

$$\|B(x,y)\| = \|\tilde{B}(x \otimes y)\| \leq \|\tilde{B}\|\|x\|\|y\|.$$

In total $\|B\| = \|\tilde{B}\|$. It remains to show that $\Psi|_{\mathcal{B}(X\times Y,Z)} : \mathcal{B}(X \times Y, Z) \to L_b(X \otimes Y, Z)$ is onto. Given a linear mapping $A \in L_b(X \otimes Y, Z)$ we have $\Psi^{-1}(A) \in B(X \times Y, Z)$ and $\Psi^{-1}(A)$ is bounded as

$$\left\|(\Psi^{-1}A)(x,y)\right\| = \left\|\Psi(\Psi^{-1}A)(x \otimes y))\right\| = \|A(x \otimes y)\| \leq \|A\| \cdot \|x \otimes y\| = \|A\|\|x\|\|y\|.$$

□

Together with Remark 4.1.8 we obtain

Corollary 4.1.10. Let X, Y, Z be Banach spaces and let $\mathcal{B}(X \times Y, Z)$ be the space of all bounded, bilinear functions on $X \times Y$ into Z. The mapping $\Psi : B(X \times Y, Z) \to L(X \otimes Y, Z)$ from Proposition 3.1.8 constitutes an isometric isomorphism on $\mathcal{B}(X \times Y, Z)$ onto $L_b(X \hat{\otimes}_\pi Y)$. In the case $Z = \mathbb{C}$ we obtain

$$(X\hat{\otimes}_\pi Y)' \cong \mathcal{B}(X \times Y).$$

The continuous extension of $\Psi(B)$ is called the *linearization* of B.

Corollary 4.1.11. Given Banach spaces X and Y, the mapping $\Phi : L_b(X, Y') \to (X\hat{\otimes}_\pi Y)'$, defined by $\Phi(T)(x \otimes y) = \langle y, T(x)\rangle$, constitutes an isometric isomorphism.

Proof. Consider the mapping $\phi : \mathcal{B}(X \times Y) \to L_B(X, Y')$, $\phi(B) = T_B$, defined by $\langle y, T_B(x)\rangle = B(x, y)$ as in Remark 4.1.7. T_B is bounded with $\|T_B\| = \|B\|$, since

$$\begin{aligned}\|T_B\| &= \sup\{\|T_B(x)\| : \ x \in X, \ \|x\| \leq 1\} \\ &= \sup\{|\langle y, T_B(x)\rangle| : \ x \in X, \ \|x\| \leq 1, \ y \in Y, \ \|y\| \leq 1\} = \|B\|.\end{aligned}$$

If $T \in L_b(X, Y')$ then $B(x, y) = \langle x, T(x)\rangle$ defines an element of $\mathcal{B}(X, Y)$ with $T_B = T$. Thus, ϕ is onto and constitutes an isometric isomorphism. Consequently, $\Phi = \Psi \circ \phi^{-1}$, where Ψ is as in Corollary 4.1.10, constitutes an isometric isomorphism as well. □

Remark 4.1.12. Let I be an index set and X be a Banach space. A family $x = (x_i)_{i\in I}$ of elements in X is called absolutely summing, if

$$\|x\|_1 = \sum_{i\in I} \|x_i\| := \lim_{A\in\mathcal{E}(I)} \sum_{i\in A} \|x_i\| < +\infty,$$

where $(\mathcal{E}(I), \preceq)$ denotes the directed set of all finite subsets of I, i.e. $A \preceq B :\Leftrightarrow A \subseteq B$. By $\ell_1(I, X)$ we denote the set of all absolutely summing tupels $(x_i)_{i\in I}$ where $x_i \in X$, $i \in I$. Moreover, for $k \in I$ the k-th canonical unit vector is defined by $e_k := (\delta_{ik})_{i\in I}$, where δ_{ik} denotes the Kronecker δ.
In the case $X = \mathbb{C}$ we denote this space by $\ell_1(I)$. In the case $I = \mathbb{N}$ we write $\ell_1(X)$. $\ell_1(I, X)$ constitutes a vector space if endowed with pointwise addition and scalar multiplication and $\|.\|_1$ is a norm on $\ell_1(I, X)$. We claim that this space even constitutes a

Banach space.
In order to prove this, let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\ell_1(I,X)$, where $x_n = (x_{n,i})_{i\in I}$. In particular, there exists a constant $M > 0$, such that $\|x_n\|_1 \leq M$ for all $n \in \mathbb{N}$. Given $j \in I$ and $\epsilon > 0$ choosing $N \in \mathbb{N}$, such that $\|x_n - x_m\|_1 < \epsilon$ if only $n, m > N$, gives

$$\|x_{n,j} - x_{m,j}\| \leq \sum_{i\in I} \|x_{n,i} - x_{m,i}\| = \|x_n - x_m\|_1 < \epsilon$$

for $n, m > N$. Hence, $(x_{n,j})_{n=1}^{\infty}$ is a Cauchy sequence in X and in consequence convergent for any $j \in I$. We define $y = (y_i)_{i\in I} \in X^I$ by

$$y_i := \lim_{n\to\infty} x_{n,i}.$$

We will show that $y \in \ell_1(I,X)$. Given $A \in \mathcal{E}(I)$ and $\epsilon > 0$ choose $n \in \mathbb{N}$, such that $\sum_{i\in A} \|y_i - x_{n,i}\| < \epsilon$. In consequence

$$\begin{aligned}\sum_{i\in A} \|y_i\| &\leq \sum_{i\in A} \|x_{n,i}\| + \sum_{i\in A} \|y_i - x_{n,i}\| \\ &\leq \sum_{i\in I} \|x_{n,i}\| + \sum_{i\in A} \|y_i - x_{n,i}\| \leq M + \epsilon.\end{aligned}$$

Therefore, $\sum_{i\in I} \|y_i\| \leq M + \epsilon$ and y is absolutely summing.
It remains to show that y is limit of the sequence $(x_n)_{n=1}^{\infty}$ with respect to $\|\cdot\|_1$. Given $\epsilon > 0$ choose $N \in \mathbb{N}$, such that $\|x_n - x_m\| \leq \epsilon$ if only $m, n > N$. For $A \in \mathcal{E}(I)$ we obtain

$$\sum_{i\in A} \|x_{n,i} - x_{m,i}\| \leq \|x_n - x_m\|_1 \leq \epsilon.$$

Taking the limit $m \longrightarrow \infty$ yields

$$\sum_{i\in A} \|x_{n,i} - y\| \leq \epsilon$$

for all $n > N$. Since A was arbitrarily chosen we get

$$\|y - x_n\|_1 = \lim_{A\in\mathcal{E}(I)} \sum_{i\in A} \|y_i - x_{n,i}\| \leq \epsilon \text{ for all } n > N.$$

Given $x = (x_i)_{i\in I}$ we define its *support* by $\text{supp}(x) := \{i \in I : \ x_i \neq 0\}$. Since x is absolutely summing, the set $S_n(x) := \{i \in I : \ \|x_i\| < 1/n\}$ has to be finite. Thus, $\text{supp}(x) = \cup_{n\in\mathbb{N}} S_n(x)$ is at most countable.

Theorem 4.1.13. Let X be a Banach space and I be an arbitrary index set. Let $F : \ell_1(I) \otimes X \to \ell_1(I,X)$ be the mapping defined by $F(a \otimes x) = (a_i x)_{i\in I}$, $a = (a_i)_{i\in I} \in \ell_1(I)$, $x \in X$. Then there exists a unique extension $\tilde{F} : \ell_1(I)\hat{\otimes}_\pi X \to \ell_1(I,X)$ of F. This extension constitutes an isometric isomorphism which satisfies $\tilde{F}(\sum_{i\in I} e_i \otimes x_i) = (x_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \ell_1(I,X)$. Here the sum converges unconditionally in $\ell_1(I)\hat{\otimes}_\pi X$.

Proof. Given $a = (a_i)_{i\in I} \in \ell_1(I)$ and $x \in X$ the tupel $(a_i x)_{i\in I} \in X^I$ belongs to $\in \ell_1(I, X)$, since

$$\sum_{i\in I} \|a_i x\|_X \leq (\sum_{i\in I} |a_i|)\|x\|_X = \|a\|_1 \|x\|_X. \tag{4.3}$$

As the mapping $(a, x) \mapsto (a_i x)_{i\in I} \in \ell_1(I, X)$ is bilinear, by Proposition 3.1.8 the linear mapping $F : \ell_1(I) \otimes X \to \ell_1(I, X)$ is well defined by $F((a_i)_{i\in I}) = (a_i x)_{i\in I}$.

For $v = \sum_{k=1}^n a_k \otimes x_k \in \ell_1(I) \otimes X$ with $a_k = (a_{k,i})_{i\in I} \in \ell_1(I)$ we have

$$\begin{aligned}\|F(v)\|_1 &= \left\|\Big(\sum_{k=1}^n a_{k,i} x_k\Big)_i\right\|_1 = \sum_{i\in I}\left\|\Big(\sum_{k=1}^n a_{k,i} x_k\Big)\right\|_X \\ &\leq \sum_{i\in I}\sum_{k=1}^n \|a_{k,i} x_k\|_X = \sum_{k=1}^n \left[\Big(\sum_{i\in I} |a_{k,i}|\Big)\|x_k\|_X\right] = \sum_{k=1}^n \|a_k\|_1 \|x_k\|_X.\end{aligned}$$

Taking the infimum over all such representations of v yields $\|F(v)\|_1 \leq \pi(v)$.

Also the reverse inequality holds true. In order to see this, fix a specific representation $v = \sum_{k=1}^n a_k \otimes x_k$ where $a_k = (a_{k,i})_{i\in I}$ and define $(v_i)_{i\in I} \in X^I$ by $v_i := \sum_{k=1}^n a_{k,i} x_k$. As a consequence of (4.3) $(v_i)_{i\in I}$ belongs to $\ell_1(I, X)$. Moreover, $F(v) = (v_i)_{i\in I}$. We will show that $\sum_{i\in I} e_i \otimes v_i$ converges to v in $\ell_1(I)\hat{\otimes}_\pi X$. For every finite set $A \subseteq I$ consider the mapping

$$\Pi_A := \begin{cases} \ell_1(I) & \to \ \ell_1(I), \\ (a_i)_{i\in I} & \mapsto \ \sum_{j\in A} a_j e_j. \end{cases}$$

Defining $(a_i')_{i\in I}$ by $a_i' = a_i$, $i \notin A$, $a_i' = 0$ else, we have

$$\|\Pi_A(a) - a\| = \|\sum_{j\in A} a_j e_j - (a_i)_{i\in I}\| = \|(a_i')_{i\in I}\|_1 = \sum_{i\in I} |a_i'| = \sum_{i\in I} |a_i| - \sum_{i\in A} |a_i|$$

implying

$$\lim_{A\in\mathcal{E}(I)} \|\Pi(a) - a\| = \sum_{i\in I} |a_i| - \lim_{A\in\mathcal{E}(I)} \sum_{i\in A} |a_i| = 0.$$

We obtain

$$\begin{aligned}\pi\left[v - \sum_{i\in A} e_i \otimes v_i\right] &= \pi\left[\sum_{k=1}^n a_k \otimes x_k - \sum_{i\in A}\sum_{k=1}^n e_i \otimes (a_{k,i} x_k)\right] \\ &= \pi\left[\sum_{k=1}^n \left(a_k \otimes x_k - \sum_{i\in A} (a_{k,i} e_i) \otimes x_k\right)\right] \\ &= \pi\left[\sum_{k=1}^n \big(a_k - \Pi_A(a_k)\big) \otimes x_k\right] \\ &\leq \sum_{k=1}^n \|a_k - \Pi_A(a_k)\|_{\ell_1(I)} \|x_k\|_X \overset{A\in\mathcal{E}(I)}{\longrightarrow} 0\end{aligned}$$

and, in consequence,

$$\begin{aligned}\pi(v) &= \pi\big(\lim_{A\in\mathcal{E}(I)} \sum_{i\in A} e_i \otimes v_i\big) = \lim_{A\in\mathcal{E}(I)} \pi\big(\sum_{i\in A} e_i \otimes v_i\big) \\ &\leq \lim_{A\in\mathcal{E}(I)} \sum_{i\in A} \|v_i\|_X = \sum_{i\in I} \|v_i\|_X = \|F(v)\|_1.\end{aligned}$$

So far we have shown that F constitutes a linear isometry from $\ell_1 \otimes X$ into $\ell_1(I, X)$. Since $\ell_1(I, X)$ is complete, there exists a unique, bounded and linear extension $\tilde{F} : \ell_1(I)\hat{\otimes}_\pi X \mapsto \ell_1(I, X)$ which is isometric, too. Given $x = (x_i)_{i\in I} \in \ell_1(I, X)$ we have

$$\sum_{i\in I} \|e_i \otimes x_i\|_\pi \leq \sum_{i\in I} \|x_i\|_X < +\infty.$$

Hence $\tilde{x} = \sum_{i\in I} e_i \otimes x_i$ converges unconditional to an $\tilde{x} \in \ell_1(I)\hat{\otimes}_\pi X$. Since $F(e_i \otimes x_i) = e_i x_i \in \ell_1(I, X)$, continuity and linearity of $\tilde{F}$ implies $\tilde{F}(\tilde{x}) = x$. In particular, $\tilde{F}$ is also onto. □

Lemma 4.1.14. Let X, Y, Z, W be Banach spaces and $S \in L_b(X, Z)$, $T \in L_b(Y, W)$. Then there exists a unique $S \otimes_\pi T \in L_b(X\hat{\otimes}_\pi Y, Z\hat{\otimes}_\pi W)$ which maps $x \otimes y$ to $(Sx) \otimes (Ty)$. This mapping satisfies $\|S \otimes_\pi T\| = \|S\|\|T\|$.

Proof. By Corollary 3.1.10 there exists a unique linear mapping $S \otimes T : X \otimes Y \to Z \otimes W$, such that $(S \otimes T)(x \otimes y) = Sx \otimes Ty$ for all $x \in X$ and $y \in Y$. Given $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we have

$$\pi\big((S \otimes T)v\big) = \pi\big(\sum_{i=1}^n (Sx_i) \otimes (Ty_i)\big) \leq \|S\|\|T\| \sum_{i=1}^n \|x_i\|\|y_i\|.$$

Taking the infimum over all such representation of v yields $\|S \otimes T\| \leq \|S\|\|T\|$. Hence, $S \otimes T$ is bounded and linear. Given $\epsilon > 0$ and choosing $x \in K_1^X(0)$, $y \in K_1^Y(0)$, such that $\|Sx\| \geq \|S\| - \epsilon$ and $\|Ty\| \geq \|T\| - \epsilon$, gives

$$\|S \otimes T\| \geq \|(Sx) \otimes (Ty)\| = \|Sx\|\|Ty\| \geq \|S\|\|T\| - \epsilon\|S\| - \epsilon\|T\| + \epsilon^2.$$

Taking the limit $\epsilon \to 0$ yields $\|S \otimes T\| \geq \|S\|\|T\|$. In consequence there exists a unique, linear extension $S \otimes_\pi T :\ X\hat{\otimes}_\pi Y \to Z\hat{\otimes}_\pi W$ with the same norm. □

Definition 4.1.15. Given normed spaces X and Z, a mapping $Q \in L_b(X, Z)$ is called *quotient operator*, if Q is onto and satisfies $\|z\| = \inf\{\|x\| :\ x \in X,\ Q(x) = z\}$ for all $z \in Z$.

Remark 4.1.16. By the second requirement of Definition 4.1.15 a quotient operator Q maps the open unit ball in X onto the open unit ball in Z. Consequently, $[x]_{\ker(Q)} \mapsto Qx$ constitutes an isometric isomorphism $Q/\ker(Q)$ from $X/\ker(Q)$ onto Z.

Lemma 4.1.17. Given normed spaces X, Y and a quotient operator $Q \in L_b(X, Y)$, the unique extension $\hat{Q} : \hat{X} \to \hat{Y}$ of Q to the completion $\hat{X}$ of X into the completion $\hat{Y}$ of Y is a quotient operator.

Proof. $\hat{X}$ being the completion of X implies the existence of an isometric mapping $\iota : X \to \hat{X}$, such that $\iota(X)$ is a dense subset of $\hat{X}$. We set $N := \overline{\iota(\ker(Q))}^{\hat{X}}$. Because of $\iota(\ker(Q)) \subseteq N$ the mapping

$$\kappa : \begin{cases} X/\ker(Q) & \to \quad \hat{X}/N, \\ [x]_{\ker(Q)} & \mapsto \quad [\iota(x)]_N \end{cases}$$

is well defined and linear. Since ι is isometric and since $d(a,A) = d(a,\overline{A})$ in every metric space, we have

$$\begin{aligned} \|[x]_{\ker}(Q)\|_{X/\ker(Q)} &= \inf\{\|x-w\|_X :\ w \in \ker(Q)\} \\ &= \inf\{\|\iota(x)-\iota(w)\|_{\hat{X}} :\ w \in \ker(Q)\} \\ &= d\big(\iota(x), \iota(\ker(Q))\big) = d\big(\iota(x), N\big) = \|[\iota(x)]_N\|_{\hat{X}/N}. \end{aligned} \tag{4.4}$$

The quotient mapping $\pi_N(\hat{x}) = [\hat{x}]_N$ being continuous implies that $\kappa(X/\ker Q) = \pi_N(\iota(X))$ is a dense subset of $\hat{X}/N$. Consequently $(\hat{X}/N, \kappa)$ is a completion of $X/\ker(Q)$.
Since the linear mapping $Q/\ker(Q) : X/\ker(Q) \to Y$ defined by $(Q/\ker(Q))([x]_{\ker(Q)}) = Q(x)$ for $x \in X$ is an isometric bijection, its extension $R : \hat{X}/N \to \hat{Y}$ is isometric and onto as well. In consequence, $\pi_N : \hat{X} \to \hat{X}/N$ being a quotient operator implies that $R \circ \pi_N : \hat{X} \to \hat{Y}$ is also a quotient operator. Because of $R \circ \pi_N(\iota(x)) = R(\kappa(x)) = Qx$ for $x \in X$, $R \circ \pi_N$ is the unique extension $\hat{Q}$ of Q. □

Lemma 4.1.18. Let X, Y, Z, W be Banach spaces and $Q : X \to Z$, $R : Y \to W$ be quotient operators, then $Q \otimes_\pi R : X\hat{\otimes}_\pi Y \to Z\hat{\otimes}_\pi W$ is a quotient operator, too

Proof. By Lemma 4.1.17 it suffices to show that $Q \otimes R : X \otimes_\pi Y \to Z \otimes_\pi W$ is a quotient operator. Given $\sum_{i=1}^n z_i \otimes w_i \in Z \otimes_\pi W$ there exist $x_i \in X$ und $y_i \in Y$, such that $Qx_i = z_i$ und $Ry_i = w_i$ for $i \in \{1, \dots, n\}$. Consequently $(Q \otimes R)(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n z_i \otimes w_i$ and $Q \otimes R$ turns out to be onto.
For $u \in Z \otimes_\pi W$ and $v \in X \otimes Y$ with $(Q \otimes R)v = u$ we have

$$\pi(u) \le \|Q\|\|R\|\pi(v) = \pi(v).$$

Given $\epsilon > 0$ choose a representation $u = \sum_{i=1}^n z_i \otimes w_i$, such that $\sum_{i=1}^n \|z_i\|\|w_i\| \le \pi(u) + \epsilon$. For each $i \in \{1, \dots, n\}$ choose $x_i \in X$ and $y_i \in Y$, such that $Qx_i = z_i$, $Ry_i = w_i$, $\|x_i\| \le (1+\epsilon)\|z_i\|$ and $\|y_i\| \le (1+\epsilon)\|w_i\|$. We obtain

$$\begin{aligned} \pi\big(\sum_{i=1}^n x_i \otimes y_i\big) &\le \sum_{i=1}^n \|x_i\|\|y_i\| \le \sum_{i=1}^n (1+\epsilon)\|z_i\|(1+\epsilon)\|w_i\| \\ &= (1+\epsilon)^2\big(\sum_{i=1}^n \|z_i\|\|w_i\|\big) \le (1+\epsilon)^2(\pi(u)+\epsilon). \end{aligned}$$

Since $(Q \otimes R)(\sum_{i=1}^n x_i \otimes y_i) = u$ and $\epsilon > 0$ were chosen arbitrarily, we conclude

$$\pi(u) = \inf\{\pi(v) :\ v \in X \otimes_\pi Y,\ (Q \otimes R)v = u\}.$$

□

Lemma 4.1.19. Every Banach space is quotient of the space $\ell_1(I)$ for a suitable index set I, i.e. there exists a quotient operator $T : \ell_1(I) \to X$.

Proof. Set $I = \{x \in X :\ \ \|x\| = 1\}$ and consider the space $\ell_1(I)$ of all absolutely summing, complex valued tupels over I. Since for $(\lambda_x)_{x\in I} \in \ell_1(I)$ we have

$$\sum_{x\in I} \|\lambda_x x\|_X = \sum_{x\in I} |\lambda_x| = \|(\lambda_x)_{x\in I}\|_{\ell_1(I)} < +\infty,$$

the operator $T : \ell_1(I) \to X$ defined by

$$T((\lambda_x)_{x\in I}) := \sum_{x\in I} \lambda_x x, \quad (\lambda_x)_{x\in I} \in \ell_1(I)$$

constitutes a linear and bounded operator with $\|T\| \leq 1$.
Furthermore, T is onto, as for each $x \in X \setminus \{0\}$ we have $x/\|x\| \in I$ and $T(\|x\| \cdot e_{\frac{x}{\|x\|}}) = x$. T is a quotient operator because of

$$\Big\| \|x\| \cdot e_{\frac{x}{\|x\|}} \Big\|_{\ell_1(I)} = \|x\|.$$

□

Theorem 4.1.20. Let X and Y be Banach spaces. For each $u \in X\hat{\otimes}_\pi Y$ and $\epsilon > 0$ there exist bounded sequences $(x_k)_{k=1}^\infty \in X$ and $(y_k)_{k=1}^\infty \in Y$, such that $\sum_{k=1}^\infty x_k \otimes y_k$ converges to u and

$$\sum_{k=1}^{\infty} \|x_k\|\|y_k\| < \pi(u) + \epsilon.$$

Proof. By Lemma 4.1.19 there exists an index set I and a quotient operator $Q : \ell_1(I) \to X$, such that $X \cong \ell_1(I)/\ker Q$. According to Lemma 4.1.18 the operator $Q \otimes_\pi \mathrm{id} : \ell_1(I)\hat{\otimes}_\pi Y \to X\hat{\otimes}_\pi Y$ as well constitutes a quotient operator. Given $u \in X\hat{\otimes}_\pi Y$ and $\epsilon > 0$ there exists $v \in \ell_1(I)\hat{\otimes}_\pi Y$, such that $(Q \otimes_\pi \mathrm{id})(v) = u$ and $\pi(v) < \pi(u) + \epsilon$. According to Theorem 4.1.13 there is an isometric isomorphism $\tilde{F} : \ell_1(I)\hat{\otimes}_\pi Y \to \ell_1(I, Y)$. If we set $(v_i)_{i\in I} := \tilde{F}(v) \in \ell_1(Y)$, then $\pi(v) = \sum_{i\in I} \|v_i\|$ and again by Theorem 4.1.13 we have $v = \sum_{i\in I} e_i \otimes v_i$. If $x_i := Q(e_i)$ and $y_i := v_i$, then $u = (Q \otimes_\pi \mathrm{id})v = \sum_{i\in I} x_i \otimes y_i$ and $\sum_{i\in I} \|x_i\|\|y_i\| \leq \sum_{i\in I} \|e_i\|\|v_i\| = \pi(v) < \pi(u) + \epsilon$. At the end of Remark 4.1.12 it was shown that $\mathrm{supp}((x_i)_{i\in I}) = \{i \in I : \; x_i \neq 0\}$ is at most countable. Consequently we can rewrite $\sum_{i\in I} x_i \otimes y_i$ as $\sum_{k=1}^\infty x_k \otimes y_k$ for suitably chosen $x_k \in X$, $y_k \in Y$. □

Corollary 4.1.21. Let X, Y be Banach spaces and $v \in X\hat{\otimes}_\pi Y$, then

$$\pi(v) = \inf\Big\{ \sum_{n=1}^{\infty} \|x_n\|\|y_n\| : \; \sum_{n=1}^{\infty} \|x_n\|\|y_n\| < +\infty, \; v = \sum_{n=1}^{\infty} \; x_n \otimes y_n \Big\}.$$

Lemma 4.1.22. Let X, Y be Banach spaces and $u \in X\hat{\otimes}_\pi Y$. There exist sequences $(x_i)_{i\in\mathbb{N}} \in X$ and $(y_i)_{i\in\mathbb{N}} \in Y$, such that $u = \sum_{i=1}^\infty x_i \otimes y_i$ and $\sum_{i=1}^\infty \|x_i\|\|y_i\| < +\infty$. Moreover these sequences can be chosen, such that $\lim_{i\to\infty} \|x_i\| = 0$ and $\sum_{i=1}^\infty \|y_i\| < +\infty$.

Proof. Given $u \in X\hat{\otimes}_\pi Y$, the previous results yields a representation $u = \sum_{i=1}^\infty x_i \otimes y_i$, such that $\sum_{i=1}^\infty \|x_i\|\|y_i\| < +\infty$. Since $0 \otimes y = 0 = x \otimes 0$ we can assume that $x_i, y_i \neq 0$ for all $i \in \mathbb{N}$. As

$$u = \sum_{i=1}^{\infty} x_i \otimes y_i = \sum_{i=1}^{\infty} (\|y_i\| x_i) \otimes (\frac{1}{\|y_i\|} y_i),$$

we can assume without loss of generality $\|y_i\| = 1$ for all $i \in \mathbb{N}$ and hence $\sum_{i=1}^\infty \|x_i\| < +\infty$.

For $k \in \mathbb{N}$ let $N(k)$ be minimal, such that $\sum_{i=N(k)+1}^{\infty} \|x_i\| < 2^{-2k}$, $N(k) = N(k+1)$ being a possibility. For $j \in \mathbb{N}$ with $N(k) < j \leq N(k+1)$ set $a_j := 2^k$ and $a_j = 1$ for $j \leq N(1)$. We obtain

$$\sum_{j=N(k)+1}^{N(k+1)} \|a_j x_j\| = 2^k \cdot \sum_{j=N(k)+1}^{N(k+1)} \|x_j\| \leq 2^k \cdot \sum_{j=N(k)+1}^{\infty} \|x_j\| = 2^{-k}, \ k \in \mathbb{N},$$

and, in turn,

$$\begin{aligned} \sum_{i=1}^{\infty} \|a_i x_i\| &= \sum_{i=1}^{N(1)} \|x_i\| + \sum_{k=1}^{\infty} \sum_{j=N(k)+1}^{N(k+1)} \|a_i x_i\| \\ &\leq \sum_{i=1}^{N(1)} \|x_i\| + \sum_{k=1}^{\infty} 2^{-k} < +\infty. \end{aligned}$$

Thus, $u = \sum_{i=1}^{\infty} ((a_i \|x_i\|)^{-1} x_i) \otimes (a_i \|x_i\| y_i)$ is a representation with the desired properties. □

4.2 The Injective Tensor Product

Consider the canonical embedding $\iota : X \otimes Y \to B(X^*, Y^*)$ defined by $\iota(x, y) = B_{x,y}$ as introduced in Proposition 3.2.1. Restricting $B_{x,y}$ to the product $X' \times Y'$ of the topological duals yields the following result.

Lemma 4.2.1. Let X, Y be Banach spaces and $\iota : X \otimes Y \to B(X^*, Y^*)$ be the canonical embedding as in Proposition 3.2.1. Then the mapping

$$\kappa : \begin{cases} X \otimes Y & \to & B(X', Y') \\ v & \mapsto & \iota(v)|_{X' \times Y'} \end{cases},$$

is one-to-one.

Proof. Since $X' \subseteq X^*$ und $Y' \subseteq Y^*$ are point separating sets for X and Y, respectively, applying Proposition 3.1.6 with $M = X'$ and $N = Y'$ for $v = \sum_{i=1}^{n} x_i \otimes y_i$, we see that $\sum_{i=1}^{n} x'(x_i) y'(y_i) = \iota(v)(x', y') = 0$ for all $x' \in X'$ and $y' \in Y'$ implies $v = 0$. □

We set $B_v := \kappa(v)$. With this notation $B_{x \otimes y} = B_{x,y}$.

Remark 4.2.2. The mapping κ even constitutes a mapping into $\mathcal{B}(X', Y')$. Indeed, given $x' \in X'$ and $y' \in Y'$ we have

$$\|\kappa(x \otimes y)(x', y')\| = \|x'(x) y'(y)\| \leq \|x'\| \|y'\| \|x\| \|y\|.$$

If $v = \sum_{i=1}^{n} x_i \otimes y_i$, this implies

$$\|(\kappa(v))(x', y')\| \leq \|x'\| \|y'\| \sum_{i=1}^{n} \|x_i\| \|y_i\|.$$

Taking the infimum over all representations of v we obtain $\|\kappa(v)\| \leq \pi(v)$.

Theorem 4.2.3. Let X, Y be Banach spaces and $B_v = \kappa(v)$ denote the bilinear form from Lemma 4.2.1, then

$$\epsilon(v) := \sup\Big\{|B_v(x', y')| :\ x' \in K_1^{X'}(0),\ y' \in K_1^{Y'}(0)\Big\}$$

defines a norm on $X \otimes Y$. Moreover, $\epsilon(x \otimes y) = \|x\| \|y\|$ for all $x \in X, y \in Y$ and $\epsilon(v) \le \pi(v)$ for all $v \in X \otimes Y$. In case we want to specify the underlying Banach spaces, we also write $\epsilon_{X,Y}(v)$.

Remark 4.2.4. Given $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we have $B_v(x', y') = \sum_{i=1}^n x'(x_i)y'(y_i)$. In consequence,

$$\epsilon(v) = \sup\Big\{|\sum_{i=1}^n x'(x_i)y'(y_i)| :\ x' \in K_1^{X'}(0),\ y' \in K_1^{Y'}(0)\Big\}.$$

Proof of Theorem 4.2.3. By Remark 4.2.2, the space $\kappa(X \otimes Y)$ is a linear subspace of $\mathcal{B}(X' \times Y')$, which is endowed with the norm

$$\|B\| = \sup\{\|B(x', y')\|_Z :\ x' \in X',\ y' \in Y',\ \|x'\|_{X'} \le 1,\ \|y'\|_{Y'} \le 1\}.$$

Since $\epsilon(v) = \|B_v\|$, ϵ indeed constitutes a norm on $X \otimes Y$. As noted at the end of Remark 4.2.2 we have $\epsilon(v) \le \pi(v)$. Finally, $\epsilon(x \otimes y) = \sup_{x' \in K_1^X(0),\ y' \in K_1^Y(0)} |x'(x)y'(y)| \le \|x\| \|y\|$. Choosing $f \in X'$ and $g \in Y'$, such that $\|f\| = 1$, $\|g\| = 1$, $f(x) = \|x\|$ and $g(y) = \|y\|$ yields $\epsilon(x \otimes y) \ge |f(x)g(y)| = \|x\| \|y\|$. $\square$

Definition 4.2.5. Let X and Y be vector spaces and ϵ as in Theorem 4.2.3.

- We call ϵ the *injective norm*. In case we have to specify, which Banach spaces this norm emerges from, we denote this norm by $\epsilon_{X,Y}$.
- We denote by $X \otimes_\epsilon Y$ the tensor product $X \otimes Y$ endowed with the injective norm and by $X \hat{\otimes}_\epsilon Y$ its completion. The Banach space $X \hat{\otimes}_\epsilon Y$ is called the *injective tensor product* of X and Y.

Proposition 4.2.6. Let X, Y, Z, W be Banach spaces and $S \in L_b(X, Z)$, $T \in L_b(Y, W)$. Then there exists a unique $S \otimes_\epsilon T \in L_b(X \hat{\otimes}_\epsilon Y, Z \hat{\otimes}_\epsilon W)$, such that $x \otimes y$ is mapped to $(Sx) \otimes (Ty)$. Hereby, $\|S \otimes_\epsilon T\| = \|S\| \|T\|$.

Proof. Consider the operator $S \otimes T : X \otimes Y \to Z \otimes W$ as defined in corollary 3.1.10. Given $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we have

$$\begin{aligned}
\epsilon_{Z,W}\big((S \otimes T)v\big) &= \sup\big\{\big|\sum_{i=1}^n z'(Sx_i) \cdot w'(Ty_i)\big| :\ z' \in Z',\ w' \in W',\ \|z'\|, \|w'\| \le 1\big\} \\
&= \sup\big\{\big|\sum_{i=1}^n (S'z')(x_i) \cdot (T'w')(y_i)\big| :\ z' \in Z',\ w' \in W',\ \|z'\|, \|w'\| \le 1\big\} \\
&\le \sup\big\{\big|\sum_{i=1}^n \|S'\| x'(x_i) \cdot \|T'\| y'(y_i)\big| :\ x' \in X',\ y' \in Y',\ \|x'\|, \|y'\| \le 1\big\} \\
&\le \|S'\| \|T'\| \epsilon_{X,Y}(v) = \|S\| \|T\| \epsilon_{X,Y}(v).
\end{aligned}$$

Hence, $S \otimes T$ is bounded with respect to the injective norm by $\|S \otimes T\| \leq \|S\|\|T\|$. Given $\delta > 0$, choosing $x \in K_1^X(0)$ and $y \in K_1^Y(0)$, such that $\|Sx\| \geq (1-\delta)\|S\|$ and $\|Ty\| \geq (1-\delta)\|T\|$ yields $\epsilon_{X,Y}(x \otimes y) \leq 1$ and

$$\begin{aligned}\epsilon_{Z,W}((S \otimes T)(x \otimes y)) &= \epsilon_{Z,W}(Sx \otimes Ty) \\ &= \sup\big\{|z'(Sx)w'(Ty)| : \ z' \in Z', w' \in W', \|z'\|, \|w'\| \leq 1\big\} \\ &= \|Sx\|\|Ty\| \geq (1-\delta)^2\|S\|\|T\|.\end{aligned}$$

Since δ was arbitrary, we conclude $\|S \otimes T\| \geq \|S\|\|T\|$. As $S \otimes T$ is bounded and linear, it has a unique extension $S \otimes_\epsilon T$ on $X \hat{\otimes}_\epsilon Y$. □

Proposition 4.2.7. Let X, Y be Banach spaces, $F \subseteq K_1^{X'}(0)$ and $G \subseteq K_1^{Y'}(0)$, such that $\|x\| = \sup\{|f(x)| : f \in F\}$ for all $x \in X$ and $\|y\| = \sup\{|g(y)| : g \in G\}$ for all $y \in Y$. Then for $v = \sum_{i=1}^n x_i \otimes y_i$ we have

$$\epsilon(v) = \sup\big\{|\sum_{i=1}^n f(x_i)g(y_i)| : \ f \in F, g \in G\big\}.$$

Proof. We simply calculate

$$\begin{aligned}\epsilon(v) &= \sup\big\{|\sum_{i=1}^n x'(x_i)y'(y_i)| : \ x' \in K_1^{X'}(0), \ y' \in K_1^{Y'}(0)\big\} \\ &= \sup\big\{|x'\big(\sum_{i=1}^n y'(y_i)x_i\big)| : \ x' \in K_1^{X'}(0), \ y' \in K_1^{Y'}(0)\big\} \\ &= \sup\big\{|f\big(\sum_{i=1}^n y'(y_i)x_i\big)| : \ f \in F, \ y' \in K_1^{Y'}(0)\big\} \\ &= \sup\big\{|y'\big(\sum_{i=1}^n f(x_i)y_i\big)| : \ f \in F, \ y' \in K_1^{Y'}(0)\big\} \\ &= \sup\big\{|g\big(\sum_{i=1}^n f(x_i)y_i\big)| : \ f \in F, \ g \in G\big\} \\ &= \sup\big\{|\sum_{i=1}^n f(x_i)g(y_i)| : \ f \in F, \ g \in G\big\}.\end{aligned}$$

□

Remark 4.2.8. If X is a Banach space and ι_X denotes the canonical embedding from X into X'', then $\|x'\| = \sup\{|x'(x)| : x \in K_1^X(0)\} = \sup\{\iota_X(x)(x') : x \in K_1^X(0)\}$. Given $v = \sum_{i=1}^n x_i' \otimes y_i' \in X' \otimes Y'$ the previous proposition yields

$$\epsilon(v) = \sup\big\{|\sum_{i=1}^n x_i'(x)y_i'(y)| : \ x \in K_1^X(0), y \in K_1^Y(0)\big\}.$$

Remark 4.2.9. Let X and Y be Banach spaces and $\iota_L : (x^* \otimes y) \mapsto L_{x^*,y}$ be the canonical embedding from Proposition 3.2.3. We define

$$\kappa_L : \begin{cases} X' \otimes Y & \to & L_b(X,Y), \\ v & \mapsto & \iota_L(v). \end{cases}$$

Given $v = \sum_{i=1}^n x_i' \otimes y_i \in X' \otimes Y$, from Proposition 4.2.7 applied with $\iota_X(K_1^X(0)) \subseteq X''$ we obtain

$$\begin{aligned} \epsilon(v) &= \sup\{|B_v(x'', y')| : \ \|x''\| \leq 1, \ \|y'\| \leq 1\} \\ &= \sup\{|\sum_{i=1}^n \iota_X(x)(x_i')y'(y_i)| : \ \|x\| \leq 1, \ \|y'\| \leq 1\} \\ &= \sup\{|\sum_{i=1}^n x_i'(x)y'(y_i)| : \ \|x\| \leq 1, \ \|y'\| \leq 1\} \\ &= \sup\{|y'(\sum_{i=1}^n x_i'(x)y_i)| : \ \|x\| \leq 1, \ \|y'\| \leq 1\} \\ &= \sup\{\big\|\sum_{i=1}^n x_i'(x)y_i\big\|_Y : \ \|x\| \leq 1\} \\ &= \sup\{\|L_v(x)\|_Y : \ \|x\| \leq 1\} = \|L_v\| \end{aligned}$$

Therefore κ_L is isometric and we can view $X' \otimes Y$ as a subspace of $L_b(X,Y)$ and, in turn, $X'\hat{\otimes}_\epsilon Y$ as a closed subpsace of $L_b(X,Y)$. The elements of that subspace, which consists of all operators which are limits in the operator norm of finite rank operators, are called *approximable operators*.

Remark 4.2.10. Given Banach spaces X and Y, a linear operator $T : X \to Y$ is called *compact operator*, if $T(K_1^X(0))$ is relatively compact in Y, i.e. its closure is compact in Y. The set of all compact operators is denoted as $K(X,Y)$.
As every finite rank operator is compact and $K(X,Y)$ is a closed linear subspace of $L_b(X,Y)$ under the operator norm (see Proposition 6.5.4 in [BKW]), it follows from the identification of $X'\hat{\otimes}_\epsilon Y$ with the set of all approximable operators, that $X'\hat{\otimes}_\epsilon Y$ is isometrically isomorphic to a closed subspace of $K(X,Y)$.

Chapter 5

Operator Theory on Tensor Products of Banach spaces

This chapter will introduce the notion of nuclear, Pietsch integral and integral operators and mainly focuses on the properties of integral operators.

5.1 Nuclear Operators

Definition 5.1.1. Given Banach spaces X and Y a linear $T : X \to Y$ is called *nuclear*, if there exist sequences $(x'_i)_{i=1}^{\infty} \in X'$ and $(y_i)_{i=1}^{\infty} \in Y$, such that $\sum_{i=1}^{\infty} \|x'_i\| \|y_i\| < +\infty$ and

$$T(x) = \sum_{i=1}^{\infty} x'_i(x) y_i$$

for all $x \in X$. By $N(X, Y)$ we denote the set of all nuclear operators from X into Y. Clearly, this set constitutes a vector space if endowed with the usual addition and scalar multiplication.

Remark 5.1.2. Since $\sum_{i=1}^{\infty} \|x'_i\| \|y_i\| < +\infty$, the series $\sum_{i=1}^{\infty} x'_i(x) y_i$ converges absolutely. Furthermore, because of $\|T(x)\| \leq \sum_{i=1}^{\infty} \|x'_i(x)\| \|y_i\| \leq \|x\| \cdot \sum_{i=1}^{\infty} \|x'_i\| \|y_i\| < +\infty$ every nuclear operator is bounded by

$$\|T\| \leq \inf\{\sum_{i=1}^{\infty} \|x'_i\| \|y_i\| : \ T = \sum_{i=1}^{\infty} x'_i(\cdot) y_i\}.$$

Proposition 5.1.3. Let X, Y be Banach spaces and $L_{x',y} \in L_b(X, Y)$ be defined by $L_{x',y}(x) = x'(x)y$. Then

$$F : \begin{cases} X' \hat{\otimes}_\pi Y & \to & N(X, Y) \\ x' \otimes y & \mapsto & L_{x',y} \end{cases} .$$

is linear and onto. Moreover, $\ker F$ is closed in $X' \hat{\otimes}_\pi Y$.

Proof. Consider the mapping $\iota_L : X^* \otimes Y \to L(X, Y)$ as in Proposition 3.2.3. Given $v = \sum_{i=1}^{n} x_i^* \otimes y_i$ we have $\iota_L(v)(x) = \sum_{i=1}^{n} x_i^*(x) y_i$, $x \in X$. For the restriction $\iota_L|_{X' \otimes Y}$ of

ι_L to $X' \otimes Y$, $u = \sum_{i=1}^n x_i' \otimes y_i \in X' \otimes Y$ and $x \in X$ we have

$$\|\iota_L(u)|_{X'\otimes Y}(x)\|_Y \leq \left\| \sum_{i=1}^n x_i'(x) y_i \right\|_Y \leq \sum_{i=1}^n \|x_i'\|_{X'} \|x\|_X \|y_i\|_Y < +\infty.$$

Hence, $\iota_L(u)|_{X'\otimes Y}$ is a bounded operator with $\|\iota_L(u)|_{X'\otimes Y}\| \leq \sum_{i=1}^n \|x_i'\| \|y_i\|$. Taking the infimum over all representations of u yields $\|\iota_L(u)\| \leq \pi(u)$. Hence $\|\iota_L\| \leq 1$. By the definition of nuclear operators it is immediately clear that $\iota_L(u)|_{X'\otimes Y}$ maps into $N(X,Y)$. Thus, there exists a unique, linear and bounded extension $F : X'\hat{\otimes}_\pi Y \to L_b(X,Y)$ and with the help of Theorem 4.1.20 we get $F(X'\hat{\otimes}_\pi Y) = N(X,Y)$:
Given a nuclear operator $T \in N(X,Y)$ there exists sequences $(x_i')_{i=1}^\infty \in X'$ and $(y_i)_{i=1}^\infty \in Y$, such that $T(x) = \sum_{i=1}^\infty x_i'(x) y_i$ and $\sum_{i=1}^\infty \|x_i'\| \|y_i\| < +\infty$. The latter implies $\sum_{i=1}^\infty x_i' \otimes y_i \in X'\hat{\otimes}_\pi Y$. As $\iota_L(\sum_{i=1}^n x_i' \otimes y_i)(x) = \sum_{i=1}^n x_i'(x) y_i$, the continuity of F yields $F(\sum_{i=1}^\infty x_i' \otimes y_i)(x) = \sum_{i=1}^\infty x_i'(x) y_i = T(x)$. Thus F is onto and $\ker F$ is closed in $X'\hat{\otimes}_\pi Y$ because F is continuous. □

Corollary 5.1.4. Given Banach spaces X and Y, $N(X,Y)$ endowed with the norm

$$\|T\|_{nuc} := \inf\{ \sum_{i=1}^\infty \|x_i'\| \|y_i\| : T = \sum_{i=1}^\infty x_i'(\cdot) y_i \} \tag{5.1}$$

is a Banach space. We call $\|\cdot\|_{nuc}$ the *nuclear norm.* Furthermore, the operator F from Proposition 5.1.3 constitutes a quotient operator.

Proof. Proposition 5.1.3 has shown that $F : X'\hat{\otimes}_\pi Y \to N(X,Y)$ is linear and onto. Equivalently, $N(X,Y)$ is isomorphic to the space $X'\hat{\otimes}_\pi Y/\ker(F)$. If T can be written as $T = \sum_{i=1}^\infty x_i'(\cdot) y_i$, then $u = \sum_{i=1}^\infty x_i' \otimes y_i \in X'\hat{\otimes}_\pi Y$ satisfies $F(u) = T$. Looking at the right side of Equation (5.1) reveals $\|T\|_{nuc} \leq \|v\|_\pi$ if $F(v) = T$ and in consequence $\|T\|_{nuc} \leq \|[u]_{\ker(F)}\|_{X'\hat{\otimes}_\pi Y/\ker F}$. Given $\epsilon > 0$ let $v \in X'\hat{\otimes}_\pi Y$ with $F(v) = T$, such that $\|v\|_\pi \leq \|T\|_{nuc} + \epsilon$. From $v - u \in \ker F$ we conclude

$$\|[u]_{\ker(F)}\|_{X'\hat{\otimes}_\pi Y/\ker F} \leq \|u + (v-u)\|_\pi = \|v\|_\pi \leq \|T\|_{nuc} + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we obtain $\|T\|_{nuc} = \|[u]_{\ker(F)}\|_{X'\hat{\otimes}_\pi Y/\ker F}$, showing that $N(X,Y)$ is isometrically isomorphic to $X'\hat{\otimes}_\pi Y/\ker F$ and F turns out to be a quotient operator and $\|\cdot\|_{nuc}$ constitutes a norm. □

Proposition 5.1.5. Let X, Y, Z, W be Banach spaces, $T \in L_b(W,X)$, $S \in L_b(X,Y)$ and $R \in L_b(Y,Z)$. If S is nuclear, then RST is nuclear and $\|RST\|_{nuc} \leq \|R\| \|S\|_{nuc} \|T\|$.

Proof. By definition there exist sequences $(x_i')_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$, such that

$$RST(w) = R\big(\sum_{i=1}^\infty x_i'(Tw) y_i\big) = \sum_{i=1}^\infty (x_i'T)(w) R y_i.$$

Thus, RST is nuclear and we have

$$\|RST\|_{nuc} \leq \sum_{i=1}^\infty \|x_i'T\| \|Ry_i\| \leq \|R\| \|T\| \sum_{i=1}^\infty \|x_i'\| \|y_i\|.$$

Taking the infimum of the right side over all such sequences $(x_i')_{i\in\mathbb{N}}$, $(y_i)_{i\in\mathbb{N}}$ we conclude $\|RST\|_{nuc} \leq \|R\| \|S\|_{nuc} \|T\|$. □

5.2 Integral and Pietsch Integral Operators

This section will examine the topological dual of the injective tensor product. This space is much harder to characterize than the dual of the projective tensor product in Corollary 4.1.10.

Definition 5.2.1. Given Banach spaces X and Y, a bilinear $B \in B(X \times Y)$ is called *integral*, if its linearization $\tilde{B}$ ($\in (X \otimes Y)^*$) is continuous with respect to $\epsilon_{X,Y}$, thus can be seen as an element of $\big(X\hat{\otimes}_\epsilon Y\big)'$.
We define the *integral norm* of such a B by $\|B\|_{int} := \|\tilde{B}\|_\epsilon$, where $\|\cdot\|_\epsilon$ denotes the operator norm with respect to the norm ϵ on $(X\hat{\otimes}_\epsilon Y)'$.
We denote by $B_{int}(X, Y)$ the set of all integral operators endowed with the integral norm.

Remark 5.2.2. Clearly, the set $B_{int}(X, Y)$ constitutes a vector space with the usual addition and scalar multiplication. Given a bilinear form B its linearization $\tilde{B}$ is uniquely determined, since $\tilde{B} \in (X\hat{\otimes}_\epsilon Y)'$ is determined by its values on all tensors of the form $x \otimes y$, $x \in X$, $y \in Y$. Thus, the mapping $B \mapsto \tilde{B}$ is isometric from $(B(X \times Y), \|\cdot\|_{int})$ into $((X\hat{\otimes}_\epsilon Y)', \|\cdot\|_\epsilon)$. On the other hand, for $S \in (X\hat{\otimes}_\epsilon Y)'$ the function $B_S(x, y) = S(x \otimes y)$ is an element of $B(X \times Y)$. By definition, the linearization of B_S is exactly S and $\|B_S\|_{int} = \|S\|$, showing

$$B_{int}(X, Y) \cong \big(X\hat{\otimes}_\epsilon Y\big)'. \tag{5.2}$$

Remark 5.2.3. Let X and Y be Banach spaces and consider $K_1^{X'}(0)$ and $K_1^{Y'}(0)$. If both sets are endowed with the weak*-topology, then by the Theorem of Banach Alaoglu the space $K := K_1^{X'}(0) \times K_1^{Y'}(0)$ is compact if endowed with the product topology. For $x \in X$ and $y \in Y$ the mappings $x' \mapsto x'(x)$ from $K_1^{X'}(0)$ into $\mathbb{C}$ and $y' \mapsto y'(y)$ from $K_1^{Y'}(0)$ into $\mathbb{C}$ are continuous. Consequently, the mapping $(x', y') \mapsto x'(x)y'(y)$ from K into $\mathbb{C}$ is continuous. Furthermore, $(x, y) \mapsto ((x', y') \mapsto x'(x)y'(y))$ is bilinear. Hence, by Proposition 3.1.8, there exists a unique, linear mapping $F : X \otimes Y \to C(K)$, such that

$$F(\sum_{i=1}^n x_i \otimes y_i) = \big[(x', y') \mapsto \sum_{i=1}^n x'(x_i)y'(y_i)\big].$$

Given $v = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we have

$$\epsilon(v) = \sup\Big\{|\sum_{i=1}^n x'(x_i)y'(y_i)| : \ x' \in K_1^{X'}(0), \ y' \in K_1^{Y'}(0)\Big\} = \|F(v)\|_\infty.$$

In consequence, $F : X \otimes_\epsilon Y \to (C(K), \|\cdot\|_\infty)$ is an isometry and F can be extended to a linear isometry $\tilde{F} : X\hat{\otimes}_\epsilon Y \to C(K)$.

This remark preludes the next result, characterizing the space of all integral bilinear form.

Theorem 5.2.4. Given Banach spaces X and Y, for a bilinear form $B \in B(X \times Y)$ its linearization $\tilde{B}$ is continuous with respect to $\epsilon_{X,Y}$ if and only if there exists a complex regular Borel measure μ on the compact space $K := K_1^{X'}(0) \times K_1^{Y'}(0)$, such that

$$B(x, y) = \int_K x'(x)y'(y) \, d\mu(x', y') \quad \text{for all } x \in X, y \in Y. \tag{5.3}$$

In this case we have $\|B\|_{int} = \inf |\mu|(K)$, taking the infimum over all measures μ which satisfy (5.3). This infimum is in fact a minimum.

Proof. By (5.2) we can characterize the elements of $(X\hat{\otimes}_\epsilon Y)'$ as linearizations $\tilde{B}$ of some integral operator $B \in B(X \times Y)$. If $\tilde{F}$ is as in Remark 5.2.3, then the composition $\tilde{B} \circ \tilde{F}^{-1}$ constitutes a bounded, linear functional on $\tilde{F}(X\hat{\otimes}_\epsilon Y) \subseteq C(K)$ of norm $\|\tilde{B}\|$. By the Theorem of Hahn Banach there exists an extension T of $\tilde{B} \circ \tilde{F}^{-1}$ on the space $C(K)$ with the same norm. By Riesz-Markov, Theorem 2.12 and Theorem 6.19 in [Ru], there exists a complex regular Borel measure μ on K, such that

$$T(f) = \int_K f(x', y')\, d\mu(x', y') \quad \text{for all } f \in C(K),$$

where $|\mu|(K) = \|\tilde{B}\| = \|B\|_{int}$. Clearly

$$\begin{aligned}\int_K x'(x)y'(y)\, d\mu(x', y') &= T\big(F(x \otimes y)\big) = (\tilde{B} \circ \tilde{F}^{-1} \circ \tilde{F})(x \otimes y) \\ &= \tilde{B}(x \otimes y) = B(x, y).\end{aligned}$$

For the converse let $B \in B(X \times Y)$ and suppose

$$B(x, y) = \int_K x'(x)y'(y)\, d\mu(x', y') \quad \text{for all } x \in X, y \in Y,$$

where μ is a complex regular Borel measure on K. Defining the linearization in the usual way yields

$$\tilde{B}\big(\sum_{i=1}^n x_i \otimes y_i\big) = \sum_{i=1}^n B(x_i, y_i).$$

Hence,

$$\tilde{B}(v) = \int_K F(v)(x', y')\, d\mu(x', y')$$

for all $v \in X \otimes Y$. Because of

$$\begin{aligned}|\tilde{B}(v)| &= \Big|\int_K (Fv)(x', y')\, d\mu(x', y')\Big| \\ &\le \int_K |(Fv)(x', y')|\, d|\mu|(x', y') \\ &\le \|Fv\|_\infty |\mu|(K) = \epsilon(v)|\mu|(K).\end{aligned}$$

$\tilde{B}$ is bounded with respect to the injective norm by $\|\tilde{B}\| \le |\mu|(K)$. With help of the first part of the proof we finally see $\|\tilde{B}\|_{int} = \min |\mu|(K)$. □

Definition 5.2.5. Given Banach spaces X and Y, $T \in L(X, Y)$ is called *integral* if the bilinear form defined by $B_T(x, y') := \langle Tx, y' \rangle$ constitutes an integral bilinear form in $B(X, Y')$.
We define the *integral norm* of T by $\|T\|_{int} := \|B_T\|_{int}$ and denote the set of all integral operators by $Int(X, Y)$. Clearly, this set constitutes a vector space with the usual addition and scalar multiplication and $\|\cdot\|_{int}$ is a norm on $Int(X, Y)$.

Proposition 5.2.6. Let W, X, Y and Z be Banach spaces, $T \in L_b(W, X)$, $R \in L_b(Y, Z)$. If $S \in L_b(X, Y)$ is integral, then $RST : W \to Z$ is integral and $\|RST\|_{int} \leq \|R\| \|S\|_{int} \|T\|$.

$$\begin{array}{ccc} W & \xrightarrow{RST} & Z \\ {\scriptstyle T}\downarrow & & \uparrow{\scriptstyle R} \\ X & \xrightarrow[S]{} & Y \end{array}$$

Proof. Consider the bounded, linear operator $T \otimes_\epsilon R' : W \hat{\otimes}_\epsilon Z' \to X \hat{\otimes}_\epsilon Y'$, given by Proposition 4.2.6, and its adjoint $(T \otimes_\epsilon R')' : (X \hat{\otimes}_\epsilon Y')' \to (W \hat{\otimes}_\epsilon Z')'$. Let $B_S \in B_{int}(X, Y')$ be the integral bilinear form defined by $B_S(x, y') = \langle Sx, y' \rangle$. By assumption its linearization $\tilde{B}_S$ is an element of $(X \hat{\otimes}_\epsilon Y')'$. Given $w \in W$ and $z' \in Z'$ we calculate

$$\begin{aligned} \big\langle (w \otimes z'), (T \otimes_\epsilon R')' \tilde{B}_S \big\rangle &= \big\langle (T \otimes_\epsilon R')(w \otimes z'), \tilde{B}_S \big\rangle = \big\langle (Tw \otimes R'z'), \tilde{B}_S \big\rangle \\ &= B_S(Tw, R'z') = \langle STw, R'z' \rangle = \langle RSTw, z' \rangle \\ &= B_{RST}(w, z') = \tilde{B}_{RST}(w \otimes z'). \end{aligned}$$

Hence, the linearization of B_{RST} is the image of $\tilde{B}_S$ under the operator $(T \otimes_\epsilon R')'$ and in consequence an element of $(W \hat{\otimes}_\epsilon Z')'$. Thus, RST is an integral operator satisfying

$$\begin{aligned} \|RST\|_{int} = \|B_{RST}\|_{int} &= \|(T \otimes R')'(\tilde{B}_S)\|_\epsilon \leq \|(T \otimes R')'\| \cdot \|\tilde{B}_S\|_\epsilon \\ &= \|T \otimes_\epsilon R'\| \|S\|_{int} \leq \|T\| \|R'\| \|S\|_{int} = \|R\| \|S\|_{int} \|T\|. \end{aligned}$$

□

Proposition 5.2.7. Let X, Y be Banach spaces and denote by ι_Y the canonical embedding from Y into Y''. An operator $T : X \to Y$ is integral if and only if $\iota_Y T : X \to Y''$ is integral. In this case we have $\|T\|_{int} = \|\iota_Y T\|_{int}$.

Proof. By Proposition 5.2.6 $\iota_Y T$ is integral with $\|\iota_Y T\|_{int} \leq \|\iota_Y\| \cdot \|T\| \leq \|T\|_{int}$, if T is integral.
In order to show the converse, assume $\iota_Y T$ to be integral and let $B := B_{\iota_Y T} \in B_{int}(X, Y''')$ be defined by $B(x, y''') = \langle \iota_Y T(x), y''' \rangle$ for $x \in X$ und $y''' \in Y'''$. We denote by $\iota_{Y'}$ the canonical embedding from Y' into Y''' and by Id_X the identity operator on X. With the help of Proposition 4.2.6 we define the bounded operator $\sigma := Id_X \otimes_\epsilon \iota_{Y'} : X \hat{\otimes}_\epsilon Y' \to X \hat{\otimes}_\epsilon Y'''$ with its adjoint $\sigma' : (X \hat{\otimes}_\epsilon Y''')' \to (X \hat{\otimes}_\epsilon Y')'$. The linearization $\tilde{B}$ of B is an element of $(X \hat{\otimes}_\epsilon Y''')'$ since $\iota_Y T$ is integral. We calculate

$$\begin{aligned} \big\langle (x \otimes y'), \sigma'(\tilde{B}) \big\rangle &= \big\langle \sigma(x \otimes y'), \tilde{B} \big\rangle = \big\langle (x \otimes \iota_{Y'}(y')), \tilde{B} \big\rangle \\ &= B(x, \iota_{Y'}(y')) = \big\langle (\iota_Y T x, \iota_{Y'}(y') \big\rangle \\ &= \big\langle y', \iota_Y(Tx) \big\rangle = \big\langle Tx, y' \big\rangle \\ &= B_T(x, y') = \langle (x \otimes y'), \tilde{B}_T \rangle. \end{aligned}$$

Therefore, with $\sigma'(\tilde{B})$ also the linearization $\tilde{B}_T$ of B_T is an element of $(X\hat{\otimes}_\epsilon Y')'$. Thus T integral, where

$$\begin{aligned}\|T\|_{int} &= \|B_T\|_{int} = \|\sigma'(\tilde{B})\| \\ &\leq \|\sigma\|\|B\|_{int} = \|Id_X \otimes_\pi \iota_Y'\|\|\iota_Y T\|_{int} \\ &\leq \|Id_X\|\|\iota_Y'\|\|\iota_Y T\|_{int} = \|\iota_Y T\|_{int}.\end{aligned}$$

□

Proposition 5.2.8. Given a finite measure space $(\Omega, \mathcal{A}, \mu)$ with positive μ, the canonical embedding $I : L_\infty(\mu) \to L_1(\mu)$ is an integral operator satisfying $\|I\|_{int} = \mu(\Omega)$.

Proof. Since $L_1(\mu)'$ can be identified with $L_\infty(\mu)$, it suffices to show that the bilinear form $B_I : L_\infty(\mu) \times L_\infty(\mu) \to \mathbb{C}$ defined by $B_I : (f, g) \mapsto \int_\Omega fg \; d\mu$ is integral, i.e. its linearization $\tilde{B}_I$ is an element of the dual space of $L_\infty(\mu)\hat{\otimes}_\epsilon L_\infty(\mu)$.
To start with, consider tensors with a representation $v = \sum_{i=1}^n f_i \otimes g_i$, f_i and g_i being measurable simple functions. In this case v can be written as $v = \sum_{j,k} \lambda_{jk}(\chi_{A_j} \otimes \chi_{B_k})$, (A_j) and (B_k) being finite sequences of pairwise disjoint, measurable sets on Ω with strictly positive measure. With the help of Remark 4.2.8 we get

$$\begin{aligned}\epsilon(v) &= \sup\{\big|\sum_{j,k} \lambda_{jk} \; \langle f, \chi_{A_j}\rangle \; \langle g, \chi_{B_k}\rangle\big| : \; f, g \in L_1(\mu), \|f\|_1, \|g\|_1 \leq 1\big\} \\ &\leq \sup\{\sum_{j,k} |\lambda_{jk}| \; |\langle f, \chi_{A_j}\rangle| \; |\langle g, \chi_{B_k}\rangle| : \; f, g \in L_1(\mu), \|f\|_1, \|g\|_1 \leq 1\big\} \\ &\leq \sup_{j,k} |\lambda_{jk}| \; \sup\big\{\big(\sum_j \int_{A_j} |f| \; d\mu\big)\big(\sum_k \int_{B_k} |g| \; d\mu\big) : \; f, g \in L_1(\mu), \|f\|_1, \|g\|_1 \leq 1\big\} \\ &\leq \sup_{j,k} |\lambda_{jk}|.\end{aligned}$$

For fixed r and s the functions $f = \mu(A_r)^{-1}\chi_{A_r}$ and $g = \mu(B_s)^{-1}\chi_{B_s}$ are vectors in $L_1(\mu)$ of norm 1 satisfying

$$\epsilon(v) \geq \big|\sum_{j,k} \lambda_{jk}\langle f, \chi_{A_j}\rangle\langle g, \chi_{B_k}\rangle\big| = |\lambda_{rs}|.$$

In consequence, $\epsilon(v) = \sup_{j,k} |\lambda_{jk}|$. Moreover,

$$\begin{aligned}|\tilde{B}_I(v)| &= \big|\sum_{j,k} \lambda_{jk} \int_\Omega \chi_{A_j}\chi_{B_k} \; d\mu\big| = \big|\sum_{jk} \lambda_{jk}\mu(A_j \cap B_k)\big| \\ &\leq \sup_{j,k} |\lambda_{jk}| \; \sum_{j,k} \mu(A_j \cap B_k) \leq \sup_{j,k} |\lambda_{jk}|\mu(\Omega) = \mu(\Omega)\epsilon(v).\end{aligned} \tag{5.4}$$

Given an arbitrary tensor $v = \sum_{i=1}^n f_i \otimes g_i \in L_\infty(\mu) \otimes_\epsilon L_\infty(\mu)$ choose sequences $(f_{ik})_{k=1}^\infty$, $(g_{ik})_{k=1}^\infty$ of simple functions, converging uniformly to f_i and g_i respectively. We claim that $v_k = \sum_{i=1}^n f_{ik} \otimes g_{ik}$ converges to v with respect to the injective norm. For every $k \in \{1, \ldots, n\}$ we have

$$\begin{aligned}\epsilon(f_{ik} \otimes g_{ik} - f_i \otimes g_i) &= \epsilon(f_{ik} \otimes g_{ik} - f_{ik} \otimes g_i + f_{ik} \otimes g_i - f_{ik} \otimes g_{ik}) \\ &\leq \epsilon\big(f_{ik} \otimes (g_{ik} - g_i)\big) + \epsilon\big((f_{ik} - f_i) \otimes g_i\big) \\ &\leq \|f_{ik}\|\|g_{ik} - g_i\| + \|f_{ik} - f_i\|\|g_{ik}\|.\end{aligned}$$

$(\|f_{ik}\|)_{k\in\mathbb{N}}, (\|g_{ik}\|)_{k\in\mathbb{N}}$ being bounded, the latter term converges to zero for $k \to \infty$. Since v is the sum of the tensors $f_i \otimes g_i$, the v_k converge to v for $k \to \infty$ with respect to ϵ. Furthermore,

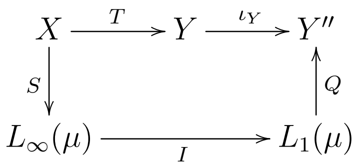
$$\tilde{B}_I(f_{ik} \otimes g_{ik}) = \int\limits_{\Omega} f_{ik} g_{ik} \, d\mu \overset{k\to\infty}{\longrightarrow} \int\limits_{\Omega} f_i g_i \, d\mu = \tilde{B}_I(f_i \otimes g_i).$$

Taking the limit $k \to \infty$ by (5.4) we have $|\tilde{B}_I(v)| \leq \mu(\Omega)\epsilon(v)$. Consequently, B_I is a bounded bilinear form on $L_\infty(\mu) \otimes_\epsilon L_\infty(\mu)$ satisfying $\|B_I\|_{int} \leq \mu(\Omega)$.
For $f = g = 1$ and $v = f \otimes g$ we have $\|v\|_\epsilon = 1$ and $\tilde{B}_I(v) = \mu(\Omega) \leq \|B_I\|_{int}$. In total $\|B_I\|_{int} = \mu(\Omega)$. □

Applying the previous result for a complex measure μ and the positive measure $|\mu|$ yields the following statement.

Corollary 5.2.9. For a complex measure μ the canonical embedding $I : L_\infty(\mu) \to L_1(\mu)$ is integral.

Theorem 5.2.10. Given Banach spaces X and Y, an operator $T : X \to Y$ is integral if and only if there exists a complex regular Borel measure μ on a compact Hausdorff space K and operators $S \in L_b(X, L_\infty(\mu))$ and $Q \in L_b(L_1(\mu), Y'')$, such that $\iota_Y T = QIS$. Here ι_Y denotes the canonical embedding from Y into Y'' and I the canonical mapping from $L_\infty(\mu)$ into $L_1(\mu)$. In this case we have $\|T\|_{int} \leq \|S\|\|Q\||\mu|(K)$. Moreover, S, Q, K and μ can be chosen, such that $\|S\|$,$\|Q\| = 1$ and $\|T\|_{int} = |\mu|(K)$.



Proof. Let $T : X \to Y$ be integral meaning that the bilinear form $B_T \in B(X, Y')$ defined by $B_T(x, y') = y'(Tx)$ is integral. By Theorem 5.2.4 there exists a complex regular Borel measure μ on the compact Hausdorff space $K := K_1^{X'} \times K_1^{Y''}$, both sets endowed with the weak*-topology, such that

$$\langle Tx, y'\rangle = B_T(x, y') = \int\limits_K x'(x) y''(y') \, d\mu(x', y'') \quad \text{for all } x \in X, \; y' \in Y',$$

and $|\mu|(K) = \|B_T\|_{int} = \|T\|_{int}$. We define $S : X \to L_\infty(\mu)$ by $Sx = [(x', y'') \mapsto x'(x)]$ and $R : Y' \to L_\infty(\mu)$ by $Ry' = [(x', y'') \mapsto y''(y')]$. By a straight forward Hahn-Banach argument S and R are linear and bounded with $\|S\|, \|R\| = 1$. Employing $L_1(\mu)' \cong$

$L_\infty(\mu)$ according to Lemma 1.3.1 we calculate

$$\begin{aligned}
(\iota_Y Tx)(y') = y'(Tx) &= \int\limits_K x'(x)y''(y')\ d\mu(x',y'') \\
&= \int\limits_K (Sx)(x',y'')(Ry')(x',y'')\ d\mu(x',y'') \\
&= \langle ISx, Ry'\rangle = \langle Ry', \iota_{L_1(\mu)} ISx\rangle \\
&= \langle y', R'\iota_{L_1(\mu)} ISx\rangle = (R'\iota_{L_1(\mu)} ISx)(y'),
\end{aligned}$$

where $R' : L_\infty(\mu)' \to Y''$ denotes the adjoint of R and $\iota_{L_1(\mu)}$ the canonical embedding from $L_1(\mu)$ into $L_1(\mu)''$. Defining $Q := R' \circ \iota_{L_1(\mu)}$ we obtain $\iota_Y T = QIS$, which constitutes the required factorization.
Conversely, if $T : X \to Y$ has a factorization $\iota_Y T = QIS$, then Propositions 5.2.6, 5.2.7 and Corollary 5.2.9 show that T is integral and $\|T\|_{int} \le \|S\|\|Q\||\mu|(K)$. $\square$

Proposition 5.2.11. Given Banach spaces X and Y, an operator $T \in L_b(X,Y)$ is integral if and only if its adjoint $T' \in L_b(Y',X')$ is integral. In this case $\|T\|_{int} = \|T'\|_{int}$.

Proof. Suppose, $T : X \to Y$ is integral. We choose Q, S, K and μ as in Theorem 5.2.10 satisfying $\|Q\|, \|S\| = 1$, $\|T\|_{int} = |\mu|(K)$ and $\iota_Y T = QIS$. We denote by $\iota_{Y'} : Y' \to Y''$ the canonical embedding and by L the canonical embedding from $L_1(\mu)$ into $L_1(\mu)''$. From

$$\langle f, LIg\rangle = \langle Ig, f\rangle = \int\limits_K fg\ d\mu \quad \text{and} \quad \langle If, g\rangle = \int\limits_K fg\ d\mu, \quad \text{for } f,g \in L_\infty(\mu)$$

and $(\iota_Y)' \circ \iota_{Y'} = \mathrm{id}_{Y'}$ we derive

$$T' = T'(\iota_Y)'\iota_{Y'} = (\iota_y T)'\iota_{Y'} = S'LIQ'\iota_{Y'}.$$

We have the diagram

$$\begin{array}{ccccccc}
Y' & \xrightarrow{\iota_{Y'}} & Y''' & \xrightarrow{(\iota_Y)'} & Y' & \xrightarrow{T'} & X' \\
 & & \big\downarrow{\scriptstyle Q'} & & & & \big\uparrow{\scriptstyle S'} \\
 & & L_\infty(\mu) = L_1(\mu)' & \xrightarrow[I]{} & L_1(\mu) & \xrightarrow[L]{} & L_\infty(\mu)' = L_1(\mu)''
\end{array}$$

Proposition 5.2.6 and Corollary 5.2.9 show that T' integral and

$$\begin{aligned}
\|T'\|_{int} = \|S'LIQ'\iota_{Y'}\|_{int} &\le \|S'\|\|L\|\|I\|_{int}\|Q'\|\|\iota_{Y'}\| \\
&\le \|S\|\|L\|\|I\|_{int}\|Q\|\|\iota_{Y'}\| \\
&\le \|I\|_{int} = |\mu|(K) = \|T\|_{int}.
\end{aligned}$$

Conversely, suppose $T \in L_b(X,Y)$ with $T' \in L_b(Y'',Y')$ being integral. By Lemma 1.1.2 we have the following diagram

$$\begin{array}{ccccc}
X & \xrightarrow{T} & Y & \xrightarrow{\iota_Y} & Y'' \\
 & {\scriptstyle \iota_X}\searrow & & \nearrow{\scriptstyle T''} & \\
 & & X'' & &
\end{array}$$

By the first part of the proof T'' is integral and by Proposition 5.2.6 so is $\iota_Y T = T''\iota_X$. By Proposition 5.2.7 the operator T is integral and $\|T\|_{int} = \|\iota_Y T\|_{int} \leq \|T''\iota_X\|_I \leq \|T'\|_{int}$. □

Corollary 5.2.12. Given Banach spaces X and Y, $B \in \mathcal{B}(X,Y)$ is integral if and only if the bounded linear operator $T_B : X \to Y'$ defined by $B(x,y) = \langle y, T_B x\rangle$ as in Remark 4.1.7 is integral. In this case $\|B\|_{int} = \|T_B\|_{int}$, i.e.

$$B_{int}(X,Y) \cong Int(X,Y').$$

Proof. If T_B is integral, the bilinear form $\tau \in B(X,Y'')$ defined by $\tau(x,y'') = \langle T_B x, y''\rangle$ is an integral bilinear form satisfying $\|\tau\|_{int} = \|T_B\|_{int}$, i.e. its linearization $\tilde{\tau}$ is an element of $(X\hat{\otimes}_\epsilon Y'')'$. Next we consider the operator $Id_X \otimes_\epsilon \iota_Y : X\hat{\otimes}_\epsilon Y \to X\hat{\otimes}_\epsilon Y''$ as given by Proposition 4.2.6. For the linearization $\tilde{B} : X \otimes Y \to \mathbb{C}$ of B we calculate

$$\begin{aligned}\langle x\otimes y, \tilde{B}\rangle = B(x,y) = \langle y, T_B x\rangle &= \langle T_B x, \iota_Y(y)\rangle = \tau(x, \iota_Y(y)) \\ &= \langle x \otimes \iota_Y(y), \tilde{\tau}\rangle = \langle (Id_X \otimes_\epsilon \iota_Y)(x\otimes y), \tilde{\tau}\rangle \\ &= \langle x\otimes y, (Id_X \otimes_\epsilon \iota_Y)'(\tilde{\tau})\rangle.\end{aligned}$$

Hence, $\tilde{B}$ is the image of $\tilde{\tau}$ under $(Id_X \otimes_\epsilon \iota_Y)'$ and as such an element of $(X\hat{\otimes}_\epsilon Y)'$, i.e. B is integral. Furthermore,

$$\begin{aligned}\|B\|_{int} &= \left\|(Id_X\hat{\otimes}_\epsilon \iota_Y)'\tilde{\tau}\right\|_{int} \leq \left\|(Id_X\hat{\otimes}_\epsilon \iota_Y)'\right\| \|\tau\|_{int} \\ &= \left\|Id_X\hat{\otimes}_\epsilon \iota_Y\right\| \|T_B\|_{int} \leq \|T_B\|_{int}\end{aligned}$$

Conversely, suppose B is an integral bilinear form on $X \times Y$. By Theorem 5.2.10 there exists a complex regular Borel measure μ on $K := K_1^{X'}(0) \times K_1^{Y'}(0)$, such that

$$B(x,y) = \int_K x'(x)y'(y)\ d\mu(x',y')$$

for all $x \in X$ and $y \in Y$, satisfying $\|B\|_{int} = |\mu|(K)$. If we define $R : X \to L_\infty(\mu)$ and $S : Y \to L_\infty(\mu)$ by $Rx = [(x',y') \mapsto x'(x)]$ and $Sy = [(x',y') \mapsto y'(y)]$, then

$$\begin{aligned}\langle y, T_B x\rangle = B(x,y) &= \int_K x'(x)y'(x)\ d\mu(x',y') \\ &= \int_K (Rx)(x',y')(Sy)(x',y')\ d\mu(x',y') \\ &= \langle ISy, Rx\rangle = \langle y, (IS)'(Rx)\rangle,\end{aligned}$$

where I denotes again the canonical embedding from $L_\infty(\mu)$ into $L_1(\mu)$. Hence, $T_B = S'I'R$. Since I is integral, Proposition 5.2.6 and Proposition 5.2.11 show that T_B is integral and

$$\begin{aligned}\|T_B\|_{int} &= \|S'I'R\|_{int} \leq \|S'\| \|I'\|_{int} \|R\| \\ &\leq \|I'\|_{int} = \|I\|_{int} \leq |\mu|(K) = \|B\|_{int}.\end{aligned}$$

□

Corollary 5.2.13. Given Banach spaces X and Y, $Int(X,Y) \ni T \mapsto \iota_Y \circ T \in Int(X,Y'')$ is isometric, where $\iota_Y : Y \to Y''$ denotes the natural embedding.

Proof. If $T \in L_b(X,Y)$ is integral, by Definition 5.2.5 the bilinear form $B \in \mathcal{B}(X \times Y')$, $B(x,y') = \langle Tx, y' \rangle$ is integral with $\|T\|_{int} = \|B\|_{int}$. In turn, by Corollary 5.2.12, the operator $S \in L_b(X,Y'')$, where $\langle y', Sx \rangle = B(x,y')$, is integral with $\|S\|_{int} = \|B\|_{int}$. Thus, $\langle y', S \rangle = \langle y', \iota_Y \circ T(x) \rangle$ for all $y' \in Y'$ and in consequence $\iota_Y \circ T = S \in Int(X,Y'')$ with $\|\iota_Y \circ T\|_{int} = \|T\|_{int}$. □

In the proof of the Theorem 5.2.10 it was essential to extend the image of $T : X \to Y$ to the bidual Y'', in order to find a suitable factorization. If we insist on staying in Y, we obtain a new class of operators, which represent a link between nuclear and integral operators.

Definition 5.2.14. Given Banach spaces X, Y, an operator $T \in L_b(X,Y)$ is called *Pietsch integral*, if there exists a complex regular Borel measure μ on a compact Hausdorff space K, as well as operators $S \in L_b(X, L_\infty(\mu))$, $Q \in L_b(L_1(\mu), Y)$ satisfying $\|Q\|, \|S\| \leq 1$, such that $T = QIS$, where I denotes the canonical embedding from $L_\infty(\mu)$ into $L_1(\mu)$. We define the *Pietsch integral norm* by $\|T\|_{pi} = \inf \|Q\| \|S\| |\mu|(K)$, taking the infimum over all such factorizations. We denote the set of all Pietsch integral operators endowed with the Pietsch integral norm by $PI(X,Y)$.

$$\begin{CD} X @>T>> Y \\ @VSVV @AAQA \\ L_\infty(\mu) @>>I> L_1(\mu) \end{CD}$$

Lemma 5.2.15. Let W, X, Y, Z be Banach spaces, $B \in L_b(W,X)$, $T \in L_b(X,Y)$ and $A \in L_b(Y,Z)$. If T is Pietsch integral, then $ATB \in L_b(W,Z)$ is Pietsch integral and $\|ATB\|_{pi} \leq \|A\| \|T\|_{pi} \|B\|$.

Proof. By definition there exists a compact Hausdorff space K, a complex regular Borel measure μ on K, as well as operators $Q \in L_b(L_1(\mu), Y)$ and $S \in L_b(X, L_\infty(\mu)$, such that the following diagram commutes.

$$\begin{CD} W @>B>> X @>T>> Y @>A>> Z \\ @. @VSVV @AAQA @. \\ @. L_\infty(\mu) @>>I> L_1(\mu) @. \end{CD}$$

Thus, $ATB = (AQ)I(SB)$ is Pietsch integral and

$$\|ATB\|_{pi} = \|AQISB\|_{pi} \leq \|AQ\| \|SB\| |\mu|(\Omega) \leq \|A\| \|Q\| \|S\| \|B\| |\mu|(\Omega).$$

Taking the infimum of the right side over all possible factorization $T = QIS$ yields $\|ATB\|_{pi} \leq \|A\| \|T\|_{pi} \|B\|$. □

Proposition 5.2.16. Given Banach spaces X and Y every nuclear operator $T : X \to Y$ is Pietsch integral and $\|T\|_{pi} \leq \|T\|_{nuc}$.

Proof. A nuclear operator T can be written as $T(x) = \sum_{i=1}^{\infty} x'_i(x) y_i$ for certain $x'_i \in X'$ and $y_i \in Y$ satisfying $\sum_{i=1}^{\infty} \|x_i\| \|y_i\| < +\infty$. As $0 \otimes y = 0 = x \otimes 0$ we can assume x_i, $y_i \neq 0$. We define $\phi_i := x'_i / \|x'_i\| \in X'$, $z_i := y_i / \|y_i\| \in Y$, $\lambda_i := \|x'_i\| \|y_i\| \in \mathbb{R}$ and obtain $T(x) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) z_i$, $x \in X$. We endow $K := K_1^{X'}(0)$ with the weak*-topology and define $\mu(E) = \sum_{i=1}^{\infty} \lambda_i \delta_{\phi_i}(E)$ for a Borel set $E \subseteq \Omega$, where $\delta_{\phi_i}(E) = 1$ in the case $\phi_i \in E$ and $\delta_{\phi_i}(E) = 0$ else. Because of

$$\mu(K) = \sum_{i=1}^{\infty} \lambda_i \delta_{\phi_i}(K) = \sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \|x'_i\| \|y_i\| < +\infty$$

μ constitutes a positive finite measure. Given $x \in X$ the mapping $S : X \to L_\infty(\mu)$ defined by $Sx = (x' \to x'(x))$ is linear and bounded by $\|S\| \leq 1$. For all $f \in L_1(\mu)$ we have

$$\sum_{i=1}^{\infty} \|\lambda_j f(\phi_j) z_j\| = \sum_{i=1}^{\infty} \lambda_j |f(\phi_j)| = \int_\Omega |f| \, d\mu < +\infty.$$

Consequently $\sum_{i=1}^{\infty} \lambda_j f(\phi_j) z_j$ is absolutely convergent and we are able to define the mapping $Q : L_1(\mu) \to Y$ by $Q(f) = \sum_{i=1}^{\infty} \lambda_j f(\phi_j) z_j$. Due to

$$\|Qf\| = \|\sum_{i=1}^{\infty} \lambda_j f(\phi_j) z_j\| \leq \sum_{i=1}^{\infty} \lambda_j |f(\phi_j)| = \|f\|_1$$

Q is bounded by $\|Q\| \leq 1$. Denoting again by $I : L_\infty(\mu) \to L_1(\mu)$ the canonical embedding

$$Q(ISx) = \sum_{i=1}^{\infty} \lambda_i (Sx)(\phi_i) z_i = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) z_i = T(x), \quad x \in X,$$

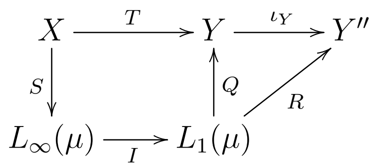
proves $T = QIS$, where Q, S, K and μ are as in Definition 5.2.14, i.e. T is Pietsch integral. Moreover,

$$\|T\|_{PI} \leq \|S\| \|Q\| |\mu|(K) \leq |\mu|(K) = \sum_{i=1}^{\infty} \|x'_i\| \|y_i\|. \tag{5.5}$$

Taking the infimum over all such representations of T shows $\|T\|_{pi} \leq \|T\|_{nuc}$. □

Theorem 5.2.17. Given Banach spaces X and Y every Pietsch integral operator $T : X \to Y$ is an integral operator with norm $\|T\|_{int} \leq \|T\|_{pi}$.
If there exists a projection $P : Y'' \to Y''$ with $\|P\| = 1$, such that $\operatorname{ran} P = \iota_Y(Y)$, then every integral operator is Pietsch integral and the integral norm coincides with the Pietsch integral norm.

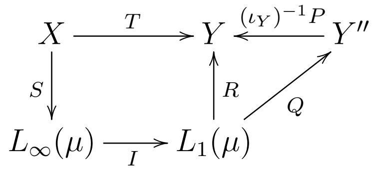
Proof. If $T : X \to Y$ is Pietsch integral, then there exists a complex regular Borel measure μ on a compact Hausdorff space K and a factorization $T = QIS$ as in Definition 5.2.14. Clearly, $\iota_Y T = RIS$ with $R := \iota_Y Q$, resulting in a factorization as in Theorem 5.2.10,



where in addition

$$\|T\|_{int} \leq \|RIS\|_{int} \leq \|\iota_Y\| \|Q\| \|I\|_{int} \|S\| = \|Q\| \|S\| |\mu|(K).$$

Taking the infimum over all such factorizations of T yields $\|T\|_{int} \leq \|T\|_{pi}$.
For the converse, suppose $T : X \to Y$ is integral and there exists a projection P on Y'' with range $\iota_Y(Y)$ and norm $\|P\| = 1$. We choose S, Q, K and μ as in Theorem 5.2.10, such that $\|T\|_{int} = |\mu|(K)$ and define $R := \iota_Y^{-1} PQ$. Then $T = RIS$ is a factorization as in Definition 5.2.14,



where

$$\|T\|_{pi} \leq \|R\| \|S\| |\mu|(\Omega) \leq |\mu|(\Omega) = \|T\|_{int}.$$

□

Remark 5.2.18. If for a Banach space Y there exists a projection as in Theorem 5.2.17, we call Y *complimented* in Y'' by a norm-1 projection. This is satisfied in particular if Y is a dual space. In fact for a Banach space X the canonical embedding $\iota_X : X \to X''$ constitutes an isometry, i.e. satisfying $\|\iota_X\| = 1$. Its adjoint operator $(\iota_X)'$ is a mapping on X''' into X' of norm one. Consequently, $\iota_{X'} \circ (\iota_X)'$ is a projection with the desired properties.

Corollary 5.2.19. Given Banach spaces X and Y we have

$$N(X, Y) \subseteq PI(X, Y) \subseteq Int(X, Y),$$

and every nuclear operator $T : X \to Y$ satisfies

$$\|T\|_{nuc} \geq \|T\|_{pi} \geq \|T\|_{int}.$$

Chapter 6

Operators on C(K)

The present Chapter is devoted to a deeper discussion of Pietsch integral operators, first introduced in [PP] in an even broader sense than presented below. The space $C(K)$ for a compact Hausdorff space K and the theory developed in Section 2.3 will prove essential for the understanding of such operators.
The structure of the following sections is orientated upon the approach found in Section VI in [DU] and Chapter 5 of [DJT]. The proofs and definitions are based on [DU], as well as on Chapters 1,2 and 5 of [DJT], and very loosely on Section 5.3 in [Ry]. For an in depth study on absolutely summing operators, integral and Pietsch integral operators in their general form, consider [DJT]. For a measure theoretical approach consider [DU]. For their connection in operator ideals consider [Pi], where this topic is extensively examined. For their connection to tensor products, take a look into [DF].

6.1 Absolutely Summing Operators

Definition 6.1.1. Let X and Y be Banach spaces. An operator $T \in L_b(X,Y)$ is called *absolutely summing* if there exists a constant $K \geq 0$, such that

$$\sum_{i=1}^{n} \|Tx_i\| \leq K \cdot \sup\{\sum_{i=1}^{n} |x'(x_i)| : \ x' \in K_1^{X'}(0)\}, \tag{6.1}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.
With $AS(X,Y)$ we will denote the set of all absolutely summing operators from X to Y. For any such operator T, the least constant K such that above equation holds true will be denoted as $\|T\|_{as}$ and is called the *absolutely summing norm* of T.
Taking $n = 1$ in (6.1) we see that $\|T\| \leq \|T\|_{as}$.

Proposition 6.1.2. Let X, Y, Z and W be Banach spaces and let $T \in L_b(W,X)$, $S \in L_b(X,Y)$, $T \in L_b(Y,Z)$. If S is absolutely summing, then $RST \in L_b(W,Z)$ is absolutely summing and $\|RST\|_{as} \leq \|R\| \|S\|_{as} \|T\|$.

Proof. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ we have

$$\begin{aligned}
\sum_{i=1}^{n} \|RST(x_i)\| &\leq \|R\| \sum_{i=1}^{n} \|S(Tx_i)\| \\
&\leq \|R\| \|S\|_{as} \cdot \sup\{\sum_{i=1}^{n} |x'(Tx_i)| : \; x' \in K_1^{X'}(0)\} \\
&= \|R\| \|S\|_{as} \cdot \sup\{\sum_{i=1}^{n} |T'x'(x_i)| : \; x' \in K_1^{X'}(0)\} \\
&\leq \|R\| \|S\|_{as} \|T'\| \cdot \sup\{\sum_{i=1}^{n} |x'(x_i)| : \; x' \in K_1^{X'}(0)\} \\
&\leq \|R\| \|S\|_{as} \|T\| \cdot \sup\{\sum_{i=1}^{n} |x'(x_i)| : \; x' \in K_1^{X'}(0)\}.
\end{aligned}$$

□

Definition 6.1.3. Let X be a Banach space. A sequence $(x_i)_{i\in\mathbb{N}} \in X$

- is called *strongly summable*, if $(x_i)_{i\in\mathbb{N}} \in \ell_1(X)$; also see Remark 4.1.12.
- is called *weakly summable*, if $(\langle x_i, x' \rangle)_{i\in\mathbb{N}} \in \ell_1$ for all $x' \in X'$.

The set of all weakly summable sequences will be denoted by $\ell_1^w(X)$. We define the weakly summable norm by

$$\|(x_i)_{i\in\mathbb{N}}\|_{1,w} := \sup\{\sum_{i=1}^{\infty} |\langle x_i, x' \rangle| : \; x' \in K_1^{X'}(0)\}.$$

Given a finite sequence $(x_i)_{i=1}^n \in X$, we can interpret it as a sequence in $\ell_1^w(X)$ by it extending to an infinite sequence with value $x_i = 0$ for $i > n$. We will denote with $(x_i)_{i=1}^n$ both the finite sequence in X as well as the corresponding element of $\ell_1^w(X)$.

Lemma 6.1.4. Given a Banach space X, $\ell_1^w(X)$ endowed with the weakly summable norm constitutes a Banach space.

Proof. It is easy to verify that $\|\cdot\|_{1,w}$ defines a norm. It remains to show completeness. Let $(x^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\ell_1^w(X)$, where $x^k = (x_i^k)_{i\in\mathbb{N}}$. Given $\epsilon > 0$ choose $N \in \mathbb{N}$, such that

$$\sum_{i=1}^{n} |\langle x_i^k - x_i^l, x' \rangle| \leq \left\|x^k - x^l\right\|_{1,w} < \epsilon \tag{6.2}$$

for all $k, l \geq N$, $n \in \mathbb{N}$ and $x' \in K_1^{X'}(0)$. Clearly, for $j \in \mathbb{N}$ and $x' \in K_1^{X'}(0)$

$$|\langle x_j^k - x_j^l, x' \rangle| \leq \sum_{i=1}^{\infty} |\langle x_i^k - x_i^l, x' \rangle| \leq \left\|x^k - x^l\right\|_{1,w} < \epsilon$$

and in turn

$$\left\|x_j - x_j^l\right\| = \sup_{x' \in K_1^{x'}(0)} |\langle x_j^k - x_j^l, x'\rangle| \leq \epsilon,$$

if only $k, l > N$. Thus, $(x_j^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X and hence converges to $x_j \in X$. We define $x := (x_j)_{j\in\mathbb{N}}$ and attempt to show that x is the limit of $(x^k)_{k\in\mathbb{N}}$ in $\ell_1^w(X)$. Taking the limit $l \to \infty$ and then the supremum over $n \in \mathbb{N}$ in the left hand side of (6.2) we obtain

$$\sum_{i=1}^{\infty} |\langle x_i^k - x_i, x'\rangle| < \epsilon \tag{6.3}$$

for all $x' \in K_1^{X'}(0)$ if only $k > N$. But this just means $\left\|x^k - x\right\|_{1,w} < \epsilon$ for all $k > N$ and we conclude $x^k - x \in \ell_1^w(X)$, consequently $x \in \ell_1^w(X)$. Moreover, as $\epsilon > 0$ was arbitrarily chosen, x is the limit of $(x^k)_{k\in\mathbb{N}}$. □

Remark 6.1.5. Given a Banach space X and a *norming set* $N \subseteq X'$, i.e.

$$\|x\| = \sup\{|x'(x)| : \ x' \in N\}$$

for all $x \in X$, in Definition 6.1.3, the value of the weakly summable norm stays the same if we only take the supremum over any norming subset of X' instead of all $x' \in K_1^{X'}(0)$. In order to show this we apply Lemma 1.2.3 to $z_i = \langle x_i, x'\rangle$ and obtain

$$\sum_{i=1}^{n} |\langle x_i, x'\rangle| = \sup\{ \ \Big|\sum_{i=1}^{n} \lambda_i \langle x_i, x'\rangle\Big| : \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\}$$

for $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ and $x' \in X'$. Consequently,

$$\begin{aligned}
\|(x_i)_{i\in\mathbb{N}}\|_{1,w} &= \sup\Big\{ \sum_{i=1}^{n} |\langle x_i, x'\rangle| : \ \|x'\|_{X'} \leq 1, \ n \in \mathbb{N}\Big\} \\
&= \sup\Big\{ \ \Big|\sum_{i=1}^{n} \langle \lambda_i x_i, x'\rangle\Big| : \ \|x'\|_{X'} \leq 1, \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\Big\} \\
&= \sup\Big\{ \ \Big|\Big\langle \sum_{i=1}^{n} \lambda_i x_i, x'\Big\rangle\Big| \ , \|x'\|_{X'} \leq 1, \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\Big\} \\
&= \sup\Big\{\Big\| \sum_{i=1}^{n} \lambda_i x_i\Big\| : \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\Big\} \\
&= \sup\Big\{ \ \Big|\Big\langle \sum_{i=1}^{n} \lambda_i x_i, x'\Big\rangle\Big| : \ x' \in N, \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\Big\} \\
&= \sup\Big\{\Big|\sum_{i=1}^{n} \langle \lambda_i x_i, x'\rangle\Big| : \ x' \in N, \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0)\Big\} \\
&= \sup\Big\{ \sum_{i=1}^{\infty} |\langle x_i, x'\rangle| : \ x' \in N\Big\}.
\end{aligned}$$

Proposition 6.1.6. Let X and Y be Banach spaces. $T \in L_b(X,Y)$ is absolutely summing if and only if $\hat{T}$, defined by $\hat{T}((x_i)_{i\in\mathbb{N}}) = (Tx_i)_{i\in\mathbb{N}}$, maps $\ell_1^w(X)$ into $\ell_1(Y)$. In this case $\hat{T} : \ell_1^w(X) \to \ell_1(Y)$ is a linear bounded operator with $\|\hat{T}\| = \|T\|_{as}$.

Proof. Suppose that T is absolutely summing and let $x = (x_i)_{i\in\mathbb{N}} \in \ell_1^w(X)$. From

$$\begin{aligned}
\|\hat{T}x\|_1 = \sum_{i=1}^{\infty} \|Tx_i\| &= \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \|Tx_i\| \\
&\le \sup_{n\in\mathbb{N}} \|T\|_{as} \sup_{\|x'\|\le 1} \sum_{i=1}^{n} |\langle x_i, x'\rangle| \\
&= \|T\|_{as} \sup_{\|x'\|\le 1} \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} |\langle x_i, x'\rangle| \\
&= \|T\|_{as} \sup_{\|x'\|\le 1} \sum_{i=1}^{\infty} |\langle x_i, x'\rangle| = \|T\|_{as} \|x\|_{1,w}
\end{aligned}$$

we conclude that $\hat{T}(\ell_1^w(X)) \subseteq \ell_1(Y)$. Clearly, $\hat{T}$ is linear and it follows from above that $\|\hat{T}\| \le \|T\|_{as}$.
Conversely, suppose that $\hat{T}(\ell_1^w(X)) \subseteq \ell_1(Y)$. Consider $x^k = (x_i^k)_{i\in\mathbb{N}} \in \ell_1^w(X)$, $k \in \mathbb{N}$, and $x = (x_i)_{i\in\mathbb{N}} \in \ell_1^w(X)$ such that x^k converges to x for $k \to \infty$ and $\hat{T}(x^k)$ converging to $y = (y_i)_{i\in\mathbb{N}} \in \ell_1(Y)$ for $k \to \infty$. For $i, k \in \mathbb{N}$ we calculate

$$\begin{aligned}
\|T(x_i) - y_i\|_Y &\le \left\|T(x_i) - T(x_i^k)\right\|_Y + \left\|T(x_i^k) - y_i\right\|_Y \\
&\le \|T\| \cdot \left\|x_i - x_i^k\right\|_X + \left\|T(x_i^k) - y_i\right\|_Y \\
&\le \|T\| \cdot \left\|x - x^k\right\|_{1,w} + \|\hat{T}(x^k) - y\|_1.
\end{aligned}$$

By our assumptions, k can be chosen such that the last term gets arbitrarily small, hence $T(x_i) = y_i$ for all $i \in \mathbb{N}$ or equivalently $\hat{T}(x) = y$. Therefore, $\hat{T}$ has a closed graph and with the help if the Closed Graph Theorem constitutes a bounded operator. For a finite sequence $(x_i)_{i=1}^n$ seen as an element of $\ell_1^w(X)$ we have

$$\sum_{i=1}^{n} \|Tx_i\| = \|\hat{T}((x_i)_{i=1}^n)\|_1 \le \|\hat{T}\| \cdot \|(x_i)_{i=1}^n\|_{1,w} = \|\hat{T}\| \cdot \sup_{\|x'\|\le 1} \sum_{i=1}^{n} |\langle x_i, x'\rangle|.$$

This means that T is absolutely summing and $\|T\|_{as} \le \|\hat{T}\|$. With the reverse inequality proven earlier we conclude $\|T\|_{as} = \|\hat{T}\|$. □

Example 6.1.7. Let K be a compact Hausdorff space and μ be a non-negative regular Borel measure on K.

(i) For each $\phi \in L_1(\mu)$ the multiplication operator

$$M_\phi := \left\{ \begin{array}{ccc} C(K) & \longrightarrow & L_1(\mu) \\ f & \mapsto & f \cdot \phi \end{array} \right.$$

is absolutely summing with $\|M_\phi\|_{as} = \|\phi\|_1$.

(ii) The canonical embedding

$$J: \ C(K) \longrightarrow L_1(\mu)$$

is absolutely summing with $\|J\|_{as} = \mu(K)$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

(iii) For each $\phi \in L_1(\mu)$, the multiplication operator

$$M_\phi := \begin{cases} L_\infty(\mu) & \longrightarrow & L_1(\mu) \\ f & \mapsto & f \cdot \phi \end{cases}$$

is absolutely summing with $\|M_\phi\|_{as} = \|\phi\|_1$.

(iv) The canonical embedding

$$I: \ L_\infty(\mu) \longrightarrow L_1(\mu)$$

is absolutely summing with $\|I\|_{as} = \mu(\Omega)$.

We will study those examples in order to verify our claim. Regarding (i), given $x \in K$ consider the point evaluation functional $\delta_x \in C(K)'$ defined by $\langle f, \delta_x \rangle = f(x)$. The set of all δ_x, $x \in X$, is a norming subset of $C(K)'$. Hence, by Remark 6.1.5 for $f_1, \ldots, f_n \in C(K)$ we have

$$\|(f_i)_{i=1}^n\|_{1,w} = \sup_{x \in K} \sum_{i=1}^n |\langle f_i, \delta_x \rangle| = \sup_{x \in K} \sum_{i=1}^n |f_i(x)| = \Big\| \sum_{i=1}^n |f_i| \Big\|_\infty.$$

From

$$\begin{aligned} \sum_{i=1}^n \|M_\phi(f_i)\|_1 &= \sum_{i=1}^n \|f_i \cdot \phi\|_1 = \sum_{i=1}^n \int_K |f_i \cdot \phi| \, d\mu = \int_K \sum_{i=1}^n |f_i \cdot \phi| \, d\mu \\ &= \int_K |\phi| \cdot \sum_{i=1}^n |f_i| \, d\mu \leq \Big(\int_K |\phi| d\mu \Big) \cdot \Big\| \sum_{i=1}^n |f_i| \Big\|_\infty \\ &= \|\phi\|_1 \cdot \|(f_i)_{i=1}^n\|_{1,w}. \end{aligned} \tag{6.4}$$

we conclude that M_ϕ is absolutely summing with $\|M_\phi\|_{as} \leq \|\phi\|_1$. Finally,

$$\|\phi\|_1 = \|\mathbb{1}_K \cdot \phi\|_1 = \|M_\phi(\mathbb{1}_K)\|_1 \leq \|M_\phi\| \leq \|\phi\|_{as}. \tag{6.5}$$

yields $\|M_\phi\|_{as} = \|\phi\|_1$.

Applying (i) for the multiplication operator with the constant 1-function immediately results in (ii).

Regarding (iii) we start with some general considerations. Let X be a Banach space and $x_1, \ldots, x_n \in X$. We calculate

$$\begin{aligned} \sup_{|a_1| \leq 1, \ldots, |a_n| \leq 1} \Big\| \sum_{i=1}^n a_i x_i \Big\| &= \sup_{|a_1| \leq 1, \ldots, |a_n| \leq 1} \ \sup_{\|x'\| \leq 1} \Big| \sum_{i=1}^n a_i \langle x_i, x' \rangle \Big| \\ &= \sup_{\|x'\| \leq 1} \sum_{i=1}^n |\langle x_i, x' \rangle| = \|(x_i)_{i=1}^n\|_{1,w}. \end{aligned}$$

Applying this for the Banach space $X = L_\infty(\mu)$ and functions $x_i = f_1, \dots, x_n = f_n \in L_\infty(\mu)$ we derive

$$\begin{aligned}\|(f_i)_{i=1}^n\|_{1,w} &= \sup\{\big\|\sum_{i=1}^n a_i f_i\big\|_\infty :\ |a_1| \le 1, \dots, |a_n| \le 1\} \\ &\le \sup\{\big|\sum_{i=1}^n a_i f_i(x)\big| :\ |a_1| \le 1, \dots, |a_n| \le 1,\ x \in K \setminus N\} \\ &\le \sup\{\big|\sum_{i=1}^n |f_i(x)|\big| :\ x \in K \setminus N\}\end{aligned}$$

for any μ-null subsets N of K implying

$$\|(f_i)_{i=1}^n\|_{1,w} \le \big\|\sum_{i=1}^n |f_i|\big\|_\infty.$$

On the other hand for $a = (a_1, \dots, a_n) \in (K_1^{\mathbb{C}}(0))^n$ we have for $x \in K \setminus N_a$

$$\big|\sum_{i=1}^n a_i f_i(x)\big| \le \big\|\sum_{i=1}^n a_i f_i\big\|_\infty,$$

with a certain μ-null set N_a. Let D be a countably dense subset of $(K_1^{\mathbb{C}}(0))^n$. For $N = \cup_{a\in D} N_a$ the above inequality holds for all $a \in D$ and all $x \in K \setminus N$. According to Lemma 1.2.3, applied with $z_i = |f_i(x)|$ and $\lambda_i = a_i$, we get

$$\sum_{i=1}^n |f_i(x)| = \sup_{a\in D} \big|\sum_{i=1}^n a_i f_i(x)\big|.$$

This yields

$$\begin{aligned}\big\|\sum_{i=1}^n |f_i|\big\|_\infty &\le \sup_{x\in K\setminus N} \sum_{i=1}^n |f_i(x)| = \sup_{a\in D} \sup_{x\in K\setminus N} \big|\sum_{i=1}^n a_i f_i(x)\big| \\ &\le \sup_{a\in D} \big\|\sum_{i=1}^n a_i f_i\big\|_\infty = \sup_{|a_1|\le 1,\dots,|a_n|\le 1} \big\|\sum_{i=1}^n a_i f_i\big\| = \|(f_i)_{i=1}^n\|_{1,w}.\end{aligned}$$

In total, we have shown that $\|(f_i)_{i=1}^n\|_{1,w} = \|\sum_{i=1}^n |f_i|\|_\infty$. By the same considerations as in (6.4) we get

$$\begin{aligned}\sum_{i=1}^n \|M_\phi(f_i)\|_1 &= \int_\Omega |\phi| \cdot \sum_{i=1}^n |f_i|\ d\mu \le \Big(\int_\Omega |\phi|\ d\mu\Big) \big\|\sum_{i=1}^n |f_i|\big\|_\infty \\ &= \|\phi\|_1 \cdot \|(f_i)_{i=1}^n\|_{1,w},\end{aligned}$$

showing that M_ϕ is absolutely summing with $\|M_\phi\|_{as} \le \|\phi\|_1$. Finally, in analogy to (6.5)

$$\|\phi\|_1 = \|\mathbb{1}_K \cdot \phi\|_1 = \|M_\phi(\mathbb{1}_K)\|_1 \le \|M_\phi\| \le \|\phi\|_{as} \tag{6.6}$$

yields $\|M_\phi\|_{as} = \|\phi\|_1$.
(iv) Follows from (iii) in the same way as (ii) followed from (i).

Lemma 6.1.8. Given a compact Hausdorff space K and $n \in \mathbb{N}$, $f_1, \dots, f_n \in C(K)$, we have

$$\Big\| \sum_{i=1}^{n} |f_i| \Big\|_\infty = \sup \Big\{ \Big\| \sum_{i=1}^{n} \lambda_i f_i \Big\|_\infty \ \lambda_1, \dots, \lambda_n \in K_1^{\mathbb{C}}(0) \setminus U_1^{\mathbb{C}}(0) \Big\}. \tag{6.7}$$

Proof. Given $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $x \in X$ we attain

$$|\lambda_1| \cdot |f_1(x)| + \ldots + |\lambda_n| \cdot |f_n(x)| \geq |\lambda_1 f_1(x) + \ldots + \lambda_n f(x)|,$$

hence

$$\||f_1| + \ldots + |f_n|\|_\infty \geq \sup\{\|\lambda_1 f_1 + \ldots + \lambda_n f_n\|_\infty : \ |\lambda_1|, \dots, |\lambda_n| = 1\}. \tag{6.8}$$

On the other hand, given $\epsilon > 0$ there exists an $x \in K_1^X(0)$, such that

$$\|(|f_1| + \ldots + |f_n|)\|_\infty - \epsilon \leq |f_1(x)| + \ldots + |f_n(x)| \tag{6.9}$$

Choosing $\lambda_i \in K_1^{\mathbb{C}}(0) \setminus U_1^{\mathbb{C}}(0)$ such that $\lambda_i f_i(x) = |f_i(x)|$, $i = 1, \dots, n$, yields

$$|f_1(x)| + \ldots + |f_n(x)| = \lambda_1 f_1(x) + \ldots + \lambda_n f_n(x) = |\lambda_1 f_1(x) + \ldots + \lambda_n f_n(x)|.$$

Taking the supremum of the right side over all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $|\lambda_1|, \dots, |\lambda_n| = 1$ and then over all $x \in K_1^X(0)$ we get in combination with (6.9)

$$\|(|f_1| + \ldots + |f_n|)\|_\infty - \epsilon \leq \sup\{\|\lambda_1 f_1 + \ldots + \lambda_n f_n\|_\infty : \ |\lambda_1|, \dots, |\lambda_n| = 1\}.$$

As ϵ was chosen arbitrarily, together with (6.8) we obtain

$$\Big\| \sum_{i=1}^{n} |f_i| \Big\|_\infty = \sup\Big\{\Big\| \sum_{i=1}^{n} \lambda_i f_i \Big\|_\infty : \ |\lambda_i| = 1\Big\}.$$

□

Theorem 6.1.9. Let K be a compact Hausdorff space and X be a Banach space. A bounded linear operator $T : C(K) \to X$ is absolutely summing if and only if its representing measure $m_T : \mathfrak{B}(K) \to X''$ as defined in Theorem 2.2.1 is of bounded variation. In this case $\|T\|_{as} = \|m_T\|$.

Proof. If m_T is of bounded variation, then by Lemma 2.2.2 m_T is a vector measure. In consequence $\int_K f \ dm_T$ ($\in X''$) is defined for $f \in B(\mathfrak{B}(K))$. Given $f_i \in C(K)$, $i = 1, \dots, n$, by (ii) from Theorem 2.2.1 and (i) from Proposition 2.1.15

$$\begin{aligned}
\sum_{i=1}^{n} \|T f_i\| &= \sum_{i=1}^{n} \sup_{\|x'\| \leq 1} \langle T f_i, x' \rangle = \sum_{i=1}^{n} \sup_{\|x'\| \leq 1} \langle x', \int_K f_i \ dm_T \rangle \\
&= \sum_{i=1}^{n} \Big\| \int_K f_i \ dm_T \Big\| \overset{2.1.15}{\leq} \sum_{i=1}^{n} \int_K |f_i| \ d|m_T| \\
&= \int_K \sum_{i=1}^{n} |f_i| \ d|m_T| \leq \Big\| \sum_{i=1}^{n} |f_i| \Big\|_\infty \|m_T\|.
\end{aligned} \tag{6.10}$$

By the Riesz representation theorem we can interpret $C(K)'$ is the space of all complex regular Borel measures, where $\langle f, \mu \rangle = \int_K f d\mu$. Applying Lemma 1.2.3 for $z_i = \int_K f_i \, d\mu$ we get

$$\sum_{i=1}^{n} | \int_K f_i \, d\mu| = \sup\{ \, \big| \sum_{i=1}^{n} \lambda_i \int_k f_i \, d\mu \big| : \; n \in \mathbb{N}, \; \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0) \setminus U_1^{\mathbb{C}}(0)\}$$

and further

$$\begin{aligned}
&\sup\{\big\| \sum_{i=1}^{n} \lambda_i f_i \big\|_\infty : \; n \in \mathbb{N}, \; \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0) \setminus U_1^{\mathbb{C}}(0)\} \\
&\quad = \sup\{\big| \int_K \sum_{i=1}^{n} \lambda_i f_i \, d\mu \big| : \; n \in \mathbb{N}, \; \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0) \setminus U_1^{\mathbb{C}}(0), \; \mu \in C(K)', \; \|\mu\| \leq 1\} \\
&\quad = \sup\{\big| \sum_{i=1}^{n} \lambda_i \int_K f_i \, d\mu \big| : \; n \in \mathbb{N}, \; \lambda_1, \ldots, \lambda_n \in K_1^{\mathbb{C}}(0) \setminus U_1^{\mathbb{C}}(0), \; \mu \in C(K)', \; \|\mu\| \leq 1\} \\
&\quad = \sup\{\sum_{i=1}^{n} | \int_K f_i \, d\mu| : \; \mu \in C(K)', \; \|\mu\| \leq 1\} \\
&\quad = \sup\{\sum_{i=1}^{n} |\langle f_i, \mu \rangle| : \; \mu \in C(K)', \; \|\mu\| \leq 1\}. \qquad (6.11)
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum_{i=1}^{n} \|T(f_i)\| &\overset{(6.10)}{\leq} \|m_T\| \cdot \big\| \sum_{i=1}^{n} |f_i| \big\|_\infty \\
&\overset{6.1.8}{=} \|m_T\| \cdot \sup\{\big\| \sum_{i=1}^{n} \lambda_i f_i \big\|_\infty : \; |\lambda_i| = 1\} \\
&\overset{(6.11)}{=} \|m_T\| \cdot \sup\{\sum_{i=1}^{n} |\langle f_i, \mu \rangle| : \; \mu \in C(K)', \; \|\mu\| \leq 1\}
\end{aligned}$$

Thus, T is absolutely summing and $\|T\|_{as} \leq \|m_T\|$.
Conversely, assume T to be absolutely summing. According to Theorem 2.2.1 $\iota_{X'}(x')m_T(A) = \langle x', m_T(A) \rangle$ defines a complex regular Borel measure on K for any $x' \in X'$. Given an arbitrary complex regular Borel measure μ on K, open and pairwise disjoint $A_i \subseteq K$ and $f_i \in C(K)$, such that $\|f_i\|_\infty \leq 1$ and $\operatorname{supp}(f_i) \subseteq A_i$ for $i = 1, \ldots l$, we have

$$\sum_{i=1}^{l} |\langle f_i, \mu \rangle| = \sum_{i=1}^{l} \Big| \int_K f_i \, d\mu \Big| \leq \|\mu\|.$$

If we take the supremum over all such measures with $\|\mu\| \leq 1$, we obtain

$$\sum_{i=1}^{l} \|T(f_i)\| \leq \|T\|_{as} \cdot \sup\{\sum_{i=1}^{l} |\langle f_i, \mu \rangle| : \; \|\mu\| \leq 1\} \leq \|T\|_{as}.$$

If $x_i' \in X'$ with $\|x_i'\| \leq 1$, $i = 1, \ldots, l$, then

$$\sum_{i=1}^{l} \Big| \int_{A_i} f_i \; d(\iota_{X'}(x_i')m_T) \Big| = \sum_{i=1}^{l} \Big| \int_{K} f_i \; d(\iota_{X'}(x_i')m_T) \Big| \overset{2.2.1}{=} \sum_{i=1}^{l} |x_i'(T(f_i))|$$
$$\leq \sum_{i=1}^{l} \|T(f_i)\| \leq \|T\|_{as}. \tag{6.12}$$

Given a finite sequence of pairwise disjoint, open Borel sets $(O_i)_{i=1}^{l}$ and $\epsilon > 0$, according to the definition of the variation we find pairwise disjoint, but not necessarily open Borel sets $O_{i,j}$, $j = 1, \ldots, k$, such that $\cup_{j=1}^{k} O_{i,j} = O_i$ and

$$\sum_{i=1}^{l} |\iota_{X'}(x_i')m_T|(O_i) \leq \sum_{i=1}^{l}\sum_{j=1}^{k} |\iota_{X'}(x_i')m_T(O_{i,j})| + \epsilon. \tag{6.13}$$

Since $\iota_{X'}(x_i')m_T$ is regular, for each $O_{i,j}$ there exists a compact set $L_{i,j}$ and an open set $A_{i,j}$, such that $L_{i,j} \subseteq O_{i,j} \subseteq A_{i,j}$ and

$$\big|\iota_{X'}(x_i')m_T\big(O_{i,j} \setminus L_{i,j}\big)\big| \leq |\iota_{X'}(x_i')m_T|\big(O_{i,j} \setminus L_{i,j}\big) \tag{6.14}$$
$$\leq |\iota_{X'}(x_i')m_T|\big(A_{i,j} \setminus L_{i,j}\big) \leq \frac{\epsilon}{lk} \tag{6.15}$$

for $i = 1, \ldots, l$, $j = 1, \ldots, k$. Starting with (6.13) we obtain

$$\sum_{i=1}^{n} |\iota_{X'}(x_i')m_T|(O_i) \leq \sum_{i=1}^{l}\sum_{j=1}^{k} |\iota_{X'}(x_i')m_T(O_{i,j})| + \epsilon$$
$$= \sum_{i=1}^{l}\sum_{j=1}^{k} |\iota_{X'}(x_i')m_T(L_{i,j}) + \iota_{X'}(x_i')m_T(O_{i,j} \setminus L_{i,j})| + \epsilon$$
$$\leq \sum_{i=1}^{l}\sum_{j=1}^{k} \big|\iota_{X'}(x_i')m_T(L_{i,j})\big| + \sum_{i=1}^{l}\sum_{j=1}^{k} \big|\iota_{X'}(x_i')m_T(O_{i,j} \setminus L_{i,j})\big| + \epsilon$$
$$\overset{(6.14)}{\leq} \sum_{i=1}^{l}\sum_{j=1}^{k} \big|\iota_{X'}(x_i')m_T(L_{i,j})\big| + 2\epsilon. \tag{6.16}$$

Since K is normal, there exist disjoint open sets $U_{i,j}$ with $L_{i,j} \subseteq U_{i,j}$. Defining $V_{i,j} = A_{i,j} \cap U_{i,j}$ we obtain a finite sequence of open and pairwise disjoint sets $V_{i,j}$ satisfying $L_{i,j} \subseteq V_{i,j} \subseteq A_{i,j}$ and

$$\big|\iota_{X'}(x_i')m_T\big|\big(V_{i,j} \setminus L_{i,j}\big) \leq \big|\iota_{X'}(x_i')m_T\big|\big(A_{i,j} \setminus L_{i,j}\big) \leq \frac{\epsilon}{lk} \tag{6.17}$$

for $i = 1, \ldots, l$, $j = 1, \ldots, k$. By Urysohn's Lemma there exist functions $g_{i,j} \in C(K)$

with $g_{i,j}(L_{i,j}) \subseteq \{1\}$ and $\mathrm{supp}(g_{i,j}) \subseteq V_{i,j}$, we obtain

$$\begin{aligned}
\sum_{i=1}^{l}\sum_{j=1}^{k} \big|\iota_{X'}(x_i')m_T(L_{i,j})\big| &= \sum_{i=1}^{l}\sum_{j=1}^{k} \Big| \int_{L_{i,j}} g_{i,j}\ d(\iota_{X'}(x_i')m_T)\Big| \\
&= \sum_{i=1}^{l}\sum_{j=1}^{k} \Big| \int_{V_{i,j}} g_{i,j}\ d(\iota_{X'}(x_i')m_T) - \int_{V_{i,j}\setminus L_{i,j}} g_{i,j}\ d(\iota_{X'}(x_i')m_T)\Big| \\
&\leq \sum_{i=1}^{l}\sum_{j=1}^{k} \Big| \int_{V_{i,j}} g_{i,j}\ d(\iota_{X'}(x_i')m_T)\Big| \\
&\qquad + \sum_{i=1}^{l}\sum_{j=1}^{k} \int_{V_{i,j}\setminus L_{i,j}} |g_{i,j}|\ d|\iota_{X'}(x_i')m_T| \\
&\overset{(6.12)}{\leq} \|T\|_{as} + \sum_{i=1}^{l}\sum_{j=1}^{k} |\iota_{X'}(x_i')m_T|\big(V_{i,j}\setminus L_{i,j}\big) \\
&\overset{(6.17)}{\leq} \|T\|_{as} + \epsilon. \qquad (6.18)
\end{aligned}$$

Combining (6.16) and (6.18) yields

$$\sum_{i=1}^{l} |\iota_{X'}(x_i')m_T|(O_i) \leq \|T\|_{as} + 3\epsilon.$$

As $\epsilon > 0$ was chosen arbitrarily, we even have

$$\sum_{i=1}^{l} |\iota_{X'}(x_i')m|(O_i) \leq \|T\|_{as} \qquad (6.19)$$

for all $x_i' \in K_1^{X'}(0)$, $i = 1, \dots, l$.
Finally, in order to show that m_T is of bounded variation, let A_i, $i = 1, \dots, l$, be pairwise disjoint Borel sets in K. Choosing $x_i' \in K_1^{X'}(0)$, such that $\|m_T(A_i)\|_{X''} - \epsilon/(2l) \leq |\langle x_i', m_T(A_i)\rangle| = |\iota_{X'}(x_i')m_T(A_i)|$ for $i = 1, \dots, l$, we get

$$\sum_{i=1}^{l} \|m_T(A_i)\|_{X''} - \epsilon/2 \leq \sum_{i=1}^{l} |\iota_{X'}(x_i')m_T(A_i)|. \qquad (6.20)$$

By regularity of $\iota_{X'}(x_i')m_T$ for $\epsilon > 0$ there exist compact sets $K_i \subseteq A_i$, such that $|\iota_{X'}(x_i')m_T(A_i)| - \epsilon/(2l) \leq |\iota_{X'}(x_i')m_T(K_i)|$ for all $i = 1, \dots, l$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{l} \|m_T(A_i)\|_{X''} - \epsilon &\overset{(6.20)}{=} \sum_{i=1}^{l} |\iota_{X'}(x_i')m_T(A_i)| - \epsilon/2 \\
&\leq \sum_{i=1}^{l} |\iota_{X'}(x_i')m_T(K_i)|.
\end{aligned}$$

Since K is a compact Hausdorff space, there exist pairwise disjoint open sets O_i satisfying $O_i \supseteq K_i$. According to (6.19) we derive

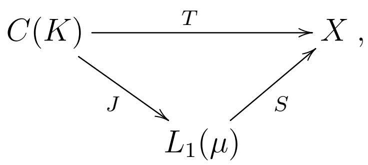
$$\sum_{i=1}^{l} \|m_T(A_i)\|_{X''} - \epsilon \leq \sum_{i=1}^{l} |\iota_{X'}(x')m_T(K_i)| \leq \sum_{i=1}^{l} |\iota_{X'}(x_i')m_T|(O_i) \overset{(6.19)}{\leq} \|T\|_{as}.$$

As $\epsilon > 0$ was arbitrary, $\sum_{i=1}^{l} \|m_T(A_i)\|_{X''} \leq \|T\|_{as}$ holds true. Taking the supremum over all such families of pairwise disjoint Borel sets $(A_i)_{i=1}^{n}$ finally yields $\|m_T\| \leq \|T\|_{as}$. □

Corollary 2.2.4 immediately yields

Corollary 6.1.10. Let K be a compact Hausdorff space and X be a Banach space. Then every absolutely summing operator $T : C(K) \to X$ is weakly compact.

Corollary 6.1.11. Let K be a compact Hausdorff space and X a Banach space. An operator $T \in L_b(C(K), X)$ is absolutely summing if and only if there exists a non-negative Borel measure μ on K and $S \in L_b(L_1(\mu), X)$, such that the diagram below commutes.



Here $J : C(K) \to L_1(\mu)$ denotes the natural embedding. Moreover, μ and S can be found such that $\|S\| \leq 1$ and $\|T\|_{as} = \mu(K)$.

Proof. Let $T : C(K) \to X$ be absolutely summing. By Lemma 6.1.10 T is weakly compact, thus by Theorem 2.2.3 the representing measure m_T takes its values in $\iota_X(X)$. As, according to Theorem 6.1.9, m_T is of bounded variation, the variation $\mu := |m_T|$ constitutes a non-negative Borel measure and $\|T\|_{as} = \|m_T\| = \mu(K)$. We define S on the set M of all simple functions by

$$S := \begin{cases} M & \longrightarrow & X \\ \sum_{i=1}^{n} a_i \mathbb{1}_{A_i} & \mapsto & \sum_{i=1}^{n} a_i \cdot \iota_X^{-1} \circ m_T(A_i) \end{cases}$$

for $a_i \in \mathbb{C}$ and pairwise disjoint $A_i \in \mathcal{A}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. By definition for a simple function f we have

$$\|S(f)\| = \|\int_K f \; d(\iota_X^{-1} \circ m_T)\| \leq \int_K |f| \; d|m_T| = \int_K |f| \; d\mu = \|f\|_{L_1(\mu)}.$$

Thus, S constitutes a linear and bounded operator on $M \subseteq L_1(\mu)$ with $\|S\| \leq 1$ and, in turn, extends to the entirety of $L_1(\mu)$. By continuity of the integral the equality $\int_K g \; d(\iota_X^{-1} \circ m_T) = S(g)$ for simple functions g extends to all bounded functions on K. In particular, for all functions $f \in C(K)$ we have

$$S(J(f)) = S(f) = \int_K f \; d(\iota_X^{-1} \circ m_T) = T(f).$$

Suppose T has a factorization indicated as above. By Example 6.1.7 J is absolutely summing and T is absolutely summing as a composition with an absolutely summing operator by Proposition 6.1.2. □

6.2 Absolutely Summing, Pietsch Integral and Nuclear Operators

Finally we are going to examine the relationship between absolutely summing, Pietsch integral and nuclear operators. We start with a preluding result.

Remark 6.2.1. Every Pietsch integral T together with its Pietsch integral norm are defined by diagrams of the following kind.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle Q} \\ L_\infty(\mu) & \xrightarrow[I]{} & L_1(\mu) \end{array}$$

Here, μ is a complex regular Borel measure on a compact Hausdorff space K. Since we have defined $L_1(\mu)$ as $L_1(|\mu|)$ in Section 3.1, the above diagram still commutes if we replace μ with $|\mu|$. Moreover, the values $\|S\|$, $\|Q\|$ and in consequence $\|S\|\|Q\||\mu|(K)$ do not change after a replacement of μ with $|\mu|$. Consequently, $\|T\|_{pi}$ does not change if we only consider non-negative regular Borel measures. The density ϕ relating μ and $|\mu|$ by $\int f\ d\mu = \int f\phi\ d|\mu|$ will not be of concern in this section, since we will only work with diagrams as above and not evaluate any explicit integrals. Hence, we can assume for the rest of this section that the regular Borel measures, which define Pietsch integral operators in Definition 5.2.14, are non-negative and maintain this assumption throughout this section.

Remark 6.2.2. Given a Pietsch integral operator $T \in L_b(X, Y)$ in Definition 5.2.14 it suffices to factorize over probability measures. Let μ be a non-negative regular Borel measure (see the previous remark) on a compact Hausdorff space K and $S \in L_b(X, L_\infty(\mu))$, $Q \in L_b(L_1(\mu), Y))$ be such that $T = QS$. Defining $\nu := \mu(K)^{-1}\mu$ we obtain a probability measure on K. Clearly, $f \in L_1(\mu)$ and $g \in L_\infty(\mu)$ if and only if $f \in L_1(\nu)$ and $g \in L_\infty(\nu)$, respectively. We define $B : X \to L_\infty(\nu)$ and $A : L_1(\nu) \to Y$ by $Bx = Sx$ and $Af = Qf$ for $x \in X$ and $f \in L_1(\nu)$, respectively. We obtain the following commuting diagram.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle B}\downarrow & & \uparrow{\scriptstyle A} \\ L_\infty(\nu) & \xrightarrow[I]{} & L_1(\nu) \end{array}$$

Since $\|Sx\|_{L_\infty(\mu)} = \|Bx\|_{L_\infty(\nu)}$ for $x \in X$, we have $\|S\| = \|B\|$. Furthermore, given $f \in L_1(\mu)$ we have

$$\frac{\|Af\|_Y}{\|f\|_{L_1(\nu)}} = \frac{\|Qf\|_Y}{\int_K |f|\ d\nu} = \frac{\|Qf\|_Y}{\mu(K)^{-1}\int_K |f|\ d\mu} = \mu(K)\frac{\|Qf\|_Y}{\|f\|_{L_1(\mu)}}.$$

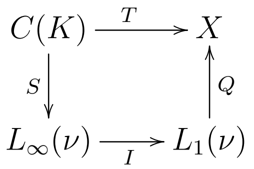
Hence, $\mu(K)\|Q\| = \|A\|$ and in total

$$\|S\|\|Q\|\mu(K) = \|B\|\|A\| = \|B\|\|A\|\nu(K).$$

Thus, the value of the Pietsch integral norm stays unchanged if we only consider regular probability measures.

Theorem 6.2.3. Let K be a compact Hausdorff space and X be a Banach space. An operator $T \in L_b(C(K), X)$ is absolutely summing if and only if it is Pietsch integral. In this case $\|T\|_{as} = \|T\|_{pi}$.

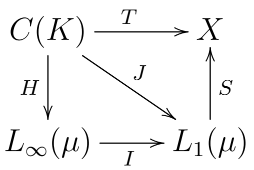
Proof. T being Pietsch integral by Definition 5.2.14 and Remark 6.2.0 there exists a compact Hausdorff space $\tilde{K}$ and a non-negative, regular Borel measure ν on $\tilde{K}$, as well as operators $S \in L_b(X, L_\infty(\nu))$ and $Q \in L_b(L_1(\nu), Y)$, such that the diagram below commutes.



Here, I denotes the natural embedding $I(f) = f$ from $L_\infty(\nu)$ into $L_1(\nu)$. By Example 6.1.7 I is absolutely summing with $\|I\|_{as} = \nu(\tilde{K})$. Consequently, by Proposition 6.1.2 T is absolutely summing and satisfies

$$\|T\|_{as} = \|QIS\|_{as} \leq \|Q\| \|I\|_{as} \|S\| = \|Q\| \|S\| \nu(\tilde{K}).$$

Taking the infimum of the right side over all possible factorizations of the above kind yields $\|T\|_{as} \leq \|T\|_{pi}$; see Remark 6.2.0.
Conversely, assume T to be absolutely summing. By Corollary 6.1.11 there exists a non-negative, regular Borel measure μ on K and an operator $S \in L_b(L_1(\mu), X)$, such that $T = SJ$, $\|S\| \leq 1$ and $\|T\|_{as} = \mu(K)$, where J denotes the natural embedding of $C(K)$ into $L_1(\mu)$. Let H denote the natural embedding of $C(K)$ into $L_\infty(\mu)$ and I the natural embedding of $L_\infty(\mu)$ into $L_1(\mu)$. We then have the following commuting diagram.

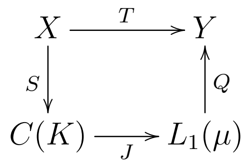


By Definition 5.2.14 T is Pietsch integral and $\|T\|_{pi} \leq \|H\| \|S\| \mu(K) \leq \mu(K) = \|T\|_{as}$. □

Theorem 6.2.4. If X, Y are Banach Spaces and $T \in L_b(X, Y)$, then the following assertions are equivalent.

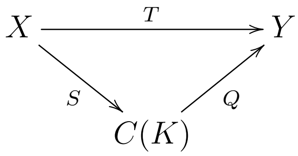
(i) T is Pietsch integral.

(ii) There exists a compact Hausdorff space K, a non-negative regular Borel measure μ on K and operators $S \in L_b(X, C(K))$, $Q \in L_b(L_1(\mu), Y)$, such that $T = QJS$, where J denotes the canonical embedding from $C(K)$ into $L_1(\mu)$.



Moreover, for every $\epsilon > 0$ the operators R and S and can be chosen such that $\|Q\|$, $\|S\| \leq 1$ and $\|T\|_{pi} \leq \mu(K) \leq \|T\|_{pi} + \epsilon$.

(iii) There exists a compact Hausdorff space K and operators $S \in L_b(X, C(K))$, $Q \in AS(C(K), Y)$, such that $T = QS$.



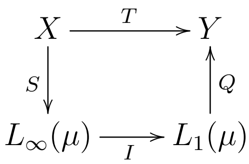
Moreover, for every $\epsilon > 0$ the operators S and Q and can be chosen such that $\|S\| \leq 1$ and $\|T\|_{pi} \leq \|Q\|_{as} \leq \|T\|_{pi} + \epsilon$.

Proof. Suppose T is factorized as in (ii). By Example 6.1.7 J is absolutely summing. Thus, by Theorem 6.2.3 it is Pietsch integral and T is Pietsch integral as a composition with a Pietsch integral operator by Lemma 5.2.15. By the same line of arguments, if T has a factorization as in (iii), it must be Pietsch integral due to the fact that Q is absolutely summing.

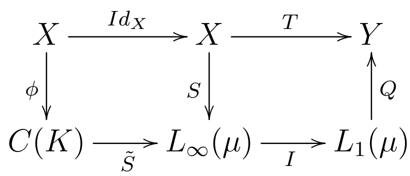
Conversely, given a Pietsch integral operator T and $\epsilon > 0$ by Definition 5.2.14 there exists is a compact Hausdorff space $\tilde{K}$, a non-negative regular Borel measure μ on $\tilde{K}$, $S \in L_b(X, L_1(\mu))$, $Q \in L_b(L_\infty(\mu), Y)$, such that

$$\|T\|_{pi} \leq \|S\|\|Q\|\mu(\tilde{K}) \leq \|T\|_{pi} + \epsilon \tag{6.21}$$

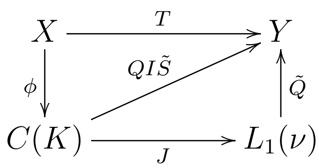
and such that the following diagram commutes.



Here I denotes the canonical embedding from $L_\infty(\mu)$ into $L_1(\mu)$. By Remark 6.2.2 we can assume that μ is a probability measure. Define $K := K_1^{X'}(0)$ as the closed unit ball in X' endowed with the weak*-topology and denote by $\phi : X \to C(K)$ the natural embedding $\phi(x) = \langle x, \cdot \rangle$. As remarked in Example 1.2.2 $L_\infty(\mu)$ is injective. Since $\phi(X)$ is a subspace of $C(K)$, we can find an extension $\tilde{S} : C(K) \to L_\infty(\mu)$ of the operator $S \circ \phi^{-1}$, such that $S = \tilde{S} \circ \phi$ and $\|S\| = \|\tilde{S}\|$.



According to Example 6.1.7 I is absolutely summing and by Theorem 6.2.3 in particular Pietsch integral with $\|I\|_{as} = \|I\|_{pi} = \mu(\tilde{K})$. It follows that $Q \circ I \circ \tilde{S}$ is Pietsch integral as well absolutely summing. The corresponding norms coincide by Theorem 6.2.3. By Corollary 6.1.11 there exist a non-negative regular Borel measure ν on K and an absolutely summing operator $\tilde{Q} \in L_b(L_1(\nu), Y)$, such that $\|\tilde{Q}\| \leq 1$ and $\|QI\tilde{S}\|_{as} = \nu(K)$.



Here J denotes the natural embedding of $C(K)$ into $L_1(\nu)$. Thus T can be factorized as stated in (ii) and (iii).
It remains to show the various claims about the norms. First recall that $\|\phi\| = 1$ and $\|\tilde{Q}\| \leq 1$, as well as the fact that $QI\tilde{S}$ is absolutely summing. We calculate

$$\|T\|_{pi} \leq \|QI\tilde{S}\|_{pi} = \|QI\tilde{S}\|_{as} \leq \|Q\| \|\tilde{S}\| \|I\|_{as} = \|Q\| \|S\| \mu(\tilde{K}) \overset{(6.21)}{\leq} \|T\|_{pi} + \epsilon.$$

This concludes the proof of (iii).
Finally, note that $\|QI\tilde{S}\|_{pi} = \|QI\tilde{S}\|_{as} = \nu(K)$. Together with the prior calculation we get

$$\|T\|_{pi} \leq \|QI\tilde{S}\|_{pi} = \nu(K) \leq \|T\|_{pi} + \epsilon,$$

which finalizes the proof if (ii). □

For the following result recall that by Corollary 2.2.4 under the assumption in Proposition 6.2.5 T is weakly compact. Correspondingly, the results from Theorem 2.2.3 hold true.

Proposition 6.2.5. Let X be a Banach space, K be a compact Hausdorff space and $T \in L_b(C(K), X)$. If the representing measure $m_T : \mathfrak{B}(K) \to X''$ of T as in Theorem 2.2.1 is of bounded variation and if there exists a function $g \in L_1(|m_T|, X)$, such that $\iota_X^{-1} \circ m_T(A) = \int_A g \, d|m_T|$ for every Borel set A of K, then T is a nuclear operator and $\|T\|_{nuc} \leq |m_T|(K)$.

Proof. By Lemma 2.3.19 for $\epsilon > 0$ there exists a sequence $(x_k)_{k=1}^{\infty} \in X^{\mathbb{N}}$ and a sequence $(A_k)_{k=1}^{\infty}$ of Borel sets, such that

$$g = \sum_{k=1}^{\infty} \mathbb{1}_{A_k} x_k \quad |m_T|\text{-almost everywhere}$$

and

$$\int_K \|g\| \; d|m_T| \leq \sum_{k=1}^{\infty} \|x_k\| |m_T|(A_k) \leq \int_K \|g\| \; d|m_T| + \epsilon. \tag{6.22}$$

Given $f \in C(K)$ we have

$$\begin{aligned} T(f) &= \int_K f \; d(\iota_X^{-1} \circ m_T) = \int_K fg \; d|m_T| \\ &= \int_K f \sum_{k=1}^{\infty} \mathbb{1}_{A_k} x_k \; d|m_T| = \sum_{k=1}^{\infty} \big(\int_{A_k} f \; d|m_T| \big) x_k. \end{aligned}$$

Defining $\phi_k(f) := \int_{A_k} f \; d|m_T|$ yields a bounded linear functional on $C(K)$ with $\|\phi_k\| = |m_T|(A_k)$. Hence, $T(f) = \sum_{k=1}^{\infty} \phi_k(f) x_k$ and

$$\sum_{k=1}^{\infty} \|\phi_k\| \|x_k\| \overset{(6.22)}{\leq} \int_K \|g\| \; d|m_T| + \epsilon = |m_T|(K) + \epsilon$$

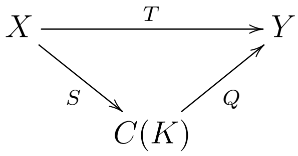
showing T is nuclear. Since $\epsilon > 0$ was arbitrarily chosen, we get $\|T\|_{nuc} \leq |m_T|(K)$. □

Corollary 6.2.6. If K is a compact Hausdorff space and X is a Banach space that satisfies RNP, then every Pietsch integral operator $T \in L_b(C(K), X)$ is nuclear with $\|T\|_{nuc} = \|T\|_{pi} = \|T\|_{as}$.

Proof. If $T \in L_b(C(K), X)$ is Pietsch integral, by Theorem 6.2.3 and Theorem 6.1.9 T is absolutely summing with $\|T\|_{pi} = \|T\|_{as} = |m_T|(K)$, m_T denoting the representing measure of T as in Definition 2.2.1. By Lemma 6.1.10 T is weakly compact and according to Theorem 2.2.3 the representing measure m_T takes its values in $\iota_X(X)$ and is of bounded variation, $\iota_X : X \to X''$ denoting the canonical embedding. Since X satisfies RNP, there exists a Bochner integrable function g with respect to $|m_T|$, such that $\iota_X^{-1} \circ m_T(A) = \int_A g \, d|m_T|$ for every Borel set A. Appealing to Proposition 6.2.5 we see that T is indeed nuclear and $\|T\|_{nuc} \leq |m_T|(K) = \|T\|_{pi}$. The reverse inequality holds true due to Theorem 5.2.19. □

Theorem 6.2.7. Let Y be a Banach space that satisfies RNP. Then for every Banach space X every Pietsch integral operator $T \in L_b(X, Y)$ is nuclear and $\|T\|_{nuc} = \|T\|_{pi}$.

Proof. Given a Pietsch integral operator $T \in L_b(X, Y)$ and $\epsilon > 0$ by Theorem 6.2.4 there exists a compact Hausdorff space K and operators $S \in L_b(X, C(K))$, $Q \in AS(C(K), Y)$, such that $\|S\| \leq 1$, $\|T\|_{pi} \leq \|Q\|_{as} \leq \|T\|_{pi} + \epsilon$ and such that the following diagram commutes.



By Theorem 6.2.3 Q is Pietsch integral and by Corollary 6.2.6 Q is nuclear with $\|Q\|_{as} = \|Q\|_{pi} = \|Q\|_{nuc}$. Hence, T is a nuclear operator as a composition of a bounded and a nuclear operator and by Proposition 5.2.16 satisfies $\|T\|_{nuc} \geq \|T\|_{pi}$. Lastly, we have

$$\|T\|_{nuc} = \|QS\|_{nuc} \leq \|Q\|_{nuc} \|S\| \leq \|Q\|_{as} \leq \|T\|_{pi} + \epsilon.$$

As ϵ was arbitrary, we get $\|T\|_{nuc} \leq \|T\|_{pi}$. □

Chapter 7

Tensor Products and the Radon-Nikodým Property

This chapter is devoted to the beautiful applications of tensor products in combination with vector measures. We will start of with results about the approximation property. The approximation property will be the final key to combine all the previous sections into what represents the ground breaking work done by Alexander Grothendieck on this topic.

7.1 The Approximation Property

In Corollary 5.1.4 we have seen that nuclear operators form a quotient space of a tensor product. The question remains, when the space of nuclear operators is isometrically isomorphic to the corresponding projective tensor product. This leads to the approximation property for Banach spaces. We will also consider further implications of the approximation property in connection to tensor products. The structure is mainly orientated upon Section 4 in [Ry]. The proofs of this present section can be found in [Du], [Ry], [FHHMZ], and [DF]. For a quick summary of the most important results on this topic, consider a look into Chapter 16 of [FHHMZ] or Chapter VIII of [DU]. For an even deeper insight into the topic of tensor products and related results covering the most striking and essential statements of the previous sections, as well as providing different viewpoints, consult [DF].

Definition 7.1.1. A Banach space X is said to satisfy the *approximation property (AP)*, if for every compact set $K \subseteq X$ and every $\epsilon > 0$, there exists a continuous, finite rank operator $T : X \to X$, such that $\|Tx - x\| \leq \epsilon$ for all $x \in K$. If such a T can be found, which additionally satisfies $\|T\| \leq 1$, then X is said to satisfy the *metric approximation property (mAP)*.

Remark 7.1.2. Most of the Banach spaces we deal with satisfy AP. The question, if every Banach space satisfies AP could not be answered for a long time. The first counterexample, which is of constructed nature, was provided by Per Enflo in [E] about 20 years after the question was first posed. The first counterexample of a Banach space that naturally appeared in analysis without AP was shown to be $L_b(\mathcal{H})$, the space of all bounded, linear operators on a Hilbert space, by Andrzej Szankowski in [Sz].

Lemma 7.1.3. For a Banach space X the following statements are equivalent.

(i) X satisfies AP.

(ii) For every Banach space Y, every operator $T \in L_b(X,Y)$, every compact $K \subseteq X$ and every $\epsilon > 0$, there exists a finite rank operator $S : X \to Y$, such that $\|Tx - Sx\| \leq \epsilon$ for all $x \in K$.

(iii) For every Banach space Y, every operator $T \in L_b(Y,X)$, every compact $K \subseteq Y$, and every $\epsilon > 0$, there exists a finite rank operator $S : Y \to X$, such that $\|Tx - Sx\| \leq \epsilon$ for all $x \in K$.

Proof. (i) $\Rightarrow$ (ii). Let $T \in L_b(X,Y)$. In order to show (ii) we can assume that $T \neq 0$. X satisfying AP implies the existence of a finite rank operator $U : X \to X$, such that $\|x - Ux\| \leq \epsilon/\|T\|$ for all $x \in K$. Defining $S := TU : X \to Y$ yields a finite rank operator that satisfies $\|Tx - Sx\| = \|T(x - Ux)\| \leq \|T\|\|x - Ux\| \leq \epsilon$ for all $x \in K$.
(i) $\Rightarrow$ (iii). Then $T(K)$ is a compact subset of X. As X satisfies AP, there exists a finite rank operator $U : X \to X$, such that $\|x - Ux\| \leq \epsilon$ for all $x \in T(K)$. Defining $S := UT : Y \to X$ yields a finite rank operator that satisfies $\|Ty - Sy\| = \|Ty - U(Ty)\| \leq \epsilon$ for every $y \in K$. (ii) $\Rightarrow$ (i) as well as (iii) $\Rightarrow$ (i) clearly follow when one considers the case $Y = X$ and $T = Id_X$. □

Remark 7.1.4. Examining the proof above, it is immanent that if X satisfies mAP, the statements above still hold true, if we add $\|S\| \leq \|T\|$ to (ii) and (iii).

Proposition 7.1.5. For a Banach space X the following statements are equivalent.

(i) X satisfies AP.

(ii) If $u = \sum_{i=1}^{\infty} x'_i \otimes x_i \in X'\hat{\otimes}_\pi X$ for bounded sequences $(x'_i)_{i\in\mathbb{N}} \in X'$ and $(x_i)_{i\in\mathbb{N}} \in X$ satisfying $\sum_{i=1}^{\infty} \|x'_i\|\|x_i\| < +\infty$, then $\sum_{i=1}^{\infty} x'_i(x)x_i = 0$ for every $x \in X$ implies $u = 0$.

(iii) Given a Banach space Y and $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in X\hat{\otimes}_\pi Y$ for bounded sequences $(x_i)_{i\in\mathbb{N}} \in X$ and $(y_i)_{i\in\mathbb{N}} \in Y$ satisfying $\sum_{i=1}^{\infty} \|x_i\|\|y_i\| < +\infty$, $\sum_{i=1}^{\infty} x'(x_i)y_i = 0$ for every $x' \in X'$ implies $u = 0$.

(iv) Given a Banach space Y and $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in X\hat{\otimes}_\pi Y$ for bounded sequences $(x_i)_{i\in\mathbb{N}} \in X$ and $(y_i)_{i\in\mathbb{N}} \in Y$ satisfying $\sum_{i=1}^{\infty} \|x_i\|\|y_i\| < +\infty$, $\sum_{i=1}^{\infty} y'(y_i)x_i = 0$ for every $y' \in Y'$ implies $u = 0$.

Proof. (i) $\Rightarrow$ (iv): Assume that X satisfies the approximation property and let $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in X\hat{\otimes}_\pi Y$ be as in statement (iv). In Lemma 4.1.22 we have seen that we can change $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$, such that $x_i \to 0$ as well as $\sum_{i=1}^{\infty} \|y_i\| < +\infty$. Looking at the proof of how these sequences have been constructed from the previous ones reveals that $\sum_{i=1}^{\infty} y'(y_i)x_i$ stays unchanged. Let $B \in \mathcal{B}(X \times Y) \cong (X\hat{\otimes}_\pi Y)'$ (Theorem 4.1.10). By Corollary 4.1.11 the mapping $\Phi : L_b(X, Y') \to (X\hat{\otimes}_\pi Y)'$, $\Phi(T)(x \otimes y) = \langle y, T(x)\rangle$, constitutes an isometric isomorphism. For (iv) it suffices to show $\langle u, \Phi(T)\rangle = 0$ for every $T \in L_b(X, Y')$.
Let $T \in L_b(X, Y')$ and let $\epsilon > 0$. As X satisfies AP and $\{x_i : i \in \mathbb{N}\} \cup \{0\}$ is compact,

by Lemma 7.1.3 we find a finite rank Operator $S : X \to Y'$, such that $\|Tx_i - Sx_i\| \leq \epsilon$, $i \in \mathbb{N}$. Since there exist $x_i' \in X'$, $y_i' \in Y'$, $i = 1, \ldots, m$, such that $Sx = \sum_{j=1}^{m} x_j'(x) y_j'$, $x \in X$,

$$\begin{aligned}\langle u, \Phi(S)\rangle &= \sum_{i=1}^{\infty} \langle y_i, S(x_i)\rangle = \sum_{i=1}^{\infty}\sum_{j=1}^{m} x_j'(x_i) y_j'(y_i)\\ &= \sum_{j=1}^{m} x_j'\Big(\sum_{i=1}^{\infty} y_j'(y_i) x_i\Big) = 0.\end{aligned}$$

We conclude

$$\begin{aligned}\big|\big\langle u, \Phi(T)\big\rangle\big| &\leq \big|\big\langle u, \Phi(T-S)\big\rangle\big| + \big|\big\langle u, \Phi(S)\big\rangle\big| = \big|\big\langle \sum_{i=1}^{\infty} x_i \otimes y_i, \Phi(T-S)\big\rangle\big|\\ &\leq \sum_{i=1}^{\infty} \big|\big\langle x_i \otimes y_i, \Phi(T-S)\big\rangle\big| = \sum_{i=1}^{\infty} \big|\langle y_i, (T-S)(x_i)\rangle\big|\\ &\leq \sum_{i=1}^{\infty} \|Tx_i - Sx_i\|\|y_i\| \leq \epsilon \sum_{i=1}^{\infty}\|y_i\|.\end{aligned}$$

As $\epsilon > 0$ was chosen arbitrarily, $\langle u, \Phi(T)\rangle = 0$ for every $T \in L_b(X, Y')$.
(iv) $\Rightarrow$ (iii). Assume $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in X \hat{\otimes}_\pi Y$ as in statement (iii). For all $x' \in X'$ and $y' \in Y'$ we have

$$0 = y'(\sum_{i=1}^{\infty} x'(x_i) y_i) = \sum_{i=1}^{\infty} x'(x_i) y'(y_i) = x'(\sum_{i=1}^{\infty} y'(y_i) x_i).$$

Consequently, $\sum_{i=1}^{\infty} y'(y_i) x_i = 0$ for all $y' \in Y'$. Appealing to (iv) yields $u = 0$.
(iii) $\Rightarrow$ (ii). Let $u = \sum_{i=1}^{\infty} x_i' \otimes x_i \in X' \hat{\otimes}_\pi X$, with $\sum_{i=1}^{\infty} x_i'(x) x_i = 0$ for all $x \in X$. For $x \in X$ and $x' \in X'$ we get

$$0 = \langle \sum_{i=1}^{\infty} x_i'(x) x_i, x'\rangle = \sum_{i=1}^{\infty} x_i'(x) x'(x_i) = \sum_{i=1}^{\infty} \iota_X(x)(x_i') x'(x_i) = \big\langle \sum_{i=1}^{\infty} x'(x_i) x_i', \iota_X(x)\big\rangle$$

Since $\iota_X(X)$ acts point separating for X' we obtain $\sum_{i=1}^{\infty} x'(x_i) x_i' = 0$ for all $x' \in X'$. We can now appeal to (iii) and conclude $\sum_{i=1}^{\infty} x_i \otimes x_i' = 0 \in X \hat{\otimes}_\pi X'$. As $X' \hat{\otimes}_\pi X$ and $X \hat{\otimes}_\pi X'$ are evidently isometrically isomorphic, we get $u = 0$.
(ii) $\Rightarrow$ (i). Assume X does not satisfy AP. We denote by $L_b^c(X, X)$ the space $L_b(X, X)$, if endowed with the topology of uniform convergence on compact subsets, which is induced by the family ρ_K of seminorms on $L_b(X, X)$, where K runs through all compact subsets of X and

$$\rho_K(T) := \sup\{\|Tx\| : \ x \in K\}.$$

Let $F \subseteq L_b^c(X, X)$ denote the subspace of finite-rank operators in $L_b^c(X, X)$. The assumption that X does not satisfy AP means that the identity operator is not an element of the closure of F in $L_b^c(X, X)$. As a consequence of the Hahn-Banach Theorem there exists a linear functional $\xi \in L_b^c(X, X)'$, such that $\xi(T) = 0$ for all $T \in F$ and $\xi(Id) = 1$.

The continuity of ξ yields a compact set $K \subseteq X$, such that $|\xi(T)| \le \rho_K(T)$ for all $T \in F$. By Lemma 1.2.5 we find a null-sequence $(x_n)_{n\in\mathbb{N}}$, such that K is contained in its closed convex hull, which implies

$$|\xi(T)| \le \sup_{n\in\mathbb{N}}\|Tx_n\| \quad \text{for all } T \in L_b^c(X,X). \tag{7.1}$$

Considering $(Tx_n)_{n\in\mathbb{N}}$ as an element of the Banach space $c_0(X)$, we define the subspace $Y = \{T(x_n)_{n\in\mathbb{N}} : \ T \in L_b^c(X,X)\}$ of $c_0(X)$. For $(y_n)_{n\in\mathbb{N}} \in Y$ there exists $T \in L_b^c(X,X)$, such that $(Tx_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$. By (7.1) $\xi_Y((y_n)_{n\in\mathbb{N}}) := \xi(T)$ yields a well defined bounded linear functional ξ_Y on Y. Applying the Hahn-Banach theorem to $c_0(X)$ we extend ξ_Y to a functional $\hat{\xi}_Y \in c_0(X)' \cong \ell_1(X')$. Hence, there exists a sequence $(x'_n)_{n\in\mathbb{N}} \in \ell_1(X')$, such that

$$\xi(T) = \hat{\xi}_Y((Tx_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} x'_n(Tx_n) \tag{7.2}$$

for all $T \in L_b^c(X,X)$. Because of $(x'_n)_{n\in\mathbb{N}} \in \ell_1(X')$ and $(x_n)_{n\in\mathbb{N}} \in c_0(X)$ we have $\sum_{i=1}^{\infty}\|x'_i\|\|x_i\| < +\infty$ and, in consequence, $u = \sum_{i=1}^{\infty} x'_i \otimes x_i \in X'\hat{\otimes}_\pi X$.
For $y \in X$, $y' \in X'$ consider the finite rank operator $T_{y,y'} \in F$ defined by $T_{y,y'}(x) = y'(x)y$. (7.2) yields

$$\begin{aligned} 0 = \Psi(T_{y,y'}) &= \sum_{n=1}^{\infty} x'_n(T_{y,y'}x_n) = \sum_{n=1}^{\infty} x'_n(y'(x_n)y) \\ &= \sum_{n=1}^{\infty} x'_n(y)y'(x_n) = y'\Big(\sum_{n=1}^{\infty} x'_n(y)x_n\Big). \end{aligned}$$

As $y' \in X'$ was chosen arbitrarily, $\sum_{i=1}^{\infty} x'_n(y)x_n = 0$ for all $y \in X$. Consequently, u satisfies all the properties in (ii). It remains to show that $u \neq 0$. Consider the bounded bilinear function $B \in \mathcal{B}(X',X)$ defined by $B(x',x) = x'(x)$. Its linearization $\tilde{B}$ is contained in the topological dual space of $X'\hat{\otimes}_\pi X$. From

$$\begin{aligned} \tilde{B}(u) = \tilde{B}\Big(\sum_{n=1}^{\infty} x'_n \otimes x_n\Big) &= \sum_{n=1}^{\infty} \tilde{B}(x'_n \otimes x_n) \\ = \sum_{n=1}^{\infty} B(x'_n, x_n) &= \sum_{n=1}^{\infty} x'_n(x_n) \\ = \Psi(Id) = 1. \end{aligned}$$

we conclude $u \neq 0$. □

Corollary 7.1.6. Let X be a Banach space. If X' satisfies AP, then so does X.

Proof. Let $(x'_i)_{i\in\mathbb{N}} \in X'$ and $(x_i)_{i\in\mathbb{N}} \in X$ be bounded sequences, such that $\sum_{i=1}^{\infty}\|x'_i\|\|x_i\| < +\infty$ and $\sum_{i=1}^{\infty} x'_i(x)x_i = 0$, for every $x \in X$. Consider $u = \sum_{i=1}^{\infty} x'_i \otimes x_i \in X'\hat{\otimes}_\pi X$. Let

$\iota_X : X \to X''$ be the natural embedding. As $\iota_X(X)$ is a separating subset of X''

$$\begin{aligned}
\forall x \in X: \ \sum_{i=1}^{\infty} x_i'(x)x_i = 0 &\Leftrightarrow \ \forall x \in X,\ x' \in X' : \ x'\Big(\sum_{i=1}^{\infty} x_i'(x)x_i\Big) = 0 \\
&\Leftrightarrow \ \forall x \in X,\ x' \in X' : \ \sum_{i=1}^{\infty} x_i'(x)x'(x_i) = 0 \\
&\Leftrightarrow \ \forall x \in X,\ x' \in X' : \ \sum_{i=1}^{\infty} \iota_X(x)(x_i')x'(x_i) = 0 \\
&\Leftrightarrow \ \forall x \in X,\ x' \in X' : \ \iota_X(x)\Big(\sum_{i=1}^{\infty} x'(x_i)x_i'\Big) = 0 \\
&\Leftrightarrow \ \forall x' \in X' : \ \sum_{i=1}^{\infty} x'(x_i)x_i' = 0.
\end{aligned}$$

If X' satisfies AP, by (iv) of Proposition 7.1.5 for the Banach spaces $Y = X$ we conclude from $\sum_{i=1}^{\infty} x'(x_i)x_i' = 0$ for all $x' \in X'$ that $u = 0$. Hence, X satisfies statement (ii) of Proposition 7.1.5, meaning that X satisfies AP. □

Corollary 7.1.7. Let X and Y be Banach spaces. If X' or Y satisfies AP, then

$$X'\hat{\otimes}_\pi Y \cong N(X,Y).$$

Proof. By Proposition 5.1.3 and Corollary 5.1.4 $x' \otimes y \mapsto L_{x',y} \in L_b(X,Y)$, where $L_{x',y}(x) = x'(x)y$, defines a quotient operator $J : X'\hat{\otimes}_\pi Y \to N(X,Y)$, such that

$$X'\hat{\otimes}_\pi Y/\ker(J) \cong N(X,Y).$$

For $u = \sum_{i=1}^{\infty} x_i' \otimes y_i \in X'\hat{\otimes}_\pi Y$ we have

$$\begin{aligned}
Ju = 0 &\Leftrightarrow \sum_{i=1}^{\infty} x_i'(x)y_i = 0 \ \ \forall x \in X \\
&\Leftrightarrow \sum_{i=1}^{\infty} \iota_X(x)(x_i')y'(y_i) = 0 \ \ \forall x \in X,\ y' \in Y' \\
&\Leftrightarrow \sum_{i=1}^{\infty} y'(y_i)x_i' = 0 \ \ \forall y' \in Y.
\end{aligned}$$

Applying (i)⇒(iv) from Proposition 7.1.5 for the case that X' satisfies AP yields $u = 0$. For the case that Y satisfies AP, analogously we conclude from (i)⇒(iii) that $0 = \sum_{i=1}^{\infty} y_i \otimes x_i' \in Y\hat{\otimes}_\pi X'$ and, in turn with Remark 4.1.3, that $u = 0$. In both cases $\ker(J) = \{0\}$ and J is an isometric isomorphism. □

In Remark 4.2.9 we have seen that $X'\hat{\otimes}_\epsilon Y$ can be identified with the space of approximable operators, i.e. the closure under the operator norm of all finite rank operators in $L_b(X,Y)$. This space is a subspace of all compact operators. In the next result we will answer the question, when the approximable operators coincide with all compact operators

Proposition 7.1.8. Let X be a Banach space.

(i) X satisfies AP if and only if for every Banach space Y every operator $T \in K(Y, X)$ is approximable.

(ii) X' satisfies AP if and only if for every Banach space Y every operator $T \in K(X, Y)$ is approximable.

Proof. To begin with we remind ourselves that by Remark 4.2.9 every finite rank operator T with representation $T(\cdot) = \sum_{i=1}^n \langle \cdot, x_i' \rangle y_i \in L_b(X, Y)$ can be identified with the tensor $\sum_{i=1}^n x_i' \otimes y_i \in X' \hat{\otimes}_\epsilon Y$. We will make use of this identification intensively throughout this proof and use the tensor representation synonymously for such operators.
(i) Suppose X satisfies AP and let $T \in K(Y, X)$ and $\epsilon > 0$, so that $K := T(K_1^X(0))$ is a relatively compact set. For $\epsilon > 0$ there exists a finite rank operator $S \in L_b(X, X)$, such that $\|x - Sx\| \le \epsilon$, $x \in K$. Consequently, $\|Ty - STy\| \le \epsilon$ for every $y \in K_1^X(0)$. Hence, ST is a finite rank operator, satisfying $\|T - ST\| \le \epsilon$.
For the converse, suppose $K \subseteq X$ is to be compact and let $\epsilon > 0$. By Corollary 1.2.7 there exists a Banach space $(Y, \|\cdot\|_Y)$, $Y \subseteq X$, such that K is compact in Y and such that the natural embedding ι from Y into X is compact. By assumption ι is approximable, i.e. for $\epsilon > 0$ there exist $y_i' \in Y'$, $x_i \in X$, such that $T = \sum_{i=1}^n y_i' \otimes x_i \in L_b(Y, X)$ satisfies

$$\|Ty - \iota(y)\|_X = \big\| \sum_{i=1}^n y_i'(y) x_i - y \big\|_X \le \epsilon \cdot \|y\|_Y \cdot \big(\max_{x \in K} \|x\|_X\big)^{-1}$$

for all $y \in Y$. Consider the topological vector space (Y', τ_M), where τ_M denotes the topology of uniform convergence on compact subsets of Y. By the theorem of Mackey-Arens we have $(Y', \tau_M)' = Y$; see Theorem 3.41 in [FHHMZ] or Satz 5.6.3 in [BKW]. As the natural embedding ι is compact and one-to-one, its adjoint $\iota' : X' \to Y'$ is compact and has weak*-dense range. From $(Y', \tau_M)' = Y = (Y', \sigma(Y', Y))'$ and the convexity of $\iota'(X')$ we conclude $\overline{\iota'(X')}^{\tau_M} = \overline{\iota'(X')}^{w*} = Y'$. Thus, there exist $x_i' \in X'$ and $\delta > 0$, $i = 1, \dots, n$, such that

$$\sup_{x \in K} |\iota'(x_i')(x) - y_i'(x)| < \delta < \epsilon \cdot \big(n \cdot \max_{i=1,\dots,n} \|x_i\|_X \cdot \max_{x \in K} \|x\|_X\big)^{-1}.$$

We obtain the following inequality.

$$\begin{aligned}
\sup_{x \in K} \|\big(\sum_{i=1}^n \iota'(x_i') \otimes x_i - \iota\big)(x)\|_X &\le \sup_{x \in K} \|\big(\sum_{i=1}^n y_i' \otimes x_i\big)(x) - \iota(x)\|_X \\
&\quad + \sup_{x \in K} \|\big(\sum_{i=1}^n y_i' \otimes x_i\big)(x) - \big(\sum_{i=1}^n \iota'(x_i') \otimes x_i\big)(x)\|_X \\
&= \sup_{x \in K} \|T(x) - \iota(x)\|_X \\
&\quad + \sup_{x \in K} \|\big(\sum_{i=1}^n \big(y_i' - \iota'(x_i')\big) \otimes x_i\big)(x)\|_X \\
&< \epsilon + n \cdot \delta \cdot \max_{i=1,\dots,n} \|x_i\|_X \cdot \max_{x \in K} \|x\|_X = 2\epsilon.
\end{aligned}$$

Given $f \in X'$ we have $\iota'(f) = f \circ \iota \in Y'$. The previous inequality can be written as

$$\sup_{x \in K} \|\big(\sum_{i=1}^{n} x_i' \otimes x_i - Id_X\big)(x)\|_X < 2\epsilon.$$

To summarize, for every compact set K and every $\epsilon > 0$ we have found a finite rank operator $S \in L_b(X, X)$, such that $\|x - Sx\| < 2\epsilon$ for every $x \in K$, i.e. X satisfies AP.
(ii) Suppose X' satisfies AP. If $T : X \to Y$ is compact, then $T' : Y' \to X'$ is compact. By the same argument as in beginning of the proof of (i), given $\epsilon > 0$ there exists a finite rank $S : X' \to X'$, such that $\|T' - ST'\| \leq \epsilon$ implying $\|T'' - T''S'\| \leq \epsilon$. As T is compact, by Lemma 1.1.4 $T''(X'') \subseteq \iota_Y(Y)$. Hence, $\iota_Y^{-1} \circ T'' \circ S'$ is a finite rank operator from X'' into Y. Since for $x \in K_1^X(0)$ we have

$$\begin{aligned}\big\|Tx - (\iota_Y^{-1} \circ T'' \circ S' \circ \iota_X)x\big\| &= \sup_{y' \in K_1^{Y'}(0)} |y'(Tx) - y'(\iota_Y^{-1} \circ T'' \circ S' \circ \iota_X(x))| \\ &= \sup_{y' \in K_1^{Y'}(0)} |(T'y')x - (ST'y')x| \\ &\leq \sup_{y' \in K_1^{Y'}(0)} \|T'y' - S(T'y')\| \leq \epsilon,\end{aligned}$$

T is approximable.
For the converse, assume every compact operator defined on X is approximable. We show that X' satisfies AP by point (i). Therefore, we have to show that every $T \in K(Y, X')$ is approximable. For any such T the operator $T' \circ \iota_X : X \to Y'$ is compact and by assumption approximable. Given $\epsilon > 0$ we therefore find a finite operator $S : X \to Y'$, such that $\|T' \circ \iota_X - S\| < \epsilon$. Then S' has finite rank and $\|(T' \circ \iota_X)' - S'\| = \|T' \circ \iota_X - S\| < \epsilon$, i.e. $(T' \circ \iota_X)'$ is approximable and also $(T' \circ \iota_X)' \circ \iota_Y$. $\langle \iota_X(x), \iota_{X'}(x')\rangle = \langle x, \iota_{X'}^{-1} \circ \iota_{X'}(x')\rangle$ for $x \in X$ and $x' \in X'$ shows $(\iota_X)'|_{\iota_{X'}(X')} = \iota_{X'}^{-1}$. By Lemma 1.1.2 $\iota_{X'} \circ T = T'' \circ \iota_Y$. Hence,

$$(T' \circ \iota_X)' \circ \iota_Y = (\iota_X)' \circ T'' \circ \iota_Y = (\iota_X)' \circ \iota_{X'} \circ T = T$$

and in turn T is approximable. □

Corollary 7.1.9. Let X, Y be Banach spaces and let $\kappa_L : X' \otimes Y \to L_b(X, Y)$ be the mapping from Remark 4.2.9. If either X' or Y satisfies AP, then

$$\hat{\kappa}_L : \begin{cases} X'\hat{\otimes}_\epsilon Y & \to \quad K(X, Y) \\ x' \otimes y & \mapsto \quad \kappa_L(x' \otimes y) \end{cases},$$

defines an isometric isomorphism.

Lemma 7.1.10. For Banach spaces X, Y and $v = \sum_{i=1}^{\infty} x_i \otimes y_i \in X \hat{\otimes}_\pi Y$, $v \mapsto \tilde{B}_v$, where $\tilde{B}_v(x' \otimes y') = \sum_{i=1}^{\infty} \langle x_i, x'\rangle\langle y_i, y'\rangle$ if $\sum_{i=1}^{\infty} \|x_i\|\|y_i\| < +\infty$, defines a linear and bounded mapping from $X\hat{\otimes}_\pi Y$ into $(X'\hat{\otimes}_\epsilon Y')'$.
For $w = \sum_{i=1}^{\infty} x_i \otimes x_i' \in X\hat{\otimes}_\pi X'$, $w \mapsto \tilde{C}_w$, where $\tilde{C}_w(x' \otimes x) = \sum_{i=1}^{\infty} \langle x_i, x'\rangle\langle x, x_i'\rangle$ if $\sum_{i=1}^{\infty} \|x_i\|\|x_i'\| < +\infty$, defines a linear and bounded mapping from $X\hat{\otimes}_\pi X'$ into $(X'\hat{\otimes}_\epsilon X)'$.

Proof. For $v = \sum_{i=1}^{\infty} x_i \otimes y_i$ with $\sum_{i=1}^{\infty} \|x_i\| \|y_i\| < +\infty$ the operator $T_v : X' \to Y$, defined by

$$T_v(x') = \sum_{i=1}^{\infty} \langle x_i, x' \rangle y_i = \sum_{i=1}^{\infty} \langle x', \iota_X(x_i) \rangle y_i \tag{7.3}$$

is a nuclear operator $T_v \in N(X', Y)$ by Proposition 5.1.3. By Corollary 5.2.19 $N(X', Y) \subseteq Int(X', Y)$. Hence, T_v is integral, i.e. the bilinear form $B_v \in \mathcal{B}(X' \times Y')$, $B_v(x', y') = \langle y', T_v x' \rangle = \sum_{i=1}^{\infty} \langle x_i, x' \rangle \langle y_i, y' \rangle$ is integral. Thus, its linearization $\tilde{B}_v$ constitutes an element of $(X' \hat{\otimes}_\epsilon Y')'$, where $\tilde{B}_v(x' \otimes y') = \sum_{i=1}^{\infty} \langle x_i, x' \rangle \langle y_i, y' \rangle$.
Given $w = \sum_{i=1}^{\infty} x_i \otimes x_i' \in X \hat{\otimes}_\pi X'$, applying the same line of arguments as before for $Y = X'$ we obtain an integral operator $T_w : X' \to X'$, $T_w(x') = \sum_{i=1}^{\infty} \langle x_i, x' \rangle x_i'$. By Corollary 5.2.12 $B \in \mathcal{B}(X', X)$ is integral where $B(x', x) = \langle x, T_w x' \rangle = \sum_{i=1}^{\infty} \langle x_i, x' \rangle \langle x, x_i' \rangle$, i.e. its linearization, which is exactly $\tilde{C}_w$, is an element of $(X' \hat{\otimes}_\epsilon X)'$. $\square$

Theorem 7.1.11. Given a Banach space X the following statements are equivalent.

(i) X satisfies mAP.

(ii) For every Banach space Y the mapping $v \mapsto \tilde{B}_v$ from $X \hat{\otimes}_\pi Y$ into $(X' \hat{\otimes}_\epsilon Y')'$ as in Lemma 7.1.10 is isometric.

(iii) The mapping $w \mapsto \tilde{C}_w$ from $X \hat{\otimes}_\pi X'$ into $(X' \hat{\otimes}_\epsilon X)'$ as in Lemma 7.1.10 is isometric.

Proof. (i) $\Rightarrow$ (ii). Let $T_v \in N(X', Y)$ be as in (7.3). By Corollary 5.1.4, Definition 5.2.5 and Corollary 5.2.19 we have $\|v\|_\pi \geq \|T_v\|_{nuc} \geq \|T_v\|_{int} = \|B_v\|_{int} = \|\tilde{B}_v\|$. For the reverse inequality, let $v \in X \hat{\otimes}_\pi Y$. Using Lemma 4.1.22 we can choose a representation $v = \sum_{i=1}^{\infty} x_i \otimes y_i$, such that x_n is a null-sequence and $0 < \sum_{i=1}^{\infty} \|y_i\| < +\infty$. By Corollary 4.1.11 the mapping $\Phi : L_b(X, Y') \to (X \hat{\otimes}_\pi Y)'$, where $\Phi(T)(x \otimes y) = \langle y, Tx \rangle$, is an isometric isomorphism. Hence, by the Hahn-Banach Theorem there exists an operator $T \in L_b(X, Y')$, such that $\|v\|_\pi = |\Phi(T)(v)|$. As $\{x_n : \ n \in \mathbb{N}\} \cup \{0\}$ is compact and X satisfies mAP, by Remark 7.1.4 there exists a finite rank operator $S \in L_b(X, Y')$, $\|S\| \leq 1$, such that

$$\|Tx_n - Sx_n\| < \epsilon \cdot \big(\sum_{i=1}^{\infty} \|y_i\| \big)^{-1}$$

for every $n \in \mathbb{N}$. From

$$|(\Phi(T) - \Phi(S))(v)| = \big| \sum_{i=1}^{\infty} \langle y_i, Tx_i - Sx_i \rangle \big| \leq \sum_{i=1}^{\infty} \|y_i\| \|Tx_i - Sx_i\| < \epsilon$$

we obtain

$$\|v\|_\pi - \epsilon = |\Phi(T)(v)| - \epsilon \leq |(\Phi(T) - \Phi(S))(v)| + |\Phi(S)(v)| - \epsilon \leq |\Psi(S)(v)|.$$

S being a finite rank operator can be written as $S(x) = \sum_{j=1}^{n} \langle x, x_j' \rangle y_j'$, $x_j' \in X'$, $y_j' \in Y'$. Since by Remark 4.2.9 the tensor $w_S := \sum_{j=1}^{n} x_j' \otimes y_j' \in X' \hat{\otimes}_\epsilon Y'$ satisfies $\|w_S\|_\epsilon = \|S\| \leq 1$, we conclude

$$\pi(v) - \epsilon \leq |\langle v, \Phi(S) \rangle| = \big| \sum_{j=1}^{n} \sum_{i=1}^{\infty} \langle x_i, x_j' \rangle \langle y_i, y_j' \rangle \big| = |\langle w_S, \tilde{B}_v \rangle| \leq \|\tilde{B}_v\|_{int}.$$

As $\epsilon > 0$ was chosen arbitrarily, we get $\|\tilde{B}_v\|_{int} = \pi(v)$.
(ii) $\Rightarrow$ (iii). Applying (i) $\Rightarrow$ (ii) for $Y = X'$ to a given $w = \sum_{i=1}^{\infty} x_i \otimes x_i' \in X\hat{\otimes}_\pi X'$ we obtain $\tilde{B}_w \in (X'\hat{\otimes}_\epsilon X'')'$, $\tilde{B}_w(x' \otimes x'') = \sum_{i=1}^{\infty} \langle x_i, x' \rangle \langle x_i', x'' \rangle$, with $\|w\| = \|\tilde{B}_w\|$. $\tilde{C}_w \in (X'\hat{\otimes}_\epsilon X)'$ being the linearization of the operator $C_w \in \mathcal{B}(X' \times X)$, $C_w(x', x) = \sum_{i=1}^{\infty} \langle x_i, x' \rangle \langle x, x_i' \rangle$, means that C_w is integral with $\|\tilde{C}_w\| = \|C_w\|_{int}$. Hence, by Corollary 5.2.12, the operator $T \in L_b(X', X')$, where $T(x') = \sum_{i=1}^{\infty} \langle x_i, x' \rangle x_i'$, is integral and $\|T\|_{int} = \|C_w\|_{int}$. By the definition of T being integral the bilinear form $B_T \in \mathcal{B}(X' \times X'')$, with $B_T(x', x'') = \langle Tx', x'' \rangle = \sum_{i=1}^{\infty} \langle x_i, x' \rangle \langle x_i', x'' \rangle$ is integral, i.e. its linearization, which is exactly $\tilde{B}_w$, is integral and $\|T\|_{int} = \|\tilde{B}_w\|$. In total, $\|w\| = \|\tilde{B}_w\| = \|T\|_{int} = \|\tilde{C}_w\|$.
(iii) $\Rightarrow$ (i). As the closed unit ball of $X' \otimes_\epsilon X$ is dense in the closed units ball of its completion $X'\hat{\otimes}_\epsilon X$, we have

$$\|w\|_\pi = \|\tilde{C}_w\| = \sup\{|\langle v, \tilde{C}_w \rangle| : \ v \in X' \otimes X, \epsilon(v) \leq 1\} \tag{7.4}$$

for every $w \in X\hat{\otimes}_\pi X'$. By Corollary 4.1.11 the mapping $\zeta : L_b(X, X) \to (X\hat{\otimes}_\pi X')'$, given by $\zeta(T)(x \otimes x') = \langle x', \iota_X \circ T(x) \rangle = \langle Tx, x' \rangle$, is isometric. In consequence,

$$\|w\|_\pi \geq \sup\{|\langle w, \zeta(T) \rangle| : \ T \in L_b(X, X), \ \|T\| \leq 1\}. \tag{7.5}$$

By Remark 4.2.9 $\kappa : X' \otimes_\epsilon X \to F$, $\kappa(v)(x) = \sum_{i=1}^{n} \langle x, x_i' \rangle x_i$ for $v = \sum_{i=1}^{n} x_i' \otimes x_i$, is isometric, F denoting the subset of $L_b(X, X)$ of operators with finite rank. It is easily seen that

$$\langle v, \tilde{C}_w \rangle = \langle w, (\zeta \circ \kappa)(v) \rangle, \quad \text{for all } v \in X \otimes_\epsilon X', \ w \in X\hat{\otimes}_\pi X'.$$

Combining (7.4) and (7.5) we obtain

$$\begin{aligned} \|w\|_\pi &= \sup\{|\langle w, \zeta(S) \rangle| : \ S \in F, \ \|S\| \leq 1\} \\ &\leq \sup\{|\langle w, \zeta(T) \rangle| : \ T \in L_b(X, X), \ \|T\| \leq 1\} \leq \|w\|_\pi \end{aligned}$$

By the geometric Hahn-Banach Theorem $\zeta(F)$ is dense in the closed unit ball of $\zeta(L_b(X, X))$ with respect to the weak*-topology $\sigma((X\hat{\otimes}_\pi X')', X\hat{\otimes}_\pi X')$. In consequence, $\zeta(Id_X)$ can be weak* approximated by elements of $\zeta(F)$, so that for $x \in X$, $x' \in X'$ and $\epsilon > 0$ there exists a finite rank operator $S : X \to X$, such that

$$|\langle Sx, x' \rangle - \langle x, x' \rangle| = \Big| \big(\zeta(S) - \zeta(Id_X)\big)(x' \otimes x) \Big| < \epsilon.$$

Therefore, the point x belongs to the closure of $Fx := \{Sx : \ S \in F\}$ with respect to the weak topology. This set being convex its weak closure coincides with its norm closure. As the strong operator topology on $L_b(X, X)$ is generated by the seminorms $T \mapsto \|Tx\|$, $x \in X$, the identity operator Id_X lies in the closure of F with respect to the strong operator topology. Hence, there exists a net $(S_i)_{i \in I} \in F$, such that $S_i x \xrightarrow{i \in I} x$, for all $x \in X$. It remains to show that this is sufficient to prove that X satisfies mAP.
Let K be a compact subset of X and let $\epsilon > 0$. Choose $x_1, \ldots, x_n \in K$, $n \in \mathbb{N}$, such that $K \subseteq \cup_{k=1}^{n} U_{\epsilon/3}(x_k)$. Choose $i_0 \in I$, such that $\|x_k - S_i x_k\| \leq \epsilon/3$ for every $k = 1, \ldots, n$, if $i \geq i_0$. If we choose for $x \in K$ a number $k \in \{1, \ldots, n\}$, such that $\|x - x_k\| < \epsilon/3$, then

$$\|x - S_{i_0} x\| \leq \|x - x_k\| + \|x_k - S_{i_0} x_k\| + \|S_{i_0} x_k - S_{i_0} x\| < \epsilon.$$

□

7.2 Applications of the Radon-Nikodým Property with Tensor Products of Banach Spaces

The last section of this Chapter is devoted to the Radon-Nikodým Property for Banach spaces and presents the work of A. Grothendieck on this topic, as can be found in [Gr1] and [Gr2]. We have already seen at the end of the previous chapter, that in case a Banach space satisfies RNP the classes of nuclear and Pietsch integral operator coincide. In Chapter 6 we have worked towards the conditions, under which the classes of integral and Pietsch integral operators coincide. Both results combined give remarkable result.

Theorem 7.2.1. Let X be a Banach space which is complemented in X'' by a norm one projection. If X satisfies AP and RNP, then X satisfies mAP.

Proof. By Theorem 5.2.17 the integral operators and the Pietsch integral operators from X into X and their norms coincide. By Theorem 6.2.7 the Pietsch integral operators and the nuclear operators from X into X and their norms coincide. Corollary 7.1.7 guarantees that the space of nuclear operators from X into X is isometrically isomorphic to the projective tensor product $X'\hat{\otimes}_\pi X$. To summarize,

$$X'\hat{\otimes}_\pi X \cong N(X,X) = PI(X,X) = Int(X,X).$$

By Theorem 5.2.12 we have $Int(X,X'') \cong B_{int}(X,X') \cong (X\hat{\otimes}_\epsilon X')'$. According to Corollary 5.2.13 the natural injection $T \mapsto \iota_X \circ T$ from $Int(X,X)$ into $Int(X,X'')$ is isometric. Consequently, as well must be the embedding from $X'\hat{\otimes}_\pi X$ into $(X\hat{\otimes}_\epsilon X')'$ which arises by the above correspondences. Looking at the proof of Lemma 7.1.10, it is easily shown that this embedding is exactly the embedding $w \mapsto \tilde{C}_w$ in (iii) from Theorem 7.1.11, in turn proving that X satisfies mAP. □

Corollary 7.2.2. If X is a reflexive Banach space that satisfies AP, then X satisfies mAP.

Proof. According to Remark 5.2.18 dual spaces are complemented in their dual by a norm one projection. By Corollary 2.4.13 reflexive Banach spaces satisfy RNP and the statement follows immediately from the preceding theorem. □

Corollary 7.2.3. If X is a separable dual space that satisfies AP, then X satisfies mAP.

Proof. Note that dual spaces are complemented in their bidual spaces by a norm one projection. Furthermore, by Corollary 2.4.9 separable dual spaces satisfy RNP. The statement now follows from the preceding theorem. □

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