Diplomarbeit

Boundary behavior of singular integrals

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Preface

Let μ be a complex Borel measure on the unit circle. In this work we study integral transforms of μ on the unit disk, namely the well-known Cauchy transform as well as the closely related Poisson and conjugate Poisson transforms. These transforms are of particular interest in function theory, as certain classes of analytic functions allow for integral representations of this type, and they are important tools in many function theoretic proofs. We are concerned with the behavior of the transforms near the support of the measure μ. In the case of measures absolutely continuous with respect to Lebesgue measure on the circle, this topic has been thoroughly studied. The results are well-known and can be found for instance in [10].
If the measure μ is singular with respect to Lebesgue measure, its Cauchy transform tends to infinity as the function argument approaches points lying in the support of the measure. The mode of growth of the transforms in this case became the object of deeper study very recently. In [2], A. Poltoratski introduced a classification of measures based on the relative speed of growth of their Poisson and conjugate Poisson transforms. Several results on this subject have been published in the recent years, and there are a number of open questions at this point.

In this thesis we give an overview of the developments in this field. In the first chapter, we provide necessary definitions and preliminary results. Chapter 2 deals with the classical results on absolutely continuous measures.
In Chapter 3, we turn to the study of the normalized Cauchy transform of a measure, which is an operator defined on the space of μ-integrable functions. The main result is a statement about the boundary behavior of the image functions of the normalized Cauchy transform that is due to A. Poltoratski [1]. In the first two sections of this chapter, we present a construction that actually originates in a different direction of study but leads to important results on the normalized Cauchy transform, and was introduced by D.N. Clark [5] and taken up by A.B. Aleksandrov [6]. The rest of the chapter follows A. Poltoratski's paper [1].
In Chapter 4 it is shown how the results from Chapter 3 can be applied in the study of the boundary behavior of the Poisson and conjugate Poisson transforms of singular measures. We present the classification from [2] and state some results characterizing those classes that were obtained by A. Poltoratski and P.W. Jones in [2], [3] and [4].

At this point I want to thank my advisor Harald Woracek for his support and helpful suggestions.

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Contents

Chapter 1

Preliminaries

1.1 The Hardy Spaces on the unit disk

We use the following notational conventions throughout.
$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk, $\mathbb{T} := \partial\mathbb{D}$ is the unit circle.
$M(\mathbb{T})$ denotes the set of all finite regular complex Borel measures on $\mathbb{T}$, and $M_+(\mathbb{T})$ is the subset of all finite positive measures.
The normalized Lebesgue measure on the unit circle is denoted by σ, and speaking of absolutely continuous or singular measures we mean with respect to σ unless explicitly stated otherwise.

We introduce the *Hardy Spaces* on the unit disk: For $0 < p \leq \infty$, H^p denotes the space of holomorphic functions on $\mathbb{D}$ which satisfy

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(r\xi)|^p \, d\sigma(\xi) \right)^{1/p} < +\infty, \tag{1.1}$$

if $0 < p < \infty$, and in the case $p = \infty$

$$\|f\|_{H^\infty} := \sup_{z\in\mathbb{D}} |f(z)| < +\infty, \tag{1.2}$$

i.e., H^∞ is the space of bounded analytic functions in $\mathbb{D}$.
If $p \geq 1$, the space H^p is a Banach space, if $0 < p < 1$, it is a complete metric space. We state some properties of the functions lying in Hardy spaces that will be relevant in the following.

(i) Every function $f \in H^p$ has boundary values almost everywhere on $\mathbb{T}$, and the boundary function $\tilde{f}$ lies in $L^p(\sigma)$, with $\|f\|_{H^p} = \|\tilde{f}\|_{L^p}$. Thus H^p is isomorphic to a closed subspace of $L^p(\sigma)$; see [13, Chapter II, Theorem 3.1]. Thanks to this correspondence, we will usually not distinguish between $f \in H^p$ and its boundary function in $L^p(\sigma)$.

(ii) Let $1 \leq p \leq \infty$. Then we have the following characterization of the elements of H^p in terms of their boundary function ([12, Theorem 4.25]): $f \in L^p(\sigma)$ is the boundary function of a function in H^p if and only if for all $n \in \mathbb{N}$

$$\int_{\mathbb{T}} \zeta^n f(\zeta) \, d\sigma(\zeta) = 0. \tag{1.3}$$

(iii) If $1 \leq p \leq \infty$, every function $f \in H^p$ has the Cauchy representation

$$f(z) = \int_{\mathbb{T}} \frac{f(\xi)}{1 - z\bar{\xi}} d\sigma(\xi), \quad z \in \mathbb{D}. \tag{1.4}$$

(iv) We set

$$H_0^p := \{f \in H^p \mid f(0) = 0\} \quad \text{and} \quad \overline{H_0^p} := \{f \in L^p(\sigma) \mid \bar{f} \in H_0^p\}.$$

In terms of boundary functions, these are the spaces ($p \geq 1$)

$$H_0^p = \{f \in L^p(\sigma) \mid (1.3) \text{ holds for all } n \in \mathbb{N}_0\}, \tag{1.5}$$

$$\overline{H_0^p} = \{f \in L^p(\sigma) \mid (1.3) \text{ holds for all } n \leq 0\}. \tag{1.6}$$

If $1 < p < \infty$, we have the decomposition $L^p(\sigma) = H^p \oplus \overline{H_0^p}$, see e.g. [15, Chapter 9].

(v) The dual space of H^p, $1 < p < \infty$, can be identified with H^q, where q is the conjugate index of p, i.e., the unique positive number such that $\frac{1}{p} + \frac{1}{q} = 1$ holds; see [14, Section 3.6]. The duality product for $f \in H^p$ and $g \in H^q$ is given by

$$\langle f, g \rangle := \int_{\mathbb{T}} f(\xi) \overline{g(\xi)} d\sigma(\xi).$$

1.2 Harmonic functions

A function u which is continuous on $\overline{\mathbb{D}}$ and twice continuously differentiable on $\mathbb{D}$ is called harmonic if $\Delta u(x + iy) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $\mathbb{D}$. We state some facts about harmonic functions we will need later on. Proofs of the statements can be found in [13], [17], [21, Chapter 11].

(i) If f is a function continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$, and we write $f(z) = u(z) + iv(z)$ with real-valued u and v, then u and v satisfy the Cauchy-Riemann differential equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.7}$$

From these equations we infer that u and v are harmonic functions:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial y^2}.$$

By the linearity of the derivative, f itself is harmonic.

(ii) Conversely, every real-valued harmonic function u on $\mathbb{D}$ is the real part of some analytic function on $\mathbb{D}$. The corresponding imaginary part, which is then called a *harmonic conjugate* of u, can be constructed from the Cauchy-Riemann differential equations. Harmonic conjugates are unique up to a constant and we call a harmonic conjugate v satisfying $v(0) = 0$ *the* harmonic conjugate of u.

(iii) For positive harmonic functions we have *Harnack's inequality* (see [22, Theorem 1.18]):

Theorem 1.2.1. *Suppose that u is positive and harmonic in the open ball $U_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$. For $z \in U_R(z_0)$, set $r := |z - z_0|$. Then the inequality*

$$\frac{R-r}{R+r}u(\xi) \leq u(z) \leq \frac{R+r}{R-r}u(\xi) \tag{1.8}$$

holds.

□

1.3 Integral transforms of measures

Let $\mu \in M(\mathbb{T})$. We define the Cauchy transform $K[\mu]$ of μ on the unit disk by

$$K[\mu](z) := \int_{\mathbb{T}} \frac{1}{1-\bar{\xi}z} d\mu(\xi). \tag{1.9}$$

The Cauchy transform is an analytic function in $\mathbb{D}$. Moreover, it lies in the Hardy spaces H^p for $p < 1$, see [19, Theorem 3.5].

We further define the Poisson- and conjugate Poisson transform of μ, $P[\mu]$ and $Q[\mu]$, respectively, through

$$P[\mu](z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} d\mu(\xi) = \int_{\mathbb{T}} P(z,\xi)\, d\mu(\xi), \tag{1.10}$$

$$Q[\mu](z) := \int_{\mathbb{T}} \frac{2\mathrm{Im}\left(\bar{\xi}z\right)}{|\xi - z|^2} d\mu(\xi) = \int_{\mathbb{T}} Q(z,\xi)\, d\mu(\xi). \tag{1.11}$$

Here, $P(z,\xi) = \frac{1-|z|^2}{|\xi-z|^2}$ is the Poisson kernel, and $Q(z,\xi) = \frac{2\mathrm{Im}\left(\bar{\xi}z\right)}{|\xi-z|^2}$ is the conjugate Poisson kernel. These transforms are harmonic functions in $\mathbb{D}$.

If $\mu \in M(\mathbb{T})$ is a real-valued measure, its Poisson and conjugate Poisson transforms are real-valued harmonic functions. The name conjugate Poisson transform is due to the fact that in the case of a real-valued measure, $Q[\mu]$ is precisely the conjugate function of $P[\mu]$: The Poisson kernel $P(z,\xi)$ is the real part of the function $\frac{\xi+z}{\xi-z}$, and the conjugate Poisson kernel $Q(z,\xi)$ is its imaginary part. Hence we have

$$P[\mu](z) + iQ[\mu](z) = \int_{\mathbb{T}} \frac{\xi+z}{\xi-z} d\mu(\xi), \tag{1.12}$$

which is analytic in $\mathbb{D}$. Obviously $Q[\mu]$ satisfies the condition $Q[\mu](0) = 0$.

For arbitrary complex measures, the Poisson and conjugate Poisson transform are not necessarily real-valued, but we still have (1.12), and also in this case $Q[\mu]$ is called conjugate function of $P[\mu]$.

Since $1 + \frac{\xi+z}{\xi-z} = \frac{2}{1-\bar{\xi}z}$, we further have the relation

$$K[\mu](z) = \frac{1}{2}\int_{\mathbb{T}} d\mu + \frac{1}{2}P[\mu](z) + \frac{i}{2}Q[\mu](z) \tag{1.13}$$

between the Cauchy and the Poisson and conjugate Poisson transforms of μ.

1.4 Further results

We state some results here that are important tools in following proofs and give references on where to find further details and proofs of the statements.

The first statement is *Fatou's jump theorem* (see [14, Section 2.4]), which establishes a relation between boundary values on $\mathbb{T}$ of the Cauchy transform from inside and outside $\mathbb{D}$. For, although we are mainly concerned with the Cauchy transform as a function on the unit disk, it is clearly defined for $z \in \hat{\mathbb{C}} \backslash \mathbb{T}$, where $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. If we set $\mathbb{D}_e := \hat{\mathbb{C}} \backslash \bar{\mathbb{D}}$, then $K[\mu]$ is analytic on both $\mathbb{D}$ and $\mathbb{D}_e$ and satisfies $K[\mu](\infty) = 0$.
We denote by $H^p(\mathbb{D}_e)$, $0 < p \leq \infty$, the set of all functions F that are analytic on $\mathbb{D}_e$ and for which $z \mapsto F(1/z)$ belongs to H^p. The restriction $K[\mu]\big|_{\mathbb{D}_e}$ lies in the spaces $H^p(\mathbb{D}_e)$ for $0 < p < 1$ and thus it has boundary values σ-almost everywhere on $\mathbb{T}$ from $\mathbb{D}_e$. For more details on the Cauchy transform on $\mathbb{D}_e$ and the proof of the statement below see Section 2.4 of [14].

Theorem 1.4.1 *Let $\mu \in M(\mathbb{T})$. Then, for σ-almost all $\xi \in \mathbb{T}$,*

$$\lim_{r \nearrow 1} K[\mu](r\xi) - \lim_{r \searrow 1} K[\mu](r\xi) = \frac{d\mu}{d\sigma}(\xi).$$

In particular, if $\mu \perp \sigma$, at σ-almost every point on $\mathbb{T}$, we have the relation

$$\lim_{r \nearrow 1} K[\mu](r\xi) = \lim_{r \searrow 1} K[\mu](r\xi). \tag{1.14}$$

□

Kolmogorov's theorem provides us with a weak type estimate for the Cauchy transform of a measure; see [14, Theorem 3.4.1].

Theorem 1.4.2 *Let $\mu \in M(\mathbb{T})$. Then there exists a constant $A > 0$ such that for every $\lambda > 0$ the estimate*

$$\sigma(\{|K[\mu]| > \lambda\}) \leq \frac{A\|\mu\|}{\lambda} \tag{1.15}$$

holds.

□

The following two statements are concerned with integral representations of harmonic functions by means of the Poisson transform and highlight the importance of this transform in harmonic function theory; see [12, Theorem 1.10, Theorem 1.5].

Theorem 1.4.3. *Let u be a nonnegative harmonic function on $\mathbb{D}$. Then there exists a measure $\mu \in M_+(\mathbb{T})$ such that*

$$u(z) = \int_{\mathbb{T}} P(z, \zeta)\, d\mu(\zeta), \quad z \in \mathbb{D}. \tag{1.16}$$

□

In addition, if a harmonic function u is continuous on the boundary, the measure is explicitly given as the absolutely continuous measure with density $u\big|_{\mathbb{T}}$:

Theorem 1.4.4. *Let u be harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then u has the representation*

$$u(z) = \int_{\mathbb{T}} P(z,\zeta)\, u(\zeta)\, d\sigma(\zeta), \quad z \in \mathbb{D}. \tag{1.17}$$

□

Let E be a subset of $\mathbb{R}^N$. We call a collection $\mathcal{F}$ of nontrivial closed balls a *Besicovitch covering* for E if every $x \in E$ is the center of a ball $B(x) \in \mathcal{F}$. For such coverings, we have the following *Besicovitch covering theorem* ([23, Chapter II, Theorem 18.1]).

Theorem 1.4.5. *Let $E \subseteq \mathbb{R}^N$ be a bounded set, and let $\mathcal{F}$ be a Besicovitch covering for E. Then there exists a positive integer c_N depending only on the dimension N such that there are c_N subcollections $\mathcal{B}_1, \ldots, \mathcal{B}_{c_N}$ of $\mathcal{F}$ with the following properties:*

(i) *for every $k = 1, \ldots, c_N$, the collection $\mathcal{B}_k$ contains countably many disjoint balls;*

(ii) $E \subseteq \bigcup_{k=1}^{c_N} \bigcup_{B \in \mathcal{B}_k} B.$

□

Finally we state the *Marcinkiewicz interpolation theorem* ([13, Chapter I, Theorem 4.5]).

Theorem 1.4.6. *Let (X,μ) and (Y,ν) be measure spaces, and let $1 < p \leq \infty$. Suppose that T is a map defined both on $L^1(X,\mu)$ and $L^p(X,\mu)$ and mapping into the space of ν-measurable functions. Suppose further that T satisfies*

(i) $|T(f+g)| \leq |Tf| + |Tg|$ *on Y;*

(ii) *T is of weak type (1,1), i.e., for each $\lambda > 0$ and $f \in L^1(X,\mu)$,*

$$\nu(\{|Tf| > \lambda\}) \leq \frac{A_0 \|f\|_{L^1(X,\mu)}}{\lambda};$$

(iii) *if $p < \infty$, then T is of weak type (p,p), i.e., for each $\lambda > 0$ and $f \in L^p(X,\mu)$,*

$$\nu(\{|Tf| > \lambda\}) \leq \frac{A_1 \|f\|^p_{L^p(X,\mu)}}{\lambda^p},$$

and if $p = \infty$, T satisfies

$$\|Tf\|_{L^\infty(Y,\nu)} \leq A_1 \|f\|_{L^\infty(X,\mu)}.$$

Then for each $1 < p\prime < p$, there is a constant $A_{p\prime}$ depending only on A_0, A_1, p and $p\prime$ such that for all $f \in L^{p'}(X,\mu)$

$$\|Tf\|_{L^{p'}(Y,\nu)} \leq A_{p\prime} \|f\|_{L^{p'}(X,\mu)}.$$

□

Chapter 2

Absolutely continuous measures

In this chapter we concern ourselves with the boundary behavior of the Poisson and conjugate Poisson transforms of an absolutely continuous measure $\mu \in M(\mathbb{T})$. We show that in this case, boundary values of both transforms exist almost everywhere on $\mathbb{T}$. The first part of the chapter is dedicated to the symmetric derivative and maximal function of a measure.

2.1 The symmetric derivative of a measure

For $\xi \in \mathbb{T}$ and $r \in (0,2]$ we set $I_r(\xi) := \mathbb{T} \cap U_r(\xi) = \{\eta \in \mathbb{T} \mid |\xi - \eta| < r\}$, i.e., $I_r(\xi)$ is an open subarc of $\mathbb{T}$ centered in ξ that is obtained by intersecting $\mathbb{T}$ with the open ball of radius $r \in (0,2]$ centered at ξ.

Definition 2.1.1. Let $\mu \in M(\mathbb{T})$. We set

$$(A_r\mu)(\xi) := \frac{\mu(I_r(\xi))}{\sigma(I_r(\xi))}, \quad \text{and} \tag{2.1}$$

$$(D\mu)(\xi) := \lim_{r \to 0} (A_r\mu)(\xi), \tag{2.2}$$

whenever this limit exists. $(D\mu)(\xi)$ is called the *symmetric derivative of* μ *at* ξ. Furthermore, we define the *maximal function* of the measure μ by

$$(M\mu)(\xi) := \sup_{0<r<\infty} (A_r|\mu|)(\xi), \tag{2.3}$$

where $|\mu|$ denotes the total variation of μ.

We are going to show that if μ is absolutely continuous with respect to the Lebesgue measure, the symmetric derivative coincides σ-almost everywhere with the Radon-Nikodym derivative of μ with respect to σ, and that for a singular measure μ the symmetric derivative vanishes almost everywhere with respect to σ. First we establish some results about the maximal function $M\mu$.

Proposition 2.1.2. *Let $\mu \in M(\mathbb{T})$ and $K \in (0,\infty]$. The function*

$$S_K\mu(\xi) := \sup_{0<r<K} (A_r|\mu|)(\xi)$$

is lower semicontinuous, i.e., for every $\lambda \in \mathbb{R}$, the set $\{S_K\mu > \lambda\}$[1] is open. In particular, the maximal function $M\mu$ is lower semicontinuous.

[1] Notation: this expression is an abbreviation denoting the set $\{\xi \in \mathbb{T} \mid (S_K\mu)(\xi) > \lambda\}$.

Proof. Since $S_K\mu$ depends only on the total variation of the measure, we may assume $\mu \geq 0$. We fix $\lambda > 0$, denote $E_\lambda := \{S_K\mu > \lambda\}$ and fix $\xi \in E_\lambda$. By the definition of the set, we find $r \in (0, K)$ and $t > \lambda$ such that $\mu(I_r(\xi)) = t\sigma(I_r(\xi))$. Furthermore, since $t/\lambda > 1$ and σ is regular, there is some $\delta \in (0, K-r)$ for which $\sigma(I_{r+\delta}(\xi)) < \frac{t}{\lambda}\sigma(I_r(\xi))$ holds.
Let $\eta \in I_\delta(\xi)$, then by the triangle inequality we have $I_r(\xi) \subseteq I_{r+\delta}(\eta)$. This yields, using the invariance of σ under rotations,

$$\mu(I_{r+\delta}(\eta)) \geq \mu(I_r(\xi)) = t\sigma(I_r(\xi)) > \lambda\sigma(I_{r+\delta}(\xi)) = \lambda\sigma(I_{r+\delta}(\eta)),$$

and since $r + \delta < K$, η lies in E_λ. As $\eta \in I_\delta(\xi)$ was arbitrary, we see that $I_\delta(\xi) \subseteq E_\lambda$, hence the set is open.

□

We proceed with a statement about the possible size of the set where the maximal function $M\mu$ is large. We will see that $M\mu$ can take large values on relatively small sets only. First, we prove the following covering lemma.

Lemma 2.1.3. *Given a finite set of arcs $I_j = I_{r_j}(\xi_j), j = 1, \ldots, m$, there exists a subset of indices $J \subseteq \{1, ..., m\}$ such that*

(i) *the collection $\{I_j, j \in J\}$ is pairwise disjoint,*

(ii) $\bigcup_{j=1}^{m} I_j \subseteq \bigcup_{j \in J} I_{3r_j}(\xi_j)$, *and*

(iii) $\sigma\left(\bigcup_{j=1}^{m} I_j\right) \leq 6 \sum_{j \in J} \sigma(I_j)$.

Proof. We order the arcs such that $r_1 \geq r_2 \geq ... \geq r_m$. Now we inductively choose the indices in J in the following way: We set $j_1 = 1$, then remove all I_j which intersect I_1. We then choose the first remaining index, if there are any remaining arcs, to be j_2, and again remove all of the remaining arcs that intersect I_{j_2}. Continuing in this way as long as possible, we arrive at a set of indices $J = \{j_1, j_2, ..., j_k\}$. Clearly, the corresponding collection of arcs is disjoint. Furthermore, we note that every discarded arc I intersects some I_{j_0} with $j_0 \in J$, whose radius r_{j_0} is larger than that of I. Thus I is contained in the arc centered at ξ_{j_0} with radius $3r_{j_0}$ by the triangle inequality. This proves (*ii*).
In order to prove the third statement, we first derive an estimate of the σ-measure of some subarc of $\mathbb{T}$ with that of a smaller subarc: For $r \in (0, 2]$, a short calculation gives $\sigma(I_r(\xi)) = \frac{2}{\pi}\arcsin\left(\frac{r}{2}\right)$. We note that for such r the inequality $\frac{r}{2} < \arcsin(\frac{r}{2}) < r$ holds, which leads to the estimate

$$\frac{r}{\pi} \leq \sigma(I_r(\xi)) \leq \frac{2r}{\pi}.$$

Considering now the arc centered at ξ with triple radius, we use the right part of this inequality to estimate $\sigma(I_{3r}(\xi))$ and then the left part for $\sigma(I_r(\xi))$ and thereby obtain

$$\sigma(I_{3r}(\xi)) \leq \frac{2}{\pi}3r \leq 6\sigma(I_r(\xi)). \tag{2.4}$$

With this estimate, using (ii) we get

$$\sigma\left(\bigcup_{j=1}^{m} I_j\right) \leq \sigma\left(\bigcup_{j \in J} I_{3r_j}(\xi_j)\right) \leq \sum_{j \in J} \sigma\left(I_{3r_j}(\xi_j)\right) \leq 6 \sum_{j \in J} \sigma(I_j).$$

□

Theorem 2.1.4. *Let $\mu \in M(\mathbb{T})$. Then for every $\lambda > 0$, the inequality*

$$\sigma(\{M\mu > \lambda\}) \leq \frac{6\|\mu\|}{\lambda} \tag{2.5}$$

holds, where $\|\mu\| = |\mu|(\mathbb{T})$ is the total variation norm of μ.

Proof. We fix $\lambda > 0$. As established in Proposition 2.1.2, $E_\lambda := \{M\mu > \lambda\}$ is an open set. Let K be a compact subset of E_λ. By the definition of $M\mu$, for each $\xi \in K$ we find an arc $I_\xi = I_{r_\xi}(\xi)$ such that $|\mu|(I_\xi) > \lambda\sigma(I_\xi)$. The family $\{I_\xi,\ \xi \in K\}$ covers K, and since K is compact, so does a finite subcollection $\{I_{\xi_j}, j = 1, \ldots, m\}$. Now, by the covering lemma, Lemma 2.1.3, we find a subset $J \subseteq \{1, \ldots, m\}$ of indices providing a disjoint subcollection $\{I_{\xi_j}, j \in J\}$ such that $K \subseteq \bigcup_{j=1}^{m} I_{r_{\xi_j}}(\xi_j) \subseteq \bigcup_{j \in J} I_{3r_{\xi_j}}(\xi_j)$. We can then estimate the σ-measure of K, making use of the inequality (iii) from the covering lemma and the pairwise disjointness of the arcs I_{ξ_j}, by

$$\sigma(K) \leq \sigma\left(\bigcup_{j=1}^{m} I_{r_{\xi_j}}(\xi_j)\right) \leq 6\sum_{j\in J}\sigma\left(I_{\xi_j}\right) \leq \frac{6}{\lambda}\sum_{j\in J}|\mu|\left(I_{\xi_j}\right) \leq \frac{6\|\mu\|}{\lambda}.$$

As σ is a regular measure, we have $\sigma(E_\lambda) = \sup\{\sigma(K) \mid K \subseteq E_\lambda, K \text{ is compact}\}$, and since above estimate holds for arbitrary compact subsets of E_λ, we arrive at the desired conclusion.

□

In the following results we establish properties of the symmetric derivative $D\mu$. A notion important in the proofs of these and subsequent statements is that of a Lebesgue point:

Definition 2.1.5. Let $f \in L^1(\sigma)$, and let $\xi \in \mathbb{T}$. We call ξ a *Lebesgue point* of f with respect to σ, if

$$\lim_{r\to 0}\frac{1}{\sigma(I_r(\xi))}\int\limits_{I_r(\xi)}|f(\zeta) - f(\xi)|\,d\sigma(\zeta) = 0. \tag{2.6}$$

Remark 2.1.6. (i) For any Lebesgue point we have

$$\lim_{r\to 0}\frac{1}{\sigma(I_r(\xi))}\int\limits_{I_r(\xi)}f(\zeta)\,d\sigma(\zeta) = f(\xi), \tag{2.7}$$

since, by the triangle inequality,

$$\left|\frac{1}{\sigma(I_r(\xi))}\int\limits_{I_r(\xi)}f d\sigma - f(\xi)\right| = \left|\frac{1}{\sigma(I_r(\xi))}\int\limits_{I_r(\xi)}f - f(\xi)\,d\sigma\right| \leq \frac{1}{\sigma(I_r(\xi))}\int\limits_{I_r(\xi)}|f - f(\xi)|\,d\sigma.$$

(ii) If $f \in C(\mathbb{T})$, every $\xi \in \mathbb{T}$ is a Lebesgue point with respect to f. Indeed, if f is continuous at $\xi \in \mathbb{T}$, for any $\varepsilon > 0$ we can find a radius $r > 0$ such that on the arc $I_r(\xi)$ the estimate $|f(\zeta) - f(\xi)| < \varepsilon$ holds, and consequently for all $r' \leq r$,

$$\frac{1}{\sigma(I_{r'}(\xi))}\int\limits_{I_{r'}(\xi)}|f(\zeta) - f(\xi)|\,d\sigma(\zeta) < \frac{1}{\sigma(I_{r'}(\xi))}\varepsilon\sigma(I_{r'}(\xi)) = \varepsilon.$$

For an arbitrary function in $L^1(\sigma)$, the existence of Lebesgue points may not seem so clear. But in fact we have the following remarkable statement.

Theorem 2.1.7. *Let $f \in L^1(\sigma)$, then σ-almost every $\xi \in \mathbb{T}$ is a Lebesgue point of f.*

Proof. For $\xi \in \mathbb{T}$ and $r > 0$ we define

$$(T_r f)(\xi) := \frac{1}{\sigma(I_r(\xi))} \int\limits_{I_r(\xi)} |f(\zeta) - f(\xi)| \, d\sigma(\zeta)$$

and

$$(Tf)(\xi) := \limsup_{r \to 0} (T_r f)(\xi).$$

We have to show that $(Tf)(\xi) = 0$ at σ-almost every $\xi \in \mathbb{T}$, equivalently, for any $\varepsilon > 0$, the set $\{Tf > \varepsilon\}$ is a σ-nullset. To this end, we fix some $\varepsilon > 0$ and $n \in \mathbb{N}$. Since $C(\mathbb{T})$ is dense in $L^1(\sigma)$, we can find a function $g_n \in C(\mathbb{T})$ with $\|f - g_n\|_{L^1(\sigma)} < \frac{1}{n}$. We choose a representative, also denoted by f, of the equivalence class of f in $L^1(\sigma)$ (two representatives are equal up to a nullset with respect to σ, thus the choice is of no concern, for we want to prove that the statement holds everywhere up to a σ-nullset) and set $h_n = f - g_n$. As g_n is continuous, $Tg_n = 0$ by Remark 2.1.6 (ii). Furthermore, for $\xi \in \mathbb{T}$ we have

$$(T_r h_n)(\xi) \leq \frac{1}{\sigma(I_r(\xi))} \int\limits_{I_r(\xi)} |h_n(\zeta)| \, d\sigma(\zeta) + |h_n(\xi)|,$$

hence, denoting by μ_n the absolutely continuous measure with density function $|h_n|$,

$$(Th_n)(\xi) \leq (M\mu_n)(\xi) + |h_n(\xi)|.$$

Since $T_r f \leq T_r g_n + T_r h_n$, this implies $Tf \leq M\mu_n + |h_n|$, and therefore

$$\{Tf > 2\varepsilon\} \subseteq \{M\mu_n > \varepsilon\} \cup \{\, |h_n| > \varepsilon\} =: E(\varepsilon, n).$$

For the first set on the right-hand side we can apply Theorem 2.1.4 to obtain

$$\sigma(\{M\mu_n > \varepsilon\}) \leq \frac{6\|\mu_n\|}{\varepsilon} = \frac{6\|h_n\|_{L^1(\sigma)}}{\varepsilon} \leq \frac{6}{\varepsilon n}.$$

Furthermore we have, setting $B = \{|h_n| > \varepsilon\}$,

$$\varepsilon\sigma(B) \leq \int\limits_B |h_n| \, d\sigma \leq \int\limits_{\mathbb{T}} |h_n| \, d\sigma = \|h_n\|_{L^1(\sigma)} \leq \frac{1}{n},$$

and therefore $\sigma(\{|h_n| > \varepsilon\}) \leq \frac{1}{\varepsilon n}$. We infer

$$\sigma\big(\{Tf > 2\varepsilon\}\big) \leq \sigma(E(\varepsilon, n)) \leq \frac{7}{\varepsilon n}. \tag{2.8}$$

As the set $\{Tf > 2\varepsilon\}$ is independent of $n \in \mathbb{N}$, we conclude that it must be contained in every $E(\varepsilon, n)$ and hence also in the intersection $\bigcap_{n \in \mathbb{N}} E(\varepsilon, n)$. But (2.8) implies that $\sigma\Big(\bigcap_{n \in \mathbb{N}} E(\varepsilon, n)\Big) = 0$, and since σ is a complete measure, also $\{Tf > 2\varepsilon\}$ is a σ-nullset. As $\varepsilon > 0$ was arbitrary, this concludes the proof.

□

Theorem 2.1.8. *Let $\mu \in M(\mathbb{T})$ be absolutely continuous with density $f \in L^1(\sigma)$. Then $D\mu = f$ almost everywhere with respect to σ, and for all Borel subsets $E \subseteq \mathbb{T}$ we have*

$$\mu(E) = \int_E (D\mu)(\zeta)\, d\sigma(\zeta).$$

Proof. By the Radon-Nikodym theorem (see [21, Theorem 6.10]), f satisfies

$$\mu(E) = \int_E f(\zeta)\, d\sigma(\zeta)$$

for every Borel set E. Let $\xi \in \mathbb{T}$ be a Lebesgue point of f, then by Remark 2.1.6 (i) and the definition of the symmetric derivative we have

$$f(\xi) = \lim_{r\to 0} \frac{1}{\sigma(I_r(\xi))} \int_{I_r(\xi)} f(\zeta)\, d\sigma(\zeta) = \lim_{r\to 0} \frac{\mu(I_r(\xi))}{\sigma(I_r(\xi))} = (D\mu)(\xi),$$

thus, $D\mu$ exists and equals f at every Lebesgue point of f.

□

Theorem 2.1.9. *Let $\mu \in M(\mathbb{T})$ be singular with respect to σ. Then*

$$(D\mu)(\xi) = 0 \quad \text{for } \sigma\text{-almost every } \xi \in \mathbb{T}.$$

Proof. As the symmetric derivative is linear, we may confine ourselves to the case $\mu \geq 0$, for the Jordan decomposition theorem (see [18, Theorem 11.2(v)]), applied to the real and imaginary parts of the measure, provides a representation $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where the $\mu_i,\ i = 1, ..., 4$, are nonnegative singular measures. We consider the function

$$(\bar{D}\mu)(\xi) := \lim_{n\to\infty} \left[\sup_{0<r<1/n} (A_r\mu)(\xi) \right].$$

The functions $\sup_{0<r<1/n} (A_r\mu)(\xi)$ are lower semicontinuous by Proposition 2.1.2. Thus they are in particular measurable, and therefore their limit function $\bar{D}\mu$ is measurable too. We fix $\lambda > 0$ and $\varepsilon > 0$. As μ is singular and regular, we find a compact subset $K \subseteq \mathbb{T}$ such that $\sigma(K) = 0$ and $\mu(K) > \|\mu\| - \varepsilon$. We define two measures by $\mu_1(E) := \mu(K \cap E)$ for Borel sets $E \subseteq \mathbb{T}$ and $\mu_2 := \mu - \mu_1$. Then $\|\mu_1\| = \mu(K)$ and consequently $\|\mu_2\| = \|\mu\| - \|\mu_1\| < \varepsilon$. For $\xi \notin K$ we have

$$(\bar{D}\mu)(\xi) = (\bar{D}\mu_2)(\xi) \leq (M\mu_2)(\xi).$$

This yields the inclusion $\{\bar{D}\mu > \lambda\} \subseteq K \cup \{M\mu_2 > \lambda\}$, and hence, using Theorem 2.1.4,

$$\sigma(\{\bar{D}\mu > \lambda\}) \leq \sigma(K) + \sigma(\{M\mu_2 > \lambda\}) \leq \frac{6\|\mu_2\|}{\lambda} \leq \frac{6\varepsilon}{\lambda}.$$

Since ε was arbitrary, we conclude $\sigma(\{\bar{D}\mu > \lambda\}) = 0$, and as also $\lambda > 0$ was arbitrary, $D\mu = 0$ at $\sigma-$almost every point.

□

Combining Theorems 2.1.8 and 2.1.9, we obtain

Theorem 2.1.10. *Let $\mu \in M(\mathbb{T})$ with Lebesgue decomposition $\mu = f\sigma + \mu_s$. Then, for σ-almost every $\xi \in \mathbb{T}$,*

$$(D\mu)(\xi) = f(\xi).$$

□

We also have the following result for the symmetric derivative on the support of a measure singular with respect to σ.

Theorem 2.1.11. *Let $\mu \in M_+(\mathbb{T})$ be a singular measure. Then for μ-almost every $\xi \in \mathbb{T}$*

$$(D\mu)(\xi) = \infty.$$

Proof. Since μ is singular, there is a Borel set $S \subseteq \mathbb{T}$ with $\sigma(S) = 0$ and $\mu(\mathbb{T} \backslash S) = 0$, and by the regularity of σ we can find open Borel sets $V_j \supseteq S$ with $\sigma(V_j) < \frac{1}{j}$, $j \in \mathbb{N}$.
For $N \in \mathbb{N}$ we denote by E_N the set of $\xi \in S$ for which there is a sequence of radii $\{r_i = r_i(\xi)\}$ with $r_i \to 0$ and $\mu(I_{r_i}(\xi)) < N\sigma(I_{r_i}(\xi))$. Then for all $\xi \in S \backslash \bigcup\limits_{N \in \mathbb{N}} E_N$, we have $(D\mu)(\xi) = \infty$. We show that this set contains a set of full μ-measure.
We fix N and j. Since V_j is open, for every $\xi \in E_N \subseteq V_j$ we can choose an open arc I_ξ centered at ξ and contained in V_j such that $\mu(I_\xi) < N\sigma(I_\xi)$ holds. By J_ξ we denote the arc centered at ξ whose radius is one third of that of I_ξ. Then we have

$$E_N \subseteq \bigcup_{\xi \in E_N} J_\xi =: W_N^j \subseteq V_j.$$

Furthermore, the estimate $\mu\big(W_N^j\big) \leq \frac{6N}{j}$ holds: To see this we employ the covering lemma 2.1.3. Let $K \subseteq W_N^j$ be compact, then finitely many of the arcs J_ξ cover K. By the covering lemma we find a finite subset $M \subseteq E_N$ such that the collection $\{J_\xi, \xi \in M\}$ is disjoint and the union $\bigcup\limits_{\xi \in M} I_\xi$ covers K. Thus we obtain the following estimate for the μ-measure of K, using the estimate (2.4) and the fact that the arcs J_ξ, $\xi \in M$ are disjoint subsets of V_j.

$$\mu(K) \leq \sum_{\xi \in M} \mu(I_\xi) < N \sum_{\xi \in M} \sigma(I_\xi) \leq 6N \sum_{\xi \in M} \sigma(J_\xi) \leq 6N\sigma(V_j) \leq \frac{6N}{j}.$$

As μ is regular, we have $\mu\big(W_N^j\big) = \sup\{\mu(K) \mid K \subseteq W_N^j,\ K \text{ compact}\} \leq \frac{6N}{j}$.
We set $\Omega_N = \bigcap_{j \in \mathbb{N}} W_N^j$. Then $E_N \subseteq \Omega_N$, since E_N is contained in every W_N^j. Furthermore, $\mu(\Omega_N) = 0$, whence $\mu\big(\bigcup\limits_{N \in \mathbb{N}} \Omega_N\big) = 0$, and at every point $\xi \in S \backslash \bigcup\limits_{N \in \mathbb{N}} \Omega_N \subseteq S \backslash \bigcup\limits_{N \in \mathbb{N}} E_N$ we have $(D\mu)(\xi) = \infty$.

□

2.2 The Poisson transform

Now we turn to the study of the Poisson transform of a measure, defined for $z \in \mathbb{D}$ by

$$P[\mu](z) = \int\limits_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi).$$

With $z = re^{i\vartheta} \in \mathbb{D}$ and $\xi = e^{it} \in \mathbb{T}$, we can rewrite the Poisson kernel as follows.

$$P(z,\xi) = \operatorname{Re}\frac{\xi + z}{\xi - z} = \frac{1 - |z|^2}{|\xi - z|^2} = \frac{1 - r^2}{1 + r^2 - 2r\cos(\vartheta - t)}. \tag{2.9}$$

For simplicity of notation, we sometimes write $P_r(\vartheta) := P\left(re^{i\vartheta}, 1\right)$.

Proposition 2.2.1. *The Poisson kernel $P(z,\xi)$ has the following properties:*

(i) $P(z,\xi) > 0$ on $\mathbb{D} \times \mathbb{T}$.

(ii) *For every $\eta \in \mathbb{T}$ and $\delta > 0$, $\int\limits_{|\xi - \eta| > \delta} P(z,\xi)\, d\sigma(\xi) \to 0$ as z tends to η.*

(iii) $\int\limits_{\mathbb{T}} P(z,\xi)\, d\sigma(\xi) = 1$ *for all $z \in \mathbb{D}$.*

Proof. Properties (i) and (ii) follow immediately from the definition of the kernel, property (iii) follows from the representation formula (1.17) from Theorem 1.4.4 if we set $u \equiv 1$.

□

Now we come to the main results on the boundary behavior of $P[\mu]$. The kernel function $P(z,\xi)$ becomes singular as z approaches $\xi \in \mathbb{T}$, but as we see in the following theorem, the integral has finite limit at σ-almost every point $\xi \in \mathbb{T}$, if ξ is approached *nontangentially*:

Definition 2.2.2. Let $\xi = e^{it} \in \mathbb{T}$. We say that z *converges to ξ nontangentially* and write $z \underset{\triangleleft}{\to} \xi$, if z tends to ξ from within the cone region $\Delta_\xi^K \subseteq \mathbb{D}$ (see Figure 2.1). K is the positive constant such that all $z = re^{i\vartheta} \in \Delta_\xi^K$ satisfy the inequality

$$|\vartheta - t| < K(1 - r).$$

We have another useful estimate for $z \in \Delta_\xi^K$:

$$\frac{|\xi - z|}{1 - |z|} < K + 1. \tag{2.10}$$

Indeed,

$$|\xi - z| = \left|e^{it} - re^{i\vartheta}\right| \le \left|e^{it} - e^{i\vartheta}\right| + \left|e^{i\vartheta} - re^{i\vartheta}\right| \le |t - \vartheta| + (1 - r) < (K+1)(1-r).$$

Δ_ξ^K is symmetric about the radius terminating at ξ and has an opening angle $\varphi \in (0,\pi)$ increasing with K. We will sometimes also write Δ_ξ^φ for this region, when the angle φ is a more convenient characteristic than the constant K.
If h is a complex-valued function on $\mathbb{D}$ and $\xi \in \mathbb{T}$, then we say that h has the nontangential limit A at ξ, if $h(z) \to A$ as $z \underset{\triangleleft}{\to} \xi$.

Theorem 2.2.3. *(Fatou.) Let $\mu \in M(\mathbb{T})$ with Lebesgue decomposition $\mu = f\sigma + \mu_s$. Then for $\sigma-$almost every $\xi \in \mathbb{T}$,*

$$P[\mu](z) \to f(\xi) \quad \text{as } z \underset{\triangleleft}{\to} \xi. \tag{2.11}$$

Proof. Let α be a distribution function for μ,

$$\alpha(t) = \mu\left(\{e^{i\tau} : \tau \in (0,t]\}\right), \quad 0 < t \le 2\pi.$$

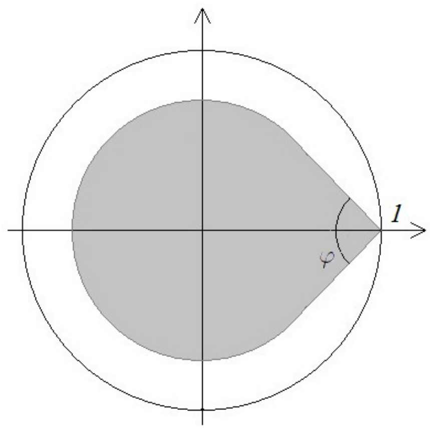

Figure 2.1: The region Δ_ξ^K for $\xi = 1$.

We note that

$$\alpha'(t) = \lim_{h\to 0} \frac{\alpha(t+h) - \alpha(t-h)}{2h} = \lim_{h\to 0} \frac{\mu\left(\{e^{i\tau} : \tau \in (t-h, t+h]\}\right)}{2\pi\sigma\left(\{e^{i\tau} : \tau \in (t-h, t+h]\}\right)} = \frac{1}{2\pi}(D\mu)\left(e^{it}\right),$$

whenever the limit exists. So, by Theorem 2.1.10, $\alpha'(t)$ exists and equals $f(\xi)/2\pi$ at σ-almost every $\xi = e^{it} \in \mathbb{T}$, and if we can show that (2.11) holds for every point at which we have $\alpha'(t) = f(\xi)/2\pi$, we are done.

Now let $\xi = e^{it_0}$ be such a point. Without loss of generality we assume $t_0 = 0$ (otherwise consider the rotated measure $\mu^\xi(B) = \mu(\xi B)$). We fix a sector $\Delta_1^K \subseteq \mathbb{D}$ with $K \in (0, \infty)$. Let $\varepsilon > 0$. We show that there is some $\delta > 0$ such that $|P[\mu](z) - 2\pi\alpha'(0)| < \varepsilon$ if only $z \in \Delta_1^K$ with $|1-z| < \delta$.

Using the properties of the Poisson kernel, we write the difference $P[\mu](z) - 2\pi\alpha'(0)$ as Riemann-Stieltjes integrals and then integrate by parts, obtaining

$$\begin{aligned} P[\mu](z) - 2\pi\alpha'(0) &= \int_{-\pi}^{\pi} P\left(z, e^{it}\right) d\alpha(t) - \int_{-\pi}^{\pi} P\left(z, e^{it}\right)\alpha'(0)\, dt \\ &= \left(\alpha(t) - \alpha'(0)\, t\right) P\left(z, e^{it}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\alpha(t) - \alpha'(0)\, t\right) \frac{\partial}{\partial t} P\left(z, e^{it}\right) dt \\ &= \frac{1 - |z|^2}{|1+z|^2}\Big(\alpha(\pi) - \alpha(-\pi) - 2\alpha'(0)\,\pi\Big) \\ &\qquad - \int_{-\pi}^{\pi} \left(\alpha(t) - \alpha'(0)\, t\right) \frac{\partial}{\partial t} P\left(z, e^{it}\right) dt. \end{aligned}$$

Clearly the first term tends to zero as z approaches 1. We choose $\delta_1 > 0$ such that the modulus of the term becomes less than $\varepsilon/3$ if $|1-z| < \delta_1$. We split the remaining integral into two parts: We choose $\eta \in (0, \pi)$ such that for $t \in (-\eta, \eta)$ we have

$$\left|\frac{\alpha(t)}{t} - \alpha'(0)\right| < \varepsilon M, \tag{2.12}$$

with $M = \frac{1}{3}\left(6\pi + 8K^3\right)^{-1}$, and write

$$\int_{-\pi}^{\pi} \left(\alpha(t) - \alpha'(0)\,t\right) \frac{\partial}{\partial t} P\left(z, e^{it}\right) dt = \left[\int_{-\eta}^{\eta} + \int_{\eta<|t|\leq\pi}\right] \left(\alpha(t) - \alpha'(0)\,t\right) \frac{\partial}{\partial t} P\left(z, e^{it}\right) dt = I + II.$$

We write $z = re^{i\vartheta}$. The partial derivative of the Poisson kernel with respect to the angle t is

$$\frac{\partial}{\partial t} P\left(z, e^{it}\right) = \frac{\partial}{\partial t} \frac{1 - r^2}{1 + r^2 - 2r\cos(\vartheta - t)} = \frac{\left(1 - r^2\right) 2r \sin(t - \vartheta)}{\left(1 + r^2 - 2r\cos(\vartheta - t)\right)^2}.$$

For $|t| \in (\eta, \pi]$, this term is bounded by $\frac{(1-r^2)2r}{(1+r^2-2r\cos(\vartheta-\eta))^2}$. The argument ϑ of z becomes small when z is sufficiently close to 1, so we can find $\delta_2 > 0$ such that $|\vartheta| < \eta/2$ if only $|1 - z| < \delta_2$. We see then that the integrand in II tends to zero uniformly in t as $z \to 1$. By eventually making δ_2 smaller we have that $|II| < \varepsilon/3$ if $|1 - z| < \delta_2$.
It remains to estimate the integral I. We use (2.12) to get

$$|I| \leq \int_{-\eta}^{\eta} \left|\frac{\alpha(t)}{t} - \alpha'(0)\right| \left|t \frac{\partial}{\partial t} P\left(z, e^{it}\right)\right| dt \leq \varepsilon M \int_{-\eta}^{\eta} \left|\frac{t\left(1 - r^2\right) 2r \sin(t - \vartheta)}{\left(1 + r^2 - 2r\cos(\vartheta - t)\right)^2}\right| dt.$$

As before we choose $\delta_3 > 0$ such that $|\vartheta| < \eta/2$ if $|1 - z| < \delta_3$. Furthermore, without loss of generality we assume that $\vartheta > 0$. Then we split the last integral into three more parts,

$$\begin{aligned} \varepsilon M \int_{-\eta}^{\eta} \left|\frac{t\left(1 - r^2\right) 2r \sin(t - \vartheta)}{\left(1 + r^2 - 2r\cos(\vartheta - t)\right)^2}\right| dt &= \varepsilon M \left[\int_{-\eta}^{0} + \int_{0}^{2\vartheta} + \int_{2\vartheta}^{\eta}\right] \left|\frac{t\left(1 - r^2\right) 2r \sin(t - \vartheta)}{\left(1 + r^2 - 2r\cos(\vartheta - t)\right)^2}\right| dt \\ &= \varepsilon M \big(A + B + C\big). \end{aligned}$$

We show that each of these integrals is bounded. In order to deal with C, note that for $t \in (2\vartheta, \eta)$ we have $t < 2(t - \vartheta)$, so we can estimate t in the integrand accordingly and then replace t by $t + \vartheta$. Using the positivity of the resulting integrand on $(0, \pi)$ we widen the range of integration and thus obtain

$$\begin{aligned} |C| &\leq 2\int_{2\vartheta}^{\eta} \frac{(t - \vartheta)\left(1 - r^2\right) 2r \sin(t - \vartheta)}{\left(1 + r^2 - 2r\cos(\vartheta - t)\right)^2} dt = 2\int_{\vartheta}^{\eta - \vartheta} t \frac{\left(1 - r^2\right) 2r \sin(t)}{\left(1 + r^2 - 2r\cos(t)\right)^2} dt \\ &\leq 2\int_{0}^{\pi} t \underbrace{\frac{\left(1 - r^2\right) 2r \sin(t)}{\left(1 + r^2 - 2r\cos(t)\right)^2}}_{= \frac{\partial}{\partial t} - P(r, e^{it})} dt = 2\left[\left(-t \frac{1 - r^2}{1 + r^2 - 2r\cos(t)}\right)\Bigg|_0^{\pi} + \int_0^{\pi} P\left(r, e^{it}\right) dt\right] \\ &= 2\left[-\frac{\pi\left(1 - r^2\right)}{(1 + r)^2} + \int_0^{\pi} P\left(r, e^{it}\right) dt\right]. \end{aligned}$$

The left term is bounded by π and tends to zero as z tends to 1, and the integral is equal to π, so the expression in brackets is bounded by 2π, hence $|C| < 4\pi$.
For deriving an estimate for $|A|$, we proceed similarly, noting that since ϑ is positive, $|t| < |t - \vartheta|$ for $t \in (-\eta, 0)$, so that we can estimate the integral analogously and arrive at $|A| < 2\pi$.

On the remaining interval $(0, 2\vartheta)$, we estimate the trigonometric functions with their respective maximal values and obtain

$$|B| \leq \int_0^{2\vartheta} t \frac{2\left(1-r^2\right)\sin(\vartheta)}{(1-r)^4} dt = \frac{2(1+r)\sin(\vartheta)}{(1-r)^3} \frac{4\vartheta^2}{2} \leq \frac{8\vartheta^3}{(1-r)^3}.$$

In the sector Δ_1^K the quotient $\frac{\vartheta}{1-r}$ is bounded by K, and we arrive at $|B| < 8K^3$. Thus,

$$|I| \leq \varepsilon M\big(|A| + |B| + |C|\big) < \varepsilon M\left(6\pi + 8K^3\right) = \frac{\varepsilon}{3}$$

by the choice of M. If we now set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then for all $z \in \Delta_1^K$ with $|1-z| < \delta$, we have the estimate $|P[\mu](z) - 2\pi\alpha'(0)| < \varepsilon$, which is what was to be proved.

□

As an immediate consequence of Theorem 2.1.11 we obtain the following statement on the boundary behavior of the Poisson transform of a singular measure.

Theorem 2.2.4. *Let $\mu \in M_+(\mathbb{T})$ be a singular measure. Then for μ-almost every $\xi \in \mathbb{T}$,*

$$P[\mu](z) \longrightarrow \infty \quad \text{as} \quad z \underset{\triangleleft}{\to} \xi.$$

Proof. Let $\xi \in \mathbb{T}$ be such that $D\mu(\xi) = \infty$. By Theorem 2.1.11, this holds for μ-almost every point on $\mathbb{T}$. For $z = re^{i\vartheta} \in \Delta_\xi^K$ we have the estimate $\frac{1-r}{|\xi-z|} > \frac{1}{K+1}$ by (2.10), hence we can estimate the Poisson kernel from below by

$$P(z,\xi) = \frac{1-r^2}{|\xi-z|^2}\frac{1-r}{1-r} > \frac{1}{(K+1)^2}\frac{1+r}{1-r} \geq \frac{1}{(K+1)^2(1-r)}.$$

We set $h = 1-r$ and consider the arc $I_h \subseteq \mathbb{T}$ centered at ξ with $\sigma(I_h) = h$, then we can estimate the Poisson transform by

$$P[\mu](z) = \int_{\mathbb{T}} P(z,\xi)\, d\mu > \int_{\mathbb{T}} \frac{1}{(K+1)^2(1-r)} d\mu \geq \int_{I_h} \frac{1}{(K+1)^2 h} d\mu = \frac{1}{(K+1)^2}\frac{\mu(I_h)}{\sigma(I_h)}.$$

As z approaches ξ, h tends to zero, and by Theorem 2.1.11 the right-hand term tends to infinity.

□

2.3 The conjugate Poisson transform

Recall the definition of the conjugate Poisson transform of a measure $\mu \in M(\mathbb{T})$,

$$Q[\mu](z) = \int_{\mathbb{T}} \frac{2\mathrm{Im}\left(\bar{\xi}z\right)}{|\xi-z|^2} d\mu(\xi).$$

The conjugate Poisson kernel $Q(z,\xi)$, with $(z,\xi) = \left(re^{i\vartheta}, e^{it}\right) \in \mathbb{D} \times \mathbb{T}$, can be written as

$$Q(z,\xi) = \mathrm{Im}\,\frac{\xi+z}{\xi-z} = \frac{2\mathrm{Im}\left(z\bar{\xi}\right)}{|\xi-z|^2} = \frac{2r\sin(\vartheta-t)}{1+r^2-2r\cos(\vartheta-t)}. \tag{2.13}$$

For the sake of convenience we denote $Q_r(\vartheta) := Q\left(re^{i\vartheta}, 1\right) = \frac{2r\sin\vartheta}{1+r^2-2r\cos\vartheta}$. Then, for $\vartheta \neq 0$,

$$\lim_{r\to 1} Q_r(\vartheta) = \lim_{r\to 1} \frac{2r\sin\vartheta}{1+r^2-2r\cos\vartheta} = \frac{\sin\vartheta}{1-\cos\vartheta} = \frac{2\sin\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\vartheta}{2}\right)}{2\sin^2\left(\frac{\vartheta}{2}\right)} = \cot(\vartheta/2) =: Q_1(\vartheta).$$

The conjugate Poisson kernel is an odd function and $\|Q_r(\vartheta)\|_{L^1(-\pi,\pi)} \to \infty$ as $r \to 1$:

$$\begin{aligned}\int_{-\pi}^{\pi} |Q_r(t)|\, dt &= \int_{-\pi}^{\pi} \left|\frac{2r\sin t}{1+r^2-2r\cos t}\right| dt = 4r\int_0^{\pi} \frac{\sin t}{1+r^2-2r\cos t} dt \\ &= 4r\int_{-1}^{1} \frac{1}{1+r^2-2r\tau} d\tau = 4r\int_{(1-r)^2}^{(1+r)^2} \frac{1}{2ru} du = 2\ln\frac{(1+r)^2}{(1-r)^2} \overset{r\to 1}{\longrightarrow} \infty.\end{aligned}$$

Thus, the behavior of the conjugate Poisson transform of a measure is quite different from that of its Poisson transform and depends on different properties of the measure in question. Since we have no absolute convergence of the integral, convergence to boundary values can only occur as a consequence of cancellation of the negative and positive portions of the integral. However, if μ is an absolutely continuous measure we have the existence of boundary values in the sense of principal value integrals almost everywhere:

Theorem 2.3.1. *Let $\mu \in M(\mathbb{T})$ be absolutely continuous with density $f \in L^1(\sigma)$. Then for σ-almost every $\xi = e^{i\vartheta} \in \mathbb{T}$, the conjugate Poisson transform $Q[\mu](z)$ tends to a boundary value $\tilde{f}(\xi)$ as z approaches ξ nontangentially, and*

$$\tilde{f}(e^{i\vartheta}) = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{|\vartheta - t|>\varepsilon} \cot\left(\frac{\vartheta - t}{2}\right) f\left(e^{it}\right) dt. \tag{2.14}$$

Proof. We may assume $\mu \geq 0$. We first prove the existence of the nontangential boundary values. To this end, we consider the function $P[\mu](z) + iQ[\mu](z) =: g(z)$, which is an analytic function in $\mathbb{D}$ with nonnegative real part by virtue of the nonnegativity of μ. Therefore, $G(z) := \frac{g(z)}{1+g(z)}$ is a bounded analytic function in $\mathbb{D}$, and thus it has nontangential boundary values $G(\xi)$ at σ-almost every $\xi \in \mathbb{T}$. Furthermore, $G(\xi) = 1$ for at most a σ-nullset by the Lusin-Privalov theorem ([11, p.212, 2.5]). Hence, $g(z) = \frac{G(z)}{1-G(z)}$ has finite nontangential limits almost everywhere, and the same must hold for its imaginary part, $Q[\mu]$.

It remains to verify the stated formula for the limit function. Let $\xi = e^{i\vartheta}$ be a Lebesgue point of f. We show that the conjugate Poisson transform of μ tends to the function in (2.14) as $z = re^{i\vartheta}$ approaches ξ radially.

We denote $F(\vartheta) = f\left(e^{i\vartheta}\right)$ and set $\varepsilon = 1-r$. We then consider the difference of $Q[\mu]\left(re^{i\vartheta}\right)$ and the integral on the right-hand side of (2.14). We first substitute t by $\vartheta - t$. Since the kernel functions Q_r and $\cot\left(\frac{t}{2}\right) = Q_1(t)$ are odd functions, we can replace F by $F - F(\vartheta)$ in the integrals without changing their value. We then rearrange the difference in two integrals according to the respective integration intervals, thus we obtain

$$
\begin{aligned}
&Q[\mu]\left(re^{i\vartheta}\right) - \frac{1}{2\pi}\int\limits_{|\vartheta-t|>\varepsilon} \cot\Big(\frac{\vartheta-t}{2}\Big)F(t)\,dt \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} Q_r(t)\,F(\vartheta-t)\,dt - \frac{1}{2\pi}\int\limits_{\varepsilon<|t|<\pi} \cot\Big(\frac{t}{2}\Big)F(\vartheta-t)\,dt \\
&= \frac{1}{2\pi}\int\limits_{-\varepsilon}^{\varepsilon} Q_r(t)\big[F(\vartheta-t)-F(\vartheta)\big]dt \\
&\qquad + \frac{1}{2\pi}\int\limits_{\varepsilon<|t|<\pi} \big(Q_r(t)-Q_1(t)\big)\big[F(\vartheta-t)-F(\vartheta)\big]dt = I + II.
\end{aligned}
$$

For $|t| \le \varepsilon$, we use the inequality $|\sin t| \le |t|$ to estimate the conjugate Poisson kernel by

$$|Q_r(t)| \le \frac{2\sin t}{(1-r)^2} \le \frac{1}{\varepsilon}.$$

Hence, for I we get

$$|I| \le \frac{1}{2\pi\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} \big|F(\vartheta-t)-F(\vartheta)\big|dt,$$

which tends to zero for $r \to 1$ since $e^{i\vartheta}$ is a Lebesgue point. We still have to deal with II. Using trigonometric identities we rewrite

$$Q_r(t) - Q_1(t) = \frac{2r\sin t}{\varepsilon^2 + 4r\sin^2(t/2)} - \frac{\sin t}{2\sin^2(t/2)} = \frac{-\varepsilon^2\sin t}{2\sin^2(t/2)\left(\varepsilon^2+4r\sin^2(t/2)\right)}.$$

For $|t| \in (0,\pi)$ we can estimate $\big|\sin\left(\frac{t}{2}\right)\big| \ge \frac{|t|}{4}$, and for r sufficiently large, say, $r \ge 1/2$, we thus have

$$|Q_r(t) - Q_1(t)| \le \frac{\varepsilon^2|\sin t|}{4\sin^4(t/2)} \le \frac{\varepsilon^2|t|}{4(t/4)^4} = \frac{64\varepsilon^2}{|t|^3}.$$

This yields

$$|II| \le \frac{1}{2\pi}\int\limits_{\varepsilon\le|t|\le\pi} \frac{64\varepsilon^2}{|t|^3}\,|F(\vartheta-t)-F(\vartheta)|\,dt.$$

We set $C := \frac{32}{\pi}$. We now consider the integral over the positive interval and integrate by parts to obtain

$$
\begin{aligned}
&C\varepsilon^2\int\limits_{\varepsilon}^{\pi} \frac{|F(\vartheta-t)-F(\vartheta)|}{t^3}dt \\
&= C\varepsilon^2\left(\int\limits_0^s |F(\vartheta-t)-F(\vartheta)|\,dt\frac{1}{s^3}\Big|_\varepsilon^\pi + 4\int\limits_\varepsilon^\pi \frac{\int_0^s |F(\vartheta-t)-F(\vartheta)|\,dt}{s^4}ds\right) \\
&= C\varepsilon^2\left(\frac{1}{\pi^3}\int\limits_0^\pi |F(\vartheta-t)-F(\vartheta)|\,dt - \frac{1}{\varepsilon^3}\int\limits_0^\varepsilon |F(\vartheta-t)-F(\vartheta)|\,dt\right. \\
&\qquad\left. + 4\int\limits_\varepsilon^\pi \frac{\int_0^s |F(\vartheta-t)-F(\vartheta)|\,dt}{s^4}ds\right).
\end{aligned}
$$

Since f is integrable, the first resulting integral is bounded and the term tends to zero as $\varepsilon \to 0$, and the second term tends to zero because $e^{i\vartheta}$ is a Lebesgue point of f. For given $\delta > 0$ choose $\eta > 0$ such that $\frac{1}{s}\int_0^s |F(\vartheta - t) - F(\vartheta)|\, dt < \delta$ if $0 < s < \eta$ (again this is possible since $e^{i\vartheta}$ is a Lebesgue point). Then, if $\varepsilon < \eta$, the last integral becomes

$$\begin{aligned}
\int\limits_{\varepsilon}^{\pi} \frac{\int_0^s |F(\vartheta - t) - F(\vartheta)|\, dt}{s^4} ds &\leq \int\limits_{\varepsilon}^{\eta} \frac{\delta}{s^3} ds + \int\limits_{\eta}^{\pi} \frac{1}{\eta^4} \int\limits_{0}^{s} |F(\vartheta - t) - F(\vartheta)|\, dt\ ds \\
&\leq \frac{\delta}{-2}\left(\frac{1}{\eta^2} - \frac{1}{\varepsilon^2}\right) + \frac{\pi}{\eta^4} \int\limits_{0}^{\pi} |F(\vartheta - t) - F(\vartheta)|\, dt \\
&\leq \frac{\delta}{2\varepsilon^2} + \frac{K\pi}{\eta^4}.
\end{aligned}$$

Hence the last of the above terms is bounded by $2C\delta + \frac{4C\varepsilon^2 K\pi}{\eta^4}$. Since δ was arbitrary, this expression becomes arbitrarily small as $\varepsilon \to 0$. We can treat the integral over the interval $(-\pi, -\varepsilon)$ in the same way and conclude $|II| \to 0$, which ends the proof.

□

Chapter 3

The normalized Cauchy transform

In this chapter we examine the *normalized Cauchy transform* V_μ associated with a measure $\mu \in M(\mathbb{T})$. It is an operator defined for $f \in L^1(\mu)$ by

$$(V_\mu f)(z) := \frac{K[f\mu](z)}{K[\mu](z)}. \tag{3.1}$$

If $\mu \in M_+(\mathbb{T})$, its Poisson transform is positive in $\mathbb{D}$ and the relation (1.13) yields $|K[\mu](z)| \geq |\mathrm{Re}\, K[\mu](z)| \geq \frac{\|\mu\|}{2}$. Therefore, if μ is a positive measure, $V_\mu f$ is a holomorphic function on $\mathbb{D}$. As the quotient of two H^p-functions ($p < 1$), it has boundary values almost everywhere on $\mathbb{T}$ with respect to σ. If μ is an arbitrary complex measure, $V_\mu f$ is a meromorphic function of bounded type and has thus boundary values σ-almost everywhere on $\mathbb{T}$. The goal of this chapter is a result on the boundary behavior of the function $V_\mu f$ which is due to A. Poltoratski [1] and states that nontangential boundary values of this function exist almost everywhere with respect to μ also. We first treat the case of a singular measure for which we provide the necessary means in the following two sections. We use a relation between singular measures and inner functions and construct a unitary operator such that the adjoint of this operator is precisely the normalized Cauchy transform of the singular measure. Using the special properties we have from this construction, we can prove Poltoratski's theorem for singular measures, and we can then reduce the proof of the general result to this special case.

3.1 Inner functions and singular measures

A function $\theta \in H^\infty$ is called *inner*, if $|\theta(\xi)| = 1$ for σ-almost every $\xi \in \mathbb{T}$. For $p \geq 1$, with each inner function we associate the space

$$\theta^*(H^p) := \left\{ f \in H^p \mid f\bar{\theta} \in \overline{H_0^p} \right\}, \tag{3.2}$$

i.e., the set of those functions $f \in H^p$ whose boundary function is of the form $f(\xi) = \bar{\xi}\theta(\xi)\overline{h(\xi)}$ for some $h \in H^p$. The motivation for this definition stems from the study of the backward shift operator $S^* : H^p \to H^p$, $(S^* f)(z) = \frac{f(z) - f(0)}{z}$, for it turns out that for $1 < p < \infty$, the spaces $\theta^*(H^p)$ are precisely the closed subspaces of H^p that are invariant under S^*: The invariant subspaces of its conjugate operator, the forward shift $S : H^p \to H^p$, $(Sf)(z) = zf(z)$, are identified as the spaces θH^p, where θ is an inner function, by Beurling's theorem (see e.g. [13, Chapter II, Section 7], [19, Section 7.3]).

Now, if A is a subspace of H^p, we have the relation

$$SA \subset A \Leftrightarrow S^* A^\perp \subset A^\perp,$$

and hence the closed invariant subspaces of S^* in H^p are of the form $(\theta H^q)^\perp$, where q is the conjugate index of p. Let us convince ourselves of the relation

$$\theta^*(H^p) = (\theta H^q)^\perp .$$

We start with the inclusion "$\subseteq$": Let $f \in \theta^*(H^p)$, then $f = \theta \bar{h}$ for some $h \in H_0^p$ almost everywhere on $\mathbb{T}$, and if $g \in H^q$, we have

$$\langle f, \theta g\rangle = \int_{\mathbb{T}} \left(\theta \bar{h}\right) \overline{(\theta g)} d\sigma = \int_{\mathbb{T}} \overline{hg} d\sigma = \overline{(hg)\,(0)} = 0,$$

since $|\theta| = 1$ σ-almost everywhere on $\mathbb{T}$ and $hg \in H_0^1$. Hence we have $f \in (\theta H^q)^\perp$.
For the other inclusion, let $f \in (\theta H^q)^\perp$, then $\int_{\mathbb{T}} f \overline{\theta \zeta^n} d\sigma = 0$ for all $n \geq 0$, and this implies $f\bar{\theta} \in \overline{H_0^p}$ by the characterization (1.6), and thus we have $f \in \theta^*(H^p)$.

□

Also, the following construction and several of the results presented here have their origin in the study of the backward shift operator S^*. See [16] and [5] for more information on this subject.

With each nonconstant inner function θ we can associate a family of nonnegative singular measures $\{\nu_\alpha\}_{\alpha \in \mathbb{T}}$, called the *Clark measures* of θ, in the following way. Let $\alpha \in \mathbb{T}$, then the function $\frac{\alpha+\theta}{\alpha-\theta}$ is analytic in $\mathbb{D}$, and $\operatorname{Re} \frac{\alpha+\theta}{\alpha-\theta} = \frac{1-|\theta|^2}{|\alpha-\theta|^2}$ is a positive harmonic function in $\mathbb{D}$. Therefore, by Theorem 1.4.3, it is the Poisson integral of some positive measure $\nu_\alpha \in M_+(\mathbb{T})$,

$$\operatorname{Re} \frac{\alpha + \theta(z)}{\alpha - \theta(z)} = P[\nu_\alpha](z). \tag{3.3}$$

By Theorems 2.2.3 and 2.2.4, the absolutely continuous part of ν_α is supported on the set of points where $P[\nu_\alpha]$ has finite nontangential limit, and the singular part is supported on the set of points ξ where $P[\nu_\alpha](z)$ tends to infinity as z nontangentially approaches ξ. Since $P[\nu_\alpha](z) = \frac{1-|\theta(z)|^2}{|\alpha-\theta(z)|^2}$ tends to infinity at all points where $\theta(\xi) = \alpha$, and tends to zero σ-almost everywhere else on $\mathbb{T}$, we see that the measures ν_α are supported on $\{\theta(\xi) = \alpha\}$. Hence, they are singular with respect to Lebesgue measure, and also pairwise mutually singular.

Remark 3.1.1. By the above approach we can relate a singular positive measure ν_1 to every inner function. In fact, the converse works too: every singular positive measure is a Clark measure for some inner function. To see this, let a singular measure $\nu \in M_+(\mathbb{T})$ be given and consider the function h defined on $\mathbb{D}$ by

$$h(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta).$$

Then $\operatorname{Re} h = P[\nu] > 0$ in $\mathbb{D}$. Since $w \mapsto \frac{w-1}{w+1}$ maps the half plane $\{\operatorname{Re} w > 0\}$ onto the unit disk, the composition function $\Theta(z) := \frac{h(z)-1}{h(z)+1}$ is an analytic function mapping $\mathbb{D}$

into $\mathbb{D}$. Solving for h yields $h(z) = \frac{1+\Theta(z)}{1-\Theta(z)}$, and thus $P[\nu](z) = \operatorname{Re} h(z) = \operatorname{Re}\frac{1+\Theta(z)}{1-\Theta(z)} = \frac{1-|\Theta(z)|^2}{|1-\Theta(z)|^2}$. Since ν is singular, its Poisson transform tends to zero σ-almost everywhere on the boundary, and we infer that $|\Theta| = 1$ almost everywhere on the boundary. So, Θ is an inner function and $\nu = \nu_1$ for Θ.

We now obtain a simple formula for the Cauchy transform of the measures ν_α.

Propostion 3.1.2. *Let θ be a nonconstant inner function and let $\{\nu_\alpha\}_{\alpha\in\mathbb{T}}$ be the associated family of singular measures. Then the following relations hold for $z \in \mathbb{D}$.*

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\nu_\alpha(\zeta) = \frac{\alpha+\theta(z)}{\alpha-\theta(z)} - 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2}, \tag{3.4}$$

$$K[\nu_\alpha](z) = \frac{1}{1-\overline{\alpha}\theta(z)} + \frac{\|\nu_\alpha\|-1}{2} - i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2}. \tag{3.5}$$

Proof. In splitting $\frac{\alpha+\theta}{\alpha-\theta}$ into its real and imaginary parts we get

$$\frac{\alpha+\theta(z)}{\alpha-\theta(z)} = \frac{1-|\theta(z)|^2}{|\alpha-\theta(z)|^2} + 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(z))}{|\alpha-\theta(z)|^2}.$$

Furthermore, we have

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\nu_\alpha(\zeta) = P[\nu_\alpha](z) + iQ[\nu_\alpha](z).$$

We know that $P[\nu_\alpha](z) = \frac{1-|\theta(z)|^2}{|\alpha-\theta(z)|^2}$, so from above relations we conclude that both $2\frac{\operatorname{Im}(\overline{\alpha}\theta)}{|\alpha-\theta|^2}$ and $Q[\nu_\alpha]$ are harmonic conjugates of $P[\nu_\alpha]$, and harmonic conjugates are unique up to an additive constant, hence

$$2\frac{\operatorname{Im}(\overline{\alpha}\theta(z))}{|\alpha-\theta(z)|^2} = Q[\nu_\alpha](z) + c.$$

Setting $z = 0$ we get $c = 2\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2}$, and thus

$$\begin{aligned}
\frac{\alpha+\theta(z)}{\alpha-\theta(z)} &= \frac{1-|\theta(z)|^2}{|\alpha-\theta(z)|^2} + 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(z))}{|\alpha-\theta(z)|^2} = P[\nu_\alpha](z) + iQ[\nu_\alpha](z) + 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2} \\
&= \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\nu_\alpha(\zeta) + 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2},
\end{aligned}$$

which proves (3.4). The formula (3.5) for the Cauchy transform now follows from this if we recall the relation $\frac{1}{1-\bar{\zeta}z} = \frac{1}{2}\left(\frac{\zeta+z}{\zeta-z}+1\right)$for $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$. For then we have

$$\begin{aligned}
K[\nu_\alpha](z) &= \int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z} d\nu_\alpha(\zeta) = \int_{\mathbb{T}} \frac{1}{2}\Big(\frac{\zeta+z}{\zeta-z}+1\Big)\nu_\alpha(\zeta) \\
&= \frac{1}{2}\left(\frac{\alpha+\theta(z)}{\alpha-\theta(z)} - 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2} + \|\nu_\alpha\|\right) \\
&= \frac{1}{2}\left(\frac{2}{1-\overline{\alpha}\theta(z)} - 1 - 2i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2} + \|\nu_\alpha\|\right) \\
&= \frac{1}{1-\overline{\alpha}\theta(z)} + \frac{\|\nu_\alpha\|-1}{2} - i\frac{\operatorname{Im}(\overline{\alpha}\theta(0))}{|\alpha-\theta(0)|^2}.
\end{aligned}$$

□

Corollary 3.1.3. *If $\theta(0) = 0$, the Clark measures are probability measures and we have*

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu_\alpha(\zeta) = \frac{\alpha + \theta(z)}{\alpha - \theta(z)}, \qquad \textit{and}$$
$$K[\nu_\alpha](z) = \frac{1}{1 - \overline{\alpha}\theta(z)}.$$

Proof. Setting $z = 0$ in (3.3), we have $\|\nu_\alpha\| = P[\nu_\alpha](0) = \mathrm{Re}\,\frac{\alpha+\theta(0)}{\alpha-\theta(0)} = 1$, so ν_α is a probability measure, and the above formulas follow immediately from Proposition 3.1.2.

□

There is an interesting connection between angular derivatives of an inner function and the point masses of the associated singular measures.

Definiton 3.1.4. Let $\theta : \mathbb{D} \to \mathbb{D}$ be analytic. We say that θ has an *angular derivative* at a point $\xi \in \mathbb{T}$, if for some $\alpha \in \mathbb{T}$ the limit

$$\lim_{\substack{z \to \xi \\ \sphericalangle}} \frac{\theta(z) - \alpha}{z - \xi}$$

exists and is finite. Whenever this is the case, we call the limit the angular derivative of θ at ξ and denote it by $\theta'(\xi)$.
Clearly, if θ has an angular derivative at ξ, we have $\theta(\xi) = \alpha$.

Lemma 3.1.5. *Let θ be a nonconstant inner function and let $\xi \in \mathbb{T}$ be such that the angular derivative $\theta'(\xi)$ exists. Then $\theta'(\xi) \neq 0$.*

Proof. We set $\alpha = \theta(\xi)$ and consider $z = r\xi$ lying on the radius terminating at ξ, where we have the estimate

$$\left| \frac{\theta(r\xi) - \alpha}{\xi(1 - r)} \right| \geq \frac{1 - |\theta(r\xi)|}{1 - r} = \frac{1 - |\theta(z)|}{1 - |z|}. \tag{3.6}$$

We now show that the expression on the right is bounded away from zero in $\mathbb{D}$. From the Schwarz lemma ([21, Theorem 12.2]) we first derive that an inner function satisfies

$$\left| \frac{\theta(z) - \theta(0)}{1 - \overline{\theta(0)}\theta(z)} \right| \leq |z|, \quad z \in \mathbb{D}. \tag{3.7}$$

Indeed, if $\theta(0) = 0$, this is precisely the statement of the Schwarz lemma. If $\theta(0) = a$, note that the function

$$b_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad z \in \mathbb{D}, \tag{3.8}$$

is an automorphism on $\mathbb{D}$ mapping a to zero (see [21, Theorem 12.4]). Hence, the composition function $b_a \circ \theta(z) = \frac{\theta(z)-a}{1-\bar{a}\theta(z)}$ is an inner function that fixes 0 and we can apply the lemma for $b_a \circ \theta$ to obtain (3.7).
Next, we note that for any two complex numbers p and q the relation $\left(1-|p|^2\right)\left(1-|q|^2\right) = |1 - \bar{p}q|^2 - |p - q|^2$ holds. This yields

$$\frac{\left(1 - |p|^2\right)\left(1 - |q|^2\right)}{|1 - \bar{p}q|^2} = 1 - \frac{|p - q|^2}{|1 - \bar{p}q|^2}.$$

We use this equation with $p = \theta(0)$ and $q = \theta(z)$, and apply the reverse triangle inequality on the left as well as the estimate (3.7) on the right side. Thus we get

$$\frac{\left(1 - |\theta(0)|^2\right)\left(1 - |\theta(z)|^2\right)}{\left(1 - \left|\overline{\theta(0)}\theta(z)\right|\right)^2} \geq \frac{\left(1 - |\theta(0)|^2\right)\left(1 - |\theta(z)|^2\right)}{\left|1 - \overline{\theta(0)}\theta(z)\right|^2} = 1 - \left|\frac{\theta(z) - \theta(0)}{1 - \overline{\theta(0)}\theta(z)}\right|^2 \geq 1 - |z|^2.$$

Rewriting this relation and making use of the fact that $|\theta(z)| \leq 1$, we arrive at the estimate

$$\frac{1 - |\theta(z)|^2}{1 - |z|^2} \geq \frac{(1 - |\theta(z)\theta(0)|)^2}{1 - |\theta(0)|^2} \geq \frac{1 - |\theta(0)|}{1 + |\theta(0)|} > 0.$$

It remains to notice that $\frac{1 - |\theta(z)|^2}{1 - |z|^2} = \frac{(1 - |\theta(z)|)(1 + |\theta(z)|)}{(1 - |z|)(1 + |z|)} \leq \frac{2(1 - |\theta(z)|)}{1 - |z|}$ to see that

$$\frac{1 - |\theta(z)|}{1 - |z|} \geq \frac{1 - |\theta(0)|}{2(1 + |\theta(0)|)} =: c > 0$$

for all $z \in \mathbb{D}$. From (3.6) we now conclude that $\left|\frac{\theta(r\xi) - \alpha}{\xi - r\xi}\right| \geq c$ for $r \in (0, 1)$ and hence $|\theta'(\xi)| = \lim_{r \to 1} \left|\frac{\theta(r\xi) - \alpha}{\xi - r\xi}\right| \geq c$.

□

Theorem 3.1.6. *Let $\theta \in H^\infty$ be a nonconstant inner function. Further, let $\alpha \in \mathbb{T}$ and let ν_α be the corresponding Clark measure, and let $\xi \in \mathbb{T}$ be given. Then ν_α has a point mass at ξ if and only if*

$$\lim_{z \underset{\sphericalangle}{\to} \xi} \theta(z) = \alpha \quad \textit{and} \quad \left|\theta'(\xi)\right| < \infty.$$

Furthermore, in this case, $\nu_\alpha(\{\xi\}) = \frac{1}{|\theta'(\xi)|}$.

Proof. We first show that for $\xi \in \mathbb{T}$ the relation

$$\lim_{z \underset{\sphericalangle}{\to} \xi} (\alpha + \theta(z)) \frac{\xi - z}{\alpha - \theta(z)} = 2\xi\nu_\alpha(\{\xi\}). \tag{3.9}$$

holds. With this it will be easy to prove the statement of the theorem. We start from the formula

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu_\alpha(\zeta) = \frac{\alpha + \theta(z)}{\alpha - \theta(z)} - 2i \frac{\text{Im}\,(\overline{\alpha}\theta(0))}{|\alpha - \theta(0)|^2}$$

we have from Proposition 3.1.2. Multiplying this equation by $(\xi - z)$ we obtain

$$\int_{\mathbb{T}} (\zeta + z) \frac{\xi - z}{\zeta - z} d\nu_\alpha(\zeta) = (\alpha + \theta(z)) \frac{\xi - z}{\alpha - \theta(z)} - (\xi - z)\, 2i \frac{\text{Im}\,(\overline{\alpha}\theta(0))}{|\alpha - \theta(0)|^2}. \tag{3.10}$$

If we let z nontangentially tend to ξ in this relation, the rightmost term tends to zero. The integrand on the left side of (3.10) satisfies

$$(\zeta + z) \frac{\xi - z}{\zeta - z} \underset{z \underset{\sphericalangle}{\to} \xi}{\longrightarrow} \begin{cases} 2\xi, & \text{if } \zeta = \xi, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for $z \in \Delta_\xi^K$, we have the estimate $|\zeta + z| \left|\frac{\xi - z}{\zeta - z}\right| \leq 2\frac{|\xi - z|}{1-|z|} \leq 2(K+1)$. Thus, by applying the Lebesgue dominated convergence theorem we obtain

$$\lim_{z \underset{\sphericalangle}{\rightarrow} \xi} \int\limits_{\mathbb{T}} (\zeta + z) \frac{\xi - z}{\zeta - z} d\nu_\alpha(\zeta) = 2\xi\nu_\alpha(\{\xi\}),$$

hence passing to the limit in (3.10) we arrive at (3.9).

Now we come to the proof of the theorem. First we assume $\nu_\alpha(\{\xi\}) > 0$, then equation (3.9) yields that $\theta(z) \rightarrow \alpha$ as $z \underset{\sphericalangle}{\rightarrow} \xi$ and

$$\theta'(\xi) = \lim_{z \underset{\sphericalangle}{\rightarrow} \xi} \frac{\theta(z) - \alpha}{\xi - z} = \frac{2\alpha}{2\xi\nu_\alpha(\{\xi\})} = \frac{\alpha\bar{\xi}}{\nu_\alpha(\{\xi\})} < \infty.$$

Conversely, if we assume $|\theta'(\xi)|$ to be finite, then by Lemma 3.1.5, the angular derivative must be nonzero. Together with the assumption that $\lim_{z \underset{\sphericalangle}{\rightarrow} \xi} \theta(z) = \alpha$, we conclude from (3.9) that

$$2\xi\nu_\alpha(\{\xi\}) = 2\alpha\frac{1}{\theta'(\xi)} \neq 0.$$

Since ν_α is a positive measure and α and ξ lie on $\mathbb{T}$, we infer $\nu_\alpha(\{\xi\}) = \frac{1}{|\theta'(\xi)|}$.

□

The formulas for ν_α we established in the previous statements take a much simpler shape if $\theta(0) = 0$. Indeed, we are able to confine ourselves to this case in the following considerations: Suppose that the inner function θ satisfies $\theta(0) = a$ for some $a \in \mathbb{D}\backslash\{0\}$. We consider the composition function

$$\Psi(z) := b_a \circ \theta(z) = \frac{\theta(z) - a}{1 - \bar{a}\theta(z)},$$

where b_a is the function defined in (3.8). Then also Ψ is an inner function, and it maps 0 to 0. If we denote by $\{\tilde{\nu}_\alpha\}_{\alpha\in\mathbb{T}}$ the family of singular measures for Ψ, these measures are connected to the singular measures $\{\nu_\alpha\}_{\alpha\in\mathbb{T}}$ associated with θ in the following way: Let $\alpha \in \mathbb{T}$ and set $\beta = b_a(\alpha)$, then $\tilde{\nu}_\beta$ is carried on the set $\{\Psi = \beta\} = \{b_a \circ \theta = \beta\} = \{\theta = \alpha\}$, hence $\tilde{\nu}_\beta$ is carried on the same set as ν_α. Moreover, if $\xi \in \mathbb{T}$ is such that $\lim_{z \underset{\sphericalangle}{\rightarrow} \xi} \theta(z) = \alpha$ and the angular derivative exists, Theorem 3.1.6 gives us the relation

$$\tilde{\nu}_\beta(\{\xi\}) = \frac{1}{|\Psi'(\xi)|} = \frac{1}{|b_a'(\theta(\xi))\theta'(\xi)|} = \frac{1}{|b_a'(\alpha)|}\nu_\alpha(\{\xi\}).$$

Furthermore, we have a one-to-one correspondence between functions in $\theta^*(H^p)$ and those in $\Psi^*(H^p)$. The operator A mapping $f \in \theta^*(H^p)$ to $\frac{1}{1-\bar{a}\theta}f$ is a bijection from $\theta^*(H^p)$ to $\Psi^*(H^p)$: Since $\frac{1}{1-\bar{a}\theta} \in H^\infty$, the image function clearly lies in H^p. Furthermore, $f \in \theta^*(H^p)$ satisfies $\int\limits_{\mathbb{T}} f\bar{\theta}\bar{h}\, d\sigma = 0$ for all $h \in H^q$. Let $h \in H^q$, then we get

$$\begin{aligned} \int\limits_{\mathbb{T}} Af\bar{\Psi}\bar{h}\, d\sigma &= \int\limits_{\mathbb{T}} \frac{1}{1-\bar{a}\theta} f \frac{\bar{\theta} - \bar{a}}{1 - a\bar{\theta}} \bar{h}\, d\sigma = \int\limits_{\mathbb{T}} f \frac{1 - \bar{a}\theta}{|1 - \bar{a}\theta|^2} \bar{\theta}\bar{h}\, d\sigma \\ &= \int\limits_{\mathbb{T}} f\bar{\theta}\frac{1}{1 - a\bar{\theta}}\bar{h}\, d\sigma = 0, \end{aligned}$$

since $\frac{1}{1-\bar{a}\theta}h$ lies in H^q. Conversely, $g \in \Psi^*(H^p)$ satisfies $\int_{\mathbb{T}} g\bar{\Psi}\bar{h}\, d\sigma = 0$ for all $h \in H^q$, and thus

$$\int_{\mathbb{T}} g\left(1-\bar{a}\theta\right)\bar{\theta}\bar{h}\, d\sigma = \int_{\mathbb{T}} g\frac{\bar{\theta}-\bar{a}}{1-a\bar{\theta}}\left(1-a\bar{\theta}\right)\bar{h}\, d\sigma = \int_{\mathbb{T}} g\bar{\Psi}\left(1-a\bar{\theta}\right)\bar{h}\, d\sigma = 0$$

since $(1-\bar{a}\theta)\,h \in H^q$ for $h \in H^q$. Hence, $f := (1-\bar{a}\theta)\,g$ lies in $\theta^*(H^p)$ and satisfies $Af = g$, so A is surjective.

These correspondences allow us to assume that θ maps 0 to 0 in what follows.

3.2 Construction of the unitary operator U_α

Let a nonconstant inner function θ satisfying $\theta(0) = 0$ be given. With each Clark measure ν_α we can associate a unitary operator $U_\alpha : \theta^*(H^2) \to L^2(\nu_\alpha)$ which we construct in the following.

For $\lambda \in \mathbb{D}$ consider the functions k_λ defined on $\mathbb{D}$ by

$$k_\lambda(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar{\lambda}z}. \tag{3.11}$$

As for $z \in \mathbb{D}$, $\left|k_\lambda(z)\right| \le \frac{1+|\theta(\lambda)|}{1-|\lambda|}$, these are bounded analytic functions on $\mathbb{D}$, moreover, they lie in the space $\theta^*(H^2)$. Indeed, if $g \in H^2$,

$$\left\langle \theta g, k_\lambda\right\rangle_{H^2} = \int_{\mathbb{T}} \theta(\zeta)\, g(\zeta)\frac{1-\theta(\lambda)\overline{\theta(\zeta)}}{1-\lambda\bar{\zeta}}d\sigma(\zeta) = \int_{\mathbb{T}} \frac{\theta(\zeta)\, g(\zeta)-\theta(\lambda)\, g(\zeta)}{1-\lambda\bar{\zeta}}d\sigma(\zeta) = 0$$

by the Cauchy formula (1.4), and hence $k_\lambda \perp \theta H^2$. Furthermore, for $f \in \theta^*(H^2)$, we have

$$\left\langle f, k_\lambda\right\rangle_{H^2} = \int_{\mathbb{T}} f(\zeta)\frac{1-\theta(\lambda)\overline{\theta(\zeta)}}{1-\lambda\bar{\zeta}}d\sigma(\zeta) = f(\lambda) - \int_{\mathbb{T}} \frac{\theta(\lambda)\, f(\zeta)\overline{\theta(\zeta)}}{1-\lambda\bar{\zeta}}d\sigma(\zeta) = f(\lambda),$$

as the last integral is zero because f lies in $\theta^*(H^2)$. Thus, $\langle ., k_\lambda\rangle_{H^2}$ is the point evaluation functional on $\theta^*(H^2)$, and the family $\{k_\lambda\}_{\lambda\in\mathbb{D}}$, has dense span in $\theta^*(H^2)$. We denote the span of $\{k_\lambda\}_{\lambda\in\mathbb{D}}$ by $\mathcal{K}$.

We define an operator U_α from $\mathcal{K}$ into the space $L^\infty(\nu_\alpha)$ by linearity and

$$(U_\alpha k_\lambda)(\xi) = \frac{1-\overline{\theta(\lambda)}\alpha}{1-\bar{\lambda}\xi}, \quad \lambda \in \mathbb{D}. \tag{3.12}$$

We now show that this operator can be extended to a unitary operator from $\theta^*(H^2)$ onto $L^2(\nu_\alpha)$. First we note that the image of U_α contains the Cauchy kernels $C_\lambda(\zeta) = \frac{1}{1-\lambda\bar{\zeta}}$. We denote the span of this family of kernels by $\mathcal{C}$. This space has the following useful property.

Lemma 3.2.1. *The family of Cauchy kernels $\{C_\lambda\}_{\lambda\in\mathbb{D}}$ has dense span in $L^2(\nu_\alpha)$.*

Proof. Suppose that there is a function $g \in L^2(\nu_\alpha)$ such that

$$\left\langle g, C_\lambda\right\rangle_{L^2(\nu_\alpha)} = \int_{\mathbb{T}} \frac{g(\zeta)}{1-\lambda\bar{\zeta}}d\nu_\alpha(\zeta) = K[g\nu_\alpha](\lambda) = 0$$

for all $\lambda \in \mathbb{D}$, i.e., the Cauchy transform of $g\nu_\alpha$ vanishes on all of $\mathbb{D}$. Since for any measure μ we have

$$K[\mu](z) = \int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z} d\mu(\zeta) = \int_{\mathbb{T}} \sum_{n\geq 0} (z\bar{\zeta})^n d\mu(\zeta) = \sum_{n\geq 0} z^n \int_{\mathbb{T}} \bar{\zeta}^n d\mu(\zeta),$$

the vanishing of the Cauchy transform implies that for each $n \geq 0$

$$\int_{\mathbb{T}} \bar{\zeta}^n g(\zeta)\, d\nu_\alpha(\zeta) = 0$$

and using the theorem of F. and M. Riesz ([12, Theorem 4.27]) we conclude that $g\nu_\alpha = \bar{\phi}\sigma$ for some $\phi \in H_0^1$. But since ν_α is singular, ϕ must be the null function and $g\nu_\alpha$ the zero measure, and we conclude that $g = 0$ almost everywhere with respect to ν_α and the C_λ have dense span in $L^2(\nu_\alpha)$.

□

Theorem 3.2.2. *The operator U_α defined by linearity and*

$$(U_\alpha k_\lambda)(\xi) = \frac{1-\overline{\theta(\lambda)}\alpha}{1-\bar{\lambda}\xi}, \quad \lambda \in \mathbb{D},$$

can be extended to a unitary operator from $\theta^*(H^2)$ onto $L^2(\nu_\alpha)$. The adjoint operator $U_\alpha^* : L^2(\nu_\alpha) \to \theta^*(H^2)$ acts as

$$(U_\alpha^* f)(z) = \frac{K[f\nu_\alpha](z)}{K[\nu_\alpha](z)}, \quad f \in L^2(\nu_\alpha). \tag{3.13}$$

Proof. We first show that U_α is an isometry from $\mathcal{K}$ to $\mathcal{C}$. To this end, we note that for any two complex numbers z and w we have the relation

$$\frac{1+z}{1-z} + \frac{1+w}{1-w} = 2\frac{1-zw}{(1-z)(1-w)}. \tag{3.14}$$

Using this, we calculate for $\lambda, \eta \in \mathbb{D}$,

$$\begin{aligned} \langle C_\lambda, C_\eta \rangle_{L^2(\nu_\alpha)} &= \int_{\mathbb{T}} \frac{1}{1-\bar{\lambda}\zeta} \frac{1}{1-\eta\bar{\zeta}} d\nu_\alpha(\zeta) \\ &= \frac{1}{2(1-\bar{\lambda}\eta)} \int_{\mathbb{T}} \left(\frac{1+\bar{\lambda}\zeta}{1-\bar{\lambda}\zeta} + \frac{1+\eta\bar{\zeta}}{1-\eta\bar{\zeta}} \right) d\nu_\alpha(\zeta). \end{aligned}$$

Applying Corollary 3.1.3 and then once again relation (3.14), the last integral becomes

$$\int_{\mathbb{T}} \left(\frac{1+\bar{\lambda}\zeta}{1-\bar{\lambda}\zeta} + \frac{1+\eta\bar{\zeta}}{1-\eta\bar{\zeta}} \right) d\nu_\alpha(\zeta) = \frac{1+\alpha\overline{\theta(\lambda)}}{1-\alpha\overline{\theta(\lambda)}} + \frac{1+\overline{\alpha}\theta(\eta)}{1-\overline{\alpha}\theta(\eta)} = 2\frac{1-\overline{\theta(\lambda)}\theta(\eta)}{\left(1-\alpha\overline{\theta(\lambda)}\right)\left(1-\overline{\alpha}\theta(\eta)\right)},$$

and inserting this above, we get

$$\begin{aligned} \langle C_\lambda, C_\eta \rangle_{L^2(\nu_\alpha)} &= \frac{1}{1-\bar{\lambda}\eta} \frac{1-\overline{\theta(\lambda)}\theta(\eta)}{\left(1-\alpha\overline{\theta(\lambda)}\right)\left(1-\overline{\alpha}\theta(\eta)\right)} = \frac{1}{\left(1-\alpha\overline{\theta(\lambda)}\right)\left(1-\overline{\alpha}\theta(\eta)\right)} k_\lambda(\eta) \\ &= \frac{1}{\left(1-\alpha\overline{\theta(\lambda)}\right)\left(1-\overline{\alpha}\theta(\eta)\right)} \langle k_\lambda, k_\eta \rangle_{H^2} = \left\langle \frac{k_\lambda}{1-\alpha\overline{\theta(\lambda)}}, \frac{k_\eta}{1-\alpha\overline{\theta(\eta)}} \right\rangle_{H^2}. \end{aligned}$$

From this relation we now derive the isometry of U_α. Let $g = \sum\limits_{l=1}^{m} c_l k_{\lambda_l}$ be an element of $\mathcal{K}$, then

$$\begin{aligned}
\|U_\alpha g\|^2_{L^2(\nu_\alpha)} &= \Big\| \sum_{l=1}^{m} c_l \big(1-\alpha\overline{\theta\,(\lambda_l)}\big) C_{\lambda_l} \Big\|^2_{L^2(\nu_\alpha)} \\
&= \sum_{j,l=1}^{m} c_j \bar{c}_l \big(1-\alpha\overline{\theta\,(\lambda_j)}\big)\big(1-\overline{\alpha}\theta\,(\lambda_l)\,\big)\langle C_{\lambda_j}, C_{\lambda_l}\rangle_{L^2(\nu_\alpha)} \\
&= \sum_{j,l=1}^{m} c_j \bar{c}_l \big(1-\alpha\overline{\theta\,(\lambda_j)}\big)\big(1-\overline{\alpha}\theta\,(\lambda_l)\,\big)\left\langle \frac{k_{\lambda_j}}{1-\alpha\overline{\theta\,(\lambda_j)}}, \frac{k_{\lambda_l}}{1-\alpha\overline{\theta\,(\lambda_l)}} \right\rangle_{H^2} \\
&= \|g\|^2_{H^2}.
\end{aligned}$$

We see that U_α is isometric from the span of $\{k_\lambda\}$ to that of $\{C_\lambda\}$. As $\mathcal{K}$ is dense in the space $\theta^*(H^2)$, the operator can be extended to an isometry on $\theta^*(H^2)$. Furthermore, since the functions $C_\lambda,\ \lambda \in \mathbb{D}$, lie in the image of U_α, the image has dense span in $L^2\,(\nu_\alpha)$ and we conclude that U_α is unitary from $\theta^*(H^2)$ to $L^2\,(\nu_\alpha)$.
It remains to verify the stated formula for the adjoint of U_α. Let $f \in L^2\,(\nu_\alpha)$ and $\lambda \in \mathbb{D}$, then

$$\begin{aligned}
(U_\alpha^* f)\,(\lambda) &= \langle U_\alpha^* f, k_\lambda\rangle_{H^2} = \langle f, U_\alpha k_\lambda\rangle_{L^2(\nu_\alpha)} = \int\limits_{\mathbb{T}} f\,(\zeta)\,\big(1-\overline{\alpha}\theta\,(\lambda)\,\big)\frac{1}{1-\lambda\bar{\zeta}} d\nu_\alpha\,(\zeta) \\
&= \big(1-\overline{\alpha}\theta\,(\lambda)\,\big) K\,[f\nu_\alpha]\,(\lambda) = \frac{K\,[f\nu_\alpha]\,(\lambda)}{K\,[\nu_\alpha]\,(\lambda)},
\end{aligned}$$

using that $K\,[\nu_\alpha] = (1-\overline{\alpha}\theta)^{-1}$ by Corollary 3.1.3.

□

Theorem 3.2.3. *Let $f \in \theta^*(H^2)$. Then $U_\alpha f$ is the $L^2\,(\nu_\alpha)$-limit of the functions*

$$h_r\,(\xi) = \left\langle f\,(\zeta)\,, \frac{1-\theta\,(r\zeta)\,\overline{\alpha}}{1-r\zeta\bar{\xi}} \right\rangle_{L^2(\sigma)}$$

as r tends to 1.

Proof. For $f \in \theta^*(H^2)$, we have the formula

$$\begin{aligned}
f\,(\lambda) &= \langle f, k_\lambda\rangle_{L^2(\sigma)} = \langle U_\alpha f, U_\alpha k_\lambda\rangle_{L^2(\nu_\alpha)} \\
&= \int\limits_{\mathbb{T}} U_\alpha f\,(\zeta)\,\frac{1-\theta\,(\lambda)\,\overline{\alpha}}{1-\lambda\bar{\zeta}} d\nu_\alpha\,(\zeta).
\end{aligned}$$

We denote $F\,(\zeta) = U_\alpha f\,(\zeta) \in L^2\,(\nu_\alpha)$. Note that since U_α is a bijection, every function in $L^2\,(\nu_\alpha)$ can be written as $U_\alpha f$ for some $f \in \theta^*(H^2)$. For every $r \in (0,1)$ we define an operator $T_r : L^2\,(\nu_\alpha) \to H^2$ by

$$(T_r F)\,(z) = f\,(rz) = \int\limits_{\mathbb{T}} F\,(\zeta)\,\frac{1-\theta\,(rz)\,\overline{\alpha}}{1-rz\bar{\zeta}} d\nu_\alpha\,(\zeta).$$

We observe that since $\|T_r F\|_{H^2} = \|f(r.)\|_{H^2} \leq \|f\|_{H^2} = \|F\|_{L^2(\nu_\alpha)}$ by the isometry of U_α, the operators T_r are uniformly bounded with $\|T_r\| \leq 1$. We now determine the adjoint T_r^*. To this end, let $F \in L^2(\nu_\alpha)$, $g \in H^2$, then

$$\begin{aligned}\langle F, T_r^* g\rangle_{L^2(\nu_\alpha)} = \langle T_r F, g\rangle_{H^2} &= \int_{\mathbb{T}}\int_{\mathbb{T}} F(\zeta)\frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\zeta}}d\nu_\alpha(\zeta)\overline{g(z)}d\sigma(z)\\ &= \int_{\mathbb{T}} F(\zeta)\int_{\mathbb{T}}\frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\zeta}}\overline{g(z)}d\sigma(z)\,d\nu_\alpha(\zeta)\\ &= \left\langle F, \overline{\int_{\mathbb{T}}\frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\zeta}}\overline{g(z)}d\sigma(z)}\right\rangle_{L^2(\nu_\alpha)}.\end{aligned}$$

Here we used Fubini's theorem ([23, Chapter III, Theorem 14.1]) to change the order of integration. This is valid since $\frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\zeta}}$ is a bounded analytic function of $z \in \mathbb{D}$ as well as bounded as a function of $\zeta \in \mathbb{T}$, and as the measures are finite, F and g are integrable with respect to ν_α and σ, respectively, therefore the function $F(\zeta)\frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\zeta}}\overline{g(z)}$ is integrable with respect to the product measure and the resulting integrals after changing the integration order exist. The above relation shows that $T_r^* g(\xi) = \left\langle g(z), \frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\xi}}\right\rangle_{H^2}$.
As for $F \in L^2(\nu_\alpha)$ we have $U_\alpha^* F = f \in H^2$, we see that

$$\lim_{r\to 1}\|T_r F - U_\alpha^* F\|_{H^2} = \lim_{r\to 1}\|f(r.) - f\|_{H^2} = 0,$$

hence the operators T_r converge to U_α^* in the strong operator topology. Consequently, the operators $T_r^*\big|_{\theta^*(H^2)}$ converge to U_α in the weak operator topology, since for any $f \in L^2(\nu_\alpha)$ and $g \in \theta^*(H^2)$ we have

$$\langle f, T_r^* g\rangle_{L^2(\nu_\alpha)} = \langle T_r f, g\rangle_{H^2} \longrightarrow \langle U_\alpha^* f, g\rangle_{H^2} = \langle f, U_\alpha g\rangle_{L^2(\nu_\alpha)}.$$

We now show that the operators T_r^* converge in the strong operator topology in H^2 as $r \to 1$. Since the family $\{T_r^*\}_{r\in(0,1)}$ is uniformly bounded with $\|T_r^*\| = \|T_r\| \leq 1$, it suffices to show that the functions $T_r^* g$ converge in the norm of $L^2(\nu_\alpha)$ for all g lying in a complete subset of H^2, which then implies strong convergence of the operators T_r^* on all of H^2.
So we consider the complete subset of reproducing kernels $C_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$, $\lambda \in \mathbb{D}$, of H^2. Then

$$T_r^* C_\lambda(\xi) = \left\langle C_\lambda(z), \frac{1-\theta(rz)\overline{\alpha}}{1-rz\bar{\xi}}\right\rangle_{H^2} = \overline{\left(\frac{1-\theta(r\lambda)\overline{\alpha}}{1-r\lambda\bar{\xi}}\right)},$$

and since $\lambda \in \mathbb{D}$, this function converges to $\frac{1-\overline{\theta(\lambda)}\alpha}{1-\bar{\lambda}\xi}$ uniformly in ξ and hence in the norm of $L^2(\nu_\alpha)$. So the operators T_r^* converge in the strong operator topology. Since strong convergence implies weak convergence and the limit is unique, we conclude that the operators $T_r^*\big|_{\theta^*(H^2)}$ converge strongly to U_α, and this yields the statement of the theorem.

□

Remark 3.2.4. The formula (3.13) we obtained for the adjoint operator is precisely the normalized Cauchy transform V_{ν_α} for the measure ν_α. We now establish a result on the mapping properties of the operators V_μ which in turn yields statements about the operators U_α. The following results are due to A.B. Aleksandrov [6].

Theorem 3.2.5. *Let $\mu \in M_+(\mathbb{T})$. The operator V_μ is of weak type (1,1), i.e., there is a constant $A > 0$ such that for every $\lambda > 0$ and every $f \in L^1(\mu)$ we have*

$$\sigma\left(\left\{ |V_\mu f| > \lambda\right\}\right) \leq \frac{A\|f\|_{L^1(\mu)}}{\lambda}. \tag{3.15}$$

Furthermore, for $p \in (1,2]$ the operator is continuous from $L^p(\mu)$ to $L^p(\sigma)$.

Proof. As μ is a positive measure, its Cauchy transform satisfies $|K[\mu]| \geq \frac{\|\mu\|}{2}$. Thus, we obtain the inclusion

$$\left\{ |V_\mu| > \lambda\right\} = \left\{\left|K[f\mu]\right| > \lambda\left|K[\mu]\right|\right\} \subseteq \left\{\left|K[f\mu]\right| > \lambda\frac{\|\mu\|}{2}\right\}.$$

The Cauchy transform of a measure satisfies a weak-L^1 estimate by Kolmogorov's theorem (Theorem 1.4.2), so above inclusion yields the following weak estimate for V_μ:

$$\sigma\left(\left\{ |V_\mu| > \lambda\right\}\right) \leq \sigma\left(\left\{\left|K[f\mu]\right| > \lambda\frac{\|\mu\|}{2}\right\}\right) \leq \frac{2\tilde{A}\|f\mu\|}{\lambda\|\mu\|} \leq \frac{2\tilde{A}\|f\|_{L^1(\mu)}\|\mu\|}{\lambda\|\mu\|} = \frac{2\tilde{A}\|f\|_{L^1(\mu)}}{\lambda}.$$

Now we prove the continuity of $V_\mu : L^2(\mu) \to L^2(\sigma)$. The statement for $p \in (1,2)$ then follows immediately from the Marcinkiewicz interpolation theorem (Theorem 1.4.6): V_μ is a linear operator, and so the weak estimate for $p = 1$ and continuity for $p = 2$ together imply continuity for all $p \in (1,2)$.
Without loss of generality we may assume that μ is a probability measure. Let us first consider the case that μ is singular. Then $\mu = \nu_1$ for some inner function θ (see Remark 3.1.1) that satisfies $\theta(0) = 0$. Theorem 3.2.2 then implies that $V_\mu = U_1^*$ is a unitary operator from $L^2(\mu)$ to $\theta^*(H^2)$ and thus $\|V_\mu f\|_{L^2(\sigma)} = \|f\|_{L^2(\mu)}$.
If μ is an arbitrary probability measure, there exists a sequence of singular probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ converging to μ in the weak* topology (see [18, Theorem 12.11]). For these measures we have $\|V_{\mu_n} f\|_{L^2(\sigma)} = \|f\|_{L^2(\mu_n)}$ on $L^2(\mu_n)$ from the first part of the proof. Furthermore, the weak* convergence yields pointwise convergence of the Cauchy transforms in $\mathbb{D}$, $K[\mu_n](z) \to K[\mu](z)$ as $n \to \infty$, as well as for $f \in C(\mathbb{T})$,

$$K[f\mu_n](z) \to K[f\mu](z)$$

pointwise as $n \to \infty$. Hence we also have $\lim\limits_{n\to\infty} V_{\mu_n} f(z) = V_\mu f(z)$ pointwise for all continuous functions. Now we can use Fatou's lemma ([21, Lemma 1.28]) to obtain an estimate for $\|V_\mu f\|_{L^2(\sigma)}$. Recall here that $V_{\mu_n} f \in H^2$ and the operators V_{μ_n} are isometries. Thus, we get for $0 < r < 1$ and $f \in C(\mathbb{T})$

$$\begin{aligned}
\int_{\mathbb{T}} |V_\mu f(r\zeta)|^2 \, d\sigma(\zeta) &= \int_{\mathbb{T}} \lim_{n\to\infty} |V_{\mu_n} f(r\zeta)|^2 \, d\sigma(\zeta) \leq \liminf_{n\to\infty} \int_{\mathbb{T}} |V_{\mu_n} f(r\zeta)|^2 \, d\sigma(\zeta) \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{T}} |V_{\mu_n} f(\zeta)|^2 \, d\sigma(\zeta) = \liminf_{n\to\infty} \int_{\mathbb{T}} |f(\zeta)|^2 \, d\mu_n(\zeta) \\
&= \int_{\mathbb{T}} |f(\zeta)|^2 \, d\mu(\zeta) = \|f\|^2_{L^2(\mu)}.
\end{aligned}$$

We now choose a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive numbers monotonically increasing to 1 and apply Fatou's lemma once more to obtain

$$\|V_\mu f\|^2_{L^2(\sigma)} = \int_{\mathbb{T}} \lim_{n\to\infty} |V_\mu f(r_n\zeta)|^2 \, d\sigma(\zeta) \leq \liminf_{n\to\infty} \int_{\mathbb{T}} |V_\mu f(r_n\zeta)|^2 \, d\sigma(\zeta) \leq \|f\|^2_{L^2(\mu)}$$

for all $f \in C(\mathbb{T})$. As the continuous functions are dense in $L^2(\mu)$, we come to the desired conclusion.

$\square$

Theorem 3.2.5 yields the following statements about the operators U_α.

Corollary 3.2.6. *Let $1 < p \leq 2$ and let $\theta \in H^\infty$ be an inner function satisfying $\theta(0) = 0$. Then $U_\alpha^{-1}(L^p(\nu_\alpha)) \subseteq \theta^*(H^p)$.*

Proof. By Theorem 3.2.5, $U_\alpha^{-1} = V_{\nu_\alpha}$, which is originally defined on $L^2(\nu_\alpha)$, can be extended to a continuous operator from $L^p(\nu_\alpha)$ to $L^p(\sigma)$. We have $U_\alpha^{-1}\left(L^2(\nu_\alpha)\right) = \theta^*(H^2) \subseteq \theta^*(H^p)$, and the density of $L^2(\nu_\alpha)$ in $L^p(\nu_\alpha)$ now yields the stated inclusion since the space $\theta^*(H^p)$ is closed.

$\square$

Corollary 3.2.7. *Let $2 < p \leq \infty$ and $\theta \in H^\infty$ an inner function satisfying $\theta(0) = 0$. Then $U_\alpha(\theta^*(H^p)) \subseteq L^p(\nu_\alpha)$.*

Proof. In this case we use duality. Let q be the conjugate index of p, then $q \in (1, 2]$. Hence we have $U_\alpha^{-1}(L^q(\nu_\alpha)) \subseteq \theta^*(H^q)$ from the previous corollary. Now note that the conjugate operator of the extension of U_α^{-1} to $L^q(\nu_\alpha)$ is the restriction of U_α to $\theta^*(H^p)$. Since for conjugate operators we have the relation $UA \subseteq B \Leftrightarrow U^* B^\perp \subseteq A^\perp$, we conclude $U_\alpha(\theta^*(H^p)) \subseteq L^p(\nu_\alpha)$.

$\square$

3.3 Boundary behavior of functions in $\theta^*(H^p)$

Now we examine the elements of the spaces $\theta^*(H^p)$, $p \geq 2$, and their nontangential boundary functions. We always assume that θ is an inner function with $\theta(0) = 0$. We eventually show that the nontangential boundary function of $f \in \theta^*(H^p)$ equals $U_\alpha f$ almost everywhere with respect to ν_α.
First we introduce the Hardy-Littlewood maximal function. Recall that the support of a measure $\mu \in M(\mathbb{T})$ is defined as $\operatorname{supp}\mu := \mathbb{T} \setminus \bigcup \left\{ I \subseteq \mathbb{T} \mid I \text{ is open and } \mu(I) = 0 \right\}$.

Definition 3.3.1. Let $\mu \in M_+(\mathbb{T})$ and $f \in L^1(\mu)$. By f_μ^M we denote the *Hardy-Littlewood maximal function* of f,

$$f_\mu^M(\xi) := \begin{cases} \sup\limits_{\varepsilon > 0} \frac{1}{\mu(K_\varepsilon(\xi))} \int\limits_{K_\varepsilon(\xi)} |f(\zeta)| \, d\mu(\zeta), & \text{if } \xi \in \operatorname{supp}\mu, \\ 0, & \text{otherwise.} \end{cases} \tag{3.16}$$

with $K_\varepsilon(\xi) = \mathbb{T} \cap B_\varepsilon(\xi)$ and $B_\varepsilon(\xi)$ being the closed ball of radius ε centered at ξ.

Our next goal is to prove the Hardy-Littlewood maximal theorem that states integrability properties of this function. First, we have to show that f_μ^M is measurable, for which we need the following auxiliary result.

Lemma 3.3.2. *Let $\mu \in M_+(\mathbb{T})$ and $r > 0$. The function*

$$\psi : \begin{cases} \mathbb{T} \to \mathbb{R}, \\ \xi \mapsto \mu(K_r(\xi)) \end{cases}$$

is upper semicontinuous, hence in particular Borel measurable.

Proof. The function ψ is upper semicontinuous if for every $t > 0$, the set $\{\psi < t\}$ is open. An equivalent condition is that for every $\xi \in \mathbb{T}$ and every sequence $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}\backslash\{\xi\}$ tending to ξ,

$$\limsup_{n\to\infty} \psi(\xi_n) \leq \psi(\xi).$$

Let $r > 0$, $\xi \in \mathbb{T}$ and a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ as above be given. As the $K_r(\xi)$ are closed arcs, we have

$$\limsup_{n\to\infty} \chi_{K_r(\xi_n)}(\zeta) \leq \chi_{K_r(\xi)}(\zeta), \quad \zeta \in \mathbb{T},$$

and with the relation $\liminf\limits_{n\to\infty}(-x_n) = -\limsup\limits_{n\to\infty} x_n$, this yields

$$1 - \chi_{K_r(\xi)} \leq \liminf_{n\to\infty} \left(1 - \chi_{K_r(\xi_n)}\right).$$

Note that for large enough indices $n \in \mathbb{N}$ we have $\xi_n \in K_r(\xi)$ and hence, by the triangle inequality, $K_r(\xi_n) \subseteq K_{2r}(\xi)$. We now use Fatou's lemma ([21, Lemma 1.28]) to estimate

$$\begin{aligned}
\mu\big(K_{2r}(\xi)\big) - \mu\big(K_r(\xi)\big) &= \int\limits_{K_{2r}(\xi)} \left(1 - \chi_{K_r(\xi)}\right) d\mu \leq \int\limits_{K_{2r}(\xi)} \liminf_{n\to\infty} \left(1 - \chi_{K_r(\xi_n)}\right) d\mu \\
&\leq \liminf_{n\to\infty} \int\limits_{K_{2r}(\xi)} \left(1 - \chi_{K_r(\xi_n)}\right) d\mu \\
&= \liminf_{n\to\infty} \big(\mu(K_{2r}(\xi)) - \mu(K_r(\xi_n))\big) \\
&= \mu\big(K_{2r}(\xi)\big) - \limsup_{n\to\infty} \mu\big(K_r(\xi_n)\big),
\end{aligned}$$

and we see that $\limsup\limits_{n\to\infty} \mu(K_r(\xi_n)) \leq \mu(K_r(\xi))$, as was to be shown.

□

Corollary 3.3.3. *Let $\mu, \nu \in M_+(\mathbb{T})$ and $r > 0$. Then for every $t \geq 0$ the set*

$$E_t := \left\{\xi \in \mathbb{T} \mid \mu(K_r(\xi)) > t\nu(K_r(\xi))\right\}$$

is Borel measurable.

Proof. The set $E_0 = \left\{\xi \in \mathbb{T} \mid \mu(K_r(\xi)) > 0\right\}$ is measurable by Lemma 3.3.2. If $t > 0$, we can write the set as

$$E_t = \bigcup_{\substack{q\in\mathbb{Q} \\ q>0}} \left(\left\{\xi \in \mathbb{T} \mid \mu(K_r(\xi)) \geq q\right\} \cap \left\{\xi \in \mathbb{T} \mid \nu(K_r(\xi)) < \frac{q}{t}\right\}\right),$$

and by Lemma 3.3.2, the sets on the right side are measurable.

□

Now we come to the Hardy-Littlewood maximal theorem.

Theorem 3.3.4. *Let $\mu \in M_+(\mathbb{T})$ and $f \in L^p(\mu)$, $1 \leq p \leq \infty$. Then the Hardy-Littlewood maximal function f_μ^M is finite μ-almost everywhere and*

(i) *if* $p = 1$, f_μ^M *belongs to the* weak L^1, *i.e., there is a constant* $c > 0$ *independent of* f *such that for every* $\lambda > 0$

$$\mu\left(\left\{f_\mu^M > \lambda\right\}\right) \leq \frac{c}{\lambda}\|f\|_{L^1(\mu)};$$

(ii) if $p > 1$, *then* f_μ^M *belongs to* $L^p(\mu)$ *and we have the estimate*

$$\|f_\mu^M\|_{L^p(\mu)} \leq c\|f\|_{L^p(\mu)},$$

where c is independent of f.

Proof. We first show that f_μ^M is Borel measurable. To this end, we consider the functions $f_{\mu,r}^M$, which are defined for $r > 0$ by

$$f_{\mu,r}^M(\xi) := \begin{cases} \frac{1}{\mu(K_r(\xi))} \int\limits_{K_r(\xi)} |f(\zeta)|\, d\mu(\zeta), & \text{if } \xi \in \operatorname{supp}\mu, \\ 0, & \text{otherwise.} \end{cases}$$

We show that these functions are Borel measurable and that

$$f_\mu^M(\xi) = \sup\left\{f_{\mu,q}^M(\xi) \mid q \in \mathbb{Q}, q > 0\right\} \tag{3.17}$$

which then implies that also f_μ^M is measurable. Thus, let $t \in \mathbb{R}$ and set

$$E_{t,r} := \left\{\xi \in \mathbb{T} \mid f_{\mu,r}^M(\xi) > t\right\}.$$

If $t < 0$, $E_{t,r} = \mathbb{T}$. If $t \geq 0$, we define the positive measure ν by $\nu(B) = \int\limits_B |f|\, d\mu$ for Borel subsets $B \subseteq \mathbb{T}$. Then $E_{t,r}$ can be written as

$$E_{t,r} = \operatorname{supp}\mu \cap \left\{\xi \in \mathbb{T} \mid \nu(K_r(\xi)) > t\mu(K_r(\xi))\right\},$$

which is measurable by Corollary 3.3.3. Hence $E_{t,r}$ is measurable and consequently the functions $f_{\mu,r}^M$ are Borel measurable.
Now we come to the proof of (3.17). From the definition of f_μ^M it follows that surely

$$f_\mu^M \geq \sup\left\{f_{\mu,q}^M \mid q \in \mathbb{Q}, q > 0\right\}$$

For the reverse inequality, let $r > 0$ and $\xi \in \operatorname{supp}\mu$ be given, and choose a decreasing sequence $\{r_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ tending to r as $n \to \infty$. Then $\chi_{K_{r_1}(\xi)} \geq \chi_{K_{r_2}(\xi)} \geq \cdots \geq \chi_{K_r(\xi)} \geq 0$, and since the $K_{r_n}(\xi)$ are closed arcs,

$$\lim_{n\to\infty} \chi_{K_{r_n}(\xi)}(\zeta) = \chi_{K_r(\xi)}(\zeta), \quad \zeta \in \mathbb{T}.$$

The bounded convergence theorem now gives us

$$\begin{aligned} \lim_{n\to\infty} \mu(K_{r_n}(\xi)) &= \mu(K_r(\xi)), \\ \lim_{n\to\infty} \int\limits_{K_{r_n}(\xi)} |f(\zeta)|\, d\mu(\zeta) &= \int\limits_{K_r(\xi)} |f(\zeta)|\, d\mu(\zeta). \end{aligned}$$

Since $\xi \in \operatorname{supp}\mu$, we have $\mu(K_r(\xi)) \neq 0$, so we obtain $\lim\limits_{n\to\infty} f_{\mu,r_n}^M(\xi) = f_{\mu,r}^M(\xi)$, and this implies

$$f_{\mu,r}^M \leq \sup\left\{f_{\mu,q}^M \mid q \in \mathbb{Q}, q > 0\right\}.$$

As $r>0$ was arbitrary, we deduce that also

$$f_\mu^M \le \sup\left\{ f_{\mu,q}^M \mid q\in\mathbb{Q}, q>0\right\}.$$

We now show the stated L^p estimates for the Hardy-Littlewood maximal function. Let $f\in L^1(\mu)$, then we have to show that f_μ^M satisfies a weak-L^1 estimate. To this end fix $\lambda>0$ and consider the set

$$E_\lambda := \left\{\xi\in\mathbb{T} \mid f_\mu^M(\xi)>\lambda\right\}.$$

For each $\xi\in E_\lambda$ we can thus find a radius $r_\xi>0$ such that $\frac{1}{\mu(K_{r_\xi}(\xi))}\int\limits_{K_{r_\xi}(\xi)}|f|\,d\mu>\lambda$. The collection

$$\mathcal{E} := \left\{K_{r_\xi}(\xi) \mid \xi\in E_\lambda\right\}$$

is a Besicovitch covering for E_λ. Therefore, we can apply the Besicovitch covering theorem, Theorem 1.4.5, to obtain a finite number of subcollections $\mathcal{B}_1,\ldots,\mathcal{B}_N$ of $\mathcal{E}$ such that each $\mathcal{B}_k$ contains countably many elements that are pairwise disjoint and $\bigcup\limits_{k=1}^N\bigcup\limits_{B\in\mathcal{B}_k}B$ covers E_λ. Now we can estimate

$$\begin{aligned}
\mu(E_\lambda) &\le \mu\left(\bigcup_{k=1}^N\bigcup_{B\in\mathcal{B}_k}B\right) \le \sum_{k=1}^N\sum_{B\in\mathcal{B}_k}\mu(B)\\
&< \sum_{k=1}^N\sum_{B\in\mathcal{B}_k}\frac{1}{\lambda}\int\limits_B|f(\zeta)|\,d\mu(\zeta) = \sum_{k=1}^N\frac{1}{\lambda}\int\limits_{\bigcup\limits_{B\in\mathcal{B}_k}B}|f(\zeta)|\,d\mu(\zeta)\\
&\le \sum_{k=1}^N\frac{1}{\lambda}\|f\|_{L^1(\mu)} = \frac{N}{\lambda}\|f\|_{L^1(\mu)},
\end{aligned}$$

which is what was to be shown.
If $f\in L^\infty(\mu)$, clearly $\|f_\mu^M\|_{L^\infty(\mu)}\le\|f\|_{L^\infty(\mu)}$. We note that for any $f,g\in L^1(\mu)$, $(f+g)_\mu^M\le f_\mu^M+g_\mu^M$. The statement for $p\in(1,\infty)$ now follows by applying the Marcinkiewicz interpolation theorem, Theorem 1.4.6, to the operator $f\mapsto f_\mu^M$.

□

The Hardy-Littlewood maximal function is of particular importance as many other functions of f can be majorized by it. The first statement of this kind is the following result relating the Hardy-Littlewood maximal function and the Poisson transform.

Lemma 3.3.5. *Let $\nu\in M_+(\mathbb{T})$, $f\in L^1(\nu)$ and $\varphi\in(0,\pi)$. Then there is a positive constant C such that for ν-almost all $\xi\in\mathbb{T}$ and for all $z\in\Delta_\xi^\varphi$ the estimate*

$$|P[f\nu](z)|\le Cf_\nu^M(\xi)\,P[\nu](z) \tag{3.18}$$

holds.

Proof. Since $|P[f\nu]|\le P[|f|\nu]$ and the maximal function depends on $|f|$ only, we may assume $f\ge0$. Let $\xi\in\mathbb{T}$ such that $\lim\limits_{z\underset{\triangleleft}{\to}\xi}P[\mu](z)\neq0$. By Theorems 2.2.3 and 2.2.4 this is satisfied at ν-alomst every point. Without loss of generality let $\xi=1$ (otherwise consider

the rotated measure $\mu^{\xi}(B) = \mu(\xi B)$).
We first prove the statement for z lying on the radius terminating at ξ. We fix $r \in (0,1)$ and consider the Poisson kernel $P_r(\vartheta) = P\left(r, e^{i\vartheta}\right) = \frac{1-r^2}{1+r^2-2r\cos\vartheta}$ as a function of the angle ϑ only. This is a symmetric function which decreases for $\vartheta \in [0,\pi]$. We approximate $P_r(\vartheta)$ by step functions: For given $m \in \mathbb{N}$ we set $I_0 = \mathbb{T}$ and choose symmetric arcs about 1 satisfying $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$. We denote the characteristic function of the arc I_k by χ_k and choose numbers $h_k \in \mathbb{R}_+$ such that h_k is the largest number for which $h_k \chi_k\left(e^{i\vartheta}\right) \le P_r(\vartheta)$ on $\mathbb{T}$. Then the h_k are increasing and the function

$$\phi\big(e^{i\vartheta}\big) = h_0 + \sum_{k=1}^{m} (h_k - h_{k-1})\, \chi_k\big(e^{i\vartheta}\big) = \sum_{k=0}^{m} c_k \chi_k\big(e^{i\vartheta}\big)$$

is a positive step function with $\phi \equiv h_k$ on I_k and $\phi \le P_r$. For given $\delta > 0$, by choosing $m \in \mathbb{N}$ sufficiently large we get a step function ϕ which satisfies

$$P_r(\vartheta) - \delta \le \phi\big(e^{i\vartheta}\big) \le P_r(\vartheta). \tag{3.19}$$

Multiplying the left inequality by f and integrating over $\mathbb{T}$ with respect to ν we obtain

$$\begin{aligned} P[f\nu](r) - \delta \|f\|_{L^1(\nu)} &\le \sum_{k=0}^{m} c_k \int_{I_k} f(\zeta)\, d\nu(\zeta) = \sum_{k=0}^{m} c_k \nu(I_k) \frac{1}{\nu(I_k)} \int_{I_k} f(\zeta)\, d\nu(\zeta) \\ &\le f_\nu^M(1) \sum_{k=0}^{m} c_k \nu(I_k) \;=\; f_\nu^M(1) \int_{\mathbb{T}} \phi(\zeta)\, d\nu(\zeta) \\ &\le f_\nu^M(1)\, P[\nu](r), \end{aligned}$$

where in the last inequality we used the right side of (3.19). Since δ was arbitrary, we conclude that for z lying on the radius terminating at 1

$$P[f\nu](z) \le f_\nu^M(1)\, P[\nu](z),$$

or equivalently,

$$\frac{P[f\nu](z)}{P[\nu](z)} \le f_\nu^M(1).$$

To finish the proof it now suffices to show that we can estimate the values $\frac{P[f\nu](z)}{P[\nu](z)}$ for arbitrary $z \in \Delta_1^{\varphi}$ by those along the radius terminating at 1. To this end we use a construction as in [10, Chapter I, Section D]. We choose $\varphi_2 \in (\varphi, \pi)$, then $\Delta_1^{\varphi} \subseteq \Delta_1^{\varphi_2}$. Furthermore, we fix a point $z_0 = r_0$ on the radius terminating at 1, and denote by $\mathbb{D}_1$ and $\mathbb{D}_2$ the disks with center at z_0 inscribed into the sectors Δ_1^{φ} and $\Delta_1^{\varphi_2}$, respectively, see Figure 3.1. Hence, $\mathbb{D}_1 \subseteq \mathbb{D}_2$, and for the respective radii of the disks we have $r_1 = (1-r_0)\sin\frac{\varphi}{2}$ and $r_2 = (1-r_0)\sin\frac{\varphi_2}{2}$. Thus the ratio $\gamma = \frac{r_1}{r_2} < 1$ depends on the angles φ and φ_2 only, but not on z_0.
Now we recall that for a positive measure $\eta \in M_+(\mathbb{T})$, the Poisson transform $P[\eta]$ is a positive harmonic function on $\mathbb{D}$, therefore Harnack's inequality (Theorem 1.2.1) gives us for z lying in any open disk $U_r(z_0) \subseteq \mathbb{D}$ with center at z_0 and radius r,

$$P[\eta](z_0)\, \frac{r - |z - z_0|}{r + |z - z_0|} \le P[\eta](z) \le P[\eta](z_0)\, \frac{r + |z - z_0|}{r - |z - z_0|}. \tag{3.20}$$

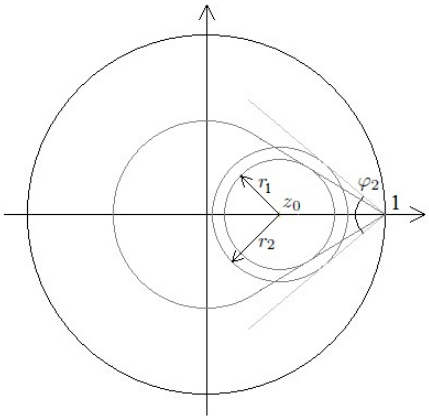

Figure 3.1: Construction of the disks $\mathbb{D}_1$ and $\mathbb{D}_2$.

We use the right-hand estimate for $P[f\nu]$ in the disk $\mathbb{D}_2$ to obtain

$$P[f\nu](z) \leq P[f\nu](z_0)\frac{r_2 + |z - z_0|}{r_2 - |z - z_0|}.$$

For $z \in \mathbb{D}_1$ we have $r_2 - |z - z_0| \geq r_2 - r_1$ and $r_2 + |z - z_0| \leq r_2 + r_1$, and noting that $\frac{r_2+r_1}{r_2-r_1} = \frac{1+\gamma}{1-\gamma}$ this leads to

$$P[f\nu](z) \leq P[f\nu](z_0)\frac{1+\gamma}{1-\gamma}, \quad z \in \mathbb{D}_1.$$

We then use the reciprocal of the left inequality in (3.20) for $P[\nu]$ in $\mathbb{D}_2$ and again estimate the factor $\frac{r_2+|z-z_0|}{r_2-|z-z_0|} \leq \frac{1+\gamma}{1-\gamma}$ for $z \in \mathbb{D}_1$ to obtain

$$\frac{1}{P[\nu](z)} \leq \frac{1}{P[\nu](z_0)}\frac{1+\gamma}{1-\gamma}, \quad z \in \mathbb{D}_1.$$

Combining the last two inequalities we arrive at

$$0 \leq \frac{P[f\nu](z)}{P[\nu](z)} \leq \left(\frac{1+\gamma}{1-\gamma}\right)^2 \frac{P[f\nu](z_0)}{P[\nu](z_0)} \tag{3.21}$$

for all $z \in \mathbb{D}_1 \subseteq \Delta_1^\varphi$. Since every $z \in \Delta_1^\varphi$ lies in $\mathbb{D}_1$ for some z_0 on the radius, we have thus obtained the desired estimate.

□

Corollary 3.3.6. *Let $\mu, \nu \in M_+(\mathbb{T})$, $f \in L^1(\mu + \nu)$ and $\varphi \in (0, \pi)$. Then for $(\mu + \nu)$-almost all $\xi \in \mathbb{T}$ and for all $z \in \Delta_\xi^\varphi$ the following estimate holds with the same constant as in Lemma 3.3.5.*

$$|P[f\mu](z)| \leq Cf_{\mu+\nu}^M(\xi)\,P[\mu + \nu](z). \tag{3.22}$$

Proof. Again we assume $f \geq 0$. We apply Lemma 3.3.5 to the measure $\mu + \nu$ to obtain that for $(\mu + \nu)$-almost every $\xi \in \mathbb{T}$ and all $z \in \Delta_\xi^\varphi$

$$P[f(\mu + \nu)](z) \leq Cf_{\mu+\nu}^M(\xi)\,P[\mu + \nu](z).$$

It remains to note that since μ and ν are positive measures and $f \geq 0$, for $z \in \mathbb{D}$ the estimate $P[f\mu](z) \leq P[f(\mu + \nu)](z)$ holds.

$\square$

We proceed with two results concerning the boundary behavior of quotients of Poisson transforms.

Proposition 3.3.7. *Let $\mu \in M(\mathbb{T})$ and $f \in L^1(\mu)$. Then for $|\mu|$-almost all $\xi \in \mathbb{T}$ we have*

$$\frac{P[f\mu](z)}{P[\mu](z)} \longrightarrow f(\xi) \quad \text{as } z \underset{\triangleleft}{\to} \xi.$$

Proof. First, let $\mu \in M_+(\mathbb{T})$. Since the transform is linear, it suffices to consider nonnegative f, as every L^1-function can be written as a linear combination of nonnegative L^1-functions.

We have proved the statement if we can show that for any $\lambda > 0$ we have

$$\mu\left(\left\{\xi \in \mathbb{T} \mid \limsup_{z \underset{\triangleleft}{\to} \xi} \left|\frac{P[f\mu](z)}{P[\mu](z)} - f(\xi)\right| > \lambda\right\}\right) = 0. \tag{3.23}$$

Note that by the linearity of the Poisson transform, $\frac{P[f\mu](z)}{P[\mu](z)} - f(\xi) = \frac{P[(f-f(\xi))\mu](z)}{P[\mu](z)}$.

We first recall that the nontangential limit of $P[\mu]$ is nonzero μ-almost everywhere by Theorems 2.2.3 and 2.2.4. Now let $\xi \in \mathbb{T}$ be such that $\lim_{z \underset{\triangleleft}{\to} \xi} P[\mu](z) > 0$.

We use the density of $C(\mathbb{T})$ in $L^1(\mu)$ to approximate f by a sequence $\{g_n\}_{n\in\mathbb{N}}$ of continuous functions such that $\|f - g_n\|_{L^1(\mu)} < \frac{1}{n}$. We choose a representative of the equivalence class of f in $L^1(\mu)$ (the choice of the representative is of no concern for the proof since the elements of an equivalence class are equal up to μ-nullsets), which we also denote by f, and set $h_n := f - g_n$. Then $\|h_n\|_{L^1(\mu)} \to 0$, and by choosing a subsequence, we can assume that $h_n \to 0$ pointwise μ-almost everywhere on $\mathbb{T}$ ([21, Theorem 3.12]).

We first consider the behavior of the quotient $\frac{P[g_n\mu]}{P[\mu]}$ for the continuous functions g_n. The continuity of g_n implies that for given $\varepsilon > 0$ we find an arc I_ξ centered at ξ such that $\sup_{\zeta\in I_\xi} |g_n(\zeta) - g_n(\xi)| < \varepsilon$. We can then split

$$P[(g_n - g_n(\xi))\mu](z) = \left[\int_{\mathbb{T}\setminus I_\xi} + \int_{I_\xi}\right] P(z,\zeta)\big(g_n(\zeta) - g_n(\xi)\big)\mu(\zeta) = I + II.$$

The sets Δ_ξ^φ and $\mathbb{T}\setminus I_\xi$ have positive distance d, so for $z \in \Delta_\xi^\varphi$ and $\zeta \in \mathbb{T}\setminus I_\xi$ we have $|z - \zeta| \geq d$. Therefore, using the boundedness of g_n, we can estimate the integrand in I by

$$\left|\frac{1-|z|^2}{|z-\zeta|^2}\left(g_n(\zeta) - g_n(\xi)\right)\right| \leq C\big(1 - |z|^2\big),$$

and we see that the integrand tends to zero uniformly in ζ as z approaches ξ. Consequently, the integral I tends to zero. Furthermore, $|II| < \varepsilon P[\mu](z)$, and so we get

$$\left|\frac{P[(g_n - g_n(\xi))\mu](z)}{P[\mu](z)}\right| \leq \frac{|I|}{P[\mu](z)} + \varepsilon.$$

Since the first summand tends to zero as z nontangentially tends to ξ and ε was arbitrary, we conclude that

$$\lim_{z \underset{\triangleleft}{\to} \xi} \frac{P[(g_n - g_n(\xi))\mu](z)}{P[\mu](z)} = 0.$$

We see that the statement of the proposition holds for continuous functions. Writing $f(\zeta) - f(\xi) = g_n(\zeta) - g_n(\xi) + h_n(\zeta) - h_n(\xi)$ and using the linearity of the Poisson transform we now obtain

$$\begin{aligned}\limsup_{z \underset{\triangleleft}{\to} \xi} \left| \frac{P[(f - f(\xi))\mu](z)}{P[\mu](z)} \right| &\leq \limsup_{z \underset{\triangleleft}{\to} \xi} \frac{P[|h_n - h_n(\xi)|\mu](z)}{P[\mu](z)} \\ &\leq \limsup_{z \underset{\triangleleft}{\to} \xi} \frac{P[|h_n|\mu](z)}{P[\mu](z)} + |h_n(\xi)| \\ &\leq (h_n)^M_\mu(\xi) + |h_n(\xi)|,\end{aligned}$$

where we used Lemma 3.3.5 in the last estimate. Now fix $\lambda > 0$. From the above estimate we infer

$$\left\{ \xi \in \mathbb{T} \mid \limsup_{z \underset{\triangleleft}{\to} \xi} \left| \frac{P[(f - f(\xi))\mu](z)}{P[\mu](z)} \right| > \lambda \right\} \subseteq \left\{ (h_n)^M_\mu > \frac{\lambda}{2} \right\} \cup \left\{ |h_n| > \frac{\lambda}{2} \right\}. \tag{3.24}$$

We now estimate the μ-measure of the sets on the right-hand side. To this end, let $\varepsilon > 0$ be given. As the functions h_n converge to zero pointwise μ-almost everywhere, by Egorov's theorem ([23, Chapter III, Theorem 2.2]), there is a set $E \subseteq \mathbb{T}$ with $\mu(\mathbb{T} \backslash E) < \frac{\varepsilon}{2}$ such that the sequence h_n tends to zero uniformly on E. Therefore we find an index $N_0 \in \mathbb{N}$ such that $|h_n| < \frac{\lambda}{2}$ on E for all n larger than N_0. Thus $\left\{ |h_n| > \frac{\lambda}{2} \right\} \subseteq \mathbb{T} \backslash E$ for $n \geq N_0$ and we get the estimate

$$\mu\Big(\left\{ |h_n| > \frac{\lambda}{2} \right\}\Big) \leq \mu(\mathbb{T} \backslash E) \leq \frac{\varepsilon}{2}, \quad n \geq N_0. \tag{3.25}$$

By making N_0 larger if necessary, we can also assume that $\|h_n\|_{L^1(\mu)} < \frac{\lambda\varepsilon}{4c}$ for all $n \geq N_0$, where c is the constant in the weak-L^1 estimate of the Hardy-Littlewood maximal function in Theorem 3.3.4 (i). Then this theorem yields

$$\mu\Big(\left\{ (h_n)^M_\mu > \frac{\lambda}{2} \right\}\Big) \leq \frac{2c}{\lambda}\|h_n\|_{L^1(\mu)} \leq \frac{2c}{\lambda}\frac{\lambda\varepsilon}{4c} = \frac{\varepsilon}{2}, \quad n \geq N_0. \tag{3.26}$$

With the estimates (3.25) and (3.26) we derive from (3.24) that

$$\mu\Bigg(\left\{ \xi \in \mathbb{T} \mid \limsup_{z \underset{\triangleleft}{\to} \xi} \left| \frac{P[(f - f(\xi))\mu](z)}{P[\mu](z)} \right| > \lambda \right\}\Bigg) < \varepsilon,$$

and as ε was arbitrary, this finishes the first part.
Now let μ be an arbitrary measure in $M(\mathbb{T})$, and let g be a μ-measurable function with $|g| = 1$ on $\mathbb{T}$ such that $\mu = g|\mu|$ (for the existence of such a function see e.g. [21, Theorem 6.12]). We then have

$$\frac{P[f\mu](z)}{P[\mu](z)} = \frac{P[fg|\mu|](z)}{P[|\mu|](z)} \cdot \frac{P[|\mu|](z)}{P[g|\mu|](z)}.$$

If we let z nontangentially approach $\xi \in \mathbb{T}$, by the first part of the proof the right-hand side converges to $(fg)(\xi) \cdot \frac{1}{g(\xi)} = f(\xi)$ at $|\mu|$-almost every $\xi \in \mathbb{T}$.

□

Lemma 3.3.8. *Let $\mu, \nu \in M(\mathbb{T})$. The following conditions are equivalent.*

(i) $\lim\limits_{z \underset{\triangleleft}{\to} \xi} \frac{P[\mu](z)}{P[\nu](z)} = 0$ for $|\nu|$-almost every $\xi \in \mathbb{T}$;

(ii) $\lim\limits_{z \underset{\triangleleft}{\to} \xi} \frac{P[\nu](z)}{P[\mu](z)} = 0$ for $|\mu|$-almost every $\xi \in \mathbb{T}$;

(iii) $\mu \perp \nu$.

Proof. We start with the implication (iii)⇒(i). Let μ and ν be mutually singular, and first assume that both are positive measures. Let f be a function on $\mathbb{T}$ such that $f = 0$ almost everywhere with respect to μ and $f = 1$ almost everywhere with respect to ν. Then $f(\nu + \mu) = \nu$ almost everywhere with respect to ν. Applying Proposition 3.3.7 we see that for $(\nu + \mu)$-almost every $\xi \in \mathbb{T}$,

$$\frac{P[f(\nu+\mu)](z)}{P[\nu+\mu](z)} \longrightarrow f(\xi)$$

as z nontangentially tends to ξ, and in particular, at ν-almost every $\xi \in \mathbb{T}$ we have

$$\frac{P[\nu](z)}{P[\nu+\mu](z)} = \frac{P[f(\nu+\mu)](z)}{P[\nu+\mu](z)} \longrightarrow f(\xi) = 1, \quad z \underset{\triangleleft}{\to} \xi.$$

Using the linearity of the Poisson transform, we infer that for ν-almost every $\xi \in \mathbb{T}$

$$\frac{P[\mu](z)}{P[\nu](z)} = \frac{P[\mu+\nu](z)}{P[\nu](z)} - \frac{P[\nu](z)}{P[\nu](z)} \longrightarrow 1 - 1 = 0$$

as z approaches ξ nontangentially.

For arbitrary measures $\mu, \nu \in M(\mathbb{T})$, we note that $\mu \perp \nu$ implies $|\mu| \perp |\nu|$ and write $\nu = h|\nu|$ with $|h| = 1$ on $\mathbb{T}$ as in the proof of the previous proposition. Then,

$$\frac{P[\mu](z)}{P[\nu](z)} = \frac{P[\mu](z)}{P[|\mu|](z)} \cdot \frac{P[|\mu|](z)}{P[|\nu|](z)} \cdot \frac{P[|\nu|](z)}{P[h|\nu|](z)}.$$

For z nontangentially tending to $\xi \in \mathbb{T}$, the first factor on the right-hand side is bounded since we have $|P[\mu]| \leq P[|\mu|]$. By Proposition 3.3.7 the last factor tends to $\frac{1}{h(\xi)}$ for $|\nu|$-almost every $\xi \in \mathbb{T}$, and by the previous part of the proof the term in the middle tends to zero almost everywhere with respect to $|\nu|$, so we arrive at (i).

For the implication (i)⇒(iii), write $\mu = f\nu + \mu_s$ with $f \in L^1(|\nu|)$ and $\mu_s \perp \nu$. Then, by assumption we have at $|\nu|$-almost every $\xi \in \mathbb{T}$

$$\frac{P[\mu](z)}{P[\nu](z)} = \frac{P[f\nu](z)}{P[\nu](z)} + \frac{P[\mu_s](z)}{P[\nu](z)} \longrightarrow 0, \quad z \underset{\triangleleft}{\to} \xi.$$

But at $|\nu|$-almost every point, the first term on the right-hand side tends to $f(\xi)$ by Proposition 3.3.7 and the second term tends to 0 by the implication (iii)⇒(i) we just proved. We conclude that $f(\xi) = 0$ at $|\nu|$-almost every $\xi \in \mathbb{T}$, and hence $\nu \perp \mu$.

The equivalence (ii)⇔(iii) is shown in the same way exchanging the roles of ν and μ.

□

We now consider the map $f \mapsto \theta\bar{f} =: \tilde{f}$ for $f \in \theta^*(H^p)$. It follows from the definition of $\theta^*(H^p)$ that $\tilde{f}$ lies in H^p_0. We set

$$\theta^*(H^p_0) := \{f \in \theta^*(H^p) \mid f(0) = 0\}.$$

If $f \in \theta^* \left(H_0^p\right)$, then $\bar{f} \in \overline{H_0^p}$. Thus, it follows from $\tilde{f}\bar{\theta} = \bar{f}\theta\bar{\theta} = \bar{f} \in \overline{H_0^p}$, that $\tilde{f} \in \theta^*(H_0^p)$. Moreover, the map $f \mapsto \tilde{f}$ induces an involution on this space. We call $f \in \theta^* \left(H_0^p\right)$ a *Hermitian element* if it satisfies $f = \tilde{f}$.

Now we are going to establish a useful property of Hermitian elements. We make use of the following auxiliary statement.

Lemma 3.3.9. *Let $\nu \in M(\mathbb{T})$ be a real-valued singular measure. Furthermore, let $F \in L^1(\nu)$ be such that $\int_{\mathbb{T}} F(\zeta)\, d\nu(\zeta) = 0$. Then the following statements are equivalent.*

(i) $\lim\limits_{r \nearrow 1} K[F\nu](r\xi) = \lim\limits_{\hat{r} \searrow 1} -\overline{K[F\nu](\hat{r}\xi)}$ *for σ-almost every $\xi \in \mathbb{T}$.*

(ii) *F is real-valued almost everywhere with respect to ν.*

Proof. The Cauchy transform of $F\nu$ lies in the spaces H^p, $0 < p < 1$ for both $\mathbb{D}$ and $\mathbb{D}_e$ and consequently has nontangential boundary values σ-almost everywhere on $\mathbb{T}$ from both within $\mathbb{D}$ and $\mathbb{D}_e$. Let $\xi \in \mathbb{T}$ be such that the boundary values from both sides exist. We first derive the following relation between those limits.

$$\begin{aligned}
\lim_{r \nearrow 1} K[F\nu](r\xi) &= \lim_{r \nearrow 1} \int_{\mathbb{T}} \frac{1}{1 - r\xi\bar{\zeta}} F(\zeta)\, d\nu(\zeta) = \lim_{r \nearrow 1} \int_{\mathbb{T}} \frac{\frac{1}{r}\bar{\xi}\zeta}{\frac{1}{r}\bar{\xi}\zeta - 1} F(\zeta)\, d\nu(\zeta) \\
&= \lim_{r \nearrow 1} \int_{\mathbb{T}} \left(\frac{\frac{1}{r}\bar{\xi}\zeta - 1}{\frac{1}{r}\bar{\xi}\zeta - 1} + \frac{1}{\frac{1}{r}\bar{\xi}\zeta - 1} \right) F(\zeta)\, d\nu(\zeta) \\
&= \int_{\mathbb{T}} F(\zeta)\, d\nu(\zeta) + \lim_{\hat{r} \searrow 1} \left(- \int_{\mathbb{T}} \frac{1}{1 - \hat{r}\bar{\xi}\zeta} F(\zeta)\, d\nu(\zeta) \right) \\
&= \int_{\mathbb{T}} F(\zeta)\, d\nu(\zeta) + \lim_{\hat{r} \searrow 1} \overline{\left(- \int_{\mathbb{T}} \frac{1}{1 - \hat{r}\xi\bar{\zeta}} \overline{F(\zeta)}\, d\nu(\zeta) \right)} \\
&= \int_{\mathbb{T}} F(\zeta)\, d\nu(\zeta) + \lim_{\hat{r} \searrow 1} -\overline{K\left[\bar{F}\nu\right](\hat{r}\xi)}.
\end{aligned}$$

By our assumption on F, the integral on the right side vanishes, hence F satisfies the relation

$$\lim_{r \nearrow 1} K[F\nu](r\xi) = \lim_{\hat{r} \searrow 1} -\overline{K\left[\bar{F}\nu\right](\hat{r}\xi)}. \tag{3.27}$$

(ii)⇒(i). Supposing that $\overline{F} = F$ almost everywhere with respect to ν, the relation (3.27) immediately yields (i).
(i)⇒(ii). Inserting (i) in (3.27) we obtain

$$\lim_{\hat{r} \searrow 1} -\overline{K[F\nu](\hat{r}\xi)} = \lim_{\hat{r} \searrow 1} -\overline{K\left[\bar{F}\nu\right](\hat{r}\xi)},$$

equivalently

$$0 = \lim_{\hat{r} \searrow 1} -\overline{K\left[\left(F - \bar{F}\right)\nu\right](\hat{r}\xi)} = 2i \lim_{\hat{r} \searrow 1} \overline{K[\operatorname{Im} F\nu](\hat{r}\xi)}$$

for σ-almost every $\xi \in \mathbb{T}$. We see that $K[\operatorname{Im} F\nu]$ has vanishing radial limits σ-almost everywhere on $\mathbb{T}$ from outside $\mathbb{D}$. As ν is singular, the Fatou jump theorem, Theorem 1.4.1, gives us

$$\lim_{r \nearrow 1} K[\operatorname{Im} F\nu](r\xi) = \lim_{r \searrow 1} K[\operatorname{Im} F\nu](r\xi),$$

hence $K[\operatorname{Im} F\nu]$ has zero limit also from within $\mathbb{D}$ at σ-almost every $\xi \in \mathbb{T}$. Applying the Lusin-Privalov theorem ([11, p.212, 2.5]) we get that $K[\operatorname{Im} F\nu] \equiv 0$ in $\mathbb{D}$. But this implies that $\operatorname{Im} F\nu$ is absolutely continuous with respect to σ (see [14, Proposition 2.1.5]), and since ν is singular, we infer that $\operatorname{Im} F = 0$ almost everywhere with respect to ν.

□

Proposition 3.3.10. *Let $f \in \theta^*(H_0^p)$, $p \geq 2$, and let $\alpha \in \mathbb{T}$. Then f is a Hermitian element if and only if $\arg U_\alpha f = \frac{\arg \alpha}{2}$ almost everywhere with respect to ν_α.*

Proof. First, let $\alpha = 1$ and let $f \in \theta^*(H_0^p)$ be given. We have to show that $U_1 f$ is real-valued ν_1-almost everywhere on $\mathbb{T}$ if and only if f is Hermitian. Now, the function $U_1 f$ lies in $L^p(\nu_1)$ by Corollary 3.2.7, and since $f(0) = 0$, it is orthogonal to constants: recall the functions k_λ introduced in Section 3.2. We have $k_0 \equiv 1$ and $U_1 k_0 \equiv 1$, so by the isometry of U_1,

$$\int_{\mathbb{T}} U_1 f(\zeta)\, d\nu_1(\zeta) = \langle U_1 f, U_1 k_0 \rangle_{L^2(\nu_1)} = \langle f, k_0 \rangle_{H^2} = f(0) = 0.$$

Consequently, the statement in the case $\alpha = 1$ can be reformulated in the following way: If $F \in L^p(\nu_1)$, $p \geq 2$, and $\int_{\mathbb{T}} F(\zeta)\, d\nu_1(\zeta) = 0$, then $U_1^* F$ is a Hermitian element if and only if F is real-valued ν_1-almost everywhere.

Thus, let $F \in L^p(\nu_1)$ be orthogonal to constants, and suppose first that F is real-valued ν_1-almost everywhere. We have to show that $U_1^* F = (1-\theta) K[F\nu_1]$ is Hermitian. We calculate $\widetilde{U_1^* F}$ and examine its radial boundary values from within $\mathbb{D}$, making use of the relation in Lemma 3.3.9 (i) that holds in this case. We thus have for σ-almost every $\xi \in \mathbb{T}$,

$$\begin{aligned}
\widetilde{U_1^* F}(\xi) = \lim_{r \nearrow 1} \widetilde{U_1^* F}(r\xi) &= \lim_{r \nearrow 1} \theta(r\xi)\big(1 - \overline{\theta(r\xi)}\big) \lim_{r \nearrow 1} \overline{K[F\nu_1](r\xi)} \\
&= \big(\theta(\xi) - 1\big) \lim_{\hat{r} \searrow 1} \big(-K[F\nu_1](\hat{r}\xi)\big) \\
&= \big(1 - \theta(\xi)\big) \lim_{\hat{r} \searrow 1} K[F\nu_1](\hat{r}\xi).
\end{aligned}$$

Since ν_1 is singular, relation (1.14) from the Fatou jump theorem, Theorem 1.4.1, yields that the last term equals $\big(1 - \theta(\xi)\big) \lim_{r \nearrow 1} K[F\nu_1](r\xi) = U_1^* F(\xi)$ at σ-almost every point. We see that $U_1^* F$ is a Hermitian element.

Conversely, if we suppose that $U_1^* F$ is Hermitian, we have

$$\begin{aligned}
\widetilde{U_1^* F}(\xi) &= \lim_{r \nearrow 1} \theta(r\xi)\big(1 - \overline{\theta(r\xi)}\big) \lim_{r \nearrow 1} \overline{K[F\nu_1](r\xi)} \\
&= \big(\theta(\xi) - 1\big) \lim_{r \nearrow 1} \overline{K[F\nu_1](r\xi)} \\
&= \big(1 - \theta(\xi)\big) \lim_{r \nearrow 1} K[F\nu_1](r\xi) = U_1^* F(\xi)
\end{aligned}$$

at σ-almost every $\xi \in \mathbb{T}$ and hence

$$\lim_{r \nearrow 1} K[F\nu_1] = -\lim_{r \nearrow 1} \overline{K[F\nu_1]}.$$

By Fatou's jump theorem, the left-hand side equals $\lim_{\hat{r} \searrow 1} K[F\nu_1]$, and we see that F satisfies condition (i) from Lemma 3.3.9. We conclude that F is real-valued almost everywhere with respect to ν_1, which proves the statement for $\alpha = 1$.

Now, if $\alpha \neq 1$, consider the inner function $\hat{\theta} := \bar{\alpha}\theta$. Then $\hat{\theta}^*(H^p) = \theta^*(H^p)$ and we have the relation $\nu_\alpha = \hat{\nu}_1$, where $\hat{\nu}_1$ is the singular measure corresponding to $\hat{\theta}$, since

$$P[\nu_\alpha] = \operatorname{Re}\frac{\alpha+\theta}{\alpha-\theta} = \operatorname{Re}\frac{\alpha}{\alpha}\frac{1+\bar{\alpha}\theta}{1-\bar{\alpha}\theta} = P[\hat{\nu}_1].$$

Furthermore, for $\lambda \in \mathbb{D}$,

$$\hat{U}_1 k_\lambda(\xi) = \frac{1-\overline{\hat{\theta}(\lambda)}}{1-\bar{\lambda}\xi} = \frac{1-\alpha\overline{\theta(\lambda)}}{1-\bar{\lambda}\xi} = U_\alpha k_\lambda(\xi),$$

and since the k_λ are a complete subset of $\theta^*(H^2)$, we have $\hat{U}_1 = U_\alpha$ on $\theta^*(H^2)$.
The Hermitian elements with respect to $\hat{\theta}$ (i.e., $g \in \theta^*(H_0^p)$ which satisfy $\hat{\tilde{g}} := \bar{\alpha}\theta\bar{g} = g$) are precisely the functions $\bar{\alpha}^{1/2}f$, where f is a Hermitian element with respect to θ:

$$\widehat{\widetilde{\bar{\alpha}^{1/2}f}} = \bar{\alpha}\theta\alpha^{1/2}\bar{f} = \bar{\alpha}^{1/2}\theta\bar{f},$$

and this equals $\bar{\alpha}^{1/2}f$ if and only if $f = \theta\bar{f}$.
Combining this with the first part of the proof we see that f is Hermitian with respect to θ if and only if $\hat{U}_1\bar{\alpha}^{1/2}f$ is real valued, and using the relation $\hat{U}_1 = U_\alpha$ and the linearity of the operators we infer

$$\begin{aligned} 0 &= \arg\left(\hat{U}_1\bar{\alpha}^{1/2}f\right) = \arg\left(\bar{\alpha}^{1/2}\hat{U}_1 f\right) \\ &= \arg\left(\bar{\alpha}^{1/2}U_\alpha f\right) = \arg\bar{\alpha}^{1/2} + \arg U_\alpha f \\ &= -\frac{\arg\alpha}{2} + \arg U_\alpha f. \end{aligned}$$

□

Note that if $f \in \theta^*(H_0^p)$, then the functions $f + \tilde{f}$ and $i(f - \tilde{f})$ are Hermitian elements. This is obvious for the first function since the map is an involution, and for the second function we compute

$$\left(i(f-\tilde{f})\right)^\sim = \theta\bar{i}\left(\bar{f} - \bar{\theta}f\right) = -i\left(\theta\bar{f} - f\right) = i\left(f - \theta\bar{f}\right).$$

Now we have everything available to prove the main theorem of the section.

Theorem 3.3.11. *Let $f \in \theta^*(H^p)$, $p \geq 2$, and let $\alpha \in \mathbb{T}$. Then for ν_α-almost every $\xi \in \mathbb{T}$,*

$$f(z) \longrightarrow (U_\alpha f)(\xi) \quad \text{as } z \underset{\sphericalangle}{\to} \xi.$$

Proof. We start with some preparatory observations. By Theorem 3.2.2, we have for any $\alpha, \beta \in \mathbb{T}$,

$$\frac{K[U_\alpha f\nu_\alpha]}{K[\nu_\alpha]} = U_\alpha^*(U_\alpha f) = f = U_\beta^*(U_\beta f) = \frac{K[U_\beta f\nu_\beta]}{K[\nu_\beta]}. \tag{3.28}$$

Furthermore, by Corollary 3.1.3, $K[\nu_\alpha] = (1-\overline{\alpha}\theta)^{-1}$, and thus

$$K[U_\alpha f\nu_\alpha] = \frac{K[\nu_\alpha]}{K[\nu_\beta]}K[U_\beta f\nu_\beta] = \frac{1-\bar{\beta}\theta}{1-\overline{\alpha}\theta}K[U_\beta f\nu_\beta]. \tag{3.29}$$

As the operators U_α are unitary, we have for any $f \in \theta^*(H^p)$ that satisfies $f(0) = 0$ (recall that $k_0 \equiv 1$ and $U_\alpha k_0 = 1$)

$$\int_{\mathbb{T}} U_\alpha f(\xi)\, d\nu_\alpha(\xi) = \langle U_\alpha f, U_\alpha k_0 \rangle_{L^2(\nu_\alpha)} = \langle f, k_0 \rangle_{H^2} = f(0) = 0, \tag{3.30}$$

In particular we infer from (1.13) that for such f

$$2K[U_\alpha f \nu_\alpha] = P[U_\alpha f \nu_\alpha] + iQ[U_\alpha f \nu_\alpha]. \tag{3.31}$$

We now make the following assumptions. Without loss of generality, let f satisfy $f(0) = 0$. Since we can write $f = \frac{f+\tilde{f}}{2} + \frac{f-\tilde{f}}{2}$, and both $f + \tilde{f}$ and $i(f - \tilde{f})$ are Hermitian elements, we can further suppose that f is a Hermitian element. We set $\alpha = 1$ (the case $\alpha \neq 1$ then follows from considering the inner function $\hat{\theta} = \overline{\alpha}\theta$ as in the proof of Proposition 3.3.10), and let $\xi \in \mathbb{T}$ be such that

(i) $\frac{P[U_1 f \nu_1](z)}{P[\nu_1](z)} \to U_1 f(\xi)$,

(ii) $\frac{P\left[|U_{-1} f|^2 \nu_{-1}\right](z)}{P[\nu_1](z)} \to 0$, and

(iii) $\theta(z) \to 1$,

as $z \underset{\triangleleft}{\to} \xi$. By Proposition 3.3.7 and Lemma 3.3.8, conditions (i) and (ii) are satisfied by ν_1-almost every $\xi \in \mathbb{T}$ (recall that ν_1 and ν_{-1} are mutually singular), and (iii) holds ν_1-almost everywhere since ν_1 is supported on the set of points at which θ has nontangential limit 1.

Using (3.28) and (3.31) we can write $f(z)$ as

$$\begin{aligned} f(z) &= \left(U_{-1}^* U_{-1} f\right)(z) = \left(1 + \theta(z)\right) K[U_{-1} f \nu_{-1}](z) \\ &= \frac{1}{2}\Big(1 + \theta(z)\Big)\Big(P[U_{-1} f \nu_{-1}](z) + iQ[U_{-1} f \nu_{-1}](z)\Big), \end{aligned} \tag{3.32}$$

so we have proved the statement of the theorem for $\alpha = 1$ if we show that this expression tends to $U_1 f(\xi)$ as z nontangentially tends to ξ. We start from the Poisson transform of $U_1 f \nu_1$ and rewrite it in terms of $P[U_{-1} f \nu_{-1}]$ and $Q[U_{-1} f \nu_{-1}]$.

As we assume f to be Hermitian, by Proposition 3.3.10 the function $U_1 f$ is real-valued almost everywhere with respect to ν_1, and $U_{-1} f$ is purely imaginary ν_{-1}-almost everywhere, so we infer from (3.31) that

$$\operatorname{Re} K[U_1 f \nu_1] = \frac{1}{2} P[U_1 f \nu_1], \quad \operatorname{Im} K[U_1 f \nu_1] = \frac{1}{2} Q[U_1 f \nu_1];$$

$$\operatorname{Re} K[U_{-1} f \nu_{-1}] = \frac{i}{2} Q[U_{-1} f \nu_{-1}], \quad \operatorname{Im} K[U_{-1} f \nu_{-1}] = \frac{-i}{2} P[U_{-1} f \nu_{-1}].$$

We now use these relations to rewrite $P[U_1 f \nu_1]$, further making use of (3.29) with $\alpha = 1$ and $\beta = -1$, and the fact that the real and imaginary parts of $\frac{1+\theta}{1-\theta}$ are the Poisson and conjugate Poisson transforms of ν_1, respectively (Corollary 3.1.3). We then obtain

$$\begin{aligned} P[U_1 f \nu_1] &= 2\operatorname{Re} K[U_1 f \nu_1] = 2\operatorname{Re}\left(\frac{1+\theta}{1-\theta} K[U_{-1} f \nu_{-1}]\right) \\ &= 2\left(\operatorname{Re}\frac{1+\theta}{1-\theta}\operatorname{Re} K[U_{-1} f \nu_{-1}] - \operatorname{Im}\frac{1+\theta}{1-\theta}\operatorname{Im} K[U_{-1} f \nu_{-1}]\right) \\ &= 2\left(P[\nu_1]\frac{i}{2} Q[U_{-1} f \nu_{-1}] - Q[\nu_1]\frac{-i}{2} P[U_{-1} f \nu_{-1}]\right) \\ &= i\Big(P[\nu_1]\, Q[U_{-1} f \nu_{-1}] + Q[\nu_1]\, P[U_{-1} f \nu_{-1}]\Big). \end{aligned}$$

Dividing this equation by $P[\nu_1]$ and applying condition (i) we get

$$i\Big(Q[U_{-1}f\nu_{-1}](z)+\frac{Q[\nu_1](z)\,P[U_{-1}f\nu_{-1}](z)}{P[\nu_1](z)}\Big)=\frac{P[U_1 f\nu_1](z)}{P[\nu_1](z)}\underset{z\underset{\triangleleft}{\to}\xi}{\longrightarrow}U_1 f(\xi)\,. \tag{3.33}$$

We continue by estimating $P[U_{-1}f\nu_{-1}]$. Since $U_{-1}f\in L^2(\nu_{-1})$, we may apply Hölder's inequality to obtain

$$\begin{aligned}\big|P[U_{-1}f\nu_{-1}](z)\big| &= \Big|\int_{\mathbb{T}}\sqrt{P(z,\zeta)}\sqrt{P(z,\zeta)}U_{-1}f(\zeta)\,d\nu_{-1}(\zeta)\Big|\\ &\leq \Big(\int_{\mathbb{T}}P(z,\zeta)\,d\nu_{-1}(\zeta)\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{T}}P(z,\zeta)\,|U_{-1}f(\zeta)|^2\,d\nu_{-1}(\zeta)\Big)^{\frac{1}{2}}\\ &= \Big(P[\nu_{-1}](z)\Big)^{\frac{1}{2}}\Big(P\Big[|U_{-1}f|^2\,\nu_{-1}\Big](z)\Big)^{\frac{1}{2}}.\end{aligned}$$

Now condition (ii) yields

$$\big|P[U_{-1}f\nu_{-1}](z)\big|=o\Big(\sqrt{P[\nu_{-1}](z)\,P[\nu_1](z)}\Big),\quad z\underset{\triangleleft}{\to}\xi.$$

Moreover,

$$P[\nu_1]\,P[\nu_{-1}]=\frac{1-|\theta|^2}{|1-\theta|^2}\frac{1-|\theta|^2}{|-1-\theta|^2}=\frac{\big(1-|\theta|^2\big)^2}{|1-\theta^2|^2},$$

whence we further get

$$|P[U_{-1}f\nu_{-1}]|=o\left(\frac{1-|\theta|^2}{|1-\theta^2|}\right),\quad z\underset{\triangleleft}{\to}\xi. \tag{3.34}$$

With this we can estimate the second term on the left side of (3.33) by

$$\left|\frac{Q[\nu_1]\,P[U_{-1}f\nu_{-1}]}{P[\nu_1]}\right|=o\left(\frac{|\mathrm{Im}\,\theta|}{|1-\theta|^2}\frac{\frac{1-|\theta|^2}{|1-\theta^2|}}{\frac{1-|\theta|^2}{|1-\theta|^2}}\right)=o\left(\frac{|\mathrm{Im}\,\theta|}{|1-\theta^2|}\right),\quad z\underset{\triangleleft}{\to}\xi.$$

Now $|\mathrm{Im}\,\theta|=|\mathrm{Im}\,(1-\theta)|$ and the quotient $\frac{|\mathrm{Im}\,(1-\theta)|}{|1-\theta^2|}=\frac{|\mathrm{Im}\,(1-\theta)|}{|(1-\theta)(1+\theta)|}$ is $O(1)$ as $z\underset{\triangleleft}{\to}\xi$. Thus, the second summand on the left-hand side of (3.33) tends to zero as z nontangentially tends to ξ and we infer

$$iQ[U_{-1}f\nu_{-1}](z)\longrightarrow U_1 f(\xi)\,,\quad z\underset{\triangleleft}{\to}\xi. \tag{3.35}$$

As $\frac{1-|\theta|^2}{|1-\theta^2|}\leq\frac{1-|\theta|^2}{1-|\theta|^2}=1$ by the reverse triangle inequality, we further get from (3.34) that

$$|P[U_{-1}f\nu_{-1}](z)|=o(1)\,,\quad z\underset{\triangleleft}{\to}\xi, \tag{3.36}$$

Applying (3.35) and (3.36) as well as condition (iii) in the expression (3.32) we obtained for f, we see that as z nontangentially tends to ξ,

$$f(z)=\frac{1}{2}\Big(1+\theta(z)\Big)\Big(P[U_{-1}f\nu_{-1}](z)+iQ[U_{-1}f\nu_{-1}](z)\Big)\longrightarrow U_1 f(\xi)\,.$$

□

Theorem 3.3.11 yields the following statement about the normalized Cauchy transform, which is an interesting analogue to Proposition 3.3.7.

Corollary 3.3.12. *Let $\mu \in M(\mathbb{T})$ be a positive singular measure and $f \in L^2(\mu)$. Then for μ-almost every $\xi \in \mathbb{T}$ we have*

$$V_\mu(z) = \frac{K[f\mu](z)}{K[\mu](z)} \longrightarrow f(\xi) \quad as \quad z \underset{\triangleleft}{\to} \xi.$$

Proof. As was shown in Remark 3.1.1, every positive singular measure μ is the measure ν_1 for some inner function. Thus, by Theorem 3.2.2 we have the relation $V_\mu = U_1^{-1}$ on $L^2(\mu)$, where U_1 is the associated unitary operator. Since the operator is continuous on $L^2(\mu)$, the statement now follows from Theorem 3.3.11.

□

3.4 The nontangential maximal function

In order to generalize the results we established in the last section to arbitrary measures, we need a maximal theorem about the nontangential maximal function of an element of $\theta^*(H^p)$, $p \geq 2$. For the definition of this function, we consider approach regions as shown

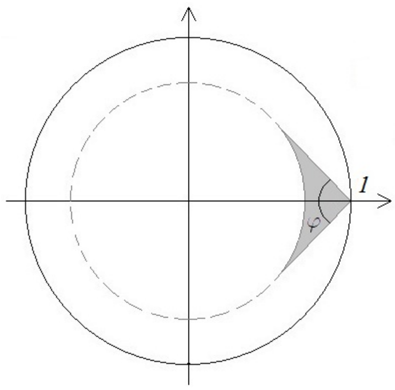

Figure 3.2: The region Γ_ξ^φ for $\xi = 1$.

in Figure 3.2, denoted by Γ_ξ^φ. For a subset $E \subseteq \mathbb{T}$, we denote $\Gamma_E^\varphi = \bigcup_{\xi \in E} \Gamma_\xi^\varphi$.

Definition 3.4.1. Let $f \in H^p$, $p > 0$. We define the *nontangential maximal function* f_φ^* of f by

$$f_\varphi^*(\xi) := \sup_{z \in \Gamma_\xi^\varphi} |f(z)|, \quad \xi \in \mathbb{T}. \tag{3.37}$$

We now use the Hardy-Littlewood maximal theorem to derive a similar statement for the nontangential maximal function.

Theorem 3.4.2. *Let $f \in \theta^*(H^p)$, $p \geq 2$. Furthermore, let $\alpha \in \mathbb{T}$ and $\varphi \in (0, \pi)$. Then f_φ^* is finite ν_α-almost everywhere, and*

(i) *if $p > 2$, $f_\varphi^* \in L^p(\nu_\alpha)$;*

(ii) if $p=2$, then f_φ^* belongs to the weak L^2 *of ν_α, i.e., for every $\lambda>0$, the inequality*

$$\nu_\alpha\left(\{f_\varphi^*>\lambda\}\right)\le\frac{c}{\lambda^2}$$

holds, where $c>0$ is a constant independent of λ.

Proof. It suffices to consider the case $\alpha=1$. Let $\xi\in\mathbb{T}$ be such that

(i) $\left|\frac{P[f\nu_1](z)}{P[\nu_1](z)}\right|\le Cf_{\nu_1}^M(\xi)$ and

(ii) $\left|P\left[|f|^2\nu_{-1}\right](z)\right|\le C\big(f^2\big)_{\nu_1+\nu_{-1}}^M(\xi)\,P[\nu_1+\nu_{-1}](z)$

for $z\in\Gamma_\xi^\varphi$. Note that by Theorem 3.3.11, for each $\alpha\in\mathbb{T}$ we have $f=U_\alpha f$ almost everywhere with respect to ν_α, therefore $f\in L^2(\nu_\alpha)$. Hence, conditions (i) and (ii) are satisfied by ν_1-almost every $\xi\in\mathbb{T}$ by Lemma 3.3.5 and Corollary 3.3.6.
Furthermore, on $\mathbb{D}$ we have the relation

$$f=U_\alpha^*U_\alpha f=U_\alpha^*f=\frac{K[f\nu_\alpha]}{K[\nu_\alpha]}=(1-\overline{\alpha}\theta)\,K[f\nu_\alpha]\,.$$

Assuming without loss of generality that $f(0)=0$, we use this relation with $\alpha=-1$ and (1.13) to write f as follows.

$$f(z)=\big(1+\theta(z)\big)K[f\nu_{-1}](z)=\frac{1}{2}\Big(1+\theta(z)\Big)\Big(P[f\nu_{-1}](z)+iQ[f\nu_{-1}](z)\Big).\qquad(3.38)$$

Our first goal is to estimate the absolute values of the Poisson and conjugate Poisson transforms on the right-hand side in terms of the Hardy-Littlewood maximal function of f at ξ. The estimates we apply are similar to those used in the proof of Theorem 3.3.11. Since $(g+h)_\varphi^*\le g_\varphi^*+h_\varphi^*$ and any $f\in\theta^*(H_0^p)$ can be written as a linear combination of Hermitian elements, we may assume f is a Hermitian element. Then we have the following relation we established in the proof of Theorem 3.3.11,

$$\frac{P[f\nu_1](z)}{P[\nu_1](z)}=i\Big(Q[f\nu_{-1}(z)]+\frac{Q[\nu_1](z)\,P[f\nu_{-1}](z)}{P[\nu_1](z)}\Big).$$

Using condition (i), we get

$$\left|Q[f\nu_{-1}(z)]+\frac{Q[\nu_1](z)\,P[f\nu_{-1}](z)}{P[\nu_1](z)}\right|\le Cf_{\nu_1}^M(\xi)\,,\quad z\in\Gamma_\xi^\varphi.\qquad(3.39)$$

We apply Hölder's inequality to obtain $|P[f\nu_{-1}]|\le\big(P[\nu_{-1}]\big)^{1/2}\big(P\big[|f|^2\nu_{-1}\big]\big)^{1/2}$. By virtue of condition (ii), this yields the estimate

$$|P[f\nu_{-1}](z)|\le\sqrt{C\,(f^2)_{\nu_1+\nu_{-1}}^M(\xi)\,P[\nu_1+\nu_{-1}](z)\,P[\nu_{-1}](z)}$$

for $z\in\Gamma_\xi^\varphi$. We further estimate the Poisson transforms in the last expression.

$$\begin{aligned}P[\nu_1+\nu_{-1}]\,P[\nu_{-1}]&=\left(\frac{1-|\theta|^2}{|1-\theta|^2}+\frac{1-|\theta|^2}{|-1-\theta|^2}\right)\frac{1-|\theta|^2}{|-1-\theta|^2}\\&=\frac{\big(1-|\theta|^2\big)^2\big(|1+\theta|^2+|1-\theta|^2\big)}{|1-\theta^2|^2\,|1+\theta|^2}\le8\frac{\big(1-|\theta|^2\big)^2}{|1-\theta^2|^2\,|1+\theta|^2}.\end{aligned}$$

This gives us, with $C_1 = \sqrt{8C}$,

$$\left|P\left[f\nu_{-1}\right]\right| \leq C_1 \frac{1-|\theta|^2}{|1-\theta^2|\,|1+\theta|}\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)}. \tag{3.40}$$

We can now estimate the second summand in (3.39) by

$$\begin{aligned}\left|\frac{Q\left[\nu_1\right]P\left[f\nu_{-1}\right]}{P\left[\nu_1\right]}\right| &\leq C_1\left|\operatorname{Im}\frac{1+\theta}{1-\theta}\right|\frac{|1-\theta|^2}{1-|\theta|^2}\frac{1-|\theta|^2}{|1-\theta^2|\,|1+\theta|}\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)} \\ &\leq C_1\frac{1}{|1+\theta|}\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)},\end{aligned}$$

so by applying the reverse triangle inequality in (3.39) and using this estimate we obtain the following estimate for $Q\left[f\nu_{-1}\right]$ on Γ^φ_ξ.

$$\left|Q\left[f\nu_{-1}\right](z)\right| \leq Cf^M_{\nu_1}(\xi) + \frac{C_1}{|1+\theta(z)|}\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)}. \tag{3.41}$$

Furthermore, (3.40) yields

$$\left|P\left[f\nu_{-1}\right](z)\right| \leq \frac{C_1}{|1+\theta(z)|}\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)}. \tag{3.42}$$

With (3.42) and (3.41) we can now estimate the Poisson and conjugate Poisson transforms in (3.38), arriving at

$$\begin{aligned}|f(z)| &\leq \frac{1}{2}|1+\theta(z)|\left(Cf^M_{\nu_1}(\xi) + 2\frac{C_1}{|1+\theta(z)|}\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)}\right) \\ &\leq Cf^M_{\nu_1}(\xi) + C_1\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)}.\end{aligned}$$

Taking the supremum over $z \in \Gamma^\varphi_\xi$, we get an estimate for the nontangential maximal function f^*_φ:

$$f^*_\varphi(\xi) \leq Cf^M_{\nu_1}(\xi) + C_1\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}(\xi)}. \tag{3.43}$$

We now show that the right-hand side lies in $L^p(\nu_1)$ for $p > 2$ and in the weak $L^2(\nu_1)$ for $p = 2$, which finishes the proof. We make use of the Hardy-Littlewood maximal theorem 3.3.4.

We first consider the case $p > 2$ and recall that the boundary function of f lies in the space $L^p(\nu_1)$ as well as $L^p(\nu_{-1})$ by Corollary 3.2.7. By the Hardy-Littlewood maximal theorem, $f^M_{\nu_1}$ then belongs to $L^p(\nu_1)$. Furthermore, since $\frac{p}{2} > 1$, $\left(f^2\right)^M_{\nu_1+\nu_{-1}}$ belongs to $L^{\frac{p}{2}}(\nu_1+\nu_{-1})$ and consequently to $L^{\frac{p}{2}}(\nu_1)$. Thus, $\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}}$ lies in $L^p(\nu_1)$, and we see from (3.43) that the nontangential maximal function also lies in $L^p(\nu_1)$.

In the case $p = 2$, the Hardy-Littlewood maximal theorem implies that $f^M_{\nu_1} \in L^2(\nu_1)$, and $\left(f^2\right)^M_{\nu_1+\nu_{-1}}$ lies in the weak L^1 of $(\nu_1+\nu_{-1})$, so we have the estimate

$$\nu_1\left(\left\{\left(f^2\right)^M_{\nu_1+\nu_{-1}} > \lambda\right\}\right) \leq (\nu_1+\nu_{-1})\left(\left\{\left(f^2\right)^M_{\nu_1+\nu_{-1}} > \lambda\right\}\right) \leq \frac{c}{\lambda},$$

and consequently

$$\nu_1\left(\left\{\sqrt{\left(f^2\right)^M_{\nu_1+\nu_{-1}}} > \lambda\right\}\right) \leq \frac{c}{\lambda^2}.$$

Since every L^2-function also satisfies the weak-L^2 inequality, we deduce that in this case f^*_φ lies in the weak L^2 and the proof is complete.

□

3.5 Boundary behavior of the normalized Cauchy transform

We now take up the discussion of the nontangential boundary behavior of functions $V_\mu f$, μ being a measure in $M(\mathbb{T})$ and $f \in L^1(\mu)$. In the case of singular positive measures ν, we saw in Section 3.3 that $V_\nu f$ nontangentially tends to f almost everywhere with respect to ν. The following lemma states that for any positive measure $\mu \in M_+(\mathbb{T})$ there is a singular positive measure *close to* μ in an appropriate sense. We will use this to generalize the statement on the boundary behavior of $V_\mu f$ to arbitrary measures from the result for singular measures we already proved.

Lemma 3.5.1. *Let $\mu \in M_+(\mathbb{T})$ and let $I \subseteq \mathbb{T}$ be an open subarc of the unit circle such that $\mu(\mathbb{T}\backslash I) = 0$. Set $J = \mathbb{T}\backslash I$. Furthermore, let $f \in L^\infty(\mu)$ and $\varphi \in (0, \pi)$ be given. Then for each $\varepsilon > 0$ there is a discrete measure μ_0 with $\mu_0(J) = 0$, and a function $f_0 \in L^\infty(\mu_0)$, with the following properties:*

(i) $\|\mu_0\| \leq \|\mu\|$,

(ii) $\|f_0\|_{L^\infty(\mu_0)} \leq \|f\|_{L^\infty(\mu)}$,

(iii) $\big|K[\mu](z) - K[\mu_0](z)\big| < \varepsilon,\ z \in \Gamma_J^\varphi$,

(iv) $\big|K[f\mu](z) - K[f_0\mu_0](z)\big| < \varepsilon,\ z \in \Gamma_J^\varphi$.

Proof. We may assume that I is bounded by the points 1 and $e^{2i\alpha}$ for some $\alpha \in (0, \pi]$ (if $\alpha = 0$, μ itself is a discrete measure and there is nothing to do). We start with the following observation: Given $\delta > 0$, there is a sequence $\{\alpha_k\}_{k\in\mathbb{N}_0} \subseteq (0, \alpha]$ which satisfies the conditions

(a) $\alpha_0 = \alpha$,

(b) $\alpha_{k+1} < \alpha_k$ for $k \geq 0$, and $\alpha_k \to 0$ as $k \to \infty$,

(c) $\mu\left(\{e^{i\alpha_k}\}\right) = 0$ and $\mu\left(\{e^{i(2\alpha-\alpha_k)}\}\right) = 0$, $k \geq 0$,

(d) for each $k \in \mathbb{N}$, setting $E_k := [\alpha_k, \alpha_{k-1}]$,

$$\sup_{\substack{z\in\Gamma_J^\varphi\\ t\in E_k}} \left|\frac{1}{1 - ze^{-it}} - \frac{1}{1 - ze^{-i\alpha_k}}\right| < \delta.$$

Such a sequence can be obtained in the following way: Since any measure can have only countably many point masses, we can find a sequence $\{\alpha_k\}_{k\in\mathbb{N}_0}$ that satisfies (a), (b) and (c) of the above conditions, and which further tends to zero sufficiently slowly such that also $\frac{\alpha_{k-1}-\alpha_k}{\alpha_k^2} \to 0$. We now consider the function

$$D(t, z, \alpha_k) := \left|\frac{1}{1 - ze^{-it}} - \frac{1}{1 - ze^{-i\alpha_k}}\right| = \left|\frac{z\big(e^{-it} - e^{-i\alpha_k}\big)}{\big(1 - ze^{-it}\big)\big(1 - ze^{-i\alpha_k}\big)}\right|, \quad t \in E_k,\ z \in \Gamma_J^\varphi.$$

If z is bounded away from the arc $\{e^{it}, t \in E_k\}$, the denominator is bounded away from zero and this function becomes small if only t is sufficiently close to α_k. But the denominator becomes small if z is close to this arc. So in order to obtain an estimate as in (d), it will be crucial to estimate the denominator from below in the part of Γ_J^φ closest to the arc $\{e^{it}, t \in (0,\alpha)\}$. We set $\omega := \frac{\varphi}{2}$ and $r_0 := \sin\omega$ and define the set Ω as

$$\Omega := \Gamma_J^\varphi \cap \left\{ z = re^{i\vartheta} \;\middle|\; r_0 \le r,\ |\vartheta| \le \frac{\pi}{2} - \omega,\ \text{and if } \vartheta > 0,\ r \le \frac{\sin\omega}{\sin(\omega+\vartheta)} \right\},$$

see Figure 3.3. Our goal now is to derive an estimate of $D(t, z, \alpha_k)$ independent of $t \in E_k$

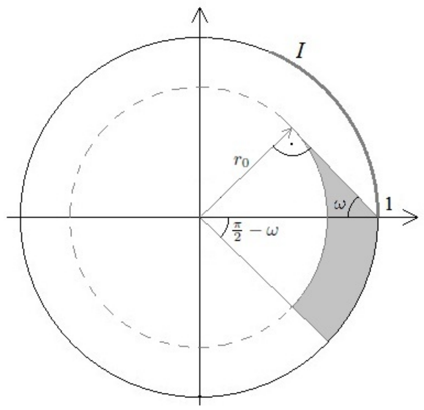

Figure 3.3: The region Ω.

and $z \in \Omega$. We define the following constants.

$$M_1 := \sup\left\{ \left|\frac{1-e^t}{t}\right|,\ |t| \le 2\pi \right\}; \tag{3.44}$$

$$m_1 := \inf\left\{ \frac{\sin(\omega+t) - \sin\omega}{t},\ |t| \le \frac{\pi}{2} - \omega \right\}; \tag{3.45}$$

$$m_2 := \inf\left\{ \frac{\sin t}{t},\ |t| \le \frac{\pi}{2} \right\}. \tag{3.46}$$

Each of the functions above is continuous and nonzero at $t = 0$, and we see that these are continuous positive functions on the respective closed intervals. Consequently M_1, m_1 and m_2 are finite and positive.
We further note that for any $\tau \in \mathbb{R}$ we get the estimate for $r \in (0,1)$

$$\left|1 - re^{i\tau}\right|^2 = \left(1 - r\cos\tau\right)^2 + r^2\sin^2\tau = 1 - 2r\cos\tau + r^2 \ge 1 - \cos^2\tau = \sin^2\tau. \tag{3.47}$$

We fix an index $k_0 \in \mathbb{N}$ such that

$$\alpha_{k_0-1} < \omega \ \text{ and } \ \alpha_{k_0} < \frac{\pi}{3}. \tag{3.48}$$

Let $k \ge k_0$. We estimate the denominator of $D(t, z, \alpha_k)$ for $z = re^{i\vartheta} \in \Omega$ and $t \in E_k$, considering three cases depending on the argument of z:
1. $\vartheta \ge \frac{\alpha_k}{2}$. In this case we get for any $t \in \mathbb{R}$, using the estimate $|z| \le \frac{\sin\omega}{\sin(\omega+\vartheta)}$ as well as (3.45),

$$\begin{aligned} \left|1 - ze^{-it}\right|\left|1 - ze^{-i\alpha_k}\right| &\ge \left(1 - |z|\right)^2 \ge \left(1 - \frac{\sin\omega}{\sin(\omega+\vartheta)}\right)^2 \\ &= \left[\frac{1}{\sin(\omega+\vartheta)} \underbrace{\frac{\sin(\omega+\vartheta) - \sin\omega}{\vartheta}}_{\ge m_1} \vartheta\right]^2 \ge m_1^2\vartheta^2 \ge \frac{m_1^2}{4}\alpha_k^2. \end{aligned}$$

2. $|\vartheta| \le \frac{\alpha_k}{2}$. We employ the estimate (3.47) to get

$$\left|1 - ze^{-it}\right|^2 \left|1 - ze^{-i\alpha_k}\right|^2 \ge \sin^2(\vartheta - t)\sin^2(\vartheta - \alpha_k). \tag{3.49}$$

As the α_k are monotonically decreasing, we can use (3.48) and we see that the difference $\alpha_k - \vartheta$ is now bounded from below and above by

$$|\alpha_k - \vartheta| \ge \alpha_k - |\vartheta| \ge \frac{\alpha_k}{2} \quad \text{and} \quad |\alpha_k - \vartheta| \le \alpha_k + |\vartheta| \le \frac{3\alpha_k}{2} < \frac{\pi}{2},$$

and letting $t \in [\alpha_k, \frac{\pi}{2} - \omega]$ we further obtain

$$t - \vartheta \ge \frac{\alpha_k}{2} \quad \text{and} \quad t - \vartheta \le \frac{\pi}{2} - \omega + \frac{\alpha_k}{2} < \frac{\pi}{2}.$$

Hence, with the constant m_2 from (3.46) we have

$$\sin(t - \vartheta) \ge \sin\frac{\alpha_k}{2} = \frac{\sin\frac{\alpha_k}{2}}{\frac{\alpha_k}{2}}\frac{\alpha_k}{2} \ge \frac{m_2}{2}\alpha_k,$$

as well as $|\sin(\alpha_k - \vartheta)| \ge \frac{m_2}{2}\alpha_k$. Applying these estimates in (3.49), we obtain

$$\left|1 - ze^{-it}\right|\left|1 - ze^{-i\alpha_k}\right| \ge \frac{m_2^2}{4}\alpha_k^2. \tag{3.50}$$

3. $\vartheta \le -\frac{\alpha_k}{2}$. Letting $t \in [\alpha_k, \alpha_{k_0 - 1}]$ we get the estimates, making use of (3.48) again,

$$\frac{\alpha_k}{2} \le \alpha_k - \vartheta \le \alpha_k + \frac{\pi}{2} - \omega < \frac{\pi}{2} \quad \text{and} \quad \frac{\alpha_k}{2} \le t - \vartheta \le \alpha_{k_0 - 1} + \frac{\pi}{2} - \omega < \frac{\pi}{2},$$

and as in the previous step we see that in this case (3.50) holds for $t \in [\alpha_k, \alpha_{k_0-1}]$. Setting $\widetilde{m} := \min\left\{\frac{m_1^2}{4}, \frac{m_2^2}{4}\right\}$, we infer that if $k \ge k_0$, for all $z \in \Omega$ and $t \in E_k$ we can estimate the denominator of $D\,(t, z, \alpha_k)$ by

$$\left|1 - ze^{-it}\right|\left|1 - ze^{-i\alpha_k}\right| \ge \widetilde{m}\alpha_k^2. \tag{3.51}$$

Note that since for $z \in \Gamma_J^\varphi \backslash \Omega$ and $t \in E_k$ we have $\left|1 - ze^{-it}\right| \ge \operatorname{dist}(\Gamma_J^\varphi \backslash \Omega, E_k) > 0$, by in case modifying $\widetilde{m} > 0$, we may assume this estimate holds for all $z \in \Gamma_J^\varphi$ and $t \in E_k$. Furthermore, for $z \in \Gamma_J^\varphi$ and $t \in E_k$ we can use (3.44) to estimate the numerator of $D\,(t, z, \alpha_k)$ by

$$\left|z\left(e^{-it} - e^{-i\alpha_k}\right)\right| \le \left|1 - e^{i(\alpha_k - t)}\right| \le M_1\left|\alpha_k - t\right| \le M_1\left(\alpha_{k-1} - \alpha_k\right). \tag{3.52}$$

We set $M := M_1\widetilde{m}^{-1}$ and note that this constant depends only on the angle φ. We now obtain the estimate for $k \ge k_0$

$$\sup_{\substack{z\in\Gamma_J^\varphi \\ t \in E_k}} D\,(t, z, \alpha_k) = \sup_{\substack{z\in\Gamma_J^\varphi \\ t \in E_k}} \left|\frac{z\left(e^{-it} - e^{-i\alpha_k}\right)}{\left(1 - ze^{-it}\right)\left(1 - ze^{-i\alpha_k}\right)}\right| \le M\frac{\alpha_{k-1} - \alpha_k}{\alpha_k^2}.$$

By our choice of the sequence $\{\alpha_k\}$, for given $\varepsilon_0 > 0$ there is an index $k_1 \ge k_0$ such that $\frac{\alpha_{k-1} - \alpha_k}{\alpha_k^2} < \varepsilon_0$ for $k > k_1$. So if we choose $\varepsilon_0 < \frac{\delta}{M}$, we get that

$$\sup_{\substack{z\in\Gamma_J^\varphi \\ t \in E_k}} D\,(t, z, \alpha_k) < \delta$$

if only $k > k_1$. We now redefine the first part of the sequence to obtain such an estimate for all indices. On the interval $[\alpha_{k_1}, \alpha]$ we have

$$\inf\left\{ \left|1 - ze^{-it}\right|,\ z \in \Gamma_J^\varphi,\ t \in [\alpha_{k_1}, \alpha] \right\} = K > 0.$$

We choose equidistant points $\beta_0 := \alpha$, β_1, ..., $\beta_L := \alpha_{k_1}$ in $[\alpha_{k_1}, \alpha]$ such that $\beta_{l-1} - \beta_l < \frac{K^2\delta}{M_1}$. Using the estimate (3.52) for the numerator, we obtain for $l = 1, \ldots, L$

$$\sup_{\substack{z\in\Gamma_J^\varphi \\ t\in[\beta_l,\beta_{l-1}]}} D(t, z, \beta_l) \leq \sup_{\substack{z\in\Gamma_J^\varphi \\ t\in[\beta_l,\beta_{l-1}]}} \frac{\left|z\left(e^{-it} - e^{-i\beta_l}\right)\right|}{K^2} \leq \frac{M_1\left(\beta_{l-1} - \beta_l\right)}{K^2} < \delta.$$

By if necessary shifting the points β_l slightly, we can assume that they satisfy condition (c) above. Now the sequence $\{\hat{\alpha}_k\}_{k\in\mathbb{N}}$ defined as

$$\hat{\alpha}_k := \begin{cases} \beta_k, & k = 0, \ldots, L, \\ \alpha_{k_1+k-L}, & k > L, \end{cases}$$

satisfies the conditions (a)-(d).

Now let $\varepsilon > 0$ be given. We choose $\delta < \min\left\{\frac{\varepsilon}{\|\mu\|}, \frac{\varepsilon}{\|\mu\|\|f\|_{L^\infty(\mu)}}\right\}$ and a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ satisfying (a)-(d). We use this sequence to partition I into subarcs I_k, $k \in \mathbb{Z}$, and then construct a discrete measure with point masses on the boundary points of those arcs that satisfies the required conditions.

For $k \in \mathbb{N}$ we denote $\alpha_{-k} = 2\alpha - \alpha_k$ and set $I_k := \{e^{it}, t \in (\alpha_k, \alpha_{k-1})\}$ and $I_{-k} = \{e^{it}, t \in (\alpha_{-k+1}, \alpha_{-k})\}$. We define a discrete positive measure μ_0 by

$$\mu_0\left(\{e^{i\alpha_k}\}\right) = \mu(I_k) \quad \text{and} \quad \mu_0\left(\{e^{i\alpha_{-k}}\}\right) = \mu(I_{-k}), \quad k \in \mathbb{N},$$

and the function f_0 by

$$f_0\left(e^{i\alpha_k}\right) = \frac{1}{\mu(I_k)} \int_{I_k} f(\zeta)\, d\mu(\zeta) \quad \text{and} \quad f_0\left(e^{-i\alpha_{-k}}\right) = \frac{1}{\mu(I_{-k})} \int_{I_{-k}} f(\zeta)\, d\mu(\zeta), \quad k \in \mathbb{N}.$$

Now we verify that μ_0 and f_0 have the properties (i)-(iv) stated in the lemma. As by the choice of $\{\alpha_k\}$, the measure μ has no point masses at the points $e^{i\alpha_k}$, $k \in \mathbb{Z}$, we have

$$\|\mu_0\| = \int_{\mathbb{T}} d\mu_0 = \sum_{k\in\mathbb{N}} \left(\mu(I_k) + \mu(I_{-k})\right) = \|\mu\|.$$

We can estimate the L^∞-norm of the function f_0 by

$$\begin{aligned} \|f_0\|_{L^\infty(\mu_0)} &= \sup_{k\in\mathbb{Z}\setminus\{0\}} \left|f_0\left(e^{i\alpha_k}\right)\right| = \sup_{k\in\mathbb{Z}\setminus\{0\}} \left| \frac{1}{\mu(I_k)} \int_{I_k} f(\zeta)\, d\mu(\zeta) \right| \\ &\leq \sup_{k\in\mathbb{Z}\setminus\{0\}} \left| \frac{1}{\mu(I_k)} \|f\|_{L^\infty(\mu)} \mu(I_k) \right| = \|f\|_{L^\infty(\mu)}. \end{aligned} \tag{3.53}$$

Furthermore, using again the fact that μ has no point masses at $e^{i\alpha_k},\ k \in \mathbb{Z}$, and that μ is a positive measure, we get for $z \in \Gamma_J^{\varphi}$

$$\begin{aligned} \big|K[\mu](z) - K[\mu_0](z)\big| &= \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\int_{I_k} \frac{1}{1-\bar{\zeta}z} d\mu(\zeta) - \mu(I_k) \frac{1}{1-ze^{-i\alpha_k}} \right) \right| \\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} \left| \int_{I_k} \frac{1}{1-\bar{\zeta}z} - \frac{1}{1-ze^{-i\alpha_k}} d\mu(\zeta) \right| \\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} \mu(I_k) \sup_{\zeta\in I_k} \left| \frac{1}{1-\bar{\zeta}z} - \frac{1}{1-ze^{-i\alpha_k}} \right| \le \|\mu\|\delta < \varepsilon. \end{aligned}$$

The same estimates and the use of the essential boundedness of f yield

$$\big|K[f\mu](z) - K[f_0\mu_0](z)\big| \le \|f\|_{L^\infty(\mu)}\|\mu\|\delta < \varepsilon,$$

and we see that the measure μ_0 and function f_0 satisfy the required conditions.

□

Theorem 3.5.2. *Let $\mu \in M(\mathbb{T})$ and $f \in L^1(\mu)$. Then for μ-almost every $\xi \in \mathbb{T}$ the limit*

$$\lim_{\substack{z\to\xi \\ \sphericalangle}} V_\mu f(z)$$

exists. Furthermore, the limit equals $f(\xi)$ almost everywhere with respect to the singular component of μ.

Proof. We do the proof in three steps: The first and crucial part is to prove the statement for $\mu \in M_+(\mathbb{T})$ and $f \in L^\infty(\mu)$. Then it is easy to pass to the case of $f \in L^1(\mu)$ for positive μ, and in the third step to the general case of an arbitrary $\mu \in M(\mathbb{T})$ and $f \in L^1(\mu)$.
So let us start with $\mu \in M_+(\mathbb{T})$, $f \in L^\infty(\mu)$. We denote $V_\mu f$ by F. Theorem 3.2.5 tells us that in this case $F \in H^2$, consequently F has nontangential limits almost everywhere with respect to σ, and hence also with respect to the absolutely continuous component of μ. We denote the singular component of μ by μ_s. What we have to show now is that boundary values of F also exist μ_s-almost everywhere, and that they equal f almost everywhere with respect to μ_s. We will do this by employing Lemma 3.5.1 to construct sequences of singular measures, for which we can apply Theorem 3.3.11, and then using a limit argument. The lemma requires a measure supported on an open subarc, so we have to consider appropriate restrictions of μ to such sets.
To this end, let $E \subseteq \mathbb{T}$ be such that $\mu_s(E) = \|\mu_s\|$ and $\sigma(E) = 0$. Since μ_s is regular and finite, we can find a sequence $\{E_N\}_{N\in\mathbb{N}}$ of closed subsets of E such that $\mu_s(E) = \mu_s\Big(\bigcup_{N\in\mathbb{N}} E_N\Big)$. Hence, $\bigcup_{N\in\mathbb{N}} E_N$ is a support of μ_s, and we may assume without loss of generality that $E = \bigcup_{N\in\mathbb{N}} E_N$.
We now fix some $N \in \mathbb{N}$. The set $\mathbb{T}\setminus E_N$ is open, therefore it can be written as $\bigcup_{k\in\mathbb{N}} I_k$ with pairwise disjoint open subarcs I_k of $\mathbb{T}$. Now we apply Lemma 3.5.1 to each of the I_k and the restriction of μ to these arcs: Given $\varepsilon > 0$ and $\varphi \in (0, \pi)$, for every $k \in \mathbb{N}$ we get a discrete measure μ_k^ε supported on I_k and a function $f_k^\varepsilon \in L^\infty(\mu_k^\varepsilon)$ such that $\|\mu_k^\varepsilon\| \le \mu(I_k)$,

$\|f_k^\varepsilon\|_{L^\infty(\mu_k^\varepsilon)} \leq \|f\|_{L^\infty(\mu)}$, and, for each $z \in \Gamma^\varphi_{\mathbb{T}\backslash I_k}$,

$$\begin{aligned}\left|K\left[\mu|_{I_k}\right](z) - K\left[\mu_k^\varepsilon\right](z)\right| &= \left|\int_{I_k} \frac{1}{1-\bar{\zeta}z} d\mu(\zeta) - \int_{I_k} \frac{1}{1-\bar{\zeta}z} d\mu_k^\varepsilon(\zeta)\right| < \frac{\varepsilon}{2^k},\\ \left|K\left[f\mu|_{I_k}\right](z) - K\left[f_k^\varepsilon\mu_k^\varepsilon\right](z)\right| &= \left|\int_{I_k} \frac{f(\zeta)}{1-\bar{\zeta}z} d\mu(\zeta) - \int_{I_k} \frac{f_k^\varepsilon(\zeta)}{1-\bar{\zeta}z} d\mu_k^\varepsilon(\zeta)\right| < \frac{\varepsilon}{2^k}.\end{aligned}$$

We define a singular measure as $\mu_\varepsilon = \sum_{k\in\mathbb{N}} \mu_k^\varepsilon + \chi_{E_N}\mu_s$, and let f_ε be such that $f_\varepsilon = f$ almost everywhere with respect to $\mu_s\big|_{E_N}$ and $f_\varepsilon = f_k^\varepsilon$ almost everywhere with respect to μ_k^ε. Then μ_ε satisfies

$$\|\mu_\varepsilon\| \leq \sum_{k\in\mathbb{N}} \mu(I_k) + \mu_s(E_N) \leq \mu(\mathbb{T}\backslash E_N) + \mu(E_N) \leq \|\mu\|$$

and since $\|f_k^\varepsilon\|_{L^\infty(\mu_k^\varepsilon)} \leq \|f\|_{L^\infty(\mu)}$, for f_ε we have $\|f_\varepsilon\|_{L^\infty(\mu_\varepsilon)} \leq \|f\|_{L^\infty(\mu)}$. We also have the following estimate for all $z \in \Gamma^\varphi_{\mathbb{T}\backslash\bigcup I_k} = \Gamma^\varphi_{E_N}$,

$$\begin{aligned}\left|K\left[\mu_\varepsilon\right](z) - K\left[\mu\right](z)\right| &= \left|\sum_{k\in\mathbb{N}}\int_{I_k} \frac{1}{1-\bar{\zeta}z} d\mu_k^\varepsilon + \int_{E_N} \frac{1}{1-\bar{\zeta}z} d\mu_s - \int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z} d\mu\right|\\ &= \left|\sum_{k\in\mathbb{N}}\left(\int_{I_k} \frac{1}{1-\bar{\zeta}z} d\mu_k^\varepsilon - \int_{I_k} \frac{1}{1-\bar{\zeta}z} d\mu\right)\right.\\ &\qquad \left. + \underbrace{\int_{E_N} \frac{1}{1-\bar{\zeta}z} d\mu_s - \int_{E_N} \frac{1}{1-\bar{\zeta}z} d\mu}_{=0}\right|\\ &\leq \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^k} = \varepsilon,\end{aligned}$$

and analogously $|K[f\mu](z) - K[f_\varepsilon\mu_\varepsilon](z)| \leq \varepsilon$. Using $|K[\mu]| \geq \frac{1}{2}\|\mu\|$, we now get the following estimate for the difference of the normalized Cauchy transforms F and $F_\varepsilon = V_{\mu_\varepsilon} f_\varepsilon$ on $\Gamma^\varphi_{E_N}$.

$$\begin{aligned}\left|F(z) - F_\varepsilon(z)\right| &= \left|\frac{K[f\mu](z)}{K[\mu](z)} - \frac{K[f_\varepsilon\mu_\varepsilon](z)}{K[\mu_\varepsilon](z)}\right|\\ &\leq \left|\frac{K[f\mu](z)}{K[\mu](z)} - \frac{K[f_\varepsilon\mu_\varepsilon](z)}{K[\mu](z)}\right| + \left|\frac{K[f_\varepsilon\mu_\varepsilon](z)}{K[\mu](z)} - \frac{K[f_\varepsilon\mu_\varepsilon](z)}{K[\mu_\varepsilon](z)}\right|\\ &= \frac{1}{|K[\mu](z)|}\left|K[f\mu](z) - K[f_\varepsilon\mu_\varepsilon](z)\right|\\ &\qquad + \frac{|F_\varepsilon(z)|}{|K[\mu](z)|}\left|K[\mu_\varepsilon](z) - K[\mu](z)\right|\\ &\leq \frac{2\varepsilon}{\|\mu\|}\big(1 + |F_\varepsilon(z)|\big).\end{aligned}$$

This yields

$$|F(z)| \leq \frac{2\varepsilon}{\|\mu\|}\big(1 + |F_\varepsilon(z)|\big) + |F_\varepsilon(z)|.$$

If we let $\xi \in E_N$ and take the supremum over $z \in \Gamma_\xi^\varphi$, we obtain

$$F_\varphi^*(\xi) \leq \frac{2\varepsilon}{\|\mu\|}\big(1 + (F_\varepsilon)_\varphi^*(\xi)\big) + (F_\varepsilon)_\varphi^*(\xi).$$

Since μ_ε is a positive singular measure and $f_\varepsilon \in L^\infty(\mu_\varepsilon)$, Theorem 3.2.5 and its Corollary 3.2.6 imply that $F_\varepsilon \in \theta^*(H^2)$, and thus by Theorem 3.4.2 its nontangential maximal function $(F_\varepsilon)_\varphi^*$ is finite almost everywhere with respect to μ_ε. As $\mu_\varepsilon\big|_{E_N} = \mu\big|_{E_N}$, $(F_\varepsilon)_\varphi^*$ is finite μ-almost everywhere on E_N. Above estimate now shows that also F_φ^* is finite μ-almost everywhere on E_N.
Furthermore, we can apply Corollary 3.3.12 for the measure μ_ε and function f_ε, which gives us that for μ_ε-almost every $\xi \in \mathbb{T}$,

$$F_\varepsilon(z) = V_{\mu_\varepsilon} f_\varepsilon(z) \longrightarrow f_\varepsilon(\xi) \quad \text{as} \quad z \underset{\sphericalangle}{\to} \xi.$$

As by construction $\mu_\varepsilon\big|_{E_N} = \mu\big|_{E_N}$ and $f_\varepsilon\big|_{E_N} = f\big|_{E_N}$, this implies that for μ-almost every $\xi \in E_N$ we have

$$\lim_{z \underset{\sphericalangle}{\to} \xi} F_\varepsilon(z) = f(\xi). \tag{3.54}$$

We fix an angle $\varphi \in (0, \pi)$ and let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers tending to zero. For each ε_j we construct a singular measure μ_{ε_j} and the functions f_{ε_j} and F_{ε_j} as above for ε. Let $\xi \in E_N$ be a point which satisfies $F_\varphi^*(\xi) < +\infty$ and at which for every $j \in \mathbb{N}$,

$$\lim_{z \underset{\sphericalangle}{\to} \xi} F_{\varepsilon_j}(z) = f(\xi). \tag{3.55}$$

As established above, this is satisfied by μ-almost every point in E_N (note that a countable union of nullsets is a nullset). Using the estimate $\big|K\left[\mu_{\varepsilon_j}\right]\big| \geq \frac{\|\mu_{\varepsilon_j}\|}{2} \geq \frac{\mu(E_N)}{2}$, we then have for $z \in \Gamma_\xi^\varphi$

$$\begin{aligned}
\big|F(z) - F_{\varepsilon_j}(z)\big| &= \left|\frac{K[f\mu](z)}{K[\mu](z)} - \frac{K\left[f_{\varepsilon_j}\mu_{\varepsilon_j}\right](z)}{K\left[\mu_{\varepsilon_j}\right](z)}\right| \\
&\leq \left|\frac{K[f\mu](z)}{K[\mu](z)} - \frac{K[f\mu](z)}{K\left[\mu_{\varepsilon_j}\right](z)}\right| + \left|\frac{K[f\mu](z)}{K\left[\mu_{\varepsilon_j}\right](z)} - \frac{K\left[f_{\varepsilon_j}\mu_{\varepsilon_j}\right](z)}{K\left[\mu_{\varepsilon_j}\right](z)}\right| \\
&= \left|\frac{F(z)}{K\left[\mu_{\varepsilon_j}\right](z)}\right| \Big|K\left[\mu_{\varepsilon_j}\right](z) - K[\mu](z)\Big| \\
&\quad + \frac{1}{\big|K\left[\mu_{\varepsilon_j}\right](z)\big|}\Big|K[f\mu](z) - K\left[f_{\varepsilon_j}\mu_{\varepsilon_j}\right](z)\Big| \\
&\leq \frac{2\varepsilon_j}{\mu(E_N)}\big(|F(z)| + 1\big) \leq \frac{2\varepsilon_j}{\mu(E_N)}\big(F_\varphi^*(\xi) + 1\big).
\end{aligned}$$

The right-hand side is finite and we see that F_{ε_j} tends to F uniformly in the region Γ_ξ^φ as $\varepsilon_j \to 0$. Thus, by the theorem on uniform convergence ([20, Theorem 7.11]) we obtain for z tending to ξ from within Γ_ξ^φ

$$\lim_{j\to\infty}\lim_{z\to\xi}|F(z) - f(\xi)| = \lim_{j\to\infty}\lim_{z\to\xi}\big|F(z) - F_{\varepsilon_j}(z)\big| = \lim_{z\to\xi}\lim_{j\to\infty}\big|F(z) - F_{\varepsilon_j}(z)\big| = 0,$$

i.e., $F(z)$ tends to $f(\xi)$ from within the set Γ_ξ^φ. Since the angle φ was arbitrary in $(0, \pi)$, we conclude that for μ-almost every $\xi \in E_N$, $F(z)$ tends to $f(\xi)$ as z nontangentially approaches ξ. This holds for each $N \in \mathbb{N}$, so recalling that $\bigcup_{N\in\mathbb{N}} E_N = E$ and that E supports μ_s, this finishes the first part of the proof.

Now let $\mu \in M_+(\mathbb{T})$ and $f \in L^1(\mu)$. As V_μ is linear, we can assume $f \geq 0$. Then the function $g = \frac{1}{1+f}$ is a bounded holomorphic function. We consider the measure $\nu = (1+f)\,\mu$ which is absolutely continuous with respect to μ, and note that $g \in L^\infty(\nu)$. By the first part of the proof, $V_\nu g$ has the nontangential limit g almost everywhere with respect to ν_s, and the support of ν_s is the same as that of μ_s. We use the linearity of the Cauchy transform and calculate

$$V_\nu g(z) = \frac{K[g\nu](z)}{K[\nu](z)} = \frac{K[\mu](z)}{K[(1+f)\,\mu](z)} = \frac{1}{1+V_\mu f(z)} \underset{z \underset{\sphericalangle}{\to} \xi}{\longrightarrow} g(\xi) = \frac{1}{1+f(\xi)},$$

hence $V_\mu f(z) \to f(\xi)$ as z nontangentially approaches ξ, for μ_s-almost every $\xi \in \mathbb{T}$.

We finally come to the general case of $\mu \in M(\mathbb{T})$ and $f \in L^1(\mu)$. Let $g \in L^1(|\mu|)$ be such that $g\,|\mu| = \mu$. Then $|g| = 1$ almost everywhere with respect to μ. We note the relation

$$\frac{V_{|\mu|} fg}{V_{|\mu|} g} = \frac{\frac{K[fg|\mu|]}{K[|\mu|]}}{\frac{K[g|\mu|]}{K[|\mu|]}} = \frac{K[fg\,|\mu|]}{K[g\,|\mu|]} = V_\mu f.$$

By the previous parts of the proof (note that g is essentially bounded), the fraction on the left-hand side tends to $\frac{(fg)(\xi)}{g(\xi)} = f(\xi)$ nontangentially at $|\mu_s|$-almost every point, and this completes the proof.

$\square$

Chapter 4

Measures with singular components

4.1 Definition of the Classes I-IV

We now continue the discussion of the boundary behavior of the Poisson and conjugate Poisson transform that we started in Chapter 2, where we showed that for absolutely continuous $\mu \in M(\mathbb{T})$ both transforms have finite boundary values almost everywhere on $\mathbb{T}$. We also already saw that the Poisson transform tends to infinity at almost every point on $\mathbb{T}$ with respect to the singular component of the measure. We now take a closer look at the mode of growth of the transforms, precisely, we want to compare the speed of growth of the Poisson and conjugate Poisson transform and try to provide classifications of the sets of measures for which either transform grows faster than or at least as fast as the other.

Definition 4.1.1. A measure $\mu \in M(\mathbb{T})$ belongs to Class N, $N = I, II, III$ or IV, if for μ-almost every $\xi \in \mathbb{T}$ the following condition holds:

$$
\begin{aligned}
(I) &\qquad P[\mu](z) = o(Q[\mu](z)) &&\text{as } z \underset{\sphericalangle}{\to} \xi; \\
(II) &\qquad Q[\mu](z) = o(P[\mu](z)) &&\text{as } z \underset{\sphericalangle}{\to} \xi; \\
(III) &\qquad P[\mu](z) = O(Q[\mu](z)) &&\text{as } z \underset{\sphericalangle}{\to} \xi; \\
(IV) &\qquad Q[\mu](z) = O(P[\mu](z)) &&\text{as } z \underset{\sphericalangle}{\to} \xi.
\end{aligned}
$$

In the following sections we examine each of those classes, collect properties of the measures therein, and give conditions which insure that a measure belongs to a certain class. As of now, a complete description of most of them is lacking.

What we can say about all four classes is that for each singular measure μ lying in Class N, all measures absolutely continuous with respect to μ also belong to the same class, i.e., for any $f \in L^1(|\mu|)$, $f\mu$ also lies in Class N. This is a consequence of Theorem 3.5.2; we provide the following somewhat stronger formulation.

Theorem 4.1.2. *Let $\mu \in M_+(\mathbb{T})$, $f \in L^1(\mu)$ and let ν be the measure absolutely continuous with respect to μ with density f. Then, at ν_s-almost every $\xi \in \mathbb{T}$,*

$$
\lim_{z \underset{\sphericalangle}{\to} \xi} \left(\frac{Q[\nu](z)}{P[\nu](z)} \right) \left(\frac{Q[\mu](z)}{P[\mu](z)} \right)^{-1} = 1.
$$

In particular, if μ_s belongs to Class N in Definition 4.1.1, ν_s belongs to the same class.

Proof. We decompose $\nu = \nu_{ac} + \nu_s = f\mu_{ac} + f\mu_s$. By Theorem 3.5.2, we have that

$$\lim_{z \underset{\sphericalangle}{\to} \xi} \frac{K[\nu](z)}{K[\mu](z)} = f(\xi) \tag{4.1}$$

at μ_s-almost every $\xi \in \mathbb{T}$, and consequently also ν_s-almost everywhere. Using (1.13), we can write this limit as

$$\begin{aligned} \lim_{z \underset{\sphericalangle}{\to} \xi} \frac{K[\nu](z)}{K[\mu](z)} &= \lim_{z \underset{\sphericalangle}{\to} \xi} \frac{P[\nu](z) + iQ[\nu](z) + \int_{\mathbb{T}} d\nu}{P[\mu](z) + Q[\mu](z) + \int_{\mathbb{T}} d\mu} \\ &= \lim_{z \underset{\sphericalangle}{\to} \xi} \frac{P[\nu](z)}{P[\mu](z)} \underbrace{\frac{1 + i\frac{Q[\nu](z)}{P[\nu](z)} + \frac{\int_{\mathbb{T}} d\nu}{P[\nu](z)}}{1 + i\frac{Q[\mu](z)}{P[\mu](z)} + \frac{\int_{\mathbb{T}} d\mu}{P[\mu](z)}}}_{=:A} . \end{aligned}$$

By Proposition 3.3.7, $\frac{P[\nu](z)}{P[\mu](z)}$ tends to $f(\xi)$ as $z \underset{\sphericalangle}{\to} \xi$ almost everywhere with respect to μ, hence also almost everywhere with respect to ν. Furthermore, we know from Theorem 2.2.4 that $P[\mu]$ and $P[\nu]$ tend to infinity at almost all $\xi \in \mathbb{T}$ with respect to μ_s and ν_s, respectively. Now let $\xi \in \mathbb{T}$ such that

(i) $f(\xi) \neq 0$;

(ii) $\frac{K[\nu](z)}{K[\mu](z)} \to f(\xi)$ and $\frac{P[\nu](z)}{P[\mu](z)} \to f(\xi)$ as $z \underset{\sphericalangle}{\to} \xi$;

(iii) $P[\mu](z) \to \infty$ and $P[\nu](z) \to \infty$ as $z \underset{\sphericalangle}{\to} \xi$.

Note that ν_s-almost every $\xi \in \mathbb{T}$ has these properties. For such ξ, we see from the above relation that the quotient A must tend to 1, equivalently,

$$B := |A - 1| = \left| \frac{i\left(\frac{Q[\nu](z)}{P[\nu](z)} - \frac{Q[\mu](z)}{P[\mu](z)}\right) + \left(\frac{\int_{\mathbb{T}} d\nu}{P[\nu](z)} - \frac{\int_{\mathbb{T}} d\mu}{P[\mu](z)}\right)}{1 + i\frac{Q[\mu](z)}{P[\mu](z)} + \frac{\int_{\mathbb{T}} d\mu}{P[\mu](z)}} \right| \longrightarrow 0 \quad \text{as } z \underset{\sphericalangle}{\to} \xi. \tag{4.2}$$

We now consider the possible cases of asymptotic behavior of $\frac{Q[\mu](z)}{P[\mu](z)}$ and the conclusions we can draw for the behavior of $\frac{Q[\nu](z)}{P[\nu](z)}$:

Case 1. $\left|\frac{Q[\mu](z)}{P[\mu](z)}\right| \to c \in [0, \infty)$ as $z \underset{\sphericalangle}{\to} \xi$.

Under this assumption, the denominator of B tends to some finite nonzero value. The right summand in the numerator tends to zero as the Poisson transforms tend to infinity. So, for B to converge to zero, we conclude that $\left|\frac{Q[\nu](z)}{P[\nu](z)} - \frac{Q[\mu](z)}{P[\mu](z)}\right| \to 0$, which implies that $\frac{Q[\nu](z)}{P[\nu](z)}$ even converges to the same value c as $\frac{Q[\mu](z)}{P[\mu](z)}$.

Case 2. $\frac{Q[\mu](z)}{P[\mu](z)} \to \infty$ as $z \underset{\sphericalangle}{\to} \xi$.

In this case we divide the numerator and denominator of B by $\frac{Q[\mu](z)}{P[\mu](z)}$ to obtain

$$B = \left| \frac{i\left(\frac{Q[\nu](z)}{P[\nu](z)}\left(\frac{Q[\mu](z)}{P[\mu](z)}\right)^{-1} - 1\right) + \left(\frac{\int_{\mathbb{T}} d\nu}{P[\nu](z)} - \frac{\int_{\mathbb{T}} d\mu}{P[\mu](z)}\right)\left(\frac{Q[\mu](z)}{P[\mu](z)}\right)^{-1}}{\left(\frac{Q[\mu](z)}{P[\mu](z)}\right)^{-1} + i + \frac{\int_{\mathbb{T}} d\mu}{P[\mu](z)}\left(\frac{Q[\mu](z)}{P[\mu](z)}\right)^{-1}} \right| .$$

The denominator now tends to 1, and in order for the whole expression to become zero we see that $\frac{Q[\nu](z)}{P[\nu](z)}\left(\frac{Q[\mu](z)}{P[\mu](z)}\right)^{-1}$ must converge to 1, thus $\frac{Q[\nu](z)}{P[\nu](z)}$ must tend to infinity at the same rate as $\frac{Q[\mu](z)}{P[\mu](z)}$.
We see that at ν_s-almost every point on $\mathbb{T}$, the measure ν satisfies the same condition (N) as μ. In particular, if μ_s belongs to Class N, i.e., condition (N) holds almost everywhere with respect to μ_s, the same condition holds almost everywhere with respect to ν_s and thus ν_s belongs to the same class.

□

By virtue of this statement, it suffices to consider positive measures in the proofs of many of the following statements, since any complex measure μ is absolutely continuous with respect to its total variation $|\mu|$.

4.2 Class I

We start with the consideration of Class I, i.e., the measures $\mu \in M(\mathbb{T})$ such that

$$P[\mu](z) = o(Q[\mu](z)) \quad \text{as} \quad z \underset{\sphericalangle}{\to} \xi \tag{I}$$

at μ-almost every $\xi \in \mathbb{T}$.
We first observe that if μ is absolutely continuous, $\mu = f\sigma$ with $f \in L^1(\sigma)$, condition (I) can only hold for a μ-nullset: For in this case, Theorem 2.3.1 tells us that $Q[\mu]$ has a finite limit at μ-almost every boundary point. So in order for (I) to be satisfied, $P[\mu]$ would have to tend to zero at μ-almost every point. But by Theorem 2.2.3, the nontangential limit of the Poisson transform equals the density f which is nonzero almost everywhere with respect to μ. Consequently, Class I does not contain any absolutely continuous measures. Now if we consider a measure $\mu \in M(\mathbb{T})$ with singular component μ_s, the Cauchy transform of μ tends to infinity at almost every boundary point with respect to the singular component. There is no immediate answer to whether the relation (I) can hold for singular measures. We shall see eventually that no singular measure $\mu \in M(\mathbb{T})$ can satisfy condition (I) on a set with positive μ-measure either, whence we deduce that Class I is empty. Before we come to the main result we have some preparatory work to do.

Let μ be a positive measure, then $P[\mu](z) > 0$ in $\mathbb{D}$, and hence $K[\mu]$ maps $\mathbb{D}$ into the right half plane $\{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$. Consequently, the rotated function $iK[\mu]$ maps $\mathbb{D}$ into the upper half plane $\mathbb{H} := \{w \in \mathbb{C} \mid \operatorname{Im} w > 0\}$. We start with a statement about the images of nontangential approach regions in $\mathbb{D}$ under the map $iK[\mu]$. We need the Poisson transform of a measure on the real line:
Let $\nu \in M(\mathbb{R})$ be a complex measure satisfying $\int\limits_{\mathbb{R}} \frac{1}{1+t^2} d|\nu|(t) < \infty$. The Poisson transform of ν in the upper half plane is defined as

$$P[\nu](z) := \int\limits_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-t|^2} d\nu(t). \tag{4.3}$$

We also introduce the following notion: Let $B \subseteq \mathbb{T}$ and $E \subseteq \mathbb{D}$. We say that E nontangentially approaches the set B, if for every $\xi \in B$ there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq E$ such that $z_n \underset{\sphericalangle}{\to} \xi$ as $n \to \infty$.

Theorem 4.2.1. *Let $\mu \in M_+(\mathbb{T})$ and denote the singular component of μ by μ_s. Furthermore let $B \subseteq \mathbb{T}$ and $E \subseteq \mathbb{D}$ be such that E approaches B nontangentially. Suppose that there exists a nonnegative Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}_+$ which satisfies*

$$\int_{\mathbb{R}} \frac{g(t)}{1+t^2} dt < \infty, \tag{4.4}$$

and such that the set $iK[\mu](E) = \{iK[\mu](z) \mid z \in E\}$ lies under the graph of g, i.e.,

$$iK[\mu](E) \subseteq \{z = x + iy \mid y \leq g(x)\}.$$

Then $\mu_s(B) = 0$.

Proof. We define the positive measure $d\nu = gdt$ on $\mathbb{R}$. By assumption on g, ν satisfies $\int_{\mathbb{R}} \frac{1}{1+t^2} d\nu(t) < \infty$.
For a point $x + iy \in iK[\mu](E)$ we have $y \leq g(x)$, and since g is Lipschitz continuous, we can find an interval $I_x := (x - c_1 y, x + c_1 y) \subseteq \mathbb{R}$, where c_1 depends only on the Lipschitz constant L of g, such that for all $t \in I_x$ we have $g(t) > \frac{y}{2}$. Indeed, using the Lipschitz condition for g, we get for $t \in I_x$,

$$g(x) - g(t) \leq |g(x) - g(t)| \leq L|x - t| \leq Lc_1 y,$$

and hence $y \leq g(x) \leq Lc_1 y + g(t)$. So if we choose $c_1 > \frac{1}{2L}$, we get $g(t) > \frac{y}{2}$ in the interval I_x. This gives us the estimate (note that the integrand is nonnegative)

$$\begin{aligned} P[\nu](x+iy) &= \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} g(t)\, dt \geq \int_{I_x} \frac{y}{(x-t)^2 + y^2} g(t)\, dt \\ &> \frac{y^2}{2} \int_{I_x} \frac{1}{(c_1 y)^2 + y^2} dt = \frac{y^2}{2} \frac{1}{(c_1^2 + 1)\, y^2} 2c_1 y = \frac{c_1}{c_1^2 + 1} y. \end{aligned}$$

We infer that for $x + iy \in iK[\mu](E)$,

$$y = O(P[\nu](x+iy)) \quad \text{as} \quad x + iy \to \infty. \tag{4.5}$$

We now want to translate this asymptotic relation on $iK[\mu](E)$ into a relation on the set E. To this end, we consider the composition function $P[\nu](iK[\mu](z))$, $z \in \mathbb{D}$. Since ν is a positive measure, this is a positive harmonic function in $\mathbb{D}$, and therefore it is the Poisson transform of some positive measure $\eta \in M_+(\mathbb{T})$ by Theorem 1.4.3. Furthermore, since μ is also positive, we infer from (1.13) that the asymptotic behavior of $y = \operatorname{Im} iK[\mu](z)$ as $z \underset{\triangleleft}{\to} \xi$ is determined by that of $P[\mu](z)$. By means of these observations, the relation (4.5) can now be translated into

$$P[\mu](z) = O(P[\eta](z)) \quad \text{as} \quad z \to \xi \in B \text{ from within } E. \tag{4.6}$$

We examine the asymptotic behavior of $P[\eta]$. By Theorem 2.2.4, we have that for μ_s-almost every $\xi \in \mathbb{T}$,

$$P[\mu](z) \longrightarrow \infty \quad \text{as } z \underset{\triangleleft}{\to} \xi, \tag{4.7}$$

hence, for μ_s-almost every $\xi \in \mathbb{T}$, the imaginary part of $iK[\mu](z)$ tends to infinity as z nontangentially approaches ξ. Furthermore, (4.3) shows that

$$P[\nu](w) \longrightarrow 0 \quad \text{as} \quad \operatorname{Im} w \to \infty, \tag{4.8}$$

since the integrand tends to zero. Now (4.7) and (4.8) imply that for μ_s-almost every ξ,

$$P[\eta](z) = P[\nu](iK[\mu](z)) \longrightarrow 0, \quad z \underset{\triangleleft}{\to} \xi.$$

Hence $\eta \perp \mu_s$ by Lemma 3.3.8. But for μ we have that

$$\frac{P[\mu](z)}{P[\mu_s](z)} = 1 + \frac{P[\mu_{ac}](z)}{P[\mu_s](z)} \longrightarrow 1, \quad z \underset{\triangleleft}{\to} \xi$$

at μ_s-almost every point by Lemma 3.3.8, and so we get

$$\frac{P[\mu](z)}{P[\eta](z)} = \frac{P[\mu](z)}{P[\mu_s](z)} \frac{P[\mu_s](z)}{P[\eta](z)} \longrightarrow \infty$$

as z approaches ξ for μ_s-almost every $\xi \in \mathbb{T}$. Since this contradicts condition (4.6), we conclude that B is a nullset with respect to μ_s.

□

Theorem 4.2.2. *Let $\mu \in M(\mathbb{T})$. Denote by B the set of $\xi \in \mathbb{T}$ for which there exists a continuous nontangential path $\gamma_\xi \subseteq \mathbb{D}$ terminating at ξ such that*

$$\frac{P[\mu](z)}{Q[\mu](z)} \longrightarrow 0$$

as z approaches ξ along γ_ξ. Then $|\mu|(F) = 0$.

Proof. We denote the absolutely continuous and singular parts of μ by μ_{ac} and μ_s, respectively.
At μ_{ac}-almost every $\xi \in \mathbb{T}$, the nontangential limit of $P[\mu]$ exists and equals $f(\xi) \neq 0$ by Theorem 2.2.3, and the nontangential limit of $Q[\mu]$ is finite by Theorem 2.3.1, so the condition of the statement can only hold for a $\mu_{ac}-$nullset.
We suppose now that $\mu_s(B) \neq 0$. By Theorem 4.1.2, it suffices to assume μ is a positive measure. By assumption, we find a function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(x) = o(x)$ as $x \to \infty$, and a set $B_0 \subseteq B$ with $\mu_s(B_0) > 0$, such that for every $\xi \in B_0$ we have

$$P[\mu](z) + \|\mu\| \leq \phi(|Q[\mu](z)|) = o(|Q[\mu](z)|) \tag{4.9}$$

and

$$P[\mu](z) \longrightarrow \infty \tag{4.10}$$

as z approaches ξ along γ_ξ. The latter is satisfied at μ_s-almost every $\xi \in \mathbb{T}$ by Theorem 2.2.4.
Our goal is to construct a suitable function g in order to apply Theorem 4.2.1. We choose a sequence $\{a_k\}_{k\in\mathbb{N}}$ of positive numbers monotonically tending to infinity such that

$$\sum_{k\in\mathbb{N}} \frac{\phi^2(a_k)}{a_k^2} < \infty. \tag{4.11}$$

This is possible as $\frac{\phi(x)}{x}$ tends to zero as $x \to \infty$, and so we can choose numbers a_k such that $\frac{\phi(a_k)}{a_k} < \frac{1}{k}$.
For $k \in \mathbb{N}$ we denote $a_{-k} := -a_k$. For each $k \in \mathbb{Z}\backslash\{0\}$ we define the function $g_k : \mathbb{R} \to \mathbb{R}_+$ by $g_k(x) = \max\{0, \phi(|a_k|) - |x - a_k|\}$. The function g_k is a hat function with maximal

value $\phi(|a_k|)$ at a_k. The support of g_k is the interval $[a_k - \phi(|a_k|), a_k + \phi(|a_k|)]$ and the area under the graph of g_k equals $\phi^2(|a_k|)$. We set $g(x) = \max_{k\in\mathbb{Z}\backslash\{0\}} \{g_k(x)\}$ and verify that this function satisfies the conditions required in Theorem 4.2.1. First we note that the hat functions g_k are Lipschitz continuous, and so g, being the maximum of Lipschitz functions, is Lipschitz continuous too. We have to show now that $\int_{\mathbb{R}} \frac{g(x)}{1+x^2}dx$ is finite. We can assume that for all $k \in \mathbb{N}$ the estimate $\frac{\phi(a_k)}{a_k} \le \frac{1}{2}$ holds, this yields $0 < \frac{a_k}{2} \le a_k - \phi(a_k)$ and so

$$\frac{1}{1+x^2} \le \frac{1}{\left(a_k - \phi(a_k)^2\right)} \le \frac{1}{\left(\frac{a_k}{2}\right)^2}$$

on the support of g_k and, by the symmetry of the domain and the function, on the support of g_{-k} as well. We get the estimate $\int_{\mathbb{R}} \frac{g_k(x)}{1+x^2}dx \le \phi^2(|a_k|)\frac{4}{a_k^2}$ for $k \in \mathbb{Z}\backslash\{0\}$, and for g this yields

$$\int_{\mathbb{R}} \frac{g(x)}{1+x^2}dx \quad \le \quad \sum_{k\in\mathbb{Z}\backslash\{0\}} \int_{\mathbb{R}} \frac{g_k(x)}{1+x^2}dx \le 8\sum_{k\in\mathbb{N}} \frac{\phi(a_k)^2}{a_k^2} < \infty$$

by (4.11). We denote by $\Delta_k,\ k \in \mathbb{Z}\backslash\{0\}$, the triangle bounded by the graph of g_k and the real axis, and set $E := (iK[\mu])^{-1}\left(\bigcup_{k\in\mathbb{Z}\backslash\{0\}} \Delta_k\right)$. We claim that this set nontangentially approaches B_0: To see this, let $\xi \in B_0$. Since by construction $g_k(a_k) = \phi(|a_k|)$, conditions (4.9) and (4.10) imply that we can find a sequence $\{z_k\} \subseteq \gamma_\xi$ nontangentially approaching ξ such that for every $k \in \mathbb{N}$, $iK[\mu](z_k)$ lies in Δ_k. Hence the sequence $\{z_k\}$ lies in E and thus E nontangentially approaches B_0. Now we have fulfilled all the requirements for Theorem 4.2.1, which yields $\mu_s(B_0) = 0$, a contradiction. B must be a nullset with respect to μ_s and the proof is complete.

□

Corollary 4.2.3. *Class* I *is empty.*

Proof. A measure $\mu \in M(\mathbb{T})$ belongs to Class I if $\lim_{z \underset{\sphericalangle}{\to} \xi} \frac{P[\mu](z)}{Q[\mu](z)} = 0$ holds for μ-almost every $\xi \in \mathbb{T}$. But, by Theorem 4.2.2, the set of points satisfying this condition is even a nullset with respect to μ (in order to apply the theorem, choose, for instance, radial approach paths).

□

4.3 Class II

We now take a look at the class of measures $\mu \in M(\mathbb{T})$ that satisfy the relation

$$Q[\mu](z) = o(P[\mu](z)) \quad \text{as} \quad z \underset{\sphericalangle}{\to} \xi \tag{II}$$

at μ-almost every $\xi \in \mathbb{T}$.

This class does not contain any measures with singular components. The following result is stated and proved in Theorem 1.2 of [3] for complex measures on the real line. However, the problems are equivalent if one uses the usual transformation between the unit disk and the upper half plane to translate the statements. The transformations can be found in [12, Chapter 5]. We state the result in the form applicable to our case.

Theorem 4.3.1. *Let $\mu \in M(\mathbb{T})$ and let $B \subseteq \mathbb{T}$ be the set of points $\xi \in \mathbb{T}$ for which $|Q[\mu](z)| = o(P[\mu](z))$ as z nontangentially approaches ξ. Then $|\mu_s|(B) = 0$, i.e., the restriction of μ on B is absolutely continuous.*

□

If $\mu \in M(\mathbb{T})$ is absolutely continuous, the Poisson transform $P[\mu](z)$ tends to a finite nonzero value μ-almost everywhere on $\mathbb{T}$. Thus, an absolutely continuous measure lies in Class II only if its conjugate Poisson transform tends to zero at μ-almost every boundary point. At this point there is no complete description of the class of measures with this property. In [4], the corresponding class of measures defined on the real line is studied, and a description of the subclass of positive absolutely continuous measures in Class II is given.

4.4 Class III

Class III is the collection of measures $\mu \in M(\mathbb{T})$ for which

$$P[\mu](z) = O(Q[\mu](z)) \quad \text{as} \quad z \underset{\triangleleft}{\to} \xi \tag{III}$$

holds at μ-almost every $\xi \in \mathbb{T}$.

If $\mu \in M(\mathbb{T})$ is absolutely continuous, both its Poisson and conjugate Poisson transforms have finite limits μ-almost everywhere on $\mathbb{T}$, and for $P[\mu]$ we further know that this limit is nonzero μ-almost everywhere. Thus, Class III contains those absolutely continuous measures whose conjugate Poisson transform has nonzero limit μ-almost everywhere on $\mathbb{T}$. It does not contain any discrete measures, as we conclude from the following result.

Proposition 4.4.1. *Let $\mu \in M(\mathbb{T})$, and let $\xi \in \mathbb{T}$ be such that $\mu(\{\xi\}) \neq 0$. Then*

$$\lim_{r\to 1} \frac{Q[\mu](r\xi)}{P[\mu](r\xi)} = 0.$$

Proof. We consider the radial behavior of $(1-r)P[\mu](r\xi)$ and $(1-r)Q[\mu](r\xi)$. We first calculate for the Poisson kernel

$$(1-r)P(r\xi,\zeta) = \begin{cases} \frac{(1-r)(1-r^2)}{|\zeta - r\xi|^2} \overset{r\to 1}{\longrightarrow} 0, & \text{if } \zeta \neq \xi, \\ \frac{(1-r)(1-r^2)}{(1-r)^2} \overset{r\to 1}{\longrightarrow} 2, & \text{if } \zeta = \xi. \end{cases}$$

Furthermore, $|(1-r)P(r\xi,\zeta)| \leq \frac{(1-r)(1-r^2)}{(1-r)^2} \leq 2$, and since the measure is finite we have a bound uniform in r for the functions $(1-r)P(r\xi,.)$ on $\mathbb{T}$. Therefore we can apply the dominated convergence theorem and obtain

$$\lim_{r\to 1}(1-r)P[\mu](r\xi) = \int_{\mathbb{T}} 2\chi_{\{\xi\}}(\zeta)\,d\mu(\zeta) = 2\mu(\{\xi\}). \tag{4.12}$$

For the conjugate Poisson kernel we get

$$(1-r)Q(r\xi,\zeta) = \begin{cases} \frac{(1-r)2\mathrm{Im}\left(r\xi\bar{\zeta}\right)}{|\zeta - r\xi|^2} \overset{r\to 1}{\longrightarrow} 0, & \text{if } \zeta \neq \xi, \\ \frac{(1-r)2\mathrm{Im}\left(r|\xi|^2\right)}{(1-r)^2} = 0, & \text{if } \zeta = \xi. \end{cases}$$

As $\left|\text{Im}\left(r\xi\bar{\zeta}\right)\right| = \left|\text{Im}\left(1 - r\xi\bar{\zeta}\right)\right| = \left|\text{Im}\left(\bar{\zeta}\left(\zeta - r\xi\right)\right)\right| \leq \left|\zeta - r\xi\right|$, we have the estimate $\left|(1-r)\, Q\left(r\xi,\zeta\right)\right| \leq \frac{(1-r)2|\zeta - r\xi|}{|\zeta - r\xi|^2} \leq 2$, and the dominated convergence theorem yields

$$\lim_{r\to 1}\,(1-r)\,Q\left[\mu\right](r\xi) = 0. \tag{4.13}$$

We infer that for $\xi \in \mathbb{T}$ with $\mu\left(\{\xi\}\right) \neq 0$

$$\lim_{r\to 1}\frac{Q\left[\mu\right](r\xi)}{P\left[\mu\right](r\xi)} = \lim_{r\to 1}\frac{(1-r)\,Q\left[\mu\right](r\xi)}{(1-r)\,P\left[\mu\right](r\xi)} = 0.$$

□

The proposition shows that along every radius terminating at a discrete point of μ we have the relation $Q\left[\mu\right] = o\left(P\left[\mu\right]\right)$, hence no discrete measure belongs to Class III.
Concerning singular continuous measures, we can draw some conclusions from the following result (the proof of the corresponding statement for measures on the real line can be found in [4, Theorem 3.1]).

Theorem 4.4.2. *Let $\mu \in M\left(\mathbb{T}\right)$ be a real-valued measure. Denote by B the set of $\xi \in \mathbb{T}$ for which there are an angle $\varphi > \frac{\pi}{2}$ and $\varepsilon > 0$ such that $Q\left[\mu\right](z) \neq 0$ for all $z \in \Delta_\xi^\varphi \cap \overline{U_{1-\varepsilon}\left(0\right)}^c$. Then B is a nullset with respect to the singular component of μ.*

□

Suppose now $\mu \in M\left(\mathbb{T}\right)$ is a positive singular measure. The theorem implies that for μ-almost every $\xi \in \mathbb{T}$, the conjugate Poisson transform $Q\left[\mu\right]$ takes the value zero arbitrarily close to ξ in any sufficiently large sector Δ_ξ^φ. Hence, if $\varphi > \frac{\pi}{2}$, we can find a sequence $\{z_k\}_{k\in\mathbb{N}} \subseteq \Delta_\xi^\varphi$ with $z_k \to \xi$ as $k \to \infty$ such that $Q\left[\mu\right](z_k) = 0$ for all $k \in \mathbb{N}$. But the Poisson transform of a positive measure is always positive in $\mathbb{D}$. Therefore we have $\lim\limits_{k\to\infty}\left|\frac{P[\mu](z_k)}{Q[\mu](z_k)}\right| = \infty$. Thus, if we consider convergence from within sufficiently large sectors, condition (III) is satisfied for no singular measure except on a nullset. Whether this statement holds true for all angles $\varphi > 0$ remains an open question at this point.

4.5 Class IV

We recall that Class IV is the class of measures $\mu \in M\left(\mathbb{T}\right)$ such that

$$Q\left[\mu\right](z) = O\left(P\left[\mu\right](z)\right) \quad \text{as} \quad z \underset{\triangleleft}{\to} \xi \tag{IV}$$

holds at μ-almost every $\xi \in \mathbb{T}$.

Class IV contains all absolutely continuous measures $\mu \in M\left(\mathbb{T}\right)$, as for such measures the transforms $P\left[\mu\right]$ and $Q\left[\mu\right]$ tend to finite values and the limit of $P\left[\mu\right]$ is nonzero μ-almost everywhere on $\mathbb{T}$.
As was pointed out in [2], all discrete measures belong to Class IV. It does not contain all singular continuous measures, as can be shown by constructing examples of singular continuous measures that do not satisfy condition (IV). Such measures are found for instance in the family of Clark measures associated with an inner function with particular boundary behavior. An inner function with the following property was constructed by W. Rudin in [8].

Proposition 4.5.1. *There is an inner function θ that takes σ-almost every radius into a path that is not nontangential, i.e., there is a set $B \subseteq \mathbb{T}$ with $\sigma(B) = 1$ such that for every $\xi \in B$, the set $\{\theta(r\xi), \ r \in (0,1)\}$ is not contained in any sector $\Delta^{\varphi}_{\theta(\xi)}$ with $\varphi \in (0, \pi)$.*

□

Denote by $\{\nu_\alpha\}_{\alpha \in \mathbb{T}}$ the Clark measures corresponding to this inner function. One can now show that for σ-almost every $\alpha \in \mathbb{T}$, the set of points $\xi \in \mathbb{T}$ such that

$$Q[\nu_\alpha](r\xi) = O(P[\nu_\alpha](r\xi)) \quad \text{as } r \to 1$$

is a nullset with respect to ν_α, see [2]. An example of a singular continuous measure on the real line that does not belong to Class IV is constructed in [4].

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