DIPLOMARBEIT

Shift-Invariant Subspaces As Linear Relations On The Hardy Space

ausgeführt am Institut für

Analysis und Scientific Computing

der Technischen Universität Wien

unter der Anleitung von

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Introduction

In the beginning of the twentieth century, mathematicians such as G. H. Hardy, F. and M. Riesz and others started working on spaces of holomorphic functions defined on a fixed domain in the complex plane. These spaces are named Hardy spaces and have a number of interesting properties. For example, A. Beurling famously proved that Hardy space functions could be factorised into inner and outer functions. This staple of complex analysis can be found in most textbooks on the subject, cf. for example [Rud87] for an overview.

However, Hardy spaces, and especially the Hardy-Hilbert space $\mathcal{H}^2(\mathbb{D})$, can also be examined against an operator theoretic background, giving alternative proofs to some well-known theorems with the help of multiplication operators. The central question of this Master's thesis is now whether this approach can be broadened even further to also work for linear relations.

We start this work with a short overview of the well-known notions we require from functional analysis and give an introduction to linear relations. These materials were covered in courses on functional analysis during my Master's studies and can mostly be found in [Wor11] and [Kal12].

In Chapter 2, we introduce the space $\mathcal{H}^2(\mathbb{D})$ and look at some of its properties, linking it to the Hilbert space of square-integrable functions on the torus in the process. Furthermore, one interesting result that we will generalise for linear relations, namely Theorem 2.1.14, is presented and Beurling's Theorem on shift-invariant subspaces of $\mathcal{H}^2(\mathbb{D})$ is proved. For further reading we suggest [Neu10] as a starting point.

After collecting some facts about complex analysis for Banach-space valued holomorphic functions, Chapter 3 expands on the one-dimensional approach from Chapter 2. Consequently, we find a number of analoguous properties for the Hardy-Hilbert space $\mathcal{H}^2(\mathbb{D}; \mathbb{C}^n)$ of vector-valued holomorphic functions. Furthermore, matrix-valued functions are discussed to generalise multiplication operators to higher dimensions and another version of Beurling's Theorem is given. We recommend the excellent book [Nag10] for a comprehensive treatment of this subject.

Finally, Chapter 4 characterises shift-invariant linear relations in Theorem 4.1.6, which extends Theorem 2.1.14. We also try to recover properties of a linear relation, such as it being an operator, from this characterisation. The chapter concludes with some examples.

All that remains to say now is thank you.

I would like to thank my advisor Michael Kaltenbäck for his continuous support and fruitful discussions that helped shape this work and for affording me the opportunity to teach alongside him in this past year.
Furthermore, I want to thank my uncle Helmuth for all his help and for all the opportunities he created for me.
Moreover, thanks to my friends and fellow students who supported me during my last six years in Vienna, and who might sometimes have slowed down this work's progress, but more than made up for it with good times.
Finally, my sincere gratitude goes to my parents who have always been there for me, who encouraged me in everything I wanted to do and without whom so much would not have been possible.

Christoph Neuner Vienna, September 2012

Chapter 1

Preliminaries

In this chapter we will collect some of the concepts on which we will build our theory. These include some well-known facts from functional analysis that are used throughout this work, as well as an introduction to linear relations, which can be understood as a way to generalise linear operators. We start with a short explanation of the used notation.

1.1 Notation

We will understand $\mathbb{N}$ as to specifically exclude the number zero and write $\mathbb{N}_0$ if we want to include it. Two subsets of the complex plane $\mathbb{C}$ are of special interest to us, namely the unit disk, $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, and its boundary, the unit circle or torus, $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. For a complex number $z \in \mathbb{C}$, the expressions $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary part of z, respectively.

Throughout this work, X, Y, Z will be Banach spaces over $\mathbb{C}$ and their norm shall be denoted by $\|.\|$. We will write X' to refer to the topological dual of X, containing all continuous linear mappings $x' : X \to \mathbb{C}$. For Hilbert spaces, we will generally write $\mathfrak{H}$ or $\mathfrak{G}$ and $(.,.)$ for their inner product. Elements of Cartesian products, i.e. ordered pairs, are to be signified by $[.,.]$ and for sequences and nets we use $(.)$. The index set will mostly be $\mathbb{N}_0$, but we will use subscripts to clarify the notation wherever that is necessary.

A mapping $T : X \to Y$ between Banach spaces X and Y will always be linear and also be called a linear operator. The space of bounded operators is signified by $\mathcal{B}(X,Y)$ — if X and Y are identical, we write $\mathcal{B}(X)$ instead. It is a standard result that $\mathcal{B}(X,Y)$ is itself a Banach space, equipped with the operator norm, $\|T\| = \sup\{\|Tx\|_Y \mid \|x\|_X \leq 1\}$. It is well-known that for linear operators boundedness is the same as continuity. Furthermore, a linear operator $T : \operatorname{dom} T \to Y$, where $\operatorname{dom} T$ is a linear subspace of X, is called closed, if its graph is closed in the product topology in $X \times Y$. The range of T is denoted by $\operatorname{ran} T$.

1.2 Basic Results from Functional Analysis

We start this section by recalling some fundamental theorems that we use later on. The proofs of these claims can be found in any basic book on functional analysis, cf. [Heu92], [Yos80] or [Wor11].

THEOREM 1.2.1 (Cauchy-Schwarz inequality). *Let $\mathfrak{H}$ be a linear space, equipped with an inner product. Then we have*

$$|(x,y)| \leq \|x\| \cdot \|y\|$$

for $x, y \in \mathfrak{H}$, with equality if and only if x and y are linearly dependent.

THEOREM 1.2.2 (Principle of Uniform Boundedness). *Let X be a Banach space and Y be a normed space. Suppose that the family $\{T_i \in \mathcal{B}(X,Y) \mid i \in I\}$ of bounded linear operators from X to Y is pointwise bounded, i.e. for every $x \in X$ we have*

$$\sup_{i \in I} \|T_i x\| < \infty,$$

then it is uniformly bounded, i.e.

$$\sup_{i \in I} \|T_i\| < \infty.$$

THEOREM 1.2.3 (Closed Graph Theorem). *Let X, Y be Banach spaces and suppose that $T : X \to Y$ is linear. If the graph of T is closed in $X \times Y$ then T must be continuous.*

THEOREM 1.2.4 (Bounded Inverse Theorem). *Let X, Y be Banach spaces and assume that $T : X \to Y$ is a bijective linear operator. If T is continuous, then so is its inverse T^{-1}.*

As a consequence of the theorems of Hahn-Banach we get the following

LEMMA 1.2.5. *Let X be a locally convex topological vector space. Then the continuous dual space X' is separating on X, i.e. for $x, y \in X$ with $x \neq y$ there exists $f \in X'$ such that $f(x) \neq f(y)$.*

Furthermore, we make use of

LEMMA 1.2.6 (Parseval's identity). *Let $\mathfrak{H}$ be a Hilbert space and let $\{h_\alpha \in \mathfrak{H} \mid \alpha \in A\}$ be an orthonormal basis. Then for every $x \in \mathfrak{H}$ we have*

$$\|x\|^2 = \sum_{\alpha \in A} |(x, h_\alpha)|^2.$$

The proof of the next lemma can be found in [Kal12], II. Alternatively, Lemma 3.3.5 encompasses it as a special case in dimension one.

LEMMA 1.2.7. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, where μ is nonnegative and finite. Let $h : \Omega \to \mathbb{C}$ be measurable. Consider the multiplication operator*

$$M_h : \begin{cases} \text{dom } M_h & \to & L^2(\mu) \\ f & \mapsto & f \cdot h \end{cases}$$

on the linear subspace $\text{dom } M_h := \{ f \in L^2(\mu) \mid f \cdot h \in L^2(\mu) \}$ *of $L^2(\mu)$. Then we have*

1. *The space* $\text{dom } (M_h)$ *is densely contained in $L^2(\mu)$ and M_h is a closed operator, i.e. the graph of M_h is closed in the product topology on $L^2(\mu) \times L^2(\mu)$.*

2. *The following statements are equivalent.*

   (a) The function h belongs to $L^\infty(\mu)$, i.e. it is essentially bounded.

   (b) M_h is defined everywhere and bounded.

   (c) M_h is bounded at least on an $L^2(\mu)$-dense linear subspace of its domain.

   (d) M_h is defined everywhere.

   In this case, M_h belongs to $\mathcal{B}(L^2(\mu))$ and $\|M_h\| = \|h\|_{L^\infty}$.

For the next results on shift operators, we follow [Nag10], I.

DEFINITION 1.2.8. Let $\mathfrak{H}$ be a Hilbert space.

1. Consider an isometry $V \in \mathcal{B}(\mathfrak{H})$ on it. We call a subspace $\mathfrak{L}$ of $\mathfrak{H}$ wandering, if $V^n \mathfrak{L} \perp \mathfrak{L}$ for all $n \in \mathbb{N}$. In this case we define $M_+(\mathfrak{L}) := \bigoplus_{n=0}^{\infty} V^n \mathfrak{L}$ in $\mathfrak{H}$.

2. If $U \in \mathcal{B}(\mathfrak{H})$ is unitary and $\mathfrak{A} \subseteq \mathfrak{H}$ is a wandering subspace, i.e. $U^n \mathfrak{A} \perp \mathfrak{A}$ holds for $n \in \mathbb{Z} \backslash \{0\}$, we define $M(\mathfrak{A}) := \bigoplus_{n=-\infty}^{\infty} U^n \mathfrak{A}$.

Applying V to $M_+(\mathfrak{L})$ gives $V M_+(\mathfrak{L}) = \bigoplus_{n=1}^{\infty} V^n \mathfrak{L} = M_+(\mathfrak{L}) \ominus \mathfrak{L}$, i.e. the orthogonal complement of $\mathfrak{L}$ in $M_+(\mathfrak{L})$. Consequently,

$$\mathfrak{L} = M_+(\mathfrak{L}) \ominus V M_+(\mathfrak{L}). \tag{1.1}$$

Notice that for the two way orthogonal sum $M(\mathfrak{A}) = \bigoplus_{n=-\infty}^{\infty} U^n \mathfrak{A}$ the space $\mathfrak{A}$ is not uniquely determined.
These considerations lead to the following

DEFINITION 1.2.9. Let $\mathfrak{H}, \mathfrak{L}, \mathfrak{A}$ and $V, U \in \mathcal{B}(\mathfrak{H})$ be as in Definition 1.2.8.

1. If $M_+(\mathfrak{L}) = \mathfrak{H}$, we call V a unilateral shift and $\mathfrak{L}$ the generating subspace of $\mathfrak{H}$ for V, which is uniquely determined on account of (1.1).

2. If $M(\mathfrak{A}) = \mathfrak{H}$, we call U a bilateral shift and $\mathfrak{A}$ a generating subspace of $\mathfrak{H}$ for U.

DEFINITION 1.2.10. Let $\mathfrak{L}$ be a closed subspace of a Hilbert space $\mathfrak{H}$ and let T be a linear operator on $\mathfrak{H}$. If we have $T\mathfrak{L} = \mathfrak{L}$, we say that $\mathfrak{L}$ reduces T. More generally, if $T\mathfrak{L} \subseteq \mathfrak{L}$ is satisfied, we call $\mathfrak{L}$ invariant under T or say that $\mathfrak{L}$ is left invariant by T.

We point out that considering only closed subspaces is no real restriction, since if $\mathfrak{L}$ satisfies $T\mathfrak{L} \subseteq \mathfrak{L}$ for a bounded operator T, then the closure $\overline{\mathfrak{L}}$ will inherit this property.

THEOREM 1.2.11 (Wold decomposition)**.** *Let V be an arbitrary isometry on the Hilbert space $\mathfrak{H}$. Then $\mathfrak{H}$ decomposes uniquely into an orthogonal sum $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ such that $\mathfrak{H}_0$ reduces V and $\mathfrak{H}_1$ is left invariant, $V \restriction \mathfrak{H}_0$ is unitary and $V \restriction \mathfrak{H}_1$ is a unilateral shift. The spaces can even be written down explicitly. In fact, if $\mathfrak{L} := \mathfrak{H} \ominus V\mathfrak{H}$, then*

$$\mathfrak{H}_0 = \bigcap_{n=0}^{\infty} V^n \mathfrak{H} \qquad \text{and} \qquad \mathfrak{H}_1 = M_+(\mathfrak{L}).$$

It is possible that one of the subspaces is absent. Consider for example a unitary V, where the Wold decomposition will be trivial, i.e. $\mathfrak{H}_0 = \mathfrak{H}$ and $\mathfrak{H}_1 = \{0\}$.

Proof of Theorem 1.2.11. By design, the subspace $\mathfrak{L}$ of $\mathfrak{H}$ is orthogonal to $V\mathfrak{H}$. Because of $V^n\mathfrak{L} \subseteq V^n\mathfrak{H} \subseteq V\mathfrak{H}$ for $n \in \mathbb{N}$, we conclude that $V^n\mathfrak{L} \perp \mathfrak{L}$, i.e. $\mathfrak{L}$ is a wandering subspace. Hence, we can form $\mathfrak{H}_1 := M_+(\mathfrak{L})$ and $\mathfrak{H}_0 := \mathfrak{H} \ominus \mathfrak{H}_1$.
Take an element h of $\mathfrak{H}$. If h is orthogonal to $\bigoplus_{n=0}^{m-1} V^n\mathfrak{L}$ for every $m \in \mathbb{N}$, then it must be orthogonal to $\mathfrak{H}_1$, and therefore $h \in \mathfrak{H}_0$. The converse is clearly true as well. Using the definition of $\mathfrak{L}$, we get

$$\begin{aligned} \bigoplus_{n=0}^{m-1} V^n\mathfrak{L} &= \mathfrak{L} \oplus V\mathfrak{L} \oplus \cdots \oplus V^{m-1}\mathfrak{L} = \\ &= (\mathfrak{H} \ominus V\mathfrak{H}) \oplus (V\mathfrak{H} \ominus V^2\mathfrak{H}) \oplus \cdots \oplus (V^{m-1}\mathfrak{H} \ominus V^m\mathfrak{H}) = \\ &= \mathfrak{H} \ominus V^m\mathfrak{H}. \end{aligned} \tag{1.2}$$

To clarify the last equality, keep in mind that $\mathfrak{H} \supseteq V\mathfrak{H} \supseteq V^2\mathfrak{H} \supseteq \dots$ form a nonincreasing sequence of closed subspaces of $\mathfrak{H}$. For nested closed inner product spaces $A \supseteq B \supseteq C$ we can consider the orthogonal projection P_B on B and express every $x \in A$ as the direct sum $x = (I - P_B)x \dotplus P_Bx$. As $(I - P_B)x \in C^\perp$, we see that $x \in (A \ominus C)$ is equivalent to $(I - P_B)x \in A \ominus B$ and $P_Bx \in B \ominus C$. Hence, $(A \ominus B) \oplus (B \ominus C) = A \ominus C$ and the last identity of (1.2) follows. As a consequence, $h \in \mathfrak{H}_0$ is also equivalent to $h \in V^m\mathfrak{H}$ for all $m \in \mathbb{N}$, and therefore $\mathfrak{H}_0 = \bigcap_{n=0}^{\infty} V^n\mathfrak{H}$. Clearly, we can omit $V^0\mathfrak{H} = \mathfrak{H}$ from the intersection, so $\mathfrak{H}_0 = \bigcap_{n=1}^{\infty} V^n\mathfrak{H}$. Hence,

$$V\mathfrak{H}_0 = V \bigcap_{n=0}^{\infty} V^n\mathfrak{H} = \bigcap_{n=0}^{\infty} V^{n+1}\mathfrak{H} = \bigcap_{m=1}^{\infty} V^m\mathfrak{H} = \mathfrak{H}_0$$

proves that $\mathfrak{H}_0$ reduces V and that $V \restriction \mathfrak{H}_0$ is unitary. Obviously, $V\mathfrak{H}_1 = \bigoplus_{n=1}^{\infty} V^n\mathfrak{L} \subseteq \mathfrak{H}_1$, i.e. $\mathfrak{H}_1$ is left invariant by V, and $V \restriction \mathfrak{H}_1$ is a unilateral shift. So we have proven that there exists a decomposition as postulated.
To show uniqueness, we suppose that there is another decomposition $\mathfrak{H} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ with the same properties. In particular, there exists a wandering subspace $\mathfrak{K}$ with respect to V, such that $\mathfrak{G}_1 = M_+(\mathfrak{K})$. But with the help of (1.1), we get

$$\mathfrak{L} = \mathfrak{H} \ominus V\mathfrak{H} = (\mathfrak{G}_0 \oplus \mathfrak{G}_1) \ominus (V\mathfrak{G}_0 \oplus V\mathfrak{G}_1) = (\mathfrak{G}_0 \oplus \mathfrak{G}_1) \ominus (\mathfrak{G}_0 \oplus V\mathfrak{G}_1) =$$

$$= (\mathfrak{G}_0 \ominus \mathfrak{G}_0) \oplus (\mathfrak{G}_1 \ominus V\mathfrak{G}_1) = \mathfrak{G}_1 \ominus V\mathfrak{G}_1 = \mathfrak{K}.$$

This shows that $\mathfrak{G}_0 = \mathfrak{H}_0$ and in turn $\mathfrak{G}_1 = \mathfrak{H}_1$. □

We also need the following two propositions.

PROPOSITION 1.2.12. *Let V be a unilateral shift on a Hilbert space $\mathfrak{H}$. Then there exists a space $\mathfrak{G}$ containing $\mathfrak{H}$ and a bilateral shift U on $\mathfrak{G}$ such that $U \upharpoonright \mathfrak{H} = V$.*

Proof. Let $\mathfrak{L} := \mathfrak{H} \ominus V\mathfrak{H}$. Clearly, we then have $\mathfrak{H} = \bigoplus_{n=0}^{\infty} V^n \mathfrak{L}$. We form a space $\mathfrak{K}$, which shall contain vectors of the form $k = (\ell_n)_{n\in\mathbb{Z}}$, such that $\ell_n \in \mathfrak{L}$ for every $n \in \mathbb{Z}$ and such that

$$\|k\|_{\mathfrak{K}}^2 := \sum_{n=-\infty}^{\infty} \|\ell_n\|_{\mathfrak{L}}^2 < \infty.$$

In this setting, U acting as $U(\ell_n)_{n\in\mathbb{Z}} = (\ell_{n-1})_{n\in\mathbb{Z}}$ is clearly a bilateral shift on $\mathfrak{K}$ and a generating subspace is given by all vectors $(\ell_n)_{n\in\mathbb{Z}}$ such that $\ell_n = 0$ for $n \in \mathbb{Z}\backslash\{0\}$ and arbitrary $\ell_0 \in \mathfrak{L}$. We can embed $\mathfrak{H}$ in $\mathfrak{K}$ by identifying $h = \sum_{n=0}^{\infty} V^n \ell_n \in \mathfrak{H}$ with the element $k_h = (\ell'_n)_{n\in\mathbb{Z}} \in \mathfrak{K}$, where $\ell'_n = \ell_n$ for $n \geq 0$ and $\ell'_n = 0$ for $n < 0$. Clearly,

$$\|k_h\|_{\mathfrak{K}}^2 = \sum_{n=0}^{\infty} \|\ell_n\|_{\mathfrak{L}}^2 = \sum_{n=0}^{\infty} \|V^n \ell_n\|_{\mathfrak{L}}^2 = \|h\|_{\mathfrak{H}}^2$$

and the identification preserves the linear and metric structure of $\mathfrak{H}$. Additionally, U is an extension of V because

$$Vh = V \sum_{n=0}^{\infty} V^n \ell_n = \sum_{n=1}^{\infty} V^n \ell_{n-1}$$

will be identified with the element $(\ell'_{n-1}) = U(\ell'_n)$. Because of this identification, we therefore have $\mathfrak{K} = \bigoplus_{n=-\infty}^{\infty} U^n \mathfrak{L}$. □

We follow [RR85], I., to prove the next result.

PROPOSITION 1.2.13. *Let $V \in \mathcal{B}(\mathfrak{H})$ be isometric and assume that $\mathfrak{L}$ and $\mathfrak{K}$ are wandering subspaces such that $M_+(\mathfrak{L}) \supseteq M_+(\mathfrak{K})$. Furthermore, suppose that $\mathfrak{L}$ is finite dimensional. Then we have* $\dim \mathfrak{L} \geq \dim \mathfrak{K}$.

Proof. Let us start by pointing out that if $P \in \mathcal{B}(\mathfrak{H})$ is an orthogonal projection and $\{e_j \in \mathfrak{H} \mid j \in J\}$ is an orthonormal basis for $\mathfrak{H}$, then $\dim P\mathfrak{H} = \sum_{j\in J} \|Pe_j\|^2$. This is easy to see: If $\{f_k \in P\mathfrak{H} \mid k \in K\}$ is an orthonormal basis of $P\mathfrak{H}$, it then follows by twice using Parseval's identity that

$$\sum_{j\in J} \|Pe_j\|^2 = \sum_{j\in J}\sum_{k\in K} |(Pe_j, f_k)|^2 = \sum_{j\in J}\sum_{k\in K} |(e_j, Pf_k)|^2 = \sum_{j\in J}\sum_{k\in K} |(e_j, f_k)|^2 =$$

$$= \sum_{k \in K} \|f_k\|^2 = \dim P\mathfrak{H}$$

as elements of $[0, \infty]$. In fact, the last equality assumes that $\mathfrak{H}$ is separable, but this is automatically satisfied in our case.
Define P as the orthogonal projection from $M_+(\mathfrak{L})$ onto $M_+(\mathfrak{K})$. Consequently, VPV^* projects $M_+(\mathfrak{L})$ onto $VM_+(\mathfrak{K})$ and $Q := (P - VPV^*)$ projects $M_+(\mathfrak{L})$ onto the orthogonal complement of $VM_+(\mathfrak{K})$ in $M_+(\mathfrak{K})$, which is $\mathfrak{K}$.
Now let $\{e_\ell \in \mathfrak{L} \mid \ell \in L\}$ be an orthonormal basis for $\mathfrak{L}$, which is finite by assumption. Therefore, $\{V^j e_\ell \in M_+(\mathfrak{L}) \mid j \in \mathbb{N}_0,\ \ell \in L\}$ is an orthonormal basis for $M_+(\mathfrak{L})$. Due to our considerations at the beginning, we have

$$\begin{aligned}
\dim \mathfrak{K} = \dim QM_+(\mathfrak{L}) &= \sum_{\ell \in L} \sum_{j=0}^{\infty} \|QV^j e_\ell\|^2 = \sum_{\ell \in L} \sum_{j=0}^{\infty} (QV^j e_\ell, V^j e_\ell) = \\
&= \lim_{n\to\infty} \sum_{\ell \in L} \sum_{j=0}^{n} ([P - VPV^*]V^j e_\ell, V^j e_\ell) = \\
&= \lim_{n\to\infty} \sum_{\ell \in L} \sum_{j=0}^{n} [(PV^j e_\ell, V^j e_\ell) - (PV^* V^j e_\ell, V^* V^j e_\ell)] = \\
&\overset{*}{=} \lim_{n\to\infty} \sum_{\ell \in L} \Big[\sum_{j=1}^{n} [(PV^j e_\ell, V^j e_\ell) - (PV^{j-1} e_\ell, V^{j-1} e_\ell)] + (Pe_\ell, e_\ell) \Big] = \\
&= \lim_{n\to\infty} \sum_{\ell \in L} (PV^n e_\ell, V^n e_\ell) = \lim_{n\to\infty} \sum_{\ell \in L} \|PV^n e_\ell\|^2 = \\
&\le \lim_{n\to\infty} \sum_{\ell \in L} \|PV^n\|^2 \|e_\ell\|^2 = \sum_{\ell \in L} \|e_\ell\|^2 = \\
&= \dim \mathfrak{L},
\end{aligned}$$

where the equality marked with an asterisk follows from $\mathfrak{L} \subseteq \ker V^*$. □

1.3 Linear Relations

Linear relations arise as a possible generalisation of linear operators. They are also useful tools when investigating multi-valued linear functions, or linear mappings only defined on a (possibly dense) subspace. We will follow [Kal12] and [Sch11] in our approach to the topic and start from a purely algebraic point of view.

DEFINITION 1.3.1. Let X, Y be vector spaces over $\mathbb{C}$. A subset R of the Cartesian product $X \times Y$ is called a linear relation (between X and Y; or on X if $X = Y$) if it is a linear subspace of $X \times Y$. We write $R \le X \times Y$ for short.

A linear operator $T : X \to Y$ certainly is a linear relation by identifying it with its graph. The converse does not hold true, as can be seen from the linear relation $R := X \times \operatorname{ls}\{y\}$ for

$y \in Y \backslash \{0\}$, which acts as the map that assigns to every $x \in X$ the same one-dimensional subspace $\mathrm{ls}\{y\}$ of Y. Certainly, it is no well-defined function. Therefore, linear relations are a generalisation of a linear operator. If in the following we call an operator a linear relation, we always refer to its graph.

DEFINITION 1.3.2. Let $R \leq X \times Y$ be a linear relation. We define

(i) the domain of R as $\mathrm{dom}\, R := \{x \in X \mid \exists y \in Y : [x, y] \in R\}$,

(ii) the range of R as $\mathrm{ran}\, R := \{y \in Y \mid \exists x \in X : [x, y] \in R\}$,

(iii) the kernel of R as $\ker R := \{x \in X \mid [x, 0] \in R\}$, and

(iv) the multi-valued part of R as $\mathrm{mul}\, R := \{y \in Y \mid [0, y] \in R\}$.

Obviously, all sets above are linear subspaces of X or Y, respectively. If we take $Rx := \{y \in Y \mid [x, y] \in R\}$ for any $x \in X$, then we get a set-valued map $R : X \to \mathcal{P}(Y)$, which maps all $x \notin \mathrm{dom}\, R$ to $\emptyset$ and has range $\bigcup_{x \in X} Rx = \bigcup_{x \in \mathrm{dom}\, R} Rx$. This characterisation can be further improved upon.

LEMMA 1.3.3. *For a linear relation $R \leq X \times Y$ and $[x, y] \in R$, we have*

$$Rx = y + \mathrm{mul}\, R.$$

Proof. Simply by definition, $y + \mathrm{mul}\, R = \{y + z \in Y \mid z \in \mathrm{mul}\, R\}$.
$\subseteq$: Choose $a \in Rx$, meaning $[x, a] \in R$. Using linearity in combination with our assumption $[x, y] \in R$, we get $[0, a - y] \in R$ or $a - y \in \mathrm{mul}\, R$. Hence, $a = y + (a - y) \in y + \mathrm{mul}\, R$.
$\supseteq$: Choose $a \in y + \mathrm{mul}\, R$. So there exists $b \in \mathrm{mul}\, R$ such that $a = y + b$. Thus using our assumption, $[0, b], [x, y] \in R$ and, again by linearity, $[x, b + y] \in R$ hold true. Therefore, $a = b + y \in Rx$. □

The lemma gives us an idea of how to measure how far a linear relation is away from being an operator. In fact, a linear operator T is characterised by $\mathrm{mul}\, T = \{0\}$. Clearly, if we see T as a linear relation, its domain, range and kernel as defined above are equivalent to the corresponding notions in operator theory.

DEFINITION 1.3.4. Let X, Y, Z be vector spaces over $\mathbb{C}$. Let $R, S \leq X \times Y$ and $T \leq Y \times Z$ be linear relations and $\alpha \in \mathbb{C}$. We define

(i) $R + S := \{[x, y] \in X \times Y \mid \exists r, s \in Y : r + s = y,\ [x, r] \in R,\ [x, s] \in S\}$, the sum of R and S,

(ii) $R \boxplus S := \{[x_r + x_s, y_r + y_s] \in X \times Y \mid [x_r, y_r] \in R,\ [x_s, y_s] \in S\}$, the subspace sum of R and S, and if it is direct, i.e. additionally satisfying $R \cap S = \{[0, 0]\}$, we write $R \dot{\boxplus} S$,

(iii) $\alpha R := \{[x, \alpha y] \in X \times Y \mid [x, y] \in R\}$, the scalar multiplication of R with α,

(iv) $R^{-1} := \{[y, x] \in Y \times X \mid [x, y] \in R\}$, the inverse of R,

(v) $TR := \{[x,z] \in X \times Z \mid \exists y \in Y : \ [x,y] \in R, \ [y,z] \in T\}$, the composition of R and T.

The class of linear relations is closed under all these operations. Moreover, the sum and composition are both associative, so

$$R + (S + Q) = (R + S) + Q \qquad \text{and} \qquad P(TR) = (PT)R$$

for $Q \leq X \times Y$, $P \leq Z \times W$, and applying the inverse to a composition reverses the order of the factors:

$$(TR)^{-1} = R^{-1}T^{-1}.$$

In the operator case, $R + S$ is a linear operator defined on $(\text{dom } R) \cap (\text{dom } S)$ and it coincides with the pointwise addition; αR is the usual multiplication of an operator with a scalar; and TR is a linear operator with domain $\{x \in \text{dom } R \mid Rx \in \text{dom } T\}$ that acts exactly like $T \circ R$.
Often, closed linear operators are of special interest in functional analysis. If we have topologies at our disposal, we arrive at the following definition.

DEFINITION 1.3.5. Let X, Y be topological vector spaces over $\mathbb{C}$. For a linear relation $R \leq X \times Y$ the closure of R with respect to the product topology on $X \times Y$ is written as $\overline{R}$. In case that $R = \overline{R}$, we call R closed.

COROLLARY 1.3.6. *Let X, Y be topological vector spaces over $\mathbb{C}$. If $R \leq X \times Y$ is closed, then* $\ker R$ *is closed in X and* $\text{mul } R$ *is closed in Y.*

Proof. Let $\pi_X : X \times Y \to X$ be the projection to the first coordinate. Clearly, π_X is linear. If we restrict π_X to $X \times \{0\}$, it is bijective, continuous and it has a continuous inverse, i.e. it is a homeomorphism. Since $X \times \{0\}$ is a closed subspace of $X \times Y$, the intersection $R \cap (X \times \{0\})$ is closed in $X \times Y$ as well. In particular, it is even a closed subspace of $X \times \{0\}$. As the kernel of R satisfies

$$\ker R = \Big(\pi_X \restriction (X \times \{0\})\Big)\big(R \cap (X \times \{0\})\big),$$

it must be closed as the homeomorphic image of a closed set. The claim involving the multi-valued part of R follows analoguously. $\square$

LEMMA 1.3.7. *Let X, Y be topological vector spaces. We define*

$$\Phi_{inv} : \begin{cases} X \times Y & \to & Y \times X \\ [x,y] & \mapsto & [y,x] \end{cases} \qquad \textit{and} \qquad \Phi_\alpha : \begin{cases} X \times Y & \to & X \times Y \\ [x,y] & \mapsto & [x,\alpha y] \end{cases}.$$

Then Φ_{inv} and Φ_α, for $\alpha \in \mathbb{C}\backslash\{0\}$, are homeomorphisms.

Proof. The mapping Φ_{inv} clearly is involutary, hence bijective. Furthermore, since it merely exchanges coordinates, it is continuous, so it is a homeomorphism.

Similarly, Φ_α is bijective with inverse $\Phi_{\frac{1}{\alpha}}$ for $\alpha \in \mathbb{C}\backslash\{0\}$. Writing Φ_α in block operator form we have

$$\Phi_\alpha = \begin{pmatrix} I_X & 0 \\ 0 & \alpha I_Y \end{pmatrix}$$

on $X \times Y$, so it is clearly bicontinuous. □

COROLLARY 1.3.8. *Let $R \leq X \times Y$ be a linear relation between topological vector spaces. Then $(\overline{R})^{-1} = \overline{R^{-1}}$ and $\overline{\alpha R} = \alpha \overline{R}$.*

Proof. The assertions easily follow from Lemma 1.3.7. First, we have $\overline{R^{-1}} = \overline{\Phi_{inv}(R)} = \Phi_{inv}(\overline{R}) = (\overline{R})^{-1}$ and secondly, we arrive at $\overline{\alpha R} = \overline{\Phi_\alpha(R)} = \Phi_\alpha(\overline{R}) = \alpha\overline{R}$. □

In the following, $R - \lambda$ is shorthand for $R - \lambda I$, where $I \leq X \times X$ is the identity relation. Regarding the point ∞, we set $(R - \infty)^{-1} := R$ with $\operatorname{ran}(R - \infty) = \operatorname{dom} R$ and $\ker(R - \infty) = \operatorname{mul} R$.

DEFINITION 1.3.9. Let X be a Banach space and let $R \leq X \times X$ be a linear relation. Then we call

(i) $\rho(R) := \big\{\lambda \in \mathbb{C} \cup \{\infty\} \;\big|\; (R - \lambda)^{-1} \in \mathcal{B}(X)\big\}$ the resolvent set,

(ii) $\sigma(R) := (\mathbb{C} \cup \{\infty\})\backslash\rho(R)$ the spectrum of R, and in particular

(iii) $\sigma_p(R) := \big\{\lambda \in \sigma(R) \;\big|\; \ker(R - \lambda) \supsetneq \{0\}\big\}$ the point spectrum or set of eigenvalues.

LEMMA 1.3.10. *Let X be a Banach space and assume that $R \leq X \times X$ is a closed linear relation. Then λ belongs to the resolvent set of R if and only if* $\ker(R - \lambda) = \{0\}$ *and* $\operatorname{ran}(R - \lambda) = X$.

Proof. Since

$$\begin{aligned} \operatorname{mul}(R - \lambda)^{-1} &= \big\{x \in X \;\big|\; [0, x] \in (R - \lambda)^{-1}\big\} = \\ &= \big\{x \in X \;\big|\; [x, 0] \in (R - \lambda)\big\} = \\ &= \ker(R - \lambda) \end{aligned}$$

and

$$\begin{aligned} \operatorname{dom}(R - \lambda)^{-1} &= \big\{x \in X \;\big|\; \exists y \in X : [x, y] \in (R - \lambda)^{-1}\big\} = \\ &= \big\{x \in X \;\big|\; \exists y \in X : [y, x] \in (R - \lambda)\big\} = \\ &= \operatorname{ran}(R - \lambda), \end{aligned}$$

the fact that $(R - \lambda)^{-1}$ is a bounded operator on X is equivalent to $\ker(R - \lambda) = \{0\}$ and $\operatorname{ran}(R - \lambda) = X$, the latter of which uses the Closed Graph Theorem 1.2.3. □

Considering linear operators between Hilbert spaces, one can define adjoint operators. So, if we look at linear relations in such a setting, a similar concept arises. Given two Hilbert spaces, $\mathfrak{H}_1$ and $\mathfrak{H}_2$, their Cartesian product becomes a Hilbert space as well, if we equip it with the sum scalar product

$$\big([x,y],[u,v]\big)_{\mathfrak{H}_1\times\mathfrak{H}_2} := (x,u)_{\mathfrak{H}_1} + (y,v)_{\mathfrak{H}_2}$$

and we can set up the decomposition

$$\mathfrak{H}_1 \times \mathfrak{H}_2 = (\mathfrak{H}_1 \times \{0\}) \oplus (\{0\} \times \mathfrak{H}_2) \cong \mathfrak{H}_1 \oplus \mathfrak{H}_2.$$

DEFINITION 1.3.11. Let $\mathfrak{H}_1, \mathfrak{H}_2$ be Hilbert spaces over $\mathbb{C}$ and $R \leq \mathfrak{H}_1 \times \mathfrak{H}_2$ be a linear relation. We call

$$R^* := \big\{[y,x] \in \mathfrak{H}_2 \times \mathfrak{H}_1 \;\big|\; (y,v)_{\mathfrak{H}_2} = (x,u)_{\mathfrak{H}_1} \text{ for all } [u,v] \in R\big\}$$

the adjoint relation of R.

LEMMA 1.3.12. *Let $R \leq \mathfrak{H}_1 \times \mathfrak{H}_2$ be a linear relation between Hilbert spaces. Then*

(i) *R^* is always a closed linear relation.*

(ii) *We have $\overline{R} = R^{**}$. In particular, R is closed iff $R = R^{**}$*

(iii) $(R^{-1})^* = (R^*)^{-1}$.

Proof. (i): By definition, an element $[y,x] \in R^*$ must fulfil $(y,v) = (x,u)$ for every $[u,v] \in R$. This condition can be rewritten to read $\big([y,x],[-v,u]\big)_{\mathfrak{H}_2\times\mathfrak{H}_1} = 0$. So R^* contains precisely those elements $[y,x] \in \mathfrak{H}_2 \times \mathfrak{H}_1$ that are orthogonal to all $[-v,u]$ where $[u,v] \in R$. Consequently, we have $R^* = \big(\Phi_{inv} \circ \Phi_{-1}(R)\big)^{\perp_{\mathfrak{H}_2\times\mathfrak{H}_1}}$ and as an orthogonal complement R^* is a closed linear subspace of $\mathfrak{H}_2 \times \mathfrak{H}_1$.
(ii): Using the same reasoning, an element $[a,b] \in R^{**}$ must fulfil $(a,l) = (b,k)$ for all $[k,l] \in R^*$, which amounts to $\big([a,b],[l,-k]\big)_{\mathfrak{H}_1\times\mathfrak{H}_2} = 0$. So R^{**} contains exactly those elements of $\mathfrak{H}_1 \times \mathfrak{H}_2$ that are orthogonal to all $[l,-k]$ for $[k,l] \in R^*$. This means $R^{**} = \big(\Phi_{-1} \circ \Phi_{inv}(R^*)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}}$. Finally, we observe that for $S \leq \mathfrak{H}_2 \times \mathfrak{H}_1$ and $T \leq \mathfrak{H}_1 \times \mathfrak{H}_2$ we have $\Phi_{inv}(S^{\perp_{\mathfrak{H}_2\times\mathfrak{H}_1}}) = \big(\Phi_{inv}(S)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}}$ and $\Phi_{-1}(T^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}}) = \big(\Phi_{-1}(T)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}}$: For the former equation, keep in mind that $[a,b] \in (\Phi_{inv}(S))^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}}$ is equivalent to $[a,b] \perp [y,x]$ for all $[y,x] \in \Phi_{inv}(S)$, i.e. $[b,a] \perp [x,y]$ for all $[x,y] \in S$. In other words, this is equivalent to $[a,b] \in \Phi_{inv}(S^{\perp_{\mathfrak{H}_2\times\mathfrak{H}_1}})$. For the latter we take into account that for $[x,y] \in T$ and $[a,b] \in \mathfrak{H}_1 \times \mathfrak{H}_2$ the expression $[x,y] \perp [a,-b]$ is equivalent to $(a,x) - (b,y) = 0$, which in turn is the same as $[x,-y] \perp [a,b]$. Combining these results, we get

$$\begin{aligned} R^{**} &= \Big(\Phi_{-1} \circ \Phi_{inv}\big((\Phi_{inv} \circ \Phi_{-1}(R))^{\perp_{\mathfrak{H}_2\times\mathfrak{H}_1}}\big)\Big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = \\ &= \Big(\big(\Phi_{-1} \circ \Phi_{inv} \circ \Phi_{inv} \circ \Phi_{-1}(R)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}}\Big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = R^{\perp\perp} = \overline{R}. \end{aligned}$$

(iii): Using the above reasoning we also get

$$\begin{aligned}(R^*)^{-1} &= \Phi_{inv}\Big(\big(\Phi_{inv}\circ\Phi_{-1}(R)\big)^{\perp_{\mathfrak{H}_2\times\mathfrak{H}_1}}\Big) = \big(\Phi_{-1}(R)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = \\ &= \big(-I_{\mathfrak{H}_1\times\mathfrak{H}_2}\Phi_{-1}(R)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = \big(\Phi_{inv}\circ\Phi_{-1}\circ\Phi_{inv}\circ\Phi_{-1}\circ\Phi_{-1}(R)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = \\ &= \big(\Phi_{inv}\circ\Phi_{-1}\circ\Phi_{inv}(R)\big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = \Big(\Phi_{inv}\circ\Phi_{-1}\big(\Phi_{inv}(R)\big)\Big)^{\perp_{\mathfrak{H}_1\times\mathfrak{H}_2}} = (R^{-1})^*.\end{aligned}$$

□

Finally, we can transfer some more notions from operator theory to linear relations.

DEFINITION 1.3.13. Let $R \leq \mathfrak{H}_1 \times \mathfrak{H}_2$ be a linear relation between two Hilbert spaces. It is called

(i) isometric, if $R^{-1} \subseteq R^*$,

(ii) unitary, if $R^{-1} = R^*$.

In the case that $\mathfrak{H}_1 = \mathfrak{H}_2$, we call R

(iii) symmetric, if $R \subseteq R^*$,

(iv) selfadjoint, if $R = R^*$.

Chapter 2

Operators on the Hardy-Hilbert Space

In this chapter we will explore a certain Hardy space, namely $\mathcal{H}^2(\mathbb{D})$, and link it to various other well-understood Hilbert spaces. First, we will concern ourselves with holomorphic functions on the disk. Secondly, the boundary values of such functions are briefly examined and then discussed in the context of square-integrable functions on the torus, $L^2(\mathbb{T})$. Considering operators on $\mathcal{H}^2(\mathbb{D})$ and $L^2(\mathbb{T})$, we find a criterion to check whether they commute with the shift operators on the respective spaces. Finally, a theorem due to Beurling characterising the shift-invariant subspaces of $\mathcal{H}^2(\mathbb{D})$ is presented.

2.1 The Hardy-Hilbert Space $\mathcal{H}^2(\mathbb{D})$

We will start this section with the definition of the object we are interested in and then explore some of its properties. We make use of [Ale10] and [Wor04] in this section.

DEFINITION 2.1.1. We call the space

$$\mathcal{H}^2(\mathbb{D}) := \left\{ f \in \mathbb{C}^{\mathbb{D}} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ on } \mathbb{D},\ (a_n) \in \mathbb{C}^{\mathbb{N}_0} \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

of all holomorphic functions on the unit disk that possess a power series expansion with square-summable complex coefficients the Hardy-Hilbert space.

Take note that we omitted the supplement "on the unit disk" from Definition 2.1.1 as it would also be possible to define Hardy-Hilbert spaces on other domains $G \subset \mathbb{C}$. One example for such a G is the upper half plane $\mathbb{C}^+$. We will mention this only in passing, however, and cite [RR94], V, where this theory is presented. For the rest of this work only the disk case will be of importance. Furthermore, up to this point we only have a linear structure on $\mathcal{H}^2(\mathbb{D})$ but, as the name suggests, we will introduce an inner product, indeed turning $\mathcal{H}^2(\mathbb{D})$ into a Hilbert space.
We first observe why the elements of $\mathcal{H}^2(\mathbb{D})$ really are holomorphic functions on $\mathbb{D}$.

LEMMA 2.1.2. *The condition* $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ *implies that the radius of convergence* ρ *of* $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ *is greater or equal to 1.*

Proof. First, $(|a_n|^2)$ and thus $(|a_n|)$ must both be a null sequences because the series $\sum_{n=0}^{\infty} |a_n|^2$ converges. Therefore, there exists an $N \in \mathbb{N}$ such that $|a_n| \leq 1$ for all $n \geq N$. Consequently, the sequence $(\sqrt[n]{|a_n|})_{n=N}^{\infty}$, and, in particular, its limes superior, will also be bounded from above by 1. We can therefore use the following well known formula to calculate the radius of convergence

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \geq 1$$

and the assertion follows. □

Secondly, we argue how an inner product can be defined on $\mathcal{H}^2(\mathbb{D})$.

LEMMA 2.1.3. *The mapping*

$$\phi : \begin{cases} \ell^2(\mathbb{N}_0) & \to & \mathcal{H}^2(\mathbb{D}) \\ (a_n) & \mapsto & f := (z \mapsto \sum_{n=0}^{\infty} a_n z^n) \end{cases} \tag{2.1}$$

is bijective and preserves the linear structures.

Proof. The function ϕ is well-defined — the holomorphy of $\phi((a_n))$ on the unit disk is due to Lemma 2.1.2 — and clearly bijective. In addition, the definitions for + and multiplication by a scalar in $\ell^2(\mathbb{N}_0)$, i.e. $(a_n) + (b_n) = (a_n + b_n)$ and $\lambda \cdot (a_n) = (\lambda \cdot a_n)$, agree with those for power series, since $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n + b_n) z^n$ and $\lambda \cdot (\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} (\lambda \cdot a_n) z^n$ hold on the disk of convergence. Consequently, ϕ is compatible with the linear structures on the two spaces. □

COROLLARY 2.1.4. *Let* ϕ *be the mapping from* (2.1). *Then*

$$(.,.)_{\mathcal{H}^2(\mathbb{D})} : \begin{cases} \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) & \to & \mathbb{C} \\ [f, g] & \mapsto & (\phi^{-1}(f), \phi^{-1}(g))_{\ell^2(\mathbb{N}_0)} \end{cases}$$

is an inner product on $\mathcal{H}^2(\mathbb{D})$. *The mapping* ϕ *is then additionally isometric.*
If $f, g \in \mathcal{H}^2(\mathbb{D})$ *have power series coefficients* (a_n) *and* (b_n), *respectively, then it can be expressed as*

$$(f, g)_{\mathcal{H}^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

Moreover, the norm induced by the inner product is

$$\|f\|_{\mathcal{H}^2(\mathbb{D})} := \sqrt{(f, f)_{\mathcal{H}^2(\mathbb{D})}} = \sqrt{\sum_{n=0}^{\infty} |a_n|^2}.$$

Proof. It is a well-known fact that the class of all square-summable complex sequences, $\ell^2(\mathbb{N}_0) = \left\{(a_n) \in \mathbb{C}^{\mathbb{N}_0} \;\middle|\; \sum_{n=0}^{\infty} |a_n|^2 < \infty\right\}$, is a Hilbert space. Its inner product is given by $\big((a_n),(b_n)\big)_{\ell^2(\mathbb{N}_0)} = \sum_{n=0}^{\infty} a_n \overline{b_n}$ for sequences $(a_n),(b_n) \in \ell^2(\mathbb{N}_0)$. All properties of $(.,.)_{\ell^2(\mathbb{N}_0)}$ are preserved under ϕ and hence, $(.,.)_{\mathcal{H}^2(\mathbb{D})}$ must be an inner product on $\mathcal{H}^2(\mathbb{D})$. The other claims are obvious. □

LEMMA 2.1.5. *The polynomial ring $\mathbb{C}[z]$ is densely contained in $\mathcal{H}^2(\mathbb{D})$ with respect to the norm $\|.\|_{\mathcal{H}^2(\mathbb{D})}$.*

Proof. Let $f \in \mathcal{H}^2(\mathbb{D})$ with power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and set $p_N(z) := \sum_{n=0}^{N} a_n z^n$. Then $\|f - p_N\|^2_{\mathcal{H}^2(\mathbb{D})} = \|\sum_{n=N+1}^{\infty} a_n z^n\|^2_{\mathcal{H}^2(\mathbb{D})} = \sum_{n=N+1}^{\infty} |a_n|^2$ converges to zero as N approaches infinity. □

The following result states that $\mathcal{H}^2(\mathbb{D})$ is even a reproducing kernel Hilbert space.

LEMMA 2.1.6. *Let $\iota_w : \mathcal{H}^2(\mathbb{D}) \to \mathbb{C} : f \mapsto f(w)$ be the point evaluation functional at $w \in \mathbb{D}$. Then ι_w is linear and continuous for every w. Moreover, ι_w coincides with $(., k_w)_{\mathcal{H}^2(\mathbb{D})}$, where $k_w \in \mathcal{H}^2(\mathbb{D})$, $k_w \neq 0$ denotes the function $z \mapsto \frac{1}{1-\overline{w}z}$.*

Proof. The linearity of ι_w is clear. First, the series $\sum_{n=0}^{\infty} |\overline{w}^n|^2 = \sum_{n=0}^{\infty} (|w|^2)^n = \frac{1}{1-|w|^2}$ converges for every $w \in \mathbb{D}$. Hence, by Lemma 2.1.2, the functions $z \mapsto \sum_{n=0}^{\infty} \overline{w}^n z^n$ are elements of $\mathcal{H}^2(\mathbb{D})$. As $\sum_{n=0}^{\infty} \overline{w}^n z^n = \frac{1}{1-\overline{w}z}$, these functions are just k_w for $w \in \mathbb{D}$. Moreover, as $\|k_w\|^2_{\mathcal{H}^2(\mathbb{D})} = \frac{1}{1-|w|^2} > 0$, all k_w are nonzero. Now let $f \in \mathcal{H}^2(\mathbb{D})$ be the function $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ and $w \in \mathbb{D}$. We calculate

$$f(w) = \sum_{n=0}^{\infty} a_n w^n = \left(\sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} \overline{\overline{w^n}} z^n\right) = \left(f, \sum_{n=0}^{\infty} (\overline{w}z)^n\right) = \left(f, \frac{1}{1-\overline{w}z}\right) = (f, k_w).$$

This shows that $\iota_w = (., k_w)_{\mathcal{H}^2(\mathbb{D})}$. By the Cauchy-Schwarz inequality, the latter is certainly continuous. □

We only mention that one can use the set of functions $\left\{k_w \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; w \in \mathbb{D}\right\}$ to define $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ by setting $K(z,w) := (k_w, k_z)_{\mathcal{H}^2(\mathbb{D})}$. This function K is then called the reproducing kernel for the Hilbert space $\mathcal{H}^2(\mathbb{D})$.
Next, let us look at one of the classical definitions of Hardy spaces and then derive an equivalent characterisation of $\mathcal{H}^2(\mathbb{D})$ from it.

DEFINITION 2.1.7. For $0 < p \leq \infty$, the Hardy class $H^p(\mathbb{D})$ includes all analytic functions $f : \mathbb{D} \to \mathbb{C}$ that fulfill

$$\|f\|_{H^p} := \begin{cases} \sup_{r\in(0,1)} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}} < \infty & \text{for } p \in (0,\infty) \\ \sup_{z\in\mathbb{D}} |f(z)| < \infty & \text{for } p = \infty \end{cases}.$$

Obviously, in the case $p = \infty$ the introduced norm $\|.\|_{H^\infty}$ is the same as the supremum norm $\|.\|_\infty$ and H^∞ contains the bounded analytic functions on $\mathbb{D}$. The space H^p is linear for $p \in (0, \infty]$. In fact, for $p \geq 1$ this is clear since $\|.\|_{H^p}$ is a norm and for $p < 1$ one proves this using the metric $d_p(f, g) := \|f - g\|_{H^p}^p$. Using Hölder's inequality, it can also be shown that $H^\infty(\mathbb{D}) \subset H^q(\mathbb{D}) \subset H^p(\mathbb{D})$ for $p < q$.
We will now show that requiring an analytic function to have square-summable power series coefficients is equivalent to demanding its mean square value on circles of radius r stay bounded as r tends to 1 from below.

PROPOSITION 2.1.8. *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an analytic function with radius of convergence $\rho \geq 1$. Then*

$$\sum_{n=0}^\infty |a_n|^2 = \sup_{r\in(0,1)} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta = \lim_{r\nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta$$

as elements of $[0, \infty]$. In particular, the right hand side is finite iff the left hand side is. Thus, $\mathcal{H}^2(\mathbb{D}) = H^2(\mathbb{D})$ and $\|.\|_{H^2(\mathbb{D})} = \|.\|_{\mathcal{H}^2(\mathbb{D})}$.

Proof. We first notice that $\sum_{n=0}^N a_n z^n$ converges to f uniformly on compact subsets of $\mathbb{D}$, since f is analytic on $\mathbb{D}$. For a fixed $r \in (0, 1)$, we use uniform convergence on the closed ball centred at zero with radius r to exchange the order of integration and the limit process and get

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \lim_{N\to\infty} \sum_{n=0}^N a_n (re^{i\theta})^n \overline{\sum_{m=0}^N a_m (re^{i\theta})^m} \, d\theta = \\ &= \lim_{N\to\infty} \sum_{n,m=0}^N a_n \overline{a_m} \, r^{n+m} \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \\ &= \sum_{n=0}^\infty |a_n|^2 r^{2n},\end{aligned}$$

since only in the case $n = m$ does $\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta$ not vanish and amount to 1. Hence, the net $\left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta\right)_{r\in(0,1)} = \left(\sum_{n=0}^\infty |a_n|^2 r^{2n}\right)_{r\in(0,1)}$ is obviously increasing as r tends to one. Thus, the limit is attained at the supremum. Finally, $\lim_{r\nearrow 1} \sum_{n=0}^\infty |a_n|^2 r^{2n} = \sum_{n=0}^\infty |a_n|^2$ follows from the monotone convergence theorem appplied to the counting measure. □

As a result we get that it is not necessary to calculate the power series coefficients of the elements in $\mathcal{H}^2(\mathbb{D})$ for the inner product and norm. In fact, integration on circles suffices.

COROLLARY 2.1.9. *Let $f, g \in \mathcal{H}^2(\mathbb{D})$. Then the norm and inner product of $\mathcal{H}^2(\mathbb{D})$ can be rewritten as*

$$\|f\|_{\mathcal{H}^2(\mathbb{D})}^2 = \lim_{r\nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta$$

and

$$(f,g)_{\mathcal{H}^2(\mathbb{D})} = \lim_{r\nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\,d\theta,$$

respectively.

Proof. The claim regarding the norm follows from Proposition 2.1.8. The second identity is a consequence of the polarisation identity, i.e.

$$\begin{aligned}
4(f,g)_{\mathcal{H}^2(\mathbb{D})} &= \|f+g\|^2_{\mathcal{H}^2(\mathbb{D})} - \|f-g\|^2_{\mathcal{H}^2(\mathbb{D})} + i\|f+ig\|^2_{\mathcal{H}^2(\mathbb{D})} - i\|f-ig\|^2_{\mathcal{H}^2(\mathbb{D})} = \\
&= \lim_{r\nearrow 1}\left[\frac{1}{2\pi}\int_0^{2\pi} \left|(f+g)\left(re^{i\theta}\right)\right|^2 - \left|(f-g)\left(re^{i\theta}\right)\right|^2\right. \\
&\qquad \left. + i\left|(f+ig)\left(re^{i\theta}\right)\right|^2 - i\left|(f-ig)\left(re^{i\theta}\right)\right|^2 d\theta\right] = \\
&= \lim_{r\nearrow 1}\left[\frac{1}{2\pi}\int_0^{2\pi} \left(|f|^2 + f\overline{g} + \overline{f}g + |g|^2 - |f|^2 + f\overline{g} + \overline{f}g - |g|^2\right.\right. \\
&\qquad \left.\left. + i|f|^2 + f\overline{g} - \overline{f}g + i|g|^2 - i|f|^2 + f\overline{g} - \overline{f}g - i|g|^2\right)\left(re^{i\theta}\right)\,d\theta\right] = \\
&= 4\lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\,d\theta.
\end{aligned}$$

□

We are now able to introduce the theorem, cf. [Neu10], IV, that will be generalised later on. Let $h : \mathbb{D} \to \mathbb{C}$ be a function, then we define on the Hardy-Hilbert space the linear relation

$$T_h := \left\{[f,g] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; g = h\cdot f\right\}.$$

Clearly, $\operatorname{mul} T_h = \{0\}$ and T_h is an operator. It multiplies every function in its domain by h, so that we can write

$$T_h : \begin{cases} \operatorname{dom} T_h & \to \quad \mathcal{H}^2(\mathbb{D}) \\ \qquad f & \mapsto \quad f\cdot h \end{cases},$$

where $\operatorname{dom} T_h = \left\{f \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; f\cdot h \in \mathcal{H}^2(\mathbb{D})\right\}$.

DEFINITION 2.1.10. We write $S := T_{id_{\mathbb{D}}} : f \mapsto (z \mapsto zf(z))$ and call it the shift operator on $\mathcal{H}^2(\mathbb{D})$ or the operator of multiplication by z.

Obviously, the operator S is defined everywhere. It should however be noted, that in general $\operatorname{dom} T_h$ could easily be a proper subspace of $\mathcal{H}^2(\mathbb{D})$. For example, since all elements of the Hardy-Hilbert space are continuous, $\operatorname{dom} T_h = \{0\}$ for a discontinuous function h.

LEMMA 2.1.11. *Let $h : \mathbb{D} \to \mathbb{C}$. Then T_h is a closed operator and the following assertions are equivalent:*

(i) $T_h \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$

(ii) dom $T_h = \mathcal{H}^2(\mathbb{D})$

Proof. First, if $([f_n, g_n])$ is a sequence in the graph of T_h converging to an element $[f,g]$ in $\mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$, then we have $g_n = f_n \cdot h$ for every $n \in \mathbb{N}$. Additionally, evaluation at a point is a norm continuous operation in $\mathcal{H}^2(\mathbb{D})$, cf. Lemma 2.1.6. So for arbitrary $w \in \mathbb{D}$ we get

$$\begin{array}{ccc} g_n(w) & = & f_n(w) \cdot h(w) \\ \downarrow & & \downarrow \\ g(w) & & f(w) \cdot h(w) \end{array}$$

This means $g = f \cdot h$ and $[f,g] \in T_h$. Hence, we showed that T_h is closed.
Secondly, the Closed Graph Theorem 1.2.3 assures us that $T_h \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$ is equivalent to dom $T_h = \mathcal{H}^2(\mathbb{D})$. □

DEFINITION 2.1.12. Let $h : \mathbb{D} \to \mathbb{C}$. If $T_h \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$, then we call h a multiplier (function) and T_h a multiplier operator. The set of all multiplier functions is denoted by $\mathfrak{M}(\mathcal{H}^2(\mathbb{D}))$.

It is easy to identify the multipliers for the Hardy-Hilbert space.

LEMMA 2.1.13. *The multiplier functions of $\mathcal{H}^2(\mathbb{D})$ are the bounded analytic functions, i.e. $\mathfrak{M}(\mathcal{H}^2(\mathbb{D})) = H^\infty(\mathbb{D})$. In this case $\|T_h\| = \|h\|_\infty$.*

Proof. $\supseteq$: Let $f \in \mathcal{H}^2(\mathbb{D})$ and $h \in H^\infty(\mathbb{D})$. Obviously, $h \cdot f$ is holomorphic with radius of convergence at least 1. We use Proposition 2.1.8 to show

$$\|h \cdot f\|^2_{\mathcal{H}^2(\mathbb{D})} = \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 \underbrace{\left| h\left(re^{i\theta}\right)\right|^2}_{\leq \|h\|^2_\infty} d\theta \leq \|h\|^2_\infty \|f\|^2_{\mathcal{H}^2(\mathbb{D})} < \infty,$$

which means $T_h \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$ with $\|T_h\| \leq \|h\|_\infty$.
$\subseteq$: Let $h : \mathbb{D} \to \mathbb{C}$ such that $T_h \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$. Since $1 \in \mathcal{H}^2(\mathbb{D})$, it immediately follows that $h = T_h 1 \in \mathcal{H}^2(\mathbb{D})$ and, therefore, h is analytic on $\mathbb{D}$. To show boundedness, we use Lemma 2.1.6 and calculate for arbitrary $f \in \mathcal{H}^2(\mathbb{D})$ and $w \in \mathbb{D}$

$$(f, \overline{h(w)} \cdot k_w) = h(w) \cdot (f, k_w) = h(w) \cdot f(w) = (T_h f)(w) = (T_h f, k_w) = (f, T_h^* k_w).$$

We conclude that $T_h^* k_w = \overline{h(w)} \cdot k_w$. Taking the norm yields $\|T_h^* k_w\| = |h(w)| \cdot \|k_w\|$, where $\|k_w\|^2 = \frac{1}{1-|w|^2} \neq 0$ as stated before. Thus,

$$|h(w)| = \frac{\|T_h^* k_w\|}{\|k_w\|} \leq \|T_h^*\| = \|T_h\|.$$

Taking the supremum over all $w \in \mathbb{D}$ results in $\|h\|_\infty \leq \|T_h\|$ and therefore, $h \in H^\infty(\mathbb{D})$. Altogether, we have shown $\|T_h\| = \|h\|_\infty$. □

It is obvious that two multiplier operators commute, since for $h_1, h_2 \in H^\infty(\mathbb{D})$ we have $T_{h_1} \circ T_{h_2} = T_{h_1 h_2} = T_{h_2 h_1} = T_{h_2} \circ T_{h_1}$.

THEOREM 2.1.14. *Let $T \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$. Then T commutes with the shift operator S if and only if there exists a function $h \in H^\infty(\mathbb{D})$ such that $T = T_h$. In this case, h is uniquely determined by T.*

Proof. As outlined above, the necessity of the condition is clear. For the converse, we first set $h := T1 \in \mathcal{H}^2(\mathbb{D})$. We begin by showing that T acts like T_h on the polynomials. For $p(z) := \sum_{n=0}^N b_n z^n$ calculate

$$\begin{aligned} Tp &= \sum_{n=0}^N b_n T(z \mapsto z^n) = \sum_{n=0}^N b_n T \circ S^n 1 = \sum_{n=0}^N b_n S^n \circ T1 = \\ &= \sum_{n=0}^N b_n S^n h = \sum_{n=0}^N b_n (z \mapsto z^n h(z)) = h \cdot p. \end{aligned}$$

Secondly, for an arbitrary function $f \in \mathcal{H}^2(\mathbb{D})$ there exists, courtesy of Lemma 2.1.5, a sequence of polynomials (p_N) that converges to f in norm and, hence, also pointwise. Using the continuity of T, we see

$$Tf = T\left(\lim_{N\to\infty} p_N\right) = \lim_{N\to\infty} Tp_N = \lim_{N\to\infty} h \cdot p_N.$$

Due to Lemma 2.1.6, evaluating a function belonging to $\mathcal{H}^2(\mathbb{D})$ at $w \in \mathbb{D}$ is a continuous operation. Hence, we arrive at

$$\left(\lim_{N\to\infty} h \cdot p_N\right)(w) = \lim_{N\to\infty} (h \cdot p_N)(w) = h(w) \cdot \lim_{N\to\infty} p_N(w) = h(w) \cdot f(w).$$

This shows $\lim_{N\to\infty} h \cdot p_N = h \cdot f$ since $w \in \mathbb{D}$ was arbitrary. Thus, we have proven $T = T_h$. This means T is a multiplier operator with the corresponding multiplier function $h \in \mathfrak{M}(\mathcal{H}^2(\mathbb{D})) = H^\infty(\mathbb{D})$, cf. Lemma 2.1.13.
The uniqueness of h is obvious, since if there were $h_1, h_2 \in H^\infty(\mathbb{D})$ such that we had $T_{h_1} = T = T_{h_2}$ we would immediately get $h_1 = T_{h_1} 1 = T1 = T_{h_2} 1 = h_2$. □

2.2 $\mathcal{H}^2(\mathbb{D})$ as a Subspace of $L^2(\mathbb{T})$

There is yet another characterisation of $\mathcal{H}^2(\mathbb{D})$. Let $L^2(\mathbb{T})$ denote the space of square-integrable functions on the unit circle with respect to the normalized Lebesgue measure on $[0, 2\pi)$. We identify $[0, 2\pi)$ with $\mathbb{T}$ via $t \mapsto e^{it}$. It is well known that

$$(f, g)_{L^2(\mathbb{T})} := \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{g\left(e^{i\theta}\right)}\, d\theta$$

with $f, g \in L^2(\mathbb{T})$ defines an inner product on $L^2(\mathbb{T})$. Let ζ^n be the trigonometric monomial $e^{i\theta} \mapsto e^{in\theta}$ on $\mathbb{T}$ for $n \in \mathbb{Z}$. It is a standard result that $\left\{\zeta^n \;\middle|\; n \in \mathbb{Z}\right\}$

forms an orthonormal basis of $L^2(\mathbb{T})$. A function $f \in L^2(\mathbb{T})$ can hence be expanded as $\sum_{n\in\mathbb{Z}}(f,\zeta^n)\zeta^n$. We set $a_n := (f,\zeta^n)$ for $n \in \mathbb{Z}$. By Parseval's identity the sequence of Fourier coefficients $(a_n)_{n\in\mathbb{Z}}$ of such a function $f \in L^2(\mathbb{T})$ is square-summable.

DEFINITION 2.2.1. Let $(\zeta^n)_{n\in\mathbb{Z}}$ be the orthonormal basis of $L^2(\mathbb{T})$ consisting of trigonometric monomials. Then we set

$$L^2_+(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) \;\middle|\; a_n = (f,\zeta^n) = 0 \text{ for all } n < 0 \right\}, \tag{2.2}$$

i.e. the set of all functions whose Fourier coefficients vanish for negative indices.

LEMMA 2.2.2. *The space $\mathcal{H}^2(\mathbb{D})$ can be embedded in $L^2(\mathbb{T})$. More precisely, $\mathcal{H}^2(\mathbb{D})$ is isometrically isomorphic to the closed linear subspace $L^2_+(\mathbb{T})$ of $L^2(\mathbb{T})$ via*

$$\psi : \begin{cases} \mathcal{H}^2(\mathbb{D}) & \to & L^2_+(\mathbb{T}) \subseteq L^2(\mathbb{T}) \\ (z \mapsto \sum_{n=0}^\infty a_n z^n) & \mapsto & (\zeta \mapsto \sum_{n=0}^\infty a_n \zeta^n) \end{cases}. \tag{2.3}$$

Proof. As we know, $f := (z \mapsto \sum_{n=0}^\infty a_n z^n) \in \mathcal{H}^2(\mathbb{D})$ is equivalent to square-summability of (a_n). This in turn is equivalent to $\tilde{f} := (\zeta \mapsto \sum_{n=0}^\infty a_n \zeta^n) \in L^2_+(\mathbb{T})$ due to Parseval's identity as mentioned above. Therefore, the mapping ψ is an isomorphism. Because of $\|\psi(f)\|^2_{L^2(\mathbb{T})} = \|\tilde{f}\|^2_{L^2(\mathbb{T})} = \sum_{n=0}^\infty |a_n|^2 = \|f\|^2_{\mathcal{H}^2(\mathbb{D})}$, it is also isometric.
Since $L^2_+(\mathbb{T})$ is the isometric image of the Banach space $\mathcal{H}^2(\mathbb{D})$, it is necessarily closed. □

The relationship between $f = (z \mapsto \sum_{n=0}^\infty a_n z^n)$ and $\tilde{f} = (\zeta \mapsto \sum_{n=0}^\infty a_n \zeta^n)$ has been analysed in depth, cf. [RR94], I and IV, or [Ale10]. We cite some fairly standard results of Hardy space theory in the following proposition, but first we need

DEFINITION 2.2.3.

1. The function $P : \mathbb{D} \times \mathbb{T} \to \mathbb{C} : (z,\zeta) \mapsto \frac{1-|z|^2}{|\zeta-z|^2}$ is called the Poisson kernel.

2. For $\zeta \in \mathbb{T}$ and $r \in (0,1)$ let $\Delta(\zeta,r) := (\mathrm{co}\{B_r(0),\zeta\})^\circ$, i.e. the interior of the convex hull of the closed ball centred at zero with radius r and the point ζ. For $f : \mathbb{D} \to \mathbb{C}$ we write $\mathrm{n.t.}\lim_{z\to\zeta} f(z) = A$, if for every $r \in (0,1)$ the values $f(z)$ converge to A as z tends to ζ within $\Delta(\zeta,r)$. A is then called the nontangential limit of f at ζ.

3. The limit $\lim_{r\nearrow 1} f(re^{i\theta}) = L$ is called the radial limit of f at $\zeta = e^{i\theta}$.

PROPOSITION 2.2.4. *Let $f \in \mathcal{H}^2(\mathbb{D})$ and $\tilde{f} \in L^2_+(\mathbb{T})$ be connected by the mapping ψ of (2.3). We set $f_r : \mathbb{T} \to \mathbb{C} : \zeta = e^{i\theta} \mapsto f(re^{i\theta})$. Then*

1. *$\tilde{f}$ is the limit of f_r in $L^2(\mathbb{T})$ as r tends to 1, i.e. $\|f_r - \tilde{f}\|_{L^2(\mathbb{T})} \to 0$.*

2. *f can be recalculated from $\tilde{f}$ by employing the Poisson formula*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\tilde{f}(e^{i\theta})\, d\theta.$$

Proof. We follow [MAR07], I, in this proof. For the first claim, calculate using Parseval's identity

$$\|f_r - \tilde{f}\|^2_{L^2(\mathbb{T})} = \left\| \sum_{n=0}^{\infty} a_n r^n e^{in\theta} - \sum_{n=0}^{\infty} a_n e^{in\theta} \right\|^2_{L^2(\mathbb{T})} = \left\| \sum_{n=0}^{\infty} a_n (r^n - 1) e^{in\theta} \right\|^2_{L^2(\mathbb{T})} =$$
$$= \sum_{n=0}^{\infty} |a_n(r^n - 1)|^2 = \sum_{n=0}^{\infty} |a_n|^2 (1 - r^n)^2.$$

For $\varepsilon > 0$ we find $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |a_n|^2 < \frac{\varepsilon}{2}$. Moreover, we also find $R \in (0,1)$ such that $\sum_{n=0}^{N} |a_n|^2 (1 - R^n)^2 < \frac{\varepsilon}{2}$. Hence, we get for $r \in (R, 1)$

$$\sum_{n=0}^{\infty} |a_n|^2 (1 - r^n)^2 = \sum_{n=0}^{N} |a_n|^2 (1 - r^n)^2 + \sum_{n=N+1}^{\infty} |a_n|^2 (1 - r^n)^2$$
$$\leq \sum_{n=0}^{N} |a_n|^2 (1 - R^n)^2 + \sum_{n=N+1}^{\infty} |a_n|^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\lim_{r \nearrow 1} \|f_r - \tilde{f}\|_{L^2(\mathbb{T})} = 0$.
Regarding the second claim, we take $w \in \mathbb{D}$ and $k_w \in \mathcal{H}^2(\mathbb{D})$ as defined in Lemma 2.1.6. Furthermore, we set $\widetilde{k_w} := (\zeta \mapsto (1 - \overline{w}\zeta)^{-1})$, which clearly satisfies

$$\psi(k_w) = \psi\left(z \mapsto \frac{1}{1 - \overline{w}z}\right) = \psi\left(z \mapsto \sum_{n=0}^{\infty} \overline{w}^n z^n\right) =$$
$$= \left(\zeta \mapsto \sum_{n=0}^{\infty} \overline{w}^n \zeta^n\right) = \left(\zeta \mapsto \frac{1}{1 - \overline{w}\zeta}\right) = \widetilde{k_w}$$

Thus, with $\zeta = e^{i\theta}$ and Lemmata 2.1.6 and 2.2.2 we get

$$\begin{aligned} f(w) &= (f, k_w)_{\mathcal{H}^2(\mathbb{D})} = \big(\psi(f), \psi(k_w)\big)_{L^2(\mathbb{T})} = (\widetilde{f}, \widetilde{k_w})_{L^2(\mathbb{T})} = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(e^{i\theta}) \overline{\frac{1}{1 - \overline{w}e^{i\theta}}}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\widetilde{f}(e^{i\theta})}{1 - we^{-i\theta}}\, d\theta. \end{aligned} \tag{2.4}$$

Next, we consider the function

$$g := \big(\zeta \mapsto (1 - w\overline{\zeta})^{-1}\big) = \left(\zeta \mapsto \sum_{n=0}^{\infty} w^n \zeta^{-n}\right).$$

Clearly, all power series coefficients of $g-1$ vanish for nonnegative indices, so this function is perpendicular to $\widetilde{f}$ in $L^2(\mathbb{T})$, i.e. with $\zeta = e^{i\theta}$

$$(\widetilde{f}, g - 1)_{L^2(\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(e^{i\theta}) \overline{\left(\frac{1}{1 - we^{-i\theta}} - 1\right)}\, d\theta = 0.$$

Hence, we can add this harmless term to equation (2.4) and arrive at

$$f(w) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(e^{i\theta})\overline{\left(\frac{1}{1-\overline{w}e^{i\theta}} + \frac{1}{1-we^{-i\theta}} - 1\right)}\, d\theta.$$

Writing $w = re^{it}$ yields

$$\frac{1}{1-\overline{w}e^{i\theta}} + \frac{1}{1-we^{-i\theta}} - 1 = \frac{1-we^{-i\theta}+1-\overline{w}e^{i\theta}-(1-\overline{w}e^{i\theta}-we^{-i\theta}+|w|^2)}{(1-\overline{w}e^{i\theta})(1-we^{-i\theta})} =$$

$$= \frac{1+|w|^2}{\left|\frac{1}{e^{i\theta}}\right|^2\cdot|e^{i\theta}-w|^2} = \frac{1+|w|^2}{|e^{i\theta}-w|^2},$$

which, for $\zeta = e^{i\theta}$ and $z = w$, is just the Poisson kernel. □

It is well known that convergence in $L^2(\mathbb{T})$ implies pointwise convergence of a subsequence almost everywhere.
The next statement requires a rather extensive proof, making use of a theorem due to Fatou. We shall omit the proof and cite [MAR07], I.

PROPOSITION 2.2.5. *Let $f \in \mathcal{H}^2(\mathbb{D})$, then it has nontangential limits almost everywhere on the unit circle. If we denote by $u(\zeta)$ the nontangential limit of f at ζ — if it exists; otherwise set for example $u(\zeta) = 0$ — then $u = \tilde{f} := \psi(f)$ in the sense of $L^2(\mathbb{T})$.*

The above proposition justifies calling $\tilde{f}$ the nontangential boundary function of f. Clearly, the radial limit $\lim_{r\nearrow 1} f(re^{it})$ then also coincides with $\tilde{f}$ almost everywhere on $\mathbb{T}$.
Given this alternative description of $\mathcal{H}^2(\mathbb{D})$ as a subspace of $L^2(\mathbb{T})$, we will attempt to recover Theorem 2.1.14 in this larger space, cf. [Neu10], VII. First, we will identify the multiplier functions of $L^2_+(\mathbb{T})$.

DEFINITION 2.2.6. We set $L^\infty_+(\mathbb{T}) := L^\infty(\mathbb{T}) \cap L^2_+(\mathbb{T})$, i.e. the space of all essentially bounded functions on $\mathbb{T}$ such that $(\widetilde{h}, (\zeta \mapsto \zeta^n))_{L^2(\mathbb{T})} = 0$ for all $n < 0$.

DEFINITION 2.2.7. We will signify by $U := M_{id_{\mathbb{T}}} : f \mapsto (\zeta \mapsto \zeta f(\zeta))$ the multiplication operator connected to the identity function on $\mathbb{T}$, cf. Lemma 1.2.7, and call it the shift operator on $L^2(\mathbb{T})$.

LEMMA 2.2.8. *The shift operator U on $L^2(\mathbb{T})$ is unitary.*

Proof. As $L^2(\mathbb{T})$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$, we can use the scalar product of $\ell^2(\mathbb{Z})$ to calculate

$$(Uf, g)_{L^2(\mathbb{T})} = \left(\sum_{n=-\infty}^{\infty} a_n\zeta^{n+1}, \sum_{n=-\infty}^{\infty} b_n\zeta^n\right)_{L^2(\mathbb{T})} = \big((a_{n-1}), (b_n)\big)_{\ell^2(\mathbb{Z})}$$

$$= \sum_{n=-\infty}^{\infty} a_{n-1}\overline{b_n} = \sum_{n=-\infty}^{\infty} a_n\overline{b_{n+1}} = (f, M_{\zeta\mapsto\overline{\zeta}}g)_{L^2(\mathbb{T})}$$

for two functions $f, g \in L^2(\mathbb{T})$ with Fourier coefficients $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$. So we have shown that $U^* = M_{\zeta\mapsto\overline{\zeta}}$. For arbitrary $f \in L^2(\mathbb{T})$ we have

$$U^*Uf = M_{\zeta\mapsto\overline{\zeta}}M_{\zeta\mapsto\zeta}f = M_{\zeta\mapsto\overline{\zeta}\zeta}f = M_{\zeta\mapsto 1}f = f = UU^*f$$

and from this $U^{-1} = U^*$. □

LEMMA 2.2.9. *Let ψ be the isometric isomorphism as defined in* (2.3). *Then*

$$\psi\big(\mathfrak{M}(\mathcal{H}^2(\mathbb{D}))\big) \subseteq L^\infty_+(\mathbb{T}).$$

Furthermore, ψ preserves the norm, i.e. $\|\psi(.)\|_{L^\infty(\mathbb{T})} = \|.\|_\infty$. Additionally, if h belongs to $\mathfrak{M}(\mathcal{H}^2(\mathbb{D}))$ with $\widetilde{h} = \psi(h) \in L^2(\mathbb{T})$ and if T_h and $M_{\widetilde{h}}$ signify the corresponding multiplier operators, then $M_{\widetilde{h}}$ leaves $L^2_+(\mathbb{T})$ invariant and

$$\psi^{-1} \circ M_{\widetilde{h}} \circ \psi = T_h. \tag{2.5}$$

Proof. Let h be a multiplier of $\mathcal{H}^2(\mathbb{D})$ with power series coefficients (a_n). According to Lemma 2.1.13, h is bounded and $T_h \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$ with $\|T_h\| = \|h\|_\infty$. The fact that $\mathfrak{M}(\mathcal{H}^2(\mathbb{D})) = H^\infty(\mathbb{D}) \subset \mathcal{H}^2(\mathbb{D})$ implies that $\widetilde{h} := \psi(h) = (\zeta \mapsto \sum_{n=0}^\infty a_n\zeta^n)$ belongs to $L^2_+(\mathbb{T})$, and therefore, also to $L^2(\mathbb{T})$. Hence, $\widetilde{h} : \mathbb{T} \to \mathbb{C}$ is measurable and square-integrable. We can thus form the multiplication operator $M_{\widetilde{h}} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, which is closed and densely defined, cf. Lemma 1.2.7, with

$$\operatorname{dom} M_{\widetilde{h}} = \left\{ f \in L^2(\mathbb{T}) \;\middle|\; \widetilde{h} \cdot f \in L^2(\mathbb{T}) \right\}.$$

We will show that $M_{\widetilde{h}}$ is bounded on the trigonometric polynomials. From this it then follows that $M_{\widetilde{h}}$ is bounded everywhere on $L^2(\mathbb{T})$.
We begin by collecting some general facts for later use. First, we notice for $n \in \mathbb{Z}$

$$\|U^n M_{\widetilde{h}} f\|_{L^2(\mathbb{T})} = \|\zeta \mapsto \zeta^n \widetilde{h}(\zeta) f(\zeta)\|_{L^2(\mathbb{T})} = \|\zeta \mapsto \widetilde{h}(\zeta) f(\zeta)\|_{L^2(\mathbb{T})} = \|M_{\widetilde{h}} f\|_{L^2(\mathbb{T})}$$

as elements of $[0, \infty]$. This implies that we have $M_{\widetilde{h}}U^n = U^n M_{\widetilde{h}}$, which means that $U^n(\operatorname{dom} M_{\widetilde{h}}) = \operatorname{dom} M_{\widetilde{h}}$ and $M_{\widetilde{h}}U^n f = U^n M_{\widetilde{h}} f$ for all $f \in \operatorname{dom} M_{\widetilde{h}}$. Furthermore, $U^n \in \mathcal{B}(L^2(\mathbb{T}))$ is unitary for all $n \in \mathbb{Z}$ due to Lemma 2.2.8.
For $q(z) = z^j$ and $j \geq 0$ we have $h(z) \cdot q(z) = \sum_{n=0}^\infty a_n z^{n+j} = \sum_{n=j}^\infty a_{n-j} z^n$. Applying ψ gives

$$\psi(h \cdot q) = \psi\left(z \mapsto \sum_{n=j}^\infty a_{n-j} z^n\right) = \left(\zeta \mapsto \sum_{n=j}^\infty a_{n-j}\zeta^n\right) = \Big(\zeta \mapsto \zeta^j \cdot \widetilde{h}(\zeta)\Big).$$

This result extends, due to linearity, to all polynomials. Therefore, the identity $\psi(h \cdot r) = \widetilde{h} \cdot \widetilde{r}$ holds true for all elements $r = (z \mapsto \sum_{n=0}^N a_n z^n)$ in the polynomial ring $\mathbb{C}[z]$ and $\widetilde{r} = (\zeta \mapsto \sum_{n=0}^N a_n \zeta^n)$. Thus, $\psi\big(\mathbb{C}[z]\big) \subseteq \operatorname{dom} M_{\widetilde{h}}$ and $\psi(h \cdot r) = M_{\widetilde{h}}\widetilde{r}$ for all $r \in \mathbb{C}[z]$. In particular, $M_{\widetilde{h}}\psi(\mathbb{C}[z]) \subseteq L^2_+(\mathbb{T})$.

As $M_{\widetilde{h}}U^n = U^n M_{\widetilde{h}}$ and as $1 \in \operatorname{dom} M_{\widetilde{h}}$, the trigonometric monomials $(\zeta^n)_{n\in\mathbb{Z}}$ clearly all belong to dom $M_{\widetilde{h}}$. Thus, we have

$$\mathcal{T} := \left\{ \sum_{n=-N}^{N} a_n \zeta^n \;\middle|\; N \in \mathbb{N}_0,\ a_{-N}, \ldots, a_N \in \mathbb{C} \right\} \subseteq \operatorname{dom} M_{\widetilde{h}}.$$

For any $\widetilde{p} \in \mathcal{T}$ with $\widetilde{p}(\zeta) = \sum_{n=-N}^{N} b_n \zeta^n$ we can define

$$\zeta^N \widetilde{p}(\zeta) = \sum_{n=-N}^{N} b_n \zeta^{n+N} = \sum_{n=0}^{2N} b_{n-N} \zeta^n =: \widetilde{q}(\zeta) \in L^2_+(\mathbb{T})$$

and $q(z) = \sum_{n=0}^{2N} b_{n-N} z^n$. This gives

$$\begin{aligned} M_{\widetilde{h}}\widetilde{p}(\zeta) &= M_{\widetilde{h}}\zeta^{-N}\widetilde{q}(\zeta) = M_{\widetilde{h}}U^{-N}\widetilde{q}(\zeta) = U^{-N}M_{\widetilde{h}}\widetilde{q}(\zeta) \\ &= U^{-N}\psi(h(z)\cdot q(z)) = U^{-N}\psi(T_h q(z)). \end{aligned} \tag{2.6}$$

Taking the norm in (2.6) and making use of the isometry of U and ψ leads to

$$\begin{aligned} \left\|M_{\widetilde{h}}\widetilde{p}\right\|_{L^2(\mathbb{T})} &= \left\|U^{-N}\psi\left(T_h q\right)\right\|_{L^2(\mathbb{T})} = \|T_h q\|_{\mathcal{H}^2(\mathbb{D})} \le \|h\|_\infty \cdot \|q\|_{\mathcal{H}^2(\mathbb{D})} = \\ &= \|h\|_\infty \cdot \left\|U^{-N}\psi(q)\right\|_{L^2(\mathbb{T})} = \|h\|_\infty \cdot \left\|U^{-N}\widetilde{q}\right\|_{L^2(\mathbb{T})} = \\ &= \|h\|_\infty \cdot \|\widetilde{p}\|_{L^2(\mathbb{T})}. \end{aligned}$$

Since this last expression is finite, $M_{\widetilde{h}}$ must be bounded on $\mathcal{T}$. As $\mathcal{T}$ is densely contained in $L^2(\mathbb{T})$ we have boundedness everywhere, i.e. $M_{\widetilde{h}} \in \mathcal{B}(L^2(\mathbb{T}))$. Because the polynomials $\psi(\mathbb{C}[z])$ are dense in $L^2_+(\mathbb{T})$ we obtain from $M_{\widetilde{h}}\psi(\mathbb{C}[z]) \subseteq L^2_+(\mathbb{T})$ and the continuity of $M_{\widetilde{h}}$ that $M_{\widetilde{h}}L^2_+(\mathbb{T}) \subseteq L^2_+(\mathbb{T})$. At the same time, this shows

$$\|\widetilde{h}\|_{L^\infty(\mathbb{T})} = \|M_{\widetilde{h}}\| \le \|h\|_\infty, \tag{2.7}$$

where the equality in (2.7) is a well-known fact about multiplication operators, cf. Lemma 1.2.7.

It is left to show the converse inequality. Given a complex polynomial $p \in \mathbb{C}[z]$, equation (2.6) then clearly reads as $(\psi \circ T_h)(p) = (M_{\widetilde{h}} \circ \psi)(p)$. Because $\mathbb{C}[z]$ is densely contained in $\mathcal{H}^2(\mathbb{D})$ due to Lemma 2.1.5, this identity extends to $\mathcal{H}^2(\mathbb{D})$, proving

$$M_{\widetilde{h}} \restriction L^2_+(\mathbb{T}) = \psi \circ T_h \circ \psi^{-1}.$$

Notice that the inverse ψ^{-1} is only defined on the image $\phi(\mathcal{H}^2(\mathbb{D})) = L^2_+(\mathbb{T})$ and that $M_{\widetilde{h}}$ leaves $L^2_+(\mathbb{T})$ invariant. Rewritten, this reads as $\psi^{-1} \circ M_{\widetilde{h}} \circ \psi = T_h$ and using that ψ is isometric results in

$$\|h\|_\infty = \|T_h\| = \|\psi^{-1} \circ M_{\widetilde{h}} \circ \psi\| \le \|M_{\widetilde{h}}\| = \|\widetilde{h}\|_{L^\infty(\mathbb{T})}. \tag{2.8}$$

Together, (2.7) and (2.8) show $\|h\|_\infty = \|\widetilde{h}\|_{L^\infty(\mathbb{T})}$ and $\psi(h) = \widetilde{h} \in L^\infty(\mathbb{T})$. Consequently, $\widetilde{h} \in L^\infty_+(\mathbb{T})$. □

COROLLARY 2.2.10. *Let ψ be the isometric isomorphisms as defined in* (2.3)*. Then ψ is multiplicative on $H^\infty(\mathbb{D})$. Moreover, $\psi(h \cdot f) = \psi(h) \cdot \psi(f)$ even holds for every $h \in H^\infty(\mathbb{D})$ and $f \in \mathcal{H}^2(\mathbb{D})$.*

Proof. Remember that according to Lemma 2.2.9

$$M_{\psi(h)} \circ \psi = \psi \circ T_h \tag{2.9}$$

for any function $h \in H^\infty(\mathbb{D})$. Hence, for $h_1, h_2 \in H^\infty(\mathbb{D})$

$$\psi(h_1)\psi(h_2) = M_{\psi(h_1)}\psi(h_2) = \big(M_{\psi(h_1)} \circ \psi\big)(h_2) = \big(\psi \circ T_{h_1}\big)h_2 = \psi(T_{h_1}h_2) = \psi(h_1h_2).$$

So ψ is multiplicative on $H^\infty(\mathbb{D})$. Since the operator equation (2.9) holds on $\mathcal{H}^2(\mathbb{D})$, we can even choose $h_2 \in \mathcal{H}^2(\mathbb{D})$ and the above calculation remains true. ☐

Next, we can formulate a result corresponding in essence to Theorem 2.1.14.

THEOREM 2.2.11. *Let $M \in \mathcal{B}(L^2(\mathbb{T}))$. Then M commutes with U and leaves $L^2_+(\mathbb{T})$ invariant if and only if there exists a function $h \in H^\infty(\mathbb{D})$ such that $M = M_{\widetilde{h}}$ for $\widetilde{h} = \psi(h)$. In this case, h is unique.*

Proof. The necessity of the condition is clear, since if $M = M_{\widetilde{h}}$ holds for a function $h \in H^\infty(\mathbb{D})$ with $\widetilde{h} := \psi(h) \in L^\infty_+(\mathbb{T})$, we immediately get

$$UM = UM_{\widetilde{h}} = M_{(\zeta \mapsto \zeta)\cdot \widetilde{h}} = M_{\widetilde{h}\cdot(\zeta\mapsto\zeta)} = M_{\widetilde{h}}U = MU.$$

Furthermore, $M_{\widetilde{h}}L^2_+(\mathbb{T}) \subseteq L^2_+(\mathbb{T})$ is a consequence of Lemma 2.2.9.
We show sufficiency. Since M leaves $L^2_+(\mathbb{T})$ invariant, we can define $T := \psi^{-1} \circ M \circ \psi$ and get $T \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$. We can also rewrite the shift operator on $\mathcal{H}^2(\mathbb{D})$ via $S = \psi^{-1} \circ U \circ \psi$, which is well defined as well, since $U(L^2_+(\mathbb{T})) \subseteq L^2_+(\mathbb{T})$. Hence, using that M commutes with U, we can calculate

$$\begin{aligned} S \circ T &= \big(\psi^{-1} \circ U \circ \psi\big) \circ \big(\psi^{-1} \circ M \circ \psi\big) \\ &= \psi^{-1} \circ U \circ M \circ \psi \\ &= \psi^{-1} \circ M \circ U \circ \psi \\ &= \big(\psi^{-1} \circ M \circ \psi\big) \circ \big(\psi^{-1} \circ U \circ \psi\big) \\ &= T \circ S. \end{aligned}$$

So T is an operator on the Hardy-Hilbert space that commutes with the the shift operator S. Courtesy of Theorem 2.1.14, there exists a function $h \in H^\infty(\mathbb{D})$ such that $\psi^{-1} \circ M \circ \psi = T = T_h$. By Lemma 2.2.9 we also have $T_h = \psi^{-1} \circ M_{\widetilde{h}} \circ \psi$. In particular, $M_{\widetilde{h}} \circ \psi = M \circ \psi$, which means that $M_{\widetilde{h}} \restriction L^2_+(\mathbb{T}) = M \restriction L^2_+(\mathbb{T})$.
The property $MU = UM$ clearly extends to $MU^n = U^nM$ for $n \geq 1$ by induction. Moreover, since $M_{\zeta\mapsto\overline{\zeta}} = U^{-1}$, we have $M = MUU^{-1} = UMU^{-1}$. Applying U^{-1} from the left, we get $U^{-1}M = MU^{-1}$, i.e. M also commutes with U^{-1}. Again, this property extends to $MU^{-n} = U^{-n}M$ for $n \geq 1$. Similarly, we have $M_{\widetilde{h}}U^n = U^nM_{\widetilde{h}}$ for all $n \in \mathbb{Z}$.

Consider the ring of trigonometric polynomials

$$\mathcal{T} := \left\{ \sum_{n=-N}^{N} a_n \zeta^n \;\middle|\; N \in \mathbb{N},\ a_{-N}, \dots, a_N \in \mathbb{C} \right\}$$

and let $p \in \mathcal{T}$. Obviously, $U^N p \in L^2_+(\mathbb{T})$ for sufficiently large $N \in \mathbb{N}$. Hence, $MU^N p = M_{\widetilde{h}} U^N p$. Applying U^{-N} gives

$$M_{\widetilde{h}} p = U^{-N} M_{\widetilde{h}} U^N p = U^{-N} M U^N p = Mp.$$

Consequently, $M = M_{\widetilde{h}}$ holds even on $\mathcal{T}$. Since the trigonometric polynomials are densely contained in $L^2(\mathbb{T})$ and since both M and $M_{\widetilde{h}}$ are continuous, this property extends to $L^2(\mathbb{T})$. Thus, M is indeed a multiplier with multiplier function $\widetilde{h}$.
The uniqueness of h is guaranteed by the second statement in Theorem 2.1.14 and Equation (2.5). □

LEMMA 2.2.12. *For every $\widetilde{h} \in L^\infty_+(\mathbb{T})$ there exists $h \in H^\infty(\mathbb{D})$ satisfying $\psi(h) = \widetilde{h}$.*

Proof. Choose any $\widetilde{h} \in L^\infty_+(\mathbb{T})$ and use Lemma 1.2.7 to construct $M_{\widetilde{h}} \in \mathcal{B}(L^2(\mathbb{T}))$ with $\|M_{\widetilde{h}}\| = \|\widetilde{h}\|_{L^\infty(\mathbb{T})}$. Clearly, this operator $M_{\widetilde{h}}$ commutes with the shift operator U. Furthermore, suppose that $\widetilde{h}$ has Fourier coefficients $(a_n)_{n\in\mathbb{Z}}$ — keep in mind that $a_n = 0$ for $n < 0$ — and take a polynomial $p \in \psi(\mathbb{C}[z]) \subseteq L^2_+(\mathbb{T})$ of the form $p(\zeta) = \sum_{n=0}^{N} b_n \zeta^n$. To avoid technicalities set $b_n := 0$ for $n \in \mathbb{Z}\backslash\{0, \dots, N\}$ and get

$$M_{\widetilde{h}} p = \widetilde{h} \cdot p = \left(\zeta \mapsto \Big(\sum_{n=0}^{\infty} a_n \zeta^n \Big) \cdot \Big(\sum_{n=0}^{N} b_n \zeta^n \Big) \right) = \left(\zeta \mapsto \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k} \Big) \zeta^n \right).$$

Since $M_{\widetilde{h}}$ maps into $L^2(\mathbb{T})$, the sequence $\big(\sum_{k=0}^{n} a_k b_{n-k} \big)_{n\in\mathbb{Z}}$ must be square-summable, so $M_{\widetilde{h}}$ maps the norm dense subset $\psi(\mathbb{C}[z])$ of $L^2_+(\mathbb{T})$ into $L^2_+(\mathbb{T})$. For $f \in L^2_+(\mathbb{T})$ we find a sequence of polynomials $(p_N)_{N\in\mathbb{N}} \subseteq \psi(\mathbb{C}[z])$ converging to f in norm. Using the continuity of $M_{\widetilde{h}}$ and the fact that $L^2_+(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$, the calculation

$$M_{\widetilde{h}} f = M_{\widetilde{h}} \left(\lim_{N\to\infty} p_N \right) = \lim_{N\to\infty} M_{\widetilde{h}} p_N$$

shows that $M_{\widetilde{h}}$ leaves $L^2_+(\mathbb{T})$ invariant.
So we can use Theorem 2.2.11, which asserts that there exists a function $h \in H^\infty(\mathbb{D})$ such that $\psi(h) = \widetilde{h}$. □

By combining the two Lemmata 2.2.9 and 2.2.12 and Corollary 2.2.10 we get the following result that fully characterises the multipliers of $L^2(\mathbb{T})$.

THEOREM 2.2.13. $\psi \restriction H^\infty(\mathbb{D}) : H^\infty(\mathbb{D}) \to L^\infty_+(\mathbb{T})$ *is linear, bijective, multiplicative and isometric, i.e.* $\|\psi(.)\|_{L^\infty(\mathbb{T})} = \|.\|_\infty$.

LEMMA 2.2.14. *Let $h \in H^\infty(\mathbb{D})$ with $\widetilde{h} = \psi(h)$. Then the following are equivalent:*

(i) We have $\operatorname{ess\,inf} |\widetilde{h}| > 0$.

(ii) $M_{\widetilde{h}}$ *is bounded from below by a* $C > 0$, *i.e.* $\|M_{\widetilde{h}} g\|_{L^2(\mathbb{T})} \geq C\|g\|_{L^2(\mathbb{T})}$ *for all* $g \in L^2(\mathbb{T})$.

(iii) T_h *is bounded from below by a* $C > 0$, *i.e.* $\|T_h f\|_{\mathcal{H}^2(\mathbb{D})} \geq C\|f\|_{\mathcal{H}^2(\mathbb{D})}$ *for every* $f \in \mathcal{H}^2(\mathbb{D})$.

(iv) $T_h^{-1} : T_h\mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ *is bounded.*

(v) $\operatorname{ran} T_h$ *is closed.*

In this case, $\frac{1}{\operatorname{ess\,inf} |\widetilde{h}|} = \|T_h^{-1}\|$ *holds and* $\operatorname{ess\,inf} |\widetilde{h}|$ *is the largest possible constant C in (ii) and (iii).*

Proof. *(i)* $\Rightarrow$ *(ii)*: Clearly,

$$\|M_{\widetilde{h}} g\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} \left|\widetilde{h}(e^{i\theta}) \cdot g(e^{i\theta})\right|^2 \, d\theta \geq \left(\operatorname{ess\,inf} |\widetilde{h}|\right)^2 \|g\|^2_{L^2(\mathbb{T})}$$

holds for $g \in L^2(\mathbb{T})$.
(ii) $\Rightarrow$ *(i)*: Assume that $E := \left\{\zeta \in \mathbb{T} \mid |\widetilde{h}(\zeta)| < C\right\}$ has positive measure. Then $g := \chi_E$, where χ is the indicator function, is not the zero function and belongs to $L^2(\mathbb{T})$. This, however, gives the contradiction

$$C^2\|g\|^2_{L^2(\mathbb{T})} \leq \|M_{\widetilde{h}} g\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} |\widetilde{h}(e^{i\theta}) \cdot \chi_E|^2 \, d\theta < C^2\|g\|^2.$$

Hence, C is an essential lower bound for $|\widetilde{h}|$ on $\mathbb{T}$, which means $C \leq \operatorname{ess\,inf} |\widetilde{h}|$. To show that C is the largest such bound, notice that assuming $C < \operatorname{ess\,inf} |\widetilde{h}|$ would imply

$$\|M_{\widetilde{h}} g\|_{L^2(\mathbb{T})} \geq \left(\operatorname{ess\,inf} |\widetilde{h}|\right) \|g\|_{L^2(\mathbb{T})} > C\|g\|_{L^2(\mathbb{T})}$$

for any nonzero $g \in L^2(\mathbb{T})$. This would mean that there was a better lower bound for $M_{\widetilde{h}}$, which is a contradiction as well.
(ii) $\Rightarrow$ *(iii)*: Since $T_h = \psi^{-1} \circ M_{\widetilde{h}} \circ \psi$ due to Lemma 2.2.9 and since ψ is isometric, we have

$$\|T_h f\|_{\mathcal{H}^2(\mathbb{D})} = \|M_{\widetilde{h}} \widetilde{f}\|_{L^2(\mathbb{T})} \geq C\|\widetilde{f}\|_{L^2(\mathbb{T})} = C\|f\|_{\mathcal{H}^2(\mathbb{D})}$$

for $f \in \mathcal{H}^2(\mathbb{D})$ and $\widetilde{f} = \psi(f)$.
(iii) $\Rightarrow$ *(ii)*: Again using Lemma 2.2.9, for $\widetilde{p} = \psi(p) \in \psi\left(\mathbb{C}[z]\right)$ we have

$$\left\|M_{\widetilde{h}} U^{-N} \widetilde{p}\right\|_{L^2(\mathbb{T})} = \|M_{\widetilde{h}} \widetilde{p}\|_{L^2(\mathbb{T})} = \|T_h p\|_{\mathcal{H}^2(\mathbb{D})} \geq C\|p\|_{\mathcal{H}^2(\mathbb{D})} = C\left\|U^{-N} \widetilde{p}\right\|_{L^2(\mathbb{T})},$$

since the shift operator U is isometric and commutes with $M_{\widetilde{h}}$. Seeing as the set

$$\left\{U^{-N}\widetilde{p} \in L^2(\mathbb{T}) \;\middle|\; N \in \mathbb{N} \text{ and } \widetilde{p} \in \psi\big(\mathbb{C}[z]\big)\right\}$$

is dense in $L^2(\mathbb{T})$, the claim follows.
$(iii) \Rightarrow (iv)$: If T_h is bounded from below, it must obviously be injective, so the operator $T_h^{-1} : T_h\mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ is well-defined. Furthermore, calculating

$$\|f\|_{\mathcal{H}^2(\mathbb{D})} = \|T_h(T_h^{-1}f)\|_{\mathcal{H}^2(\mathbb{D})} \geq C\|T_h^{-1}f\| \tag{2.10}$$

for $f \in T_h\mathcal{H}^2(\mathbb{D})$ shows that T_h^{-1} is bounded. Since C is the largest possible constant such that (2.10) is satisfied, $\frac{1}{C}$ is the smallest possible bound for T_h^{-1}, so $\|T_h^{-1}\| \leq \frac{1}{C}$.
$(iv) \Rightarrow (iii)$: If T_h^{-1} is bounded,

$$\|f\|_{\mathcal{H}^2(\mathbb{D})} = \|T_h^{-1}(T_hf)\|_{\mathcal{H}^2(\mathbb{D})} \leq \frac{1}{C}\|T_hf\|_{\mathcal{H}^2(\mathbb{D})} \tag{2.11}$$

clearly holds for $f \in \mathcal{H}^2(\mathbb{D})$, and thus, T_h is bounded from below. Because $\frac{1}{C}$ is the smallest possible constant such that (2.11) hold, C is the largest possible lower bound for T_h.
$(iv) \Leftrightarrow (v)$: This is a consequence of the Closed Graph Theorem 1.2.3 □

Notice that the function $h \in H^\infty(\mathbb{D})$ in Lemma 2.2.14 is not itself required to be bounded from below, only demanding that its boundary function stays away from zero. This restriction would be too narrow, because, for example, $h(z) := z$ has a zero at the origin and $T_h = S$ has closed range.

2.3 Characterisation of Shift-Invariant Subspaces of $\mathcal{H}^2(\mathbb{D})$

We follow the approach presented in [Neu10], VII.

DEFINITION 2.3.1. We say a function $h \in H^\infty(\mathbb{D})$ is

1. inner if $|\psi(h)| = 1$ almost everywhere on $\mathbb{T}$, and
2. outer if T_h has dense image in $\mathcal{H}^2(\mathbb{D})$, i.e. if $\overline{T_h\mathcal{H}^2(\mathbb{D})} = \mathcal{H}^2(\mathbb{D})$.

Take note that the classical definition of inner and outer functions of Hardy spaces are different from the one given above. In particular, the requirement that an outer function belongs to $H^\infty(\mathbb{D})$ is relaxed in favor of membership of the respective Hardy space, cf. [Rud87], XVII, or [RR94], IV.

PROPOSITION 2.3.2. *Let h belong to $H^\infty(\mathbb{D})$. Then h is inner iff T_h is an isometry.*

Proof. By definition, h being inner means that $\widetilde{h} := \psi(h)$ has modulus one almost everywhere on the unit circle. Hence, the operator $M_{\widetilde{h}}$ is clearly isometric. Because of $T_h = \psi^{-1} \circ M_{\widetilde{h}} \circ \psi$, cf. Lemma 2.2.9, the same is true for T_h.

Conversely, if T_h is an isometry, then so is $M_{\widetilde{h}} = \psi \circ T_h \circ \psi^{-1}$ on $L^2_+(\mathbb{T})$. Due to Theorem 2.2.11, $M_{\widetilde{h}}$ commutes with the — also isometric — shift U. Hence,

$$\|M_{\widetilde{h}} U^{-N} \widetilde{p}\| = \|U^{-N} M_{\widetilde{h}} \widetilde{p}\| = \|M_{\widetilde{h}} \widetilde{p}\| = \|\widetilde{p}\| = \|U^{-N} \widetilde{p}\|$$

shows that $M_{\widetilde{h}}$ is isometric on the dense subset $\left\{U^{-N}\widetilde{p} \mid p \in \mathbb{C}[z], N \in \mathbb{N}\right\}$ of $L^2(\mathbb{T})$. Because of continuity, it is isometric on the whole of $L^2(\mathbb{T})$. Choosing in particular $\chi_B \in L^2(\mathbb{T})$, i.e the characteristic function of a Borel set $B \subseteq \mathbb{T}$, we have

$$\frac{1}{2\pi i} \int_B |\widetilde{h}(\zeta)|^2 \, d\zeta = \|M_{\widetilde{h}} \chi_B\|^2_{L^2(\mathbb{T})} = \|\chi_B\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi i} \int_B 1 \, d\zeta.$$

Letting B run over all Borel sets of $\mathbb{T}$, this implies $|\widetilde{h}(\zeta)|^2 = 1$ for almost every $\zeta \in \mathbb{T}$. Hence, h is inner. $\square$

PROPOSITION 2.3.3. *Let $h \in H^\infty(\mathbb{D})$ be outer. Then it has no zeros in $\mathbb{D}$. If $|h|$ is bounded from below by a constant $c > 0$, then $\frac{1}{h} \in H^\infty(\mathbb{D})$ and it is outer as well.*

Proof. Assume that $h(w) = 0$ for $w \in \mathbb{D}$ and $\overline{T_h \mathcal{H}^2(\mathbb{D})} = \mathcal{H}^2(\mathbb{D})$. Since the constant function with value 1 belongs to $\mathcal{H}^2(\mathbb{D})$, there must be a sequence $(f_n) \subseteq \mathcal{H}^2(\mathbb{D})$ such that $1 = \lim_{n\to\infty} h \cdot f_n$ in the norm of $\mathcal{H}^2(\mathbb{D})$. But because of Lemma 2.1.6, point evaluation is a norm continuous operation, so we get $1 = \lim_{n\to\infty} h(w) \cdot f_n(w) = 0$ at $w \in \mathbb{D}$. Thus, h cannot have a zero in $\mathbb{D}$.
For the second claim, $\frac{1}{h}$ is clearly well-defined and holomorphic. Furthermore, the boundedness from below of $|h|$ implies $\left\|\frac{1}{h}\right\|_\infty \leq \frac{1}{c}$. This shows that $\frac{1}{h}$ belongs to $H^\infty(\mathbb{D})$. Therefore, $T_{\frac{1}{h}} \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$ and, using that T_h is continuous, the calculation

$$\overline{T_{\frac{1}{h}} \mathcal{H}^2(\mathbb{D})} = \overline{T_{\frac{1}{h}} \overline{T_h \mathcal{H}^2(\mathbb{D})}} \supseteq \overline{T_{\frac{1}{h}} T_h \overline{\mathcal{H}^2(\mathbb{D})}} = \mathcal{H}^2(\mathbb{D})$$

proves that $\frac{1}{h}$ is outer. $\square$

THEOREM 2.3.4 (Beurling)**.** *Let $\mathcal{L} \neq \{0\}$ be a closed subspace of $\mathcal{H}^2(\mathbb{D})$. Then $\mathcal{L}$ is left invariant by S if and only if it has the form*

$$\mathcal{L} = T_h \mathcal{H}^2(\mathbb{D})$$

for an inner function $h \in H^\infty(\mathbb{D})$.

Proof. If $\mathcal{L}$ can be represented with the help of an inner function $h \in H^\infty(\mathbb{D})$ as $T_h \mathcal{H}^2(\mathbb{D})$, then due to Theorem 2.1.14 we have $S\mathcal{L} = ST_h\mathcal{H}^2(\mathbb{D}) = T_h S \mathcal{H}^2(\mathbb{D}) \subseteq T_h \mathcal{H}^2(\mathbb{D}) = \mathcal{L}$.
Conversely, if $\mathcal{L} \neq \{0\}$ is a closed, shift-invariant subspace of $\mathcal{H}^2(\mathbb{D})$, let us define $n_0 := \max\left\{j \in \mathbb{N}_0 \mid \left(z \mapsto \frac{f(z)}{z^j}\right) \in \mathcal{H}^2(\mathbb{D}) \text{ for all } f \in \mathcal{L}\right\}$, i.e. all functions contained in $\mathcal{L}$ have a zero at the origin of order at least n_0 and for some function in $\mathcal{L}$ the origin is a zero of order exactly n_0. Now let $f \in \mathcal{L}$ be such that $f = (z \mapsto z^{n_0} \sum_{n=0}^\infty a_n z^n)$ and $a_0 \neq 0$. The operator S increases the order of the zero at the origin by one. Thus, $S(g) \neq f$ for all $g \in \mathcal{L}$. Therefore, $S : \mathcal{L} \to \mathcal{L}$ is isometric, but not surjective. Since

$S\mathcal{L}$ is the isometric image of a closed subspace, it is closed as well. So we can form $\mathcal{L} \ominus S\mathcal{L}$, which is nonzero since $\mathcal{L}$ is nontrivial. We therefore find $h \in \mathcal{L} \ominus S\mathcal{L}$ such that $\|h\|_{\mathcal{H}^2(\mathbb{D})} = 1$. Our aim is to show that h is inner, i.e. that $\widetilde{h} := \psi(h)$ has modulus one almost everywhere on the unit circle.
For $n > 0$ we clearly have $S^n h \in S\mathcal{L}$ and $h \perp S\mathcal{L}$. Thus, with the help of ψ from (2.3) and $\psi \circ S = U \circ \psi$ we arrive at

$$\begin{aligned} 0 = (S^n h, h)_{\mathcal{H}^2(\mathbb{D})} &= \Big(U^n \psi(h), \psi(h)\Big)_{L^2(\mathbb{T})} = \Big(U^n \widetilde{h}, \widetilde{h}\Big)_{L^2(\mathbb{T})} = \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \widetilde{h}(e^{i\theta}) \overline{\widetilde{h}(e^{i\theta})}\, d\theta = \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} |\widetilde{h}(e^{i\theta})|^2 \, d\theta. \end{aligned} \tag{2.12}$$

After conjugating the above calculation we get

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} |\widetilde{h}(e^{i\theta})|^2 \, d\theta = 0$$

for $n \neq 0$. Since the trigonometric polynomials form an orthonormal basis of $L^2(\mathbb{T})$, this forces the Fourier series of the function $|\widetilde{h}(\zeta)|^2$ to be a constant in the sense of $L^2(\mathbb{T})$. Because of $1 = \|h\|_{\mathcal{H}^2(\mathbb{D})} = \|\psi(h)\|_{L^2(\mathbb{T})}$ this constant must be of modulus one. Hence, h belongs to $H^\infty(\mathbb{D})$ and is inner.
Next, we form T_h. From $(z \mapsto z^n h(z)) = S^n h \in \mathcal{L}$ we conclude that, due to linearity, $h \cdot p \in \mathcal{L}$ for all $p \in \mathbb{C}[z]$. Since according to Lemma 2.1.5 the polynomial ring $\mathbb{C}[z]$ is dense in $\mathcal{H}^2(\mathbb{D})$, it follows that $T_h \mathcal{H}^2(\mathbb{D}) \subseteq \mathcal{L}$. Because h is inner, this makes T_h an isometry according to Proposition 2.3.2 and thus, $T_h \mathcal{H}^2(\mathbb{D})$ is a closed subspace of $\mathcal{L}$.
To show that $T_h \mathcal{H}^2(\mathbb{D})$ cannot be a proper subspace of $\mathcal{L}$, we prove that the orthogonal complement of $T_h \mathcal{H}^2(\mathbb{D})$ in $\mathcal{L}$ contains only the zero function. Let $g \in \mathcal{L} \ominus T_h \mathcal{H}^2(\mathbb{D})$. On the one hand, the function $S^n h$ belongs to $T_h \mathcal{H}^2(\mathbb{D})$, which means that $(S^n h, g) = 0$ for $n \in \mathbb{N}_0$. On the other hand, $S^n g \in S\mathcal{L}$ for $n \in \mathbb{N}$. Since h is orthogonal to $S\mathcal{L}$, we get $(S^n g, h) = 0$ for $n \in \mathbb{N}$. Similarly to (2.12), we get

$$\begin{aligned} 0 = (S^n h, g) &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \widetilde{h}(e^{i\theta}) \overline{\widetilde{g}(e^{i\theta})}\, d\theta, && n \geq 0 \\ 0 = \overline{(S^n g, h)} &= \overline{\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \widetilde{g}(e^{i\theta}) \overline{\widetilde{h}(e^{i\theta})}\, d\theta} = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \widetilde{h}(e^{i\theta}) \overline{\widetilde{g}(e^{i\theta})}\, d\theta, && n > 0. \end{aligned}$$

Together these equations imply that $\psi(h)\overline{\psi(g)}$ must vanish almost everywhere on $\mathbb{T}$. Since h is inner, i.e. $|\psi(h)| = 1$ almost everywhere, this forces $\psi(g) = 0$ almost everywhere. Therefore, $g \equiv 0$ and we are finished. □

Beurling's Theorem can be sharpened in the sense that the resulting inner function satisfies a uniqueness condition of sorts.

PROPOSITION 2.3.5. *Let $h_1, h_2 \in H^\infty(\mathbb{D})$ be inner with $T_{h_1}\mathcal{H}^2(\mathbb{D}) = T_{h_2}\mathcal{H}^2(\mathbb{D})$. Then there exists $c \in \mathbb{T}$ such that $h_2 = c \cdot h_1$ holds.*

Proof. The functions h_1 and h_2 belong to $\mathcal{H}^2(\mathbb{D})$ and, thus, also to $T_{h_1}(\mathcal{H}^2(\mathbb{D})) = T_{h_2}(\mathcal{H}^2(\mathbb{D}))$. Hence, there exists $f \in \mathcal{H}^2(\mathbb{D})$ such that

$$h_1 = h_2 \cdot f. \tag{2.13}$$

We can therefore consider $\frac{h_1}{h_2} = f$. This function is well-defined, because whenever h_2 has a root at some point in $\mathbb{D}$, equation (2.13) guarantees that h_1 vanishes at there as well, even respecting the order of the root of h_2. Hence, all singularities resulting from the denominator h_2 are removed by the numerator h_1. As a Hardy-Hilbert space function, $\frac{h_1}{h_2}$ is mapped into $L^2_+(\mathbb{T}) \subseteq L^2(\mathbb{T})$ by ψ. So all Fourier coefficients of $\psi\left(\frac{h_1}{h_2}\right)$ vanish for negative indices. This and the same arguments applied to $\frac{h_2}{h_1} \in \mathcal{H}^2(\mathbb{D})$ show that for $n < 0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\left[\psi\left(\frac{h_1(e^{i\theta})}{h_2(e^{i\theta})}\right) + \psi\left(\frac{h_2(e^{i\theta})}{h_1(e^{i\theta})}\right)\right]\, d\theta = 0. \tag{2.14}$$

Our aim now is to proof that (2.14) holds even for $n \neq 0$. To this end, we show that

$$\psi\left(\frac{h_1}{h_2}\right) = \frac{1}{\psi\left(\frac{h_2}{h_1}\right)} \tag{2.15}$$

holds, which easily follows if we can verify $\psi\left(\frac{h_1}{h_2}\right) = \frac{\psi(h_1)}{\psi(h_2)}$. Using Corollary 2.2.10 with $h_2 \in H^\infty(\mathbb{D})$ and $\frac{h_1}{h_2} \in \mathcal{H}^2(\mathbb{D})$ yields

$$\psi(h_2) \cdot \psi\left(\frac{h_1}{h_2}\right) = \psi\left(h_2 \cdot \frac{h_1}{h_2}\right) = \psi(h_1).$$

Since h_2 is inner, we have $|\psi(h_2)| \equiv 1$ on $\mathbb{T}$, i.e. $\psi(h_2)$ is not the zero function. Hence, we can divide by $\psi(h_2)$ and arrive at our desired result. Equation (2.15) now shows that

$$\psi\left(\frac{h_1}{h_2}\right) + \psi\left(\frac{h_2}{h_1}\right) = \psi\left(\frac{h_1}{h_2}\right) + \frac{1}{\psi\left(\frac{h_1}{h_2}\right)} = \psi\left(\frac{h_1}{h_2}\right) + \overline{\psi\left(\frac{h_1}{h_2}\right)},$$

where the last equality is a consequence of $\frac{\psi(h_1)(\zeta)}{\psi(h_2)(\zeta)} \in \mathbb{T}$ for almost every $\zeta \in \mathbb{T}$. Hence, this is real and we can conjugate equation (2.14) and extend it to hold for all $n \neq 0$. So it follows that $\psi\left(\frac{h_1}{h_2}\right) + \psi\left(\frac{h_2}{h_1}\right) \equiv c' \in \mathbb{C}$ almost everywhere on $\mathbb{T}$, implying $\frac{h_1}{h_2} + \frac{h_2}{h_1} \equiv c'$. Setting $H := \frac{h_2}{h_1}$ gets us $H + \frac{1}{H} = c'$. This means that we have the quadratic equation $H^2 - c'H + 1 = 0$ on $\mathbb{D}$ and solving for H clearly shows $H(z) = \frac{c'}{2} \pm \sqrt{\frac{c'^2}{4} - 1} =: c$. As H is holomorphic, it must be constant and $h_1 \cdot c = h_2$. Finally,

$$|c| = |\psi(c)| = |\psi(H)| = \left|\psi\left(\frac{h_2}{h_1}\right)\right| = \frac{|\psi(h_2)|}{|\psi(h_1)|} = 1$$

proves $c \in \mathbb{T}$. □

COROLLARY 2.3.6 (Inner-Outer-Factorization of Multiplier Functions). *For every nonzero function $h \in H^\infty(\mathbb{D})$ there exist $h_1, h_2 \in H^\infty(\mathbb{D})$, with h_1 inner and h_2 outer, such that $h = h_1 \cdot h_2$. The functions h_1 and h_2 are uniquely determined by h up to multiplication by unimodular constants.*

Proof. Let T_h be the multiplier operator for the multiplier h and set $\mathcal{L} := \overline{T_h\mathcal{H}^2(\mathbb{D})}$. As we know, the shift operator S is continuous and it commutes with any given multiplier operator, so

$$S\mathcal{L} = S\overline{T_h\mathcal{H}^2(\mathbb{D})} \subseteq \overline{ST_h\mathcal{H}^2(\mathbb{D})} = \overline{T_hS\mathcal{H}^2(\mathbb{D})} \subseteq \overline{T_h\mathcal{H}^2(\mathbb{D})} = \mathcal{L}.$$

Additionally, $h = T_h 1 \in \mathcal{L}$. Therefore, $\mathcal{L} \neq \{0\}$ is a closed, shift-invariant subspace of $\mathcal{H}^2(\mathbb{D})$. By Beurling's Theorem 2.3.4 there exists an inner function $h_1 \in H^\infty(\mathbb{D})$ such that $T_{h_1}\mathcal{H}^2(\mathbb{D}) = \mathcal{L} = \overline{T_h\mathcal{H}^2(\mathbb{D})}$. It remains to show how we can find a suitable outer function h_2.
Due to Proposition 2.3.2 we first notice that $T_{h_1} : \mathcal{H}^2(\mathbb{D}) \to \mathcal{L}$ bijective. By the Bounded Inverse Theorem 1.2.4, $T_{h_1}^{-1} : \mathcal{L} \to \mathcal{H}^2(\mathbb{D})$ is continuous and thus, $T_{h_1}^{-1} \circ T_h \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D})\big)$. Furthermore, $T_{h_1}^{-1}$ commutes with S, because we get

$$S \circ T_{h_1}^{-1} = T_{h_1}^{-1} \circ T_{h_1} \circ S \circ T_{h_1}^{-1} = T_{h_1}^{-1} \circ S \circ T_{h_1} \circ T_{h_1}^{-1} = T_{h_1}^{-1} \circ S$$

on $\mathcal{L}$. Hence, we arrive at

$$S \circ T_{h_1}^{-1} \circ T_h = T_{h_1}^{-1} \circ S \circ T_h = T_{h_1}^{-1} \circ T_h \circ S.$$

Due to Theorem 2.1.14, $T_{h_1}^{-1} \circ T_h$ must be a multiplier operator, i.e. there exists a uniquely determined function $h_2 \in H^\infty(\mathbb{D})$ such that $T_{h_2} = T_{h_1}^{-1} \circ T_h$. This is obviously equivalent to $T_{h_1} \circ T_{h_2} = T_h$. Applying this relation to the constant function with value 1 shows $h = h_1 \cdot h_2$. Finally, since $T_{h_1} : \mathcal{H}^2(\mathbb{D}) \to \mathcal{L}$ is an isometry, so is $T_{h_1}^{-1} : \mathcal{L} \to \mathcal{H}^2(\mathbb{D})$ and the calculation

$$\overline{T_{h_2}\mathcal{H}^2(\mathbb{D})} = \overline{T_{h_1}^{-1} \circ T_h\mathcal{H}^2(\mathbb{D})} = T_{h_1}^{-1}\overline{T_h\mathcal{H}^2(\mathbb{D})} = T_{h_1}^{-1}\mathcal{L} = \mathcal{H}^2(\mathbb{D})$$

shows that h_2 is outer.
Regarding uniqueness, we assume that there are two decompositions $h = h_1 \cdot h_2 = h_1' \cdot h_2'$, where h_1, h_1' are inner and h_2, h_2' are outer. By Proposition 2.3.2, T_{h_1} und $T_{h_1'}$ are isometric, so

$$\begin{aligned} T_{h_1}\mathcal{H}^2(\mathbb{D}) &= T_{h_1}\overline{T_{h_2}\mathcal{H}^2(\mathbb{D})} = \overline{T_{h_1} \circ T_{h_2}\mathcal{H}^2(\mathbb{D})} = \overline{T_h\mathcal{H}^2(\mathbb{D})} = \\ &= \overline{T_{h_1'} \circ T_{h_2'}\mathcal{H}^2(\mathbb{D})} = T_{h_1'}\overline{T_{h_2'}\mathcal{H}^2(\mathbb{D})} = T_{h_1'}\mathcal{H}^2(\mathbb{D}) \end{aligned}$$

Proposition 2.3.5 provides us with a constant $c \in \mathbb{T}$ such that $h_1 = c \cdot h_1'$. This implies $h_1 \cdot h_2 = \frac{1}{c} \cdot h_1 \cdot \widetilde{h_2}$. Since h_1 is certainly not the zero function, we see that $h_2 = \frac{1}{c} \cdot h_2'$. Hence, the postulated factorization is unique up to unimodular factors. □

The classical Hardy space theory greatly extends Corollary 2.3.6 and factorizes not just the bounded analytic functions into an inner and an outer function, but all Hardy space functions. These standard results can be found in most books on the topic, cf. for example [Rud87], XVII, or [RR94], IV.

Chapter 3

Vector-Valued Analytic Functions and the Space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$

Linear relations on the Hardy-Hilbert space are subspaces of the Cartesian product of $\mathcal{H}^2(\mathbb{D})$. We will identify this product with another structure and develop our theory in this new setting. It turns out that the right way to move forward here are vector-valued analytic functions and we will therefore start this chapter with some results on them. After this, the one-dimensional theory of the previous chapter will be expanded to the multi-dimensional case, considering matrix-valued multiplier operators on vector-valued function spaces. For linear relations, the range of these functions will then be the Hilbert space $\mathbb{C}^2$.

3.1 Holomorphy in a Banach Space Setting

First, some notions from complex analysis are transferred to our setting of vector valued functions. We cite [HP57], III.10, and [Kle07], II, for the following results on holomorphy.

DEFINITION 3.1.1. Let G be a domain in the complex plane and X, Y be Banach spaces.

1. A function $f : G \to X$ is said to be holomorphic in G if

$$\lim_{z\to w} \frac{1}{z-w}\big(f(z) - f(w)\big) =: f'(w) \in X$$

   exists for every $w \in G$ with respect to the norm of X. It is called weakly holomorphic in G, if $\hat{x} \circ f : G \to \mathbb{C}$ is holomorphic in G in the classical sense for every $\hat{x} \in X'$.

2. An operator valued function $T : G \to \mathcal{B}(X, Y)$ is said to be holomorphic in G if

$$\lim_{z\to w} \frac{1}{z-w}\big(T(z) - T(w)\big) =: T'(w) \in \mathcal{B}(X, Y)$$

exists for every $w \in G$ with respect to the operator norm of $\mathcal{B}(X,Y)$. It is called strongly holomorphic if $z \mapsto T(z)x$ is a holomorphic function for every $x \in X$, and weakly holomorphic if $z \mapsto \hat{y}(Tx)$ is holomorphic in G in the classical sense for every $\hat{y} \in Y'$ and $x \in X$.

In fact, these definitions carry with them a certain redundancy. It can be shown that the introduced notions are all equivalent, which is the content of the next lemma and two propositions.

LEMMA 3.1.2. *Let $f : G \to X$ and $T : G \to \mathcal{B}(X,Y)$. Then we have*

(i) f is holomorphic $\Rightarrow$ f is weakly holomorphic

(ii) T is holomorphic $\Rightarrow$ T is strongly holomorphic $\Rightarrow$ T is weakly holomorphic

Proof. To show (i), let f be holomorphic. Since we have

$$(\hat{x} \circ f)'(w) = \lim_{z\to w} \frac{(\hat{x} \circ f)(z) - (\hat{x} \circ f)(w)}{z - w} = \hat{x}\left(\lim_{z\to w} \frac{f(z) - f(w)}{z - w}\right) = \hat{x}(f'(w))$$

for every $\hat{x} \in X'$, it must clearly be weakly holomorphic.
Regarding (ii): For an operator T, we notice that if $\lim_{z\to w} \frac{T(z)-T(w)}{z-w}$ exists in the operator topology, then so will $\lim_{z\to w} \frac{T(z)x-T(w)x}{z-w}$ for every $x \in X$ in the topology on Y. For the second implication, proceed by applying (i) to the function $z \mapsto T(z)x$ for every $x \in X$. □

PROPOSITION 3.1.3 (Cauchy Integral Formula)**.** *Let X be a Banach space and $f : G \to X$ be a holomorphic function defined on a domain G of $\mathbb{C}$. Suppose that $w \in G$ and that the open disk with radius r around w, i.e. $U_r(w)$, is completely contained in G. Then for every $z \in U_r(w)$ the equation*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U_r(w)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \in X$$

holds, where the contour integral is taken counter-clockwise. Furthermore, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial U_r(w)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \in X$$

for every $n \in \mathbb{N}$.

Proof. For every $\hat{x} \in X'$, the function $\hat{x} \circ f : G \to \mathbb{C}$ is holomorphic in the classical sense, so the classical Cauchy integral formula holds. Furthermore, since f is differentiable, it must also be continuous in G. By using Banach space valued nets it can be shown that the Riemann integral of continuous Banach space valued functions exists. In fact, this is a straightforward translation of the proof for the classical, complex valued case. With the same argument involving nets it can be shown that a continuous linear functional can be pulled out of the integral. For every $\hat{x} \in X'$ we thus have

$$\hat{x}(f(z)) = (\hat{x} \circ f)(z) = \frac{1}{2\pi i} \int_{\partial U_r(w)} \frac{(\hat{x} \circ f)(\zeta)}{\zeta - z} \, d\zeta = \hat{x} \left(\frac{1}{2\pi i} \int_{\partial U_r(w)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right)$$

and we arrive at

$$\hat{x} \left(f(z) - \frac{1}{2\pi i} \int_{\partial U_r(w)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \right) = 0$$

for every $\hat{x} \in X'$. Since X' is separating, the claim follows.
For the second formula, we fix $n \in \mathbb{N}$. For every $\hat{x} \in X'$ we have the classical result

$$(\hat{x} \circ f)^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial U_r(w)} \frac{(\hat{x} \circ f)(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta = \hat{x} \left(\frac{n!}{2\pi i} \int_{\partial U_r(w)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \right).$$

Additionally, from

$$(\hat{x} \circ f)'(z) = \hat{x}(f'(z))$$

we infer inductively that

$$(\hat{x} \circ f)^{(n)}(z) = \hat{x}(f^{(n)}(z))$$

holds for every $n \in \mathbb{N}$. Since this works for every $\hat{x} \in X'$, the claim follows again because X' is separating. ☐

PROPOSITION 3.1.4. *Let $f : G \to X$ and $T : G \to \mathcal{B}(X, Y)$*

(i) If f is weakly holomorphic, then it is holomorphic.

(ii) If T is weakly holomorphic, then it is holomorphic.

Proof. As a first step, we will show that if T is strongly holomorphic, then it is also holomorphic. By our additional assumption, $z \mapsto T(z)x$ is holomorphic for every $x \in X$. We need to show that $\lim_{z \to w} \frac{T(z) - T(w)}{z - w}$ exists with respect to the operator norm in $\mathcal{B}(X, Y)$. Equivalently, we can verify the Cauchy condition

$$\lim_{z_1, z_2 \to w} \frac{T(z_1) - T(w)}{z_1 - w} - \frac{T(z_2) - T(w)}{z_2 - w} = 0. \tag{3.1}$$

For $z_1 \neq z_2$, $z_1 \neq w$ and $z_2 \neq w$ we define the operator

$$\mathcal{T}(z_1, z_2) := \frac{1}{z_1 - z_2} \left[\frac{T(z_1) - T(w)}{z_1 - w} - \frac{T(z_2) - T(w)}{z_2 - w} \right].$$

Clearly, since it is made up of linear operators, we have $\mathcal{T}(z_1, z_2) \in \mathcal{B}(X, Y)$. Using the Cauchy Integral Formula

$$T(z)x = \frac{1}{2\pi i} \int_{\partial U_r(w)} \frac{T(\zeta)x}{\zeta - z}\, d\zeta$$

four times in the definition of $\mathcal{T}(z_1, z_2)$, we get for every $x \in X$

$$\begin{aligned}
\|\mathcal{T}(z_1, z_2)x\| &= \left\| \frac{1}{z_1 - z_2} \frac{1}{2\pi i} \int_{\partial U_r(w)} T(\zeta)x \left(\frac{\frac{1}{\zeta - z_1} - \frac{1}{\zeta - w}}{z_1 - w} - \frac{\frac{1}{\zeta - z_2} - \frac{1}{\zeta - w}}{z_2 - w} \right) d\zeta \right\| = \\
&= \left\| \frac{1}{2\pi i} \int_{\partial U_r(w)} T(\zeta)x \left(\frac{1}{(\zeta - z_1)(\zeta - z_2)(\zeta - w)} \right) d\zeta \right\| \leq \\
&\leq \frac{2\pi r}{2\pi} \frac{4}{r^3} \max_{\zeta \in \partial U_r(w)} \|T(\zeta)x\| =: C_x
\end{aligned}$$

if $z_1, z_2 \in U_{\frac{r}{2}}(w)$. Hence, for all z_1, z_2 sufficiently close to w, we get $\|\mathcal{T}(z_1, z_2)x\| \leq C_x$ for every $x \in X$. Due to the principle of uniform boundedness, this implies the existence of a constant $C > 0$ such that $\|\mathcal{T}(z_1, z_2)\| \leq C$ holds. This, in turn, implies (3.1).
In order to show (i), we identify X with a closed subspace of its bidual $X'' = \mathcal{B}(X', \mathbb{C})$, so $f : G \to X \subseteq \mathcal{B}(X', \mathbb{C})$. By our assumption, $\hat{x} \circ f : G \to \mathbb{C}$ is holomorphic for every $\hat{x} \in X'$, i.e. we see f as a strongly holomorphic operator valued function. According to the first step of the proof, this shows that f is holomorphic.
Finally, (ii) follows: If $z \mapsto \hat{y}(T(z)x)$ is a holomorphic function from G to Y for every $\hat{y} \in Y'$, by (i) it must be holomorphic. This means that $z \mapsto T(z)x$ is a holomorphic operator function for every $x \in X$, i.e. T is strongly holomorphic. As we have seen, this means that T is holomorphic. □

The above proposition provides a convenient way to check holomorphy. In the proof we used the Cauchy Integral Formula, which in turn relies on the fact that X' is a separating set. A lot of the well-known Cauchy theory in complex analysis can be developed also in the case of vector valued functions. Usually, one composes a given holomorphic function $f : G \to X$ with continuous linear functionals and uses the classical results for the functions $\hat{x} \circ f$, which are holomorphic as we have shown. Again, since X' is separating, the functionals are then removed by the same trick.
We formulate one particular result for our convenience.

LEMMA 3.1.5. *Let $(a_n)_{n \in \mathbb{N}_0}$ be a sequence in a Banach space X. Consider the power series $z \mapsto \sum_{n=0}^{\infty} z^n a_n$ and define three subsets M_1, M_2, M_3 of $\mathbb{C}$ via*

- $M_1 := \left\{ z \in \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} \|z^n a_n\|_X < \infty \right\}$,
- $M_2 := \left\{ z \in \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} z^n a_n \text{ converges in } X \right\}$, *and*
- $M_3 := \left\{ z \in \mathbb{C} \;\middle|\; \sup_{n \in \mathbb{N}_0} \|z^n a_n\| < \infty \right\}$.

If we set $R_i := \sup_{M_i} |z|$ for $i \in \{1, 2, 3\}$, then $R_1 = R_2 = R_3$.

Proof. Obviously, we have $M_1 \subseteq M_2 \subseteq M_3$ and thus, $R_1 \leq R_2 \leq R_3$.
For $|z| < R_3$ there exists $w \in M_3$ such that $|z| < |w| \leq R_3$ and $C := \sup_{n\in\mathbb{N}_0} \|w^n a_n\|$ is finite. Because of

$$\|z^n a_n\| = \left|\frac{z}{w}\right|^n \|w^n a_n\| \leq C \left|\frac{z}{w}\right|^n$$

we can use the comparison test to determine

$$\sum_{n=0}^{\infty} \|z^n a_n\| \leq C \sum_{n=0}^{\infty} \left|\frac{z}{w}\right|^n = C\frac{1}{1-\left|\frac{z}{w}\right|} < \infty.$$

Hence, $z \in M_1$ and $|z| \leq R_1$. This implies $R_3 \leq R_1$ and we are finished. □

LEMMA 3.1.6. *Let X, Y be Banach spaces and $\Theta : \mathbb{D} \to \mathcal{B}(X,Y)$ an operator valued function. Then the following are equivalent:*

(i) Θ is holomorphic on $\mathbb{D}$.

(ii) Θ has a power series expansion $\Theta(z) = \sum_{n=0}^{\infty} z^n \Theta_n$ on $\mathbb{D}$ with $\Theta_n : X \to Y$ being bounded linear operators and where the power series is convergent in the strong, weak and operator norm topology.

Proof. (i) ⇒ (ii): We take arbitrary $x \in X$ and $\hat{y} \in Y'$ and consider the function $f_{x,\hat{y}} := \big(z \mapsto \hat{y}(\Theta(z)x)\big)$ with domain $\mathbb{D}$ and range in $\mathbb{C}$. Due to Lemma 3.1.2, $f_{x,\hat{y}}$ is holomorphic in the classical sense. Hence, it is expandable in a power series $\sum_{n=0}^{\infty} \alpha_n^{x,\hat{y}} z^n$ convergent on $\mathbb{D}$ and with $\alpha_n^{x,\hat{y}} \in \mathbb{C}$ for all $n \in \mathbb{N}_0$. Setting $w = z = 0$ and choosing $r \in (0,1)$, Proposition 3.1.3 yields

$$\begin{aligned}\alpha_n^{x,\hat{y}} &= \frac{1}{n!} f_{x,\hat{y}}^{(n)}(0) = \frac{1}{2\pi i}\int_{\partial U_r(0)} \frac{f_{x,\hat{y}}(\zeta)}{\zeta^{n+1}}\, d\zeta = \\ &= \frac{1}{2\pi i}\int_{\partial U_r(0)} \frac{1}{\zeta^{n+1}} \hat{y}(\Theta(\zeta)x)\, d\zeta = \\ &= \hat{y}\Bigg(\underbrace{\left[\frac{1}{2\pi i}\int_{\partial U_r(0)} \frac{1}{\zeta^{n+1}}\Theta(\zeta)\, d\zeta\right]}_{=:\,\Theta_n \in \mathcal{B}(X,Y)} x\Bigg) = \hat{y}(\Theta_n x).\end{aligned}$$

Note that Θ_n is well defined, because Θ is continuous due to it being holomorphic and, therefore, the $\mathcal{B}(X,Y)$-valued integral inside the square brackets exists. In summary, we have shown that $f_{x,\hat{y}}(z) = \sum_{n=0}^{\infty} z^n \hat{y}(\Theta_n x)$ holds on $\mathbb{D}$ for all $x \in X$ and $\hat{y} \in Y'$, i.e. Θ has a weakly converging power series expansion $\Theta(z) = \sum_{n=0}^{\infty} z^n \Theta_n$ on the unit disc.
Let $\iota : X \to X''$ be the canonical embedding of X into its bidual X''. For some arbitrary but fixed $z \in \mathbb{D}$ we choose any $x \in X$ and consider the family of operators

$$\left\{\iota(z^n \Theta_n x) : Y' \to \mathbb{C} \;\middle|\; n \in \mathbb{N}_0\right\}.$$

Because of

$$|\iota(z^n\Theta_n x)(\hat{y})| = |\hat{y}(z^n\Theta_n x)| \leq C_{\hat{y}}$$

for $n \in \mathbb{N}_0$, we see that this family is pointwise bounded. By the Principle of Uniform Boundedness, Theorem 1.2.2, we get that $\sup_{n\in\mathbb{N}_0} \|\iota(z^n\Theta_n x)\|$ is finite. Since the mapping ι is isometric, we obtain $\sup_{n\in\mathbb{N}_0} \|z^n\Theta_n x\| < \infty$, which is valid for all $x \in X$ and $z \in \mathbb{D}$. Due to Lemma 3.1.5, the power series expansion of Θ converges strongly on $\mathbb{D}$. This in turn implies that $\{z^n\Theta_n \mid n \in \mathbb{N}_0\}$ is a pointwise bounded family of operators as well. Applying Theorem 1.2.2 a second time yields $\sup_{n\in\mathbb{N}_0} \|z^n\Theta_n\| < \infty$. By Lemma 3.1.5, the power series expansion of Θ thus converges in the norm of $\mathcal{B}(X,Y)$.
(ii) $\Rightarrow$ (i): Let Θ have a power series expansion $\Theta(z) = \sum_{n=0}^{\infty} z^n\Theta_n$. In particular, this series converges weakly, i.e. $f_{x,\hat{y}} := \left(z \mapsto \sum_{n=0}^{\infty} z^n\hat{y}(\Theta_n x)\right)$ is analytic in the classical sense. Thus, $f_{x,\hat{y}}$ is holomorphic in the classical sense, which means that Θ is weakly holomorphic. Due to Proposition 3.1.4 it is therefore holomorphic. □

3.2 The Space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$

We are now ready to introduce the Hardy space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$.

DEFINITION 3.2.1. For $n \in \mathbb{N}$ we consider $\mathbb{C}^n$ and define

$$\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) := \left\{ f \in (\mathbb{C}^n)^{\mathbb{D}} \;\middle|\; f(z) = \sum_{k=0}^{\infty} z^k a_k \text{ on } \mathbb{D},\ (a_k) \in (\mathbb{C}^n)^{\mathbb{N}_0},\ \sum_{k=0}^{\infty} \|a_k\|_{\mathbb{C}^n}^2 < \infty \right\}$$

to contain all holomorphic and $\mathbb{C}^n$-valued functions on the unit disk. Furthermore, they are required to possess power series expansions where the coefficients are elements of $\mathbb{C}^n$ and are square-summable. The power series expansion itself is understood to be convergent with respect to the (usual Euclidean) norm $\|.\|_{\mathbb{C}^n}$.

Obviously, the case $n = 1$ is just $\mathcal{H}^2(\mathbb{D})$ from Definition 2.1.1. It should be noted, that one could also define $\mathcal{H}^2(\mathbb{D};\mathfrak{H})$ for an an infinite dimensional separable Hilbert space, cf. [Nag10], V, but we will stick to dealing with finite dimensional Hilbert spaces. In the next chapter, the case $n = 2$ will then be important for dealing with linear relations.
The following lemma assures us that a function with a power series expansion that has square-summable coefficients is automatically holomorphic on the unit disk.

LEMMA 3.2.2. *Let $(a_k) \in (\mathbb{C}^n)^{\mathbb{N}_0}$. The condition $\sum_{k=0}^{\infty} \|a_k\|_{\mathbb{C}^n}^2 < \infty$ implies that the radius of convergence ρ of $z \mapsto \sum_{k=0}^{\infty} z^k a_k$ is greater or equal to 1.*

Proof. The sequence $(\|a_k\|_{\mathbb{C}^n}^2)$ must be a null sequence because the series $\sum_{k=0}^{\infty} \|a_k\|^2$ converges. Thus, $(\|a_k\|_{\mathbb{C}^n})$ also tends to zero. Therefore, there exists $N \in \mathbb{N}$ such that $\|a_k\|_{\mathbb{C}^n} \leq 1$ for all $k \geq N$. Consequently, the sequence $(\sqrt[k]{\|a_k\|_{\mathbb{C}^n}})_{k=N}^{\infty}$, and, in particular, its limes superior, will also be bounded from above by 1. We can therefore use the following well-known formula to calculate the radius of convergence

$$\rho = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\|a_k\|_{\mathbb{C}^n}}} \geq 1$$

and the assertion follows. □

We recall that $\ell^2(\mathbb{N}_0;\mathbb{C}^n)$ is the Hilbert space of square-summable sequences where the elements in each sequence are vectors from $\mathbb{C}^n$. Furthermore, the space $\ell^2(\mathbb{N}_0;\mathbb{C}^n)$ is equipped with the scalar product $\big((a_k),(b_k)\big)_{\ell^2(\mathbb{N}_0;\mathbb{C}^n)} := \sum_{k=0}^{\infty}(a_k,b_k)_{\mathbb{C}^n}$, which is equivalent to the sum scalar product from $\ell^2(\mathbb{N}_0)$, i.e. for $(a_k) = (a_{1,k},\dots a_{n,k})^\top$ and $(b_k) = (b_{1,k},\dots,b_{n,k})^\top \in \ell^2(\mathbb{N}_0;\mathbb{C}^n)$ we take the coordinate sequences, which obviously all belong to $\ell^2(\mathbb{N}_0)$ and therefore

$$\begin{aligned}\big((a_k)_{k\in\mathbb{N}_0},(b_k)_{k\in\mathbb{N}_0}\big)_{\ell^2(\mathbb{N}_0;\mathbb{C}^n)} &:= \sum_{k=0}^{\infty}(a_k,b_k)_{\mathbb{C}^n} = \sum_{k=0}^{\infty}\sum_{j=1}^{n} a_{j,k}\overline{b_{j,k}} = \\ &= \sum_{j=1}^{n}\sum_{k=0}^{\infty} a_{j,k}\overline{b_{j,k}} = \sum_{j=1}^{n}\big((a_{j,k})_{k\in\mathbb{N}_0},(b_{j,k})_{k\in\mathbb{N}_0}\big)_{\ell^2(\mathbb{N}_0)}.\end{aligned} \tag{3.2}$$

We make use of these well-known facts in the following

PROPOSITION 3.2.3. *For every $n\in\mathbb{N}$ we have*

$$\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \cong \ell^2(\mathbb{N}_0;\mathbb{C}^n).$$

The mapping

$$\Phi_n : \begin{cases} \ell^2(\mathbb{N}_0;\mathbb{C}^n) & \to \quad \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \\ (a_n) & \mapsto \quad f := (z \mapsto \sum_{n=0}^{\infty} z^n a_n) \end{cases} \tag{3.3}$$

is bijective and preserves the linear structure. Moreover,

$$(.,.)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} : \begin{cases} \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)\times\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) & \to \quad \mathbb{C} \\ [f,g] & \mapsto \quad \big(\Phi_n^{-1}(f),\Phi_n^{-1}(g)\big)_{\ell^2(\mathbb{N}_0;\mathbb{C}^n)} \end{cases}$$

is an inner product on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ such that Φ_n is additionally isometric.

Proof. The function Φ_n is well-defined — the holomorphy of $\Phi_n((a_n))$ on the unit disk is due to Lemmata 3.2.2 and 3.1.6 — and clearly bijective. In addition, the definitions for + and multiplication by a scalar in $\ell^2(\mathbb{N}_0;\mathbb{C}^n)$ agree with those for power series. Consequently, Φ_n is compatible with the linear structures on the two spaces and $(.,.)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}$ is indeed an inner product on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ for which Φ_n is isometric. □

PROPOSITION 3.2.4. *For each $n\in\mathbb{N}$ we have*

$$\big(\mathcal{H}^2(\mathbb{D})\big)^n \cong \mathcal{H}^2(\mathbb{D};\mathbb{C}^n).$$

Furthermore, the scalar product and its induced norm on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ are equivalent to the sum scalar product and the corresponding norm of $(\mathcal{H}^2(\mathbb{D}))^n$, i.e. for $f,g\in\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with coordinate functions $f_1,\dots,f_n,g_1,\dots,g_n\in\mathcal{H}^2(\mathbb{D})$, respectively, we have

$$(f,g)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \sum_{j=1}^{n}(f_j,g_j)_{\mathcal{H}^2(\mathbb{D})} \quad \textit{and} \quad \|f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \sqrt{\sum_{j=1}^{n}\|f_j\|^2_{\mathcal{H}^2(\mathbb{D})}}.$$

Proof. Let $f_1,\dots,f_n \in \mathcal{H}^2(\mathbb{D})$ with power series coefficients $(a_{1,k})_{k\in\mathbb{N}_0},\dots,(a_{n,k})_{k\in\mathbb{N}_0}$ belonging to $\mathbb{C}^{N_0}$, respectively. Combining $a_{1,k},\dots,a_{n,k}\in\mathbb{C}$ for each $k\in\mathbb{N}_0$ to a vector $c_k\in\mathbb{C}^n$ and defining $h := (z\mapsto \sum_{k=0}^\infty z^k c_k)$ we first notice that

$$\left(z\mapsto \sum_{k=0}^\infty z^k c_k\right) = \left(z\mapsto \sum_{k=0}^\infty z^k \begin{pmatrix} a_{1,k}\\ \vdots \\ a_{n,k}\end{pmatrix}\right) = \left(z\mapsto \begin{pmatrix} \sum_{k=0}^\infty a_{1,k}z^k\\ \vdots \\ \sum_{k=0}^\infty a_{n,k}z^k\end{pmatrix}\right) = $$
$$= \left(z\mapsto \begin{pmatrix} f_1(z)\\ \vdots \\ f_n(z)\end{pmatrix}\right).$$

Using (3.2) for calculating

$$\sum_{k=0}^\infty \|c_k\|^2_{\mathbb{C}^n} = \sum_{k=0}^\infty \|(a_{1,k},\dots,a_{n,k})^\top\|^2_{\mathbb{C}^n} = \sum_{k=0}^\infty\sum_{j=1}^n |a_{j,k}|^2 = \sum_{j=1}^n\sum_{k=0}^\infty |a_{j,k}|^2 \qquad (3.4)$$

shows that h has square-summable power series coefficients $(c_k)_{k\in\mathbb{N}_0}\in\left(\mathbb{C}^n\right)^{\mathbb{N}_0}$ under our assumptions. Lemma 3.2.2 implies that h is analytic on the unit disk. Furthermore, $(f_1,\dots,f_n)^\top \mapsto h$ is clearly linear and, hence, a vector space homomorphism from $(\mathcal{H}^2(\mathbb{D}))^n$ to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Since $(\mathcal{H}^2(\mathbb{D}))^n$ is equipped with the sum scalar product, (3.4) implies

$$\|h\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \sum_{j=1}^n \|f_j\|^2_{\mathcal{H}^2(\mathbb{D})} = \|(f_1,\dots,f_n)^\top\|^2_{(\mathcal{H}^2(\mathbb{D}))^n}.$$

Thus, the above assignment is isometric and in turn injective. For surjectivity, we pick $h\in\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with coefficients (c_k). The functions $g_j(z) := \sum_{k=0}^\infty z^k c_{j,k}$, whereby $j\in\{1,\dots,n\}$ and $c_{j,k}$ signifies the j-th coordinate of c_k, are all clearly elements of $\mathcal{H}^2(\mathbb{D})$, since $\|c_{j,k}\|^2_{\mathcal{H}^2(\mathbb{D})} \le \|c_k\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}$ for all k and j. Hence, we found a preimage of h. □

LEMMA 3.2.5. *The set*

$$\mathbb{C}[z;\mathbb{C}^n] := \left\{ p:\mathbb{D}\to\mathbb{C}^n \;\middle|\; p(z) = \sum_{k=0}^K z^k a_k \text{ for } K\in\mathbb{N}_0 \text{ and } a_k\in\mathbb{C}^n \text{ for } 0\le k\le K\right\},$$

i.e. the ring of vector-valued polynomials on the unit disk, is densely contained in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with respect to the norm $\|.\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}$.

Proof. Let $f\in\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with power series $f(z) = \sum_{k=0}^\infty a_k z^k$ for $a_k\in\mathbb{C}^n$ and define $p_N(z) := \sum_{k=0}^N z^k a_k$. Then $\|f-p_N\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \|\sum_{k=N+1}^\infty z^k a_k\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \sum_{k=N+1}^\infty \|a_k\|^2_{\mathbb{C}^n}$ converges to zero as N approaches infinity. □

As we have mentioned earlier, $\mathcal{H}^2(\mathbb{D})$ is a reproducing kernel Hilbert space. For higher dimensions we arrive at the following result.

LEMMA 3.2.6. *Let $\iota_{n,w} : \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \to \mathbb{C}^n : f \mapsto f(w)$ be the point evaluation functional at $w \in \mathbb{D}$. Then $\iota_{n,w}$ is linear and continuous for every w with $\|\iota_{n,w}\| \le \sqrt{\frac{1}{1-|w|^2}}$. Moreover, for $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with coordinate functions $f_1,\dots,f_n \in \mathcal{H}^2(\mathbb{D})$ we have*

$$\iota_{n,w}(f) = \begin{pmatrix} (f_1,k_w)_{\mathcal{H}^2(\mathbb{D})} \\ \vdots \\ (f_n,k_w)_{\mathcal{H}^2(\mathbb{D})} \end{pmatrix} \in \mathbb{C}^n,$$

where the k_w are the reproducing kernel functions defined in Lemma 2.1.6.

Proof. The linearity of $\iota_{n,w}$ is clear. By Lemma 2.1.6, the functions k_w are contained in $\mathcal{H}^2(\mathbb{D})$ and are nonzero for every $w \in \mathbb{D}$.
Now let $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ be the function $z \mapsto \sum_{k=0}^\infty z^k a_k$ and $w \in \mathbb{D}$. We denote the coordinate functions of f by $f_1,\dots,f_n \in \mathcal{H}^2(\mathbb{D})$. Using Lemma 2.1.6 in each component we calculate

$$f(w) = \sum_{k=0}^\infty w^k a_k = \begin{pmatrix} \left(\sum_{k=0}^\infty a_{1,k} z^k, \sum_{k=0}^\infty \overline{w^k} z^k\right)_{\mathcal{H}^2(\mathbb{D})} \\ \vdots \\ \left(\sum_{k=0}^\infty a_{n,k} z^k, \sum_{k=0}^\infty \overline{w^k} z^k\right)_{\mathcal{H}^2(\mathbb{D})} \end{pmatrix} = \begin{pmatrix} (f_1,k_w)_{\mathcal{H}^2(\mathbb{D})} \\ \vdots \\ (f_n,k_w)_{\mathcal{H}^2(\mathbb{D})} \end{pmatrix}.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|\iota_{n,w}(f)\|_{\mathbb{C}^n}^2 &= \left\| \begin{pmatrix} (f_1,k_w)_{\mathcal{H}^2(\mathbb{D})} \\ \vdots \\ (f_n,k_w)_{\mathcal{H}^2(\mathbb{D})} \end{pmatrix} \right\|_{\mathbb{C}^n}^2 = \sum_{j=1}^n \left|(f_j,k_w)_{\mathcal{H}^2(\mathbb{D})}\right|^2 \\ &\le \|k_w\|_{\mathcal{H}^2(\mathbb{D})}^2 \sum_{j=1}^n \|f_j\|_{\mathcal{H}^2(\mathbb{D})}^2 = \|k_w\|_{\mathcal{H}^2(\mathbb{D})}^2 \|f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}^2 \end{aligned}$$

and thus, $\iota_{n,w}$ is bounded with $\|\iota_{n,w}\| \le \|k_w\|_{\mathcal{H}^2(\mathbb{D})} = \sqrt{\frac{1}{1-|w|^2}}$. □

LEMMA 3.2.7. *Let $\iota_{n,w}$ be the point evaluation functional on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ at $w \in \mathbb{D}$ and define a function $K_{n,w} : \mathbb{C}^n \to \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ by $K_{n,w} := \iota_{n,w}^*$. Then*

(i) For every $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $\alpha \in \mathbb{C}^n$ we have the relation $(f, K_{n,w}(\alpha)) = \alpha^H f(w)$ and we can explicitly calculate $K_{n,w}(\alpha) = \sum_{k=0}^\infty z^k(\overline{w}^k\alpha)$.

(ii) $K_{n,w}$, and thus also $\iota_{n,w}$, has operator norm $\sqrt{\frac{1}{1-|w|^2}}$.

(iii) $\sqrt{1-|w|^2}K_{n,w} : \mathbb{C}^n \to K_{n,w}(\mathbb{C}^n)$ is unitary and $\|K_{n,w}^{-1}\| = \sqrt{1-|w|^2}$.

Proof. Take an arbitrary $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with power series coefficients $(a_k)_{k\in\mathbb{N}_0}$. Then clearly

$$\alpha^H f(w) = \alpha^H \iota_{n,w}(f) = (\iota_{n,w}(f), \alpha)_{\mathbb{C}^n} = (f, K_{n,w}(\alpha))_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}$$

and, by the definition of the scalar product on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ in Proposition 3.2.4,

$$\alpha^H f(w) = \alpha^H \sum_{k=0}^{\infty} w^k a_k = \sum_{k=0}^{\infty} w^k \alpha^H a_k = \left(\sum_{k=0}^{\infty} z^k a_k, \sum_{k=0}^{\infty} z^k (\overline{w}^k \alpha)\right)$$

Moreover,

$$\iota_{n,w} K_{n,w}(\alpha) = \iota_{n,w}\left(\sum_{k=0}^{\infty} z^k (\overline{w}^k \alpha)\right) = \sum_{k=0}^{\infty} w^k \overline{w}^k \alpha = \left(\sum_{k=0}^{\infty} (|w|^2)^k\right)\alpha = \frac{1}{1-|w|^2}\alpha.$$

This means that

$$\iota_{n,w} K_{n,w} = \frac{1}{1-|w|^2} I_{\mathbb{C}^n},$$

or in other words, $\iota_{n,w} K_{n,w} = K_{n,w}^* K_{n,w}$ is a diagonal matrix with entry $(1-|w|^2)^{-1}$. This certainly implies that $\sqrt{1-|w|^2} K_{n,w}$ is isometric. It is therefore unitary as a mapping from $\mathbb{C}^n$ to $K_{n,w}(\mathbb{C}^n)$, and

$$\left(\sqrt{1-|w|^2} K_{n,w}\right)^{-1} = \sqrt{1-|w|^2} K_{n,w}^* \restriction K_{n,w}(\mathbb{C}^n)$$

Now $\|\sqrt{1-|w|^2} K_{n,w}\| = \|(\sqrt{1-|w|^2} K_{n,w})^{-1}\| = 1$ implies $\|K_{n,w}^{-1}\| = \sqrt{1-|w|^2}$ and $\|\iota_{n,w}\| = \|K_{n,w}\| = (\sqrt{1-|w|^2})^{-1}$. □

We will use the two lemmata above in much the same way as we used the reproducing kernel of $\mathcal{H}^2(\mathbb{D})$ in the one-dimensional case. It should be noted that for $n = 1$ we have $k_w = K_{1,w}(1)$ as 1 generates $\mathbb{C}$.
The scalar product in higher dimensions was defined via summation of power series coefficients. However, just as in the one-dimensional case, we can alternatively integrate on circles to achieve the same.

LEMMA 3.2.8. *Let $f : \mathbb{D} \to \mathbb{C}^n$ be holomorphic and have the power series coefficients $(a_k)_{k\in\mathbb{N}_0} \in (\mathbb{C}^n)^{N_0}$. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \left\| f\left(re^{i\theta}\right)\right\|_{\mathbb{C}^n}^2 d\theta = \sum_{k=0}^{\infty} r^{2k} \|a_k\|_{\mathbb{C}^n}^2$$

as elements of $[0,\infty]$. The condition that f belongs to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ is satisfied if and only if this expression stays finite as r tends to one from below. In this case, the norm of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ can be calculated via

$$\|f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}^2 = \lim_{r\nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} \left\| f\left(re^{i\theta}\right)\right\|_{\mathbb{C}^n}^2 d\theta.$$

If additionally $g \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, then the inner product satisfies

$$(f,g)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \lim_{r\nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} \Big(f\big(re^{i\theta}\big), g\big(re^{i\theta}\big)\Big)_{\mathbb{C}^n}\, d\theta.$$

Proof. Let $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ with power series coefficients (a_k). First note that $\big(z \mapsto \sum_{k=0}^N z^k a_k\big)_{N\in\mathbb{N}}$ converges to f uniformly on compact subsets of $\mathbb{D}$, since f is analytic on $\mathbb{D}$. For a fixed $r \in (0,1)$, we use uniform convergence on the closed ball centred at zero with radius r to exchange the order of integration and the limit process and get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \left\|f\left(re^{i\theta}\right)\right\|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{N\to\infty}\left(\sum_{k=0}^N (re^{i\theta})^k a_k\right)\overline{\left(\sum_{j=0}^N (re^{i\theta})^j a_j\right)}\, d\theta = \\ &= \lim_{N\to\infty}\sum_{k,j=0}^N r^{k+j}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-j)\theta}\, d\theta\right) a_k\overline{a_j} = \\ &= \sum_{k=0}^\infty r^{2k}\|a_k\|^2_{\mathbb{C}^n},\end{aligned}$$

since only in the case $k = j$ does $\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-j)\theta}\, d\theta$ not vanish and amount to 1. Hence, the net $\left(\frac{1}{2\pi}\int_0^{2\pi} \left\|f\left(re^{i\theta}\right)\right\|^2 d\theta\right)_{r\in(0,1)} = \left(\sum_{k=0}^\infty r^{2k}\|a_k\|^2_{\mathbb{C}^n}\right)_{r\in(0,1)}$ is obviously increasing as r tends to one. Thus, the limit is attained at the supremum. Finally, $\lim_{r\nearrow 1}\sum_{m=0}^\infty r^{2k}\|a_k\|^2_{\mathbb{C}^n} = \sum_{k=0}^\infty \|a_k\|^2_{\mathbb{C}^n}$ follows from the monotone convergence theorem appplied to the counting measure.
To prove the claim regarding the scalar product, we use the polarisation identity as in Corollary 2.1.9 to show

$$\begin{aligned}4(f,g)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} &= \|f+g\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} - \|f-g\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} \\ &\quad + i\|f+ig\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} - i\|f-ig\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \\ &= \lim_{r\nearrow 1}\left[\frac{1}{2\pi}\int_0^{2\pi} \left\|(f+g)\left(re^{i\theta}\right)\right\|^2_{\mathbb{C}^n} - \left\|(f-g)\left(re^{i\theta}\right)\right\|^2_{\mathbb{C}^n}\right. \\ &\quad \left. + i\left\|(f+ig)\left(re^{i\theta}\right)\right\|^2_{\mathbb{C}^n} - i\left\|(f-ig)\left(re^{i\theta}\right)\right\|^2_{\mathbb{C}^n} d\theta\right] = \\ &= \lim_{r\nearrow 1}\left[\frac{1}{2\pi}\int_0^{2\pi} \Big((f(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} + (f(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n}\right. \\ &\quad + (g(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} + (g(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n} - (f(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} \\ &\quad + (f(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n} + (g(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} - (g(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n} \\ &\quad + i(f(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} + (f(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n} - (g(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n}\end{aligned}$$

$$+ i(g(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n} - i(f(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} + (f(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n}$$

$$\left. - (g(re^{i\theta}), f(re^{i\theta}))_{\mathbb{C}^n} - i(g(re^{i\theta}), g(re^{i\theta}))_{\mathbb{C}^n}\Big) \, d\theta \right] =$$

$$= 4 \lim_{r\nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left(f\left(re^{i\theta}\right), g\left(re^{i\theta}\right)\right)_{\mathbb{C}^n} \, d\theta.$$

□

Given $n, m \in \mathbb{N}$, we can look at what multiplier operators look like in higher dimensions. Obviously, for a function $\Theta : \mathbb{D} \to \mathbb{C}^{m\times n}$ we can define the linear relation

$$T_\Theta := \left\{[f,g] \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \times \mathcal{H}^2(\mathbb{D};\mathbb{C}^m) \;\middle|\; g = \Theta \cdot f\right\}.$$

The equality $g = \Theta \cdot f$ is assumed to hold pointwise. T_Θ is an operator, because $\operatorname{mul} T_\Theta = \{0\}$. Hence, it makes sense to write

$$T_\Theta : \begin{cases} \operatorname{dom} T_\Theta & \to \quad \mathcal{H}^2(\mathbb{D};\mathbb{C}^m) \\ \qquad f & \mapsto \quad \Theta \cdot f \end{cases},$$

where $\operatorname{dom} T_\Theta = \left\{f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \;\middle|\; \Theta \cdot f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)\right\}$.
We notice that for $f = (z \mapsto \sum_{k=0}^\infty z^k a_k) \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ the function product $id_{\mathbb{D}} \cdot f = (z \mapsto z \cdot f(z)) = \left(z \mapsto \sum_{k=0}^\infty z^{k+1} a_k\right)$ belongs to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ as well. Consequently, the following operator is well-defined on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$:

DEFINITION 3.2.9. For $n \in \mathbb{N}$, we call

$$S_n : \begin{cases} \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) & \to \quad \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \\ \qquad f & \mapsto \quad (z \mapsto z \cdot f(z)) \end{cases}$$

the shift operator, or operator of multiplication by z, on the space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$.

Alternatively, we could write $S_n = T_\Theta$ for

$$\Theta = \begin{pmatrix} id_{\mathbb{D}} & & & \\ & id_{\mathbb{D}} & & \\ & & \ddots & \\ & & & id_{\mathbb{D}} \end{pmatrix} : \mathbb{D} \mapsto \mathbb{C}^{n\times n}.$$

LEMMA 3.2.10. *Let $\Theta : \mathbb{D} \to \mathbb{C}^{m\times n}$. Then T_Θ is a closed operator and the following assertions are equivalent:*

(i) $T_\Theta \in \mathcal{B}(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m))$

(ii) $\operatorname{dom} T_\Theta = \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$

Proof. First, if $([f_k, g_k])_{k\in\mathbb{N}}$ is a sequence in the graph of T_Θ converging to an element $[f, g]$ in the Hilbert space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \times \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$, then we have $g_k = \Theta \cdot f_k$ for every $k \in \mathbb{N}$. Additionally, evaluation at a point is a norm continuous operation in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$, cf. Lemma 3.2.6. So for arbitrary $w \in \mathbb{D}$ we get

$$\begin{array}{ccc} g_k(w) & = & \Theta(w) \cdot f_k(w) \\ \downarrow & & \downarrow \\ g(w) & & \Theta(w) \cdot f(w) \end{array}$$

Thus, $g = \Theta \cdot f$ and $[f, g] \in T_\Theta$, i.e. T_Θ is closed.
Secondly, the condition $T_\Theta \in \mathcal{B}(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m))$ is equivalent to $\operatorname{dom} T_\Theta = \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ because of the Closed Graph Theorem 1.2.3. □

DEFINITION 3.2.11. Let $\Theta : \mathbb{D} \to \mathbb{C}^{m\times n}$. If $T_\Theta \in \mathcal{B}(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m))$, then we call Θ an $(m\times n)$-matrix-valued multiplier function and T_Θ an $(m\times n)$-matrix-valued multiplier operator. The set of all $(m \times n)$-matrix-valued multiplier functions is denoted by $\mathfrak{M}_{m\times n}(\mathbb{D})$.

DEFINITION 3.2.12. Consider $\Theta : \mathbb{D} \to \mathbb{C}^{m\times n}$. Suppose that there is a power series expansion $\Theta(z) = \sum_{k=0}^{\infty} z^k \Theta_k$ with $\Theta_k \in \mathbb{C}^{m\times n}$ that is convergent on $\mathbb{D}$. Furthermore, suppose that there exists a constant $C > 0$ such that $\|\Theta\|_\infty := \sup_{z\in\mathbb{D}} \|\Theta(z)\| \le C$, where $\|\Theta(z)\|$ is the matrix norm of $\Theta(z)$ when $\mathbb{C}^n$ and $\mathbb{C}^m$ are both equipped with the Euclidean norm. Then Θ is called a bounded analytic function (on $\mathbb{D}$). The set of all bounded analytic $(m \times n)$-matrix-valued functions is denoted by $H^\infty_{m\times n}(\mathbb{D})$.

As we have seen in Lemma 3.1.6, it does not matter whether we demand strong, weak or norm convergence for the power series expansion. However, since $\mathbb{C}^{m\times n} \cong \mathbb{C}^{m\cdot n}$ is finite dimensional, this is not surprising as all norms in this space are equivalent.

LEMMA 3.2.13. *The $(m\times n)$-matrix-valued multiplier functions are the bounded analytic $(m \times n)$-matrix-valued functions, i.e. $\mathfrak{M}_{m\times n}(\mathbb{D}) = H^\infty_{m\times n}(\mathbb{D})$. In this case we have $\|T_\Theta\| = \|\Theta\|_\infty$.*

Proof. $\supseteq$: Let $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $\Theta \in H^\infty_{m\times n}(\mathbb{D})$. If g signifies the product $\Theta \cdot f$, then we have $g_i = \sum_{k=1}^{n} \Theta_{i,k} \cdot f_k$ for $i = 1, \ldots, m$. Since f and Θ are both analytic on $\mathbb{D}$, the same must be true for their coordinate functions f_j and $\Theta_{i,j}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The product and sum of analytic functions is analytic and the radius of convergence is clearly at least 1 for each g_i. Hence, g is analytic as well and has radius of convergence at least 1. We use Lemma 3.2.8 to show

$$\begin{aligned} \|\Theta \cdot f\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} &= \lim_{r\nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left\| \Theta\left(re^{i\theta}\right) \cdot f\left(re^{i\theta}\right) \right\|^2_{\mathbb{C}^m} d\theta \\ &\le \lim_{r\nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} \|\Theta\|^2_\infty \cdot \left\| f\left(re^{i\theta}\right) \right\|^2_{\mathbb{C}^n} d\theta = \|\Theta\|^2_\infty \cdot \|f\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}. \end{aligned}$$

This means $T_\Theta \in \mathcal{B}(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m))$ with $\|T_\Theta\| \le \|\Theta\|_\infty$.

$\subseteq$: Let $\Theta : \mathbb{D} \to \mathbb{C}^{m\times n}$ such that $T_\Theta \in \mathcal{B}(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m))$. By e_j, where $j = 1,\dots,n$, we denote the functions $z \mapsto (0,\dots,1,\dots,0)^\top \in \mathbb{C}^n$ that have 1 in the j-th coordinate and zero elsewhere. Clearly, e_j belongs to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Now set $\Theta_j := T_\Theta e_j$ and by our assumptions it follows that $\Theta_j \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. Clearly, we can therefore write $\Theta = (\Theta_1,\dots,\Theta_n)$ and each of the Θ_j is analytic, i.e. $\Theta_j(z) = \sum_{k=0}^\infty z^k\Theta_{j,k}$ on $\mathbb{D}$, where $(\Theta_{j,k})_{k\in\mathbb{N}_0}$ denote the respective power series coefficients. Hence, setting $(\Theta_k)_{k\in\mathbb{N}_0} := \big((\Theta_{1,k},\dots,\Theta_{n,k})\big)_{k\in\mathbb{N}_0}$ gets us

$$\Theta(z) = \left(\sum_{k=0}^\infty z^k\Theta_{1,k},\dots,\sum_{k=0}^\infty z^k\Theta_{n,k}\right) = \sum_{k=0}^\infty z^k(\Theta_{1,k},\dots,\Theta_{n,k}) = \sum_{k=0}^\infty z^k\Theta_k$$

for every $z\in\mathbb{D}$. Therefore, Θ is analytic with power series coefficients $(\Theta_k)_{k\in\mathbb{N}_0}$. To show boundedness, take an arbitrary $\alpha\in\mathbb{C}^m$ and $f\in\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Let $K_{i,w}$ for $i\in\{m,n\}$ be the function defined in Lemma 3.2.7 and calculate for $\alpha\in\mathbb{C}^m$

$$\begin{aligned}\big(f, T_\Theta^* K_{m,w}(\alpha)\big) &= \big(T_\Theta f, K_{m,w}(\alpha)\big) = \big(\Theta f, K_{m,w}(\alpha)\big) = \\ &= \alpha^H\Theta(w)f(w) = \big(\Theta(w)^H\alpha\big)^H f(w) = \\ &= \big(f, K_{n,w}(\Theta(w)^H\alpha)\big).\end{aligned}$$

Hence, since f was arbitrary, $T_\Theta^*K_{m,w}(\alpha) = K_{n,w}(\Theta(w)^H\alpha)$. From this we conclude, using Lemma 3.2.7,

$$\begin{aligned}\|\Theta(w)\| = \|\Theta(w)^H\| &= \sup_{\|\alpha\|_{\mathbb{C}^m}=1}\|\Theta(w)^H\alpha\|_{\mathbb{C}^n} = \\ &= \sup_{\|\alpha\|_{\mathbb{C}^m}=1}\|K_{n,w}^{-1}K_{n,w}\Theta(w)^H\alpha\|_{\mathbb{C}^n} = \\ &= \sup_{\|\alpha\|_{\mathbb{C}^m}=1}\|K_{n,w}^{-1}T_\Theta^*K_{m,w}(\alpha)\|_{\mathbb{C}^n} \\ &\le \sup_{\|\alpha\|_{\mathbb{C}^m}=1}\|K_{n,w}^{-1}\|\|T_\Theta^*\|\|K_{m,w}\|\|\alpha\|_{\mathbb{C}^m} = \\ &= \|K_{n,w}^{-1}\|\cdot\|K_{m,w}\|\cdot\|T_\Theta\| = \|T_\Theta\|.\end{aligned}$$

Since $w\in\mathbb{D}$ was arbitrary, Θ is bounded and $\|\Theta\|_\infty = \sup_{w\in\mathbb{D}}\|\Theta(w)\| \le \|T_\Theta\|$. □

THEOREM 3.2.14. *Let $T\in\mathcal{B}(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n),\mathcal{H}^2(\mathbb{D};\mathbb{C}^m))$. Then $TS_n = S_mT$ if and only if there exists a function $\Theta\in H^\infty_{m\times n}(\mathbb{D})$ such that $T = T_\Theta$. In this case, Θ is uniquely determined by T.*

Proof. Concerning the necessity of the statement, let $\operatorname{diag}(\lambda,k)$ signify the $k\times k$-dimensional diagonal matrix with entry λ and let $(a_{i,j})\in\mathbb{C}^{m\times n}$. From

$$(a_{i,j})\operatorname{diag}(\lambda,n) = (\lambda a_{i,j}) = \operatorname{diag}(\lambda,m)(a_{i,j})$$

we get

$$TS_n = T_\Theta T_{\operatorname{diag}(id_{\mathbb{D}},n)} = T_{\Theta\operatorname{diag}(id_{\mathbb{D}},n)} = T_{\operatorname{diag}(id_{\mathbb{D}},m)\Theta} = T_{\operatorname{diag}(id_{\mathbb{D}},m)}T_\Theta = S_mT.$$

To prove sufficiency, let e_j signify the functions $z \mapsto (0,\dots,1,\dots,0)^\top \in \mathbb{C}^n$ for $j = 1,\dots,n$, belonging to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. We set $\Theta_j := Te_j \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ and collect these functions as $\Theta := (\Theta_1,\dots,\Theta_n) : \mathbb{D} \to \mathbb{C}^{m\times n}$. Then Θ is an analytic $(m \times n)$-matrix-valued function. First, we show that T acts like T_Θ on the polynomials. Let $p(z) := \sum_{k=0}^N z^k b_k$ be a polynomial belonging to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Clearly, we can write $p(z) = \sum_{j=1}^n p_j(z)$ with $p_j(z) := \sum_{k=0}^N z^k b_{k,j} e_j$, i.e. we decompose a polynomial in such a way that p_j is $\mathbb{C}^n$-valued but only has its j-th coordinate different from zero. Using the linearity of T, we can calculate Tp for each function p_j separately, i.e.

$$\begin{aligned} Tp_j &= \sum_{k=0}^N T(z \mapsto z^k b_{k,j} e_j) = \sum_{k=0}^N b_{k,j} T(z \mapsto z^k e_j) = \\ &= \sum_{k=0}^N b_{k,j} T \circ S_n^k e_j = \sum_{k=0}^N b_{k,j} S_m^k \circ T e_j = \\ &= \sum_{k=0}^N b_{k,j} S_m^k \Theta_j = \sum_{k=0}^N b_{k,j} (z \mapsto z^k \Theta_j(z)) = \Theta_j p_j. \end{aligned}$$

In total, this means

$$Tp = T \sum_{j=1}^n p_j = \sum_{j=1}^n Tp_j = \sum_{j=1}^n \Theta_j p_j = \Theta p.$$

Due to Lemma 3.2.5, for every function $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ there exists a sequence of polynomials $(p_N)_{N\in\mathbb{N}}$ converging to f in norm and, thus, also pointwise. Since T is continuous, we get

$$Tf = T \lim_{N\to\infty} p_N = \lim_{N\to\infty} Tp_N = \lim_{N\to\infty} \Theta p_N.$$

According to Lemma 3.2.6 point evaluation at any point $w \in \mathbb{D}$ is a continuous operation on vector-valued Hardy-Hilbert spaces. This together with the fact that $\Theta(w)$ is bounded and linear for every $w \in \mathbb{D}$ implies

$$\begin{aligned} \left(\lim_{N\to\infty} \Theta p_N\right)(w) &= \iota_{m,w}\left(\lim_{N\to\infty} \Theta p_N\right) = \lim_{N\to\infty} \iota_{m,w}\left(\Theta p_N\right) = \\ &= \lim_{N\to\infty} \Theta(w) p_N(w) = \Theta(w) \lim_{N\to\infty} p_N(w) = \\ &= \Theta(w) f(w). \end{aligned}$$

As w was arbitrary, we see that $\lim_{N\to\infty} \Theta p_N = \Theta f$, and since this works for every $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, we get $T = T_\Theta$. In particular, T is an $(m \times n)$-matrix valued multiplier function with, according to Lemma 3.2.13, corresponding $(m \times n)$-matrix valued multiplier function $\Theta \in \mathfrak{M}_{m\times n}(\mathbb{D}) = H^\infty_{m\times n}(\mathbb{D})$.
The uniqueness of Θ is easy to see since if there were $\Theta_1, \Theta_2 \in H^\infty_{m\times n}(\mathbb{D})$ such that $T_{\Theta_1} = T = T_{\Theta_2}$ holds, we can apply these operators to the functions $(z \mapsto e_j)$, where the vectors $e_j \in \mathbb{C}^n$ form the canonical basis of $\mathbb{C}^n$, and get that the columns of Θ_1 and Θ_2 are identical, so $\Theta_1 = \Theta_2$. $\square$

3.3 $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ as a Subspace of $L^2(\mathbb{T};\mathbb{C}^n)$

There is a way to extend a bounded analytic function Θ to the unit circle at least almost everywhere. First, we need to introduce an approriate adaption of Definition 2.2.1.

DEFINITION 3.3.1. Let $(\zeta^k)_{k\in\mathbb{Z}}$ be the orthonormal basis of $L^2(\mathbb{T})$ consisting of trigonometric monomials and denote by $f_1,\ldots,f_n:\mathbb{T}\to\mathbb{C}$ the coordinate functions of $f:\mathbb{T}\to\mathbb{C}^n$. Then we set

$$L^2_+(\mathbb{T};\mathbb{C}^n) := \left\{ f\in L^2(\mathbb{T};\mathbb{C}^n) \;\middle|\; a_{i,k} = (f_i,\zeta^k)_{L^2(\mathbb{T})} = 0 \text{ for all } k<0 \text{ and } 1\le i\le n \right\},$$

i.e. the set of all functions whose Fourier coefficients a_k vanish for negative indices.

LEMMA 3.3.2. *The space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ can be embedded in $L^2(\mathbb{T};\mathbb{C}^n)$. More precisely, $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ is isometrically isomorphic to $L^2_+(\mathbb{T};\mathbb{C}^n)$, which is a closed linear subspace of $L^2(\mathbb{T};\mathbb{C}^n)$ via*

$$\Psi_n : \begin{cases} \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) & \to & L^2_+(\mathbb{T};\mathbb{C}^n) \\ \left(z\mapsto \sum_{k=0}^\infty z^k a_k\right) & \mapsto & \left(\zeta\mapsto \sum_{k=0}^\infty \zeta^k a_k\right) \end{cases} . \tag{3.5}$$

Proof. As we know, the fact that $f := \left(z\mapsto \sum_{k=0}^\infty z^k a_k\right)$ belongs to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ is equivalent to square-summability of the power series coefficients of f. This in turn is equivalent to $\widetilde{f} := \left(\zeta\mapsto \sum_{k=0}^\infty \zeta^k a_k\right) \in L^2_+(\mathbb{T};\mathbb{C}^n)$ due to Parseval's identity. Therefore, the mapping Ψ_n is an isomorphism. Because of

$$\|\Psi_n(f)\|^2_{L^2(\mathbb{T};\mathbb{C}^n)} = \|\widetilde{f}\|^2_{L^2(\mathbb{T};\mathbb{C}^n)} = \sum_{k=0}^\infty \|a_k\|^2_{\mathbb{C}^n} = \|f\|^2_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)},$$

it is also isometric.
Since $L^2_+(\mathbb{T};\mathbb{C}^n)$ is the isometric image of the Banach space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, it is necessarily closed. □

As in the one-dimensional case, Ψ_n relates functions connected via nontangential limits.

PROPOSITION 3.3.3. *Let $f\in\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $\tilde{f}\in L^2_+(\mathbb{T};\mathbb{C}^n)$ be such that $\Psi_n(f)=\tilde{f}$ is satisfied and set $f_r:\mathbb{T}\to\mathbb{C}^n : \zeta = e^{i\theta}\mapsto f(re^{i\theta})$.*

1. *f_r converges to f with respect to the norm $\|.\|_{L^2(\mathbb{T};\mathbb{C}^n)}$.*
2. *Given $\tilde{f}$, one can recover f via the Poisson formula*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\tilde{f}(e^{i\theta})\, d\theta,$$

where we integrate the vector-valued function $P(z,.)\tilde{f}(.)$ component wise.

Proof. Let f have the power series expansion coefficients $(a_k)_{k\in\mathbb{N}_0}$. Parseval's identity yields

$$\|f_r-\widetilde{f}\|^2_{L^2(\mathbb{T};\mathbb{C}^n)} = \left\|\sum_{k=0}^{\infty} r^k e^{ik\theta}a_k - \sum_{k=0}^{\infty} e^{ik\theta}a_k\right\|^2_{L^2(\mathbb{T};\mathbb{C}^n)} = \left\|\sum_{k=0}^{\infty}(r^k-1)e^{ik\theta}a_k\right\|^2_{L^2(\mathbb{T};\mathbb{C}^n)} =$$
$$= \sum_{k=0}^{\infty}\|(r^k-1)a_k\|^2_{\mathbb{C}^n} = \sum_{k=0}^{\infty}(1-r^k)^2\|a_k\|^2_{\mathbb{C}^n}.$$

For $\varepsilon>0$ we find $N\in\mathbb{N}$ such that $\sum_{k=N+1}^{\infty}\|a_k\|^2<\frac{\varepsilon}{2}$. We additionally can choose $R\in(0,1)$ such that $\sum_{k=0}^{N}(1-R^k)^2\|a_k\|^2<\frac{\varepsilon}{2}$. Hence, for $r\in(R,1)$ we get

$$\begin{aligned}\sum_{k=0}^{\infty}(1-r^k)^2\|a_k\|^2 &= \sum_{k=0}^{N}(1-r^k)^2\|a_k\|^2 + \sum_{k=N+1}^{\infty}(1-r^k)^2\|a_k\|^2\\ &\le \sum_{k=0}^{N}(1-R^k)^2\|a_k\|^2 + \sum_{k=N+1}^{\infty}\|a_k\|^2 < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

and thus, $\lim_{r\nearrow 1}\|f_r-\widetilde{f}\|_{L^2(\mathbb{T};\mathbb{C}^n)}=0$.
For the second claim we take $w\in\mathbb{D}$ and remember the function $K_{n,w}:\mathbb{C}^n\to\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ from Lemma 3.2.7. In particular, $K_{n,w}(\alpha)=\left(z\mapsto\sum_{k=0}^{\infty}z^k(\overline{w}^k\alpha)\right)$ holds for every $\alpha\in\mathbb{C}^n$. Hence, by setting $\widetilde{K}_{n,w}(\alpha)=\left(\zeta\mapsto\sum_{k=0}^{\infty}\zeta^k(\overline{w}^k\alpha)\right)$ we define an analoguous function $\widetilde{K}_{n,w}:\mathbb{C}^n\to L^2_+(\mathbb{T};\mathbb{C}^n)$. Obviously $\Psi_n(K_{n,w}(\alpha))=\widetilde{K}_{n,w}(\alpha)$ holds. With $\zeta=e^{i\theta}$ as well as Lemma 3.2.7 we arrive at

$$\begin{aligned}\alpha^H f(w) &= (f,K_{n,w}(\alpha))_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \big(\Psi_n(f),\Psi(K_{n,w}(\alpha))\big)_{L^2(\mathbb{T};\mathbb{C}^n)} = \big(\widetilde{f},\widetilde{K}_{n,w}(\alpha)\big) =\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\widetilde{f}(e^{i\theta}),\frac{1}{1-\overline{w}e^{i\theta}}\alpha\right)_{\mathbb{C}^n}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\alpha^H\widetilde{f}(e^{i\theta})}{1-we^{-i\theta}}\,d\theta.\end{aligned} \tag{3.6}$$

We define a scalar function

$$g:=(\zeta\mapsto(1-w\overline{\zeta})^{-1})=\left(\zeta\mapsto\sum_{k=0}^{\infty}w^k\zeta^{-k}\right),$$

which obviously belongs to $L^2(\mathbb{T})$. Clearly, all power series coefficients of $g-1$ vanish for nonnegative indices, so this function is perpendicular to $\alpha^H\widetilde{f}$ in $L^2(\mathbb{T})$ for every $\alpha\in\mathbb{C}^n$. Consequently, writing $\zeta=e^{i\theta}$ shows that

$$0=(\alpha^H\widetilde{f},g-1)_{L^2(\mathbb{T};\mathbb{C}^n)}=\frac{1}{2\pi}\int_0^{2\pi}(\alpha^H\widetilde{f}(e^{i\theta}))\overline{\left(\frac{1}{1-we^{-i\theta}}-1\right)}\,d\theta$$

is harmless and we can add this expression to (3.6), receiving

$$\alpha^H f(w)=\frac{1}{2\pi}\int_0^{2\pi}\alpha^H\widetilde{f}(e^{i\theta})\overline{\left(\frac{1}{1-\overline{w}e^{i\theta}}+\frac{1}{1-we^{-i\theta}}-1\right)}\,d\theta$$

As we have already mentioned in the proof of Proposition 2.2.4 the expression underneath the conjugation bar is just the Poisson kernel. Finally, we can extract α^H from the integral which means integrating the vector valued function $\zeta \mapsto f(\zeta)P(z,\zeta)$ componentwise first and then multiplying it with the vector α^H from the left. Since $\alpha \in \mathbb{C}^n$ was arbitrary, the claim follows. $\square$

We include the next result for the sake of completeness and suggest [Nik02], I.3.11, for a proof.

PROPOSITION 3.3.4. *Let $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, then it has nontangential limits almost everywhere on $\mathbb{T}$ and $\operatorname{n.t.lim}_{z\to\zeta} f(z) = \tilde{f}(\zeta)$ holds in the sense of $L^2(\mathbb{T};\mathbb{C}^n)$.*

For the following considerations, we need a vector-valued version of Lemma 1.2.7. The upcoming proof mostly expands on [Kal12], II., but some ideas are taken from [BH12] and [Tho03], II.

LEMMA 3.3.5. *Let $(\Omega,\mathcal{A},\mu)$ be a measure space, where μ is a finite nonnegative measure, and let $\Theta : \Omega \to \mathbb{C}^{m\times n}$ be measurable. Set*

$$M_\Theta : \begin{cases} \operatorname{dom} M_\Theta & \to \quad L^2(\Omega;\mathbb{C}^m) \\ \qquad g & \mapsto \quad \Theta g \end{cases},$$

where

$$\operatorname{dom} M_\Theta := \{ g \in L^2(\Omega;\mathbb{C}^n) \mid \Theta g \in L^2(\Omega;\mathbb{C}^m) \}$$

is a linear subspace of $L^2(\Omega;\mathbb{C}^n)$. Then we have

1. *The space $\operatorname{dom} M_\Theta$ is dense in $L^2(\Omega;\mathbb{C}^n)$ and M_Θ is a closed operator, i.e. the graph of M_Θ is closed in $L^2(\Omega;\mathbb{C}^n) \times L^2(\Omega;\mathbb{C}^m)$, when this Cartesian product is equipped with the product topology.*

2. *The following statements are equivalent:*

   (a) Θ belongs to $L^\infty(\Omega;\mathbb{C}^{m\times n})$, i.e. it is essentially bounded.

   (b) $M_\Theta \in \mathcal{B}\big(L^2(\Omega;\mathbb{C}^n), L^2(\Omega;\mathbb{C}^m)\big)$, i.e. it is defined everywhere and bounded.

   (c) M_Θ is bounded at least on a dense subspace $\mathcal{L}$ of $\operatorname{dom} M_\Theta$.

   (d) $\operatorname{dom} M_\Theta = L^2(\Omega;\mathbb{C}^n)$.

   In this case, M_Θ maps $L^2(\Omega;\mathbb{C}^n)$ into $L^2(\Omega;\mathbb{C}^m)$ and $\|M_\Theta\| = \|\Theta\|_{L^\infty(\Omega;\mathbb{C}^{m\times n})} := \operatorname{ess\,sup}_{\omega\in\Omega} \|\Theta(\omega)\| := \inf \{ C \geq 0 \mid \mu(\{\omega \in \Omega \mid \|\Theta(\omega)\| > C\}) = 0 \}$.

Proof. For the first claim, take $f \in L^2(\Omega;\mathbb{C}^n)$ and set $\Delta_k := \{\omega \in \Omega \mid \|\Theta(\omega)\| \leq k\}$ as well as $f_k := \chi_{\Delta_k} \cdot f$, where χ_{Δ_k} is the indicator function of the set Δ_k. Since f is square integrable,

$$\int_\Omega \|\Theta(\omega) f_k(\omega)\|^2_{\mathbb{C}^m}\, d\mu = \int_{\Delta_k} \|\Theta(\omega) f(\omega)\|^2_{\mathbb{C}^m}\, d\mu \leq \int_{\Delta_k} \|\Theta(\omega)\|^2 \cdot \|f(\omega)\|^2_{\mathbb{C}^n}\, d\mu$$

$$\leq k^2 \int_\Omega \|f_k(\omega)\|^2_{\mathbb{C}^n}\ d\mu$$

shows that all functions f_k belong to dom M_Θ. Furthermore,

$$\|f - f_k\|^2_{L^2(\Omega;\mathbb{C}^n)} = \int_\Omega \|f(\omega) - \chi_{\Delta_k}\cdot f(\omega)\|^2_{\mathbb{C}^n}\ d\mu = \int_\Omega \chi_{\Omega\setminus\Delta_k}\cdot \|f(\omega)\|^2_{\mathbb{C}^n}\ d\mu.$$

Since $\chi_{\Omega\setminus\Delta_k}\cdot\|f(.)\|^2_{\mathbb{C}^n}$ converges to zero pointwise, we apply the Dominated Convergence Theorem with majorant $\|f(.)\|^2_{\mathbb{C}^n}$, which yields that (f_k) converges to f in $L^2(\Omega;\mathbb{C}^n)$. Hence, dom M_Θ is dense in $L^2(\Omega,\mathbb{C}^n)$.

Secondly, we show the closedness of M_Θ. To this end we identify the operator with its graph. Let $[f_j,\Theta f_j] \in M_\Theta$ for $j \in \mathbb{N}$ and suppose that $([f_j,\Theta f_j])$ converges to $[f,g]$ in $L^2(\Omega;\mathbb{C}^n)\times L^2(\Omega;\mathbb{C}^m)$. We need to show that $\Theta f = g$. It is a well-known fact that convergence of (f_j) in $L^2(\Omega;\mathbb{C}^n)$ implies the existence of a subsequence (f_{j_k}) converging pointwise almost everywhere on Ω to f. The same considerations involving (Θf_{j_k}) and $L^2(\Omega;\mathbb{C}^m)$ get us yet another subsequence $(f_{j_{k_i}})$ such that $(\Theta f_{j_{k_i}})$ converges pointwise almost everywhere to g. Clearly, $(f_{j_{k_i}})$ still converges to f and therefore, $(\Theta f_{j_{k_i}})$ converges to Θf. Thus, $\Theta f = g$ almost everywhere on Ω and M_Θ is closed.

Next, we show the four equivalences of the second claim.

(a) $\Rightarrow$ (b): For $f \in L^2(\Omega;\mathbb{C}^n)$ we have

$$\int_\Omega \|\Theta(\omega)f(\omega)\|^2_{\mathbb{C}^m}\ d\mu \leq \int_\Omega \|\Theta(\omega)\|^2\cdot\|f(\omega)\|^2_{\mathbb{C}^n}\ d\mu \leq \operatorname{ess\,sup}_{\omega\in\Omega}\|\Theta(\omega)\|^2\cdot\|f\|^2_{L^2(\Omega;\mathbb{C}^n)}.$$

Hence, $\Theta f \in L^2(\Omega;\mathbb{C}^m)$, meaning f belongs to dom M_Θ, and $\|M_\Theta\| \leq \|\Theta\|_{L^\infty(\Omega;\mathbb{C}^{m\times n})}$, i.e. $M_\Theta \in \mathcal{B}\big(L^2(\Omega;\mathbb{C}^n), L^2(\Omega;\mathbb{C}^m)\big)$.

(b) $\Rightarrow$ (a): Let $C \in [0,\infty)$ be arbitrary with $C < \|\Theta\|_{L^\infty(\Omega;\mathbb{C}^{m\times n})}$, where $\|\Theta\|_{L^\infty(\Omega;\mathbb{C}^{m\times n})}$ is at first understood to be infinite if Θ is not essentially bounded. The set

$$N := \big\{\omega \in \Omega \;\big|\; \|\Theta(\omega)\| > C\big\} \subseteq \Omega$$

has positive measure by the definition of the essential supremum. Let $A := \big\{x_k \;\big|\; k \in \mathbb{N}\big\}$ be a countable dense subset of the unit sphere in $\mathbb{C}^n$. For every $\omega \in N$ there must be an $x_k \in A$ such that $\|\Theta(\omega)x_k\| > C$ by the definition of the operator norm. If we set

$$N_k := \big\{\omega \in \Omega \;\big|\; \|\Theta(\omega)x_k\|_{\mathbb{C}^m} > C\big\}$$

for $k \in \mathbb{N}$, we get $N = \bigcup_{k\in\mathbb{N}} N_k$. Since Θ is measurable, the function $(\omega \mapsto \Theta(\omega)x_k)$ is measurable, too, for any $k \in \mathbb{N}$. Because the norm is continuous, $(\omega \mapsto \|\Theta(\omega)x_k\|_{\mathbb{C}^m})$ is measurable as well. Therefore, every N_k belongs to the σ-algebra $\mathcal{A}$. As $\mu(N) > 0$, there exists a $K \in \mathbb{N}$ such that N_K has positive measure. Now define $f := \chi_{N_K} x_K : \Omega \to \mathbb{C}^n$, where χ_{N_K} is the characteristic function of the set N_K and $x_K \in \mathbb{C}^n$ is the corresponding unit vector. It follows from the above considerations that f is measurable. Because of

$$\int_\Omega \|f(\omega)\|^2_{\mathbb{C}^n}\ d\mu = \int_{N_K} \|x_K\|^2_{\mathbb{C}^n}\ d\mu = \int_{N_K} 1\ d\mu = \mu(N_K) > 0$$

and the fact that μ is a finite measure, we get $f \in L^2(\Omega;\mathbb{C}^n)$ and that it is not the zero function. Consequently,

$$\begin{aligned}\|M_\Theta\|^2 \cdot \|f\|^2_{L^2(\Omega;\mathbb{C}^n)} \geq \|M_\Theta f\|^2_{L^2(\Omega;\mathbb{C}^m)} &= \int_\Omega \|\Theta(\omega)f(\omega)\|^2_{\mathbb{C}^m}\, d\mu = \\ &= \int_{N_K} \|\Theta(\omega)x_K\|^2_{\mathbb{C}^m}\, d\mu \geq C^2 \int_{N_K} 1\, d\mu = \\ &= C^2\mu(N_K) = C^2\|f\|^2_{L^2(\Omega;\mathbb{C}^n)}.\end{aligned}$$

From $\|f\|_{L^2(\Omega;\mathbb{C}^n)} > 0$ we conclude $C \leq \|M_\Theta\| < \infty$. Since $C < \|\Theta\|_{L^\infty(\Omega;\mathbb{C}^{m\times n})}$ was arbitrary, we finally get $\|\Theta\|_{L^\infty(\Omega;\mathbb{C}^{m\times n})} \leq \|M_\Theta\| < \infty$.
(b) $\Leftrightarrow$ (d): This is a consequence of the Closed Graph Theorem 1.2.3.
(b) $\Rightarrow$ (c): This is trivially true.
(c) $\Rightarrow$ (d): Since $M_\Theta \upharpoonright \mathcal{L}$ is bounded on the dense subspace $\mathcal{L}$, it has a continuous extension $C \in \mathcal{B}\big(L^2(\Omega;\mathbb{C}^n), L^2(\Omega;\mathbb{C}^m)\big)$. If a net $([f_i, Cf_i])_{i\in I}$ in C converges to $[f,g] \in L^2(\Omega;\mathbb{C}^n) \times L^2(\Omega;\mathbb{C}^m)$, then f belongs to $L^2(\Omega;\mathbb{C}^n) = \operatorname{dom} C$ and since C is continous, $f_i \to f$ implies $Cf_i \to Cf$. The uniqueness of limits shows that $Cf = g$, i.e. $[f,g] \in C$, making C a closed operator. Therefore, $\overline{M_\Theta \upharpoonright \mathcal{L}} \subseteq \overline{C} = C$.
On the other hand, take $f \in L^2(\Omega;\mathbb{C}^n)$ and a net $(f_i)_{i\in I}$ belonging to $\mathcal{L}$ such that $f_i \to f$. Clearly, this implies that the net $\big((M_\Theta \upharpoonright \mathcal{L})f_i\big)_{i\in I} = (Cf_i)_{i\in I}$ has the limit $Cf \in L^2(\Omega;\mathbb{C}^m)$ by the definition of C. Hence, the net $\big([f_i, (M_\Theta \upharpoonright \mathcal{L})f_i]\big)_{i\in I}$ converges to $[f, Cf]$. This limit must belong to $\overline{M_\Theta \upharpoonright \mathcal{L}}$. Therefore, $C \subseteq \overline{M_\Theta \upharpoonright \mathcal{L}}$.
So we have shown $\overline{M_\Theta \upharpoonright \mathcal{L}} = C$. As we know from the first claim, M_Θ is closed and hence, $M_\Theta = \overline{M_\Theta} \supseteq \overline{M_\Theta \upharpoonright \mathcal{L}} = C$. Therefore, $\operatorname{dom} M_\Theta \supseteq \operatorname{dom} C = L^2(\Omega;\mathbb{C}^n)$. □

DEFINITION 3.3.6. We set $L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n}) := L^\infty(\mathbb{T};\mathbb{C}^{m\times n}) \cap L^2_+(\mathbb{T};\mathbb{C}^{m\times n})$, i.e. the space of all essentially bounded $(m \times n)$-matrix valued functions such that all their negative Fourier coefficients vanish.

DEFINITION 3.3.7. Similarly to Ψ_n in (3.5) we define for $n, m \in \mathbb{N}$

$$\Psi_{m\times n} : \begin{cases} \mathcal{H}^2(\mathbb{D};\mathbb{C}^{m\times n}) & \to \quad L^2_+(\mathbb{T};\mathbb{C}^{m\times n}) \\ (z \mapsto \sum_{k=0}^\infty z^k\Theta_k) & \mapsto \quad (\zeta \mapsto \sum_{k=0}^\infty \zeta^k\Theta_k) \end{cases}, \tag{3.7}$$

which is also an isometric isomorphism, when $L^2(\mathbb{T};\mathbb{C}^{m\times n})$ is provided with the norm $\|.\|_{L^2(\mathbb{T};\mathbb{C}^{m\times n})} \mathrel{\widehat{=}} \|.\|_{L^2(\mathbb{T};\mathbb{C}^{m\cdot n})}$.

DEFINITION 3.3.8. For $n \in \mathbb{N}$ we write

$$U_n : \begin{cases} L^2(\mathbb{T};\mathbb{C}^n) & \to \quad L^2(\mathbb{T};\mathbb{C}^n) \\ f & \mapsto \quad (\zeta \mapsto \zeta f(\zeta)) \end{cases}$$

for the shift operator on $L^2(\mathbb{T};\mathbb{C}^n)$.

Clearly, we have $U_n = M_{\operatorname{diag}(\zeta\mapsto\zeta, n)} = M_{\operatorname{diag}(id_{\mathbb{T}}, n)}$.

LEMMA 3.3.9. *The shift operator U_n on $L^2(\mathbb{T};\mathbb{C}^n)$ is unitary for every $n \in N$.*

Proof. Since $L^2(\mathbb{T};\mathbb{C}^n)$ is isometrically isomorphic to $\ell^2(\mathbb{Z};\mathbb{C}^n)$, we can calculate

$$(U_n f, g)_{L^2(\mathbb{T};\mathbb{C}^n)} = \left(\sum_{k=-\infty}^{\infty} \zeta^{k+1} a_k, \sum_{k=-\infty}^{\infty} \zeta^k b_k\right) = \big((a_{k-1}),(b_k)\big)_{\ell^2(\mathbb{Z};\mathbb{C}^n)} =$$
$$= \sum_{k=-\infty}^{\infty} b_k^H a_{k-1} = \sum_{k=-\infty}^{\infty} b_{k+1}^H a_k = (f, M_{\mathrm{diag}(\zeta\mapsto\overline{\zeta},n)} g)_{L^2(\mathbb{T};\mathbb{C}^n)}$$

for $f,g \in L^2(\mathbb{T};\mathbb{C}^n)$ with Fourier coefficients $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$. Hence, we have shown that $U_n^* = M_{\mathrm{diag}(\zeta\mapsto\overline{\zeta},n)}$. This implies for every $f \in L^2(\mathbb{T};\mathbb{C}^n)$ that

$$U_n^* U_n f = M_{diag(\zeta\mapsto\overline{\zeta},n)} M_{\mathrm{diag}(\zeta\mapsto\zeta,n)} f = M_{\mathrm{diag}(\zeta\mapsto\zeta\overline{\zeta},n)} f = M_{I_n} f = I_n f = f = U_n U_n^* f,$$

i.e. $U_n^{-1} = U_n^*$. □

These concepts open up a way to recover the $(m \times n)$-matrix valued multiplier functions in this higher dimensional setting just as in the one dimensional case.

LEMMA 3.3.10. *Let $\Psi_{m\times n}$ be the isometric isomorphism from* (3.7)*. Then*

$$\Psi_{m\times n}\big(\mathfrak{M}_{m\times n}(\mathbb{D})\big) \subseteq L_+^\infty(\mathbb{T};\mathbb{C}^{m\times n}).$$

Furthermore, $\Psi_{m\times n}$ preserves the norm, i.e. $\|\Psi_{m\times n}(.)\|_{L^\infty(\mathbb{T};\mathbb{C}^{m\times n})} = \|.\|_\infty$. Additionally, if $\Theta \in \mathfrak{M}_{m\times n}(\mathbb{D})$, or equivalently $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ due to Lemma 3.2.13, with $\widetilde{\Theta} = \Psi_{m\times n}(\Theta)$ and if T_Θ and $M_{\widetilde{\Theta}}$ signify the corresponding $(m \times n)$-matrix valued multiplier operators, mapping $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ into $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ and $L^2(\mathbb{T};\mathbb{C}^n)$ into $L^2(\mathbb{T};\mathbb{C}^m)$, we have that $M_{\widetilde{\Theta}}$ maps $L^2_+(\mathbb{T};\mathbb{C}^n)$ into $L^2_+(\mathbb{T};\mathbb{C}^m)$ as well as

$$\Psi_m^{-1} \circ M_{\widetilde{\Theta}} \circ \Psi_n = T_\Theta. \tag{3.8}$$

Proof. Let $\Theta \in \mathfrak{M}_{m\times n}(\mathbb{D}) = H^\infty_{m\times n}(\mathbb{D})$ and notice that Θ is bounded and has a power series expansion with power series coefficients (Θ_k) with $\Theta_k \in \mathbb{C}^{m\times n}$ for $k \in \mathbb{N}_0$. Furthermore, $T_\Theta \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)\big)$ with $\|T_\Theta\| = \|\Theta\|_\infty$, all according to Lemma 3.2.13.
The fact that Θ belongs to $H^\infty_{m\times n}(\mathbb{D}) \subseteq \mathcal{H}^2(\mathbb{D};\mathbb{C}^{m\times n})$ means that $\widetilde{\Theta} := \Psi_{m\times n}(\Theta) = (\zeta \mapsto \sum_{k=0}^\infty \zeta^k \Theta_k)$ belongs to $L^2_+(\mathbb{T};\mathbb{C}^{m\times n}) \subseteq L^2(\mathbb{T};\mathbb{C}^{m\times n})$. Hence, we can form $M_{\widetilde{\Theta}}$, which maps $L^2(\mathbb{T};\mathbb{C}^n)$ into $L^2(\mathbb{T};\mathbb{C}^m)$. Furthermore, it is a closed and densely defined operator due to Lemma 3.3.5 and its domain is

$$\mathrm{dom}\, M_{\widetilde{\Theta}} = \left\{ f \in L^2(\mathbb{T};\mathbb{C}^n) \;\middle|\; \widetilde{\Theta} f \in L^2(\mathbb{T};\mathbb{C}^m) \right\}.$$

Our next aim is to show that $M_{\widetilde{\Theta}}$ is bounded on the $\mathbb{C}^n$-valued polynomials $\mathbb{C}[z;\mathbb{C}^n]$ and then infer boundedness everywhere on $L^2(\mathbb{T};\mathbb{C}^n)$.
First, we know that $U_i^k \in \mathcal{B}(L^2(\mathbb{T};\mathbb{C}^i))$ is unitary for all $k \in \mathbb{Z}$ and every dimension $i \in \mathbb{N}$ due to Lemma 3.3.9. Furthermore, $\|U_m^k M_{\widetilde{\Theta}} f\|_{L^2(\mathbb{T};\mathbb{C}^m)} = \|M_{\widetilde{\Theta}} f\|_{L^2(\mathbb{T};\mathbb{C}^m)}$, as elements of

$[0,\infty]$, implies $M_{\widetilde{\Theta}}U_n^k f = U_m^k M_{\widetilde{\Theta}} f$ for all $f \in \operatorname{dom} M_{\widetilde{\Theta}}$ and $U_n^k \operatorname{dom} M_{\widetilde{\Theta}} = \operatorname{dom} M_{\widetilde{\Theta}}$, i.e. $M_{\widetilde{\Theta}}U_n^k = U_m^k M_{\widetilde{\Theta}}$.
Let $B = \{e_i \in \mathbb{C}^n \mid i = 1,\dots,n\}$ be the canonical basis of $\mathbb{C}^n$. Now define for $j \geq 0$ and $x \in B$ the function $q := z \mapsto z^j x$, which clearly belongs to $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Applying T_Θ gives $\Theta(z)q(z) = \sum_{k=0}^\infty z^{k+j}\Theta_k x = \sum_{k=j}^\infty z^k \Theta_{k-j}x$. Using Ψ_m gives

$$\begin{aligned}\Psi_m(T_\Theta q) = \Psi_m\big(\Theta q\big) = \Psi_m\left(z \mapsto \sum_{k=j}^{\infty} z^k \Theta_{k-j}x\right) = \left(\zeta \mapsto \sum_{k=j}^{\infty} \zeta^k \Theta_{k-j}x\right) = \\ = \big(\zeta \mapsto \zeta^j \widetilde{\Theta}(\zeta)x\big) = \big(\zeta \mapsto \widetilde{\Theta}(\zeta)\widetilde{q}(\zeta)\big)\end{aligned}$$

with $\widetilde{q} = \Psi_n(q)$. Because of linearity, we get $\Psi_m(T_\Theta r) = \widetilde{\Theta}\cdot\widetilde{r}$ for all elements r of the ring of $\mathbb{C}^n$-valued polynomials. Therefore $\Psi_n\big(\mathbb{C}[z;\mathbb{C}^n]\big) \subseteq \operatorname{dom} M_{\widetilde{\Theta}}$ and $\Psi_m(T_\Theta r) = M_{\widetilde{\Theta}}\widetilde{r}$ for all $r \in \mathbb{C}[z;\mathbb{C}^n]$. In particular, $M_{\widetilde{\Theta}}\Psi_n(\mathbb{C}[z;\mathbb{C}^n]) \subseteq L^2_+(\mathbb{T};\mathbb{C}^m)$.
The trigonometric polynomials $(\zeta^k x)$ for $k \in \mathbb{Z}$ and $x \in B$ all belong to $\operatorname{dom} M_{\widetilde{\Theta}}$. Hence, we can look at the set

$$\mathcal{T}_n := \left\{ \sum_{k=-N}^{N} \zeta^k a_k \mid N \in \mathbb{N}_0, a_{-N},\dots,a_N \in \mathbb{C}^n \right\} \subseteq \operatorname{dom} M_{\widetilde{\Theta}}.$$

For $\widetilde{p} = (\zeta \mapsto \sum_{k=-N}^N \zeta^k b_k) \in \mathcal{T}_n$ define $\widetilde{q}$ via

$$U_n^N \widetilde{p} = \left(\zeta \mapsto \sum_{k=-N}^{N} \zeta^{k+N} b_k\right) = \left(\zeta \mapsto \sum_{k=0}^{2N} \zeta^k b_{k-N}\right) =: \widetilde{q} \in L^2_+(\mathbb{T};\mathbb{C}^n).$$

Hence,

$$M_{\widetilde{\Theta}}\widetilde{p} = M_{\widetilde{\Theta}}U_n^{-N}\widetilde{q} = U_m^{-N}M_{\widetilde{\Theta}}\widetilde{q} = U_m^{-N}\Psi_m(T_\Theta q). \tag{3.9}$$

Taking the norm of (3.9) and using that U_i and Ψ_i are isometric for all $i \in \mathbb{N}$, we arrive at

$$\begin{aligned}\|M_{\widetilde{\Theta}}\widetilde{p}\|_{L^2(\mathbb{T};\mathbb{C}^m)} &= \|U_m^{-N}\Psi_m(T_\Theta q)\|_{L^2(\mathbb{T};\mathbb{C}^m)} = \|T_\Theta q\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} \leq \|T_\Theta\| \cdot \|q\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \\ &= \|\Theta\|_\infty \cdot \|U_n^{-N}\Psi_n(q)\|_{L^2(\mathbb{T};\mathbb{C}^n)} = \|\Theta\|_\infty \cdot \|U_n^{-N}\widetilde{q}\|_{L^2(\mathbb{T};\mathbb{C}^n)} = \\ &= \|\Theta\|_\infty \cdot \|\widetilde{p}\|_{L^2(\mathbb{T};\mathbb{C}^n)}\end{aligned}$$

Since the last expression on the right is finite, the operator $M_{\widetilde{\Theta}}$ is bounded on $\mathcal{T}_n$. As $\mathcal{T}_n$ is densely contained in $L^2(\mathbb{T};\mathbb{C}^n)$, this means that $M_{\widetilde{\Theta}}$ is bounded everywhere, i.e. $M_{\widetilde{\Theta}} \in \mathcal{B}(L^2(\mathbb{T};\mathbb{C}^n), L^2(\mathbb{T};\mathbb{C}^m))$. As the polynomials $\Psi_n(\mathbb{C}[z;\mathbb{C}^n])$ are dense in $L^2_+(\mathbb{T};\mathbb{C}^n)$ we obtain from $M_{\widetilde{\Theta}}\Psi_n(\mathbb{C}[z;\mathbb{C}^n]) \subseteq L^2_+(\mathbb{T};\mathbb{C}^m)$ and the continuity of $M_{\widetilde{\Theta}}$ that $M_{\widetilde{\Theta}}L^2_+(\mathbb{T};\mathbb{C}^n) \subseteq L^2_+(\mathbb{T};\mathbb{C}^m)$. Additionally, we have proved that

$$\|\widetilde{\Theta}\|_{L^\infty(\mathbb{T};\mathbb{C}^{m\times n})} = \|M_{\widetilde{\Theta}}\| \leq \|\Theta\|_\infty. \tag{3.10}$$

To show the converse of (3.10), notice from (3.9) we get $\Psi_m \circ T_\Theta = M_{\widetilde{\Theta}} \circ \Psi_n$ on the set of $\mathbb{C}^n$-valued polynomials contained in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Since this set is a dense subset, this identity is even valid on the whole of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, showing

$$M_{\widetilde{\Theta}} \restriction L^2_+(\mathbb{T};\mathbb{C}^n) = \Psi_m \circ T_\Theta \circ \Psi_n^{-1}.$$

We make a note of the facts that Ψ_n^{-1} is only defined on $\Psi_n(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n))$, which equals $L^2_+(\mathbb{T};\mathbb{C}^n)$, and that $M_{\widetilde{\Theta}}$ maps $L^2_+(\mathbb{T};\mathbb{C}^n)$ into $L^2_+(\mathbb{T};\mathbb{C}^m)$. This implies

$$T_\Theta = \Psi_m^{-1} \circ M_{\widetilde{\Theta}} \circ \Psi_n$$

and

$$\|\Theta\|_\infty = \|T_\Theta\| = \|\Psi_m^{-1} \circ M_{\widetilde{\Theta}} \circ \Psi_n\| \le \|M_{\widetilde{\Theta}}\| = \|\widetilde{\Theta}\|_{L^\infty(\mathbb{T};\mathbb{C}^{m\times n})}. \tag{3.11}$$

Combining (3.10) and (3.11) gives $\|\widetilde{\Theta}\|_{L^\infty(\mathbb{T};\mathbb{C}^{m\times n})} = \|\Theta\|_\infty$. Furthermore, this also shows that $\Psi_{m\times n}(\Theta) = \widetilde{\Theta} \in L^\infty(\mathbb{T};\mathbb{C}^n)$. Therefore, $\widetilde{\Theta} \in L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n})$. □

LEMMA 3.3.11. *Consider the mappings Ψ_i and $\Psi_{i\times j}$ defined in* (3.5) *and* (3.7)*. Let $\Theta_1 \in H^\infty_{m\times n}(\mathbb{D})$ and $\Theta_2 \in H^\infty_{k\times m}(\mathbb{D})$. Furthermore, let $h \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Then we have the two identities*

$$\begin{array}{c}\Psi_{k\times n}(\Theta_2\Theta_1) = \Psi_{k\times m}(\Theta_2)\Psi_{m\times n}(\Theta_1)\\ \Psi_m(\Theta_1 h) = \Psi_{m\times n}(\Theta_1)\Psi_n(h)\end{array}.$$

Proof. We remind ourselves of the mapping $\psi = \Psi_1$ as defined in (2.3). Let

$$\{e_i \mid i = 1,\dots,n\}, \quad \{f_j \mid j = 1,\dots,m\} \quad \text{and} \quad \{g_\ell \mid \ell = 1,\dots,k\}$$

be the respective canonical bases of $\mathbb{C}^n$, $\mathbb{C}^m$ and $\mathbb{C}^k$. Clearly, $f_j^H\Theta_1 e_i, g_\ell^H\Theta_2 f_j$ and $g_\ell^H\Theta_2\Theta_2 e_i$ all belong to $H^\infty(\mathbb{D})$. So according to Corollary 2.2.10 we have

$$\begin{aligned} g_\ell^H\Psi_{k\times n}(\Theta_2\Theta_1)e_i &= \big(\zeta \mapsto g_\ell^H(\widetilde{\Theta_2\Theta_1})(\zeta)e_i\big) = \psi\big(g_\ell^H\Theta_2\Theta_1 e_i\big) = \\ &= \psi\Bigg(\sum_{j=1}^m g_\ell^H\Theta_2 f_j f_j^H\Theta_1 e_i\Bigg) = \sum_{j=1}^m \psi\big([g_\ell^H\Theta_2 f_j]\ [f_j^H\Theta_1 e_i]\big) = \\ &= \sum_{j=1}^m \psi\big(g_\ell^H\Theta_2 f_j\big)\ \psi\big(f_j^H\Theta_1 e_i\big) = \\ &= \sum_{j=1}^m \big(\zeta \mapsto g_\ell^H\widetilde{\Theta_2}(\zeta)f_j\big)\ \big(\zeta \mapsto f_j^H\widetilde{\Theta_1}(\zeta)e_i\big) = \\ &= \sum_{j=1}^m \big[g_\ell^H\Psi_{k\times m}(\Theta_2)f_j\big]\ \big[f_j^H\Psi_{m\times n}(\Theta_1)e_i\big] = \\ &= g_\ell^H\Psi_{k\times m}(\Theta_2)\ \Big[\sum_{j=1}^m f_j f_j^H\Big]\ \Psi_{m\times n}(\Theta_1)e_i = \end{aligned}$$

$$= g_\ell^H \Psi_{k\times m}(\Theta_2)\Psi_{m\times n}(\Theta_1)e_i.$$

Take note that $\sum_{j=1}^m f_j f_j^H$ is the identity matrix on $\mathbb{C}^m$. The above calculation works for $\ell = 1,\dots,k$ and $i = 1,\dots,n$ and because of the basis property, the first claim is proved.
For the second claim, notice that $f_j^H\Theta_1 e_i$ belongs to $H^\infty(\mathbb{D})$ and that $e_i^H h$ belongs to $\mathcal{H}^2(\mathbb{D})$. Thus, again with Corollary 2.2.10, we conclude that

$$\begin{aligned} f_j^H\Psi_m(\Theta_1 h) &= \big(\zeta \mapsto f_j^H(\widetilde{\Theta_1 h})(\zeta)\big) = \psi(f_j^H\Theta_1 h) = \\ &= \psi\Big(\sum_{i=1}^n f_j^H\Theta_1 e_i e_i^H h\Big) = \sum_{i=1}^n \psi\big([f_j^H\Theta_1 e_i]\ [e_i^H h]\big) = \\ &= \sum_{i=1}^n \psi\big(f_j^H\Theta_1 e_i\big)\ \psi\big(e_i^H h\big) = \\ &= \sum_{i=1}^n \big[f_j^H\Psi_{m\times n}(\Theta_1)e_i\big]\ \big[e_i^H\Psi_n(h)\big] = \\ &= f_j^H\Psi_{m\times n}(\Theta_1)\big[\sum_{i=1}^n e_i\ e_i^H\big]\Psi_n(h) = \\ &= f_j^H\Psi_{m\times n}(\Theta_1)\Psi_n(h). \end{aligned}$$

Again, $\sum_{i=1}^n e_i e_i^H$ is the identity matrix on $\mathbb{C}^n$. As before, this works for $j = 1,\dots m$ and because of the basis property, the second claim follows as well. □

THEOREM 3.3.12. *Let $M \in \mathcal{B}\big(L^2(\mathbb{T};\mathbb{C}^n), L^2(\mathbb{T};\mathbb{C}^m)\big)$. Then $U_m M = M U_n$ and M mapping $L^2_+(\mathbb{T};\mathbb{C}^n)$ into $L^2_+(\mathbb{T};\mathbb{C}^m)$ both hold if and only if there exists a function $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ such that $M = M_{\widetilde{\Theta}}$ for $\widetilde{\Theta} := \Psi_{m\times n}(\Theta)$. In this case Θ is uniquely determined by M.*

Proof. The necessity of the conditon follows from Lemma 3.3.10 and

$$U_m M = M_{\mathrm{diag}(id_{\mathbb{T}},m)} M_{\widetilde{\Theta}} = M_{\mathrm{diag}(id_{\mathbb{T}},m)\widetilde{\Theta}} = M_{\widetilde{\Theta}\,\mathrm{diag}(id_{\mathbb{T}},n)} = M_{\widetilde{\Theta}} M_{\mathrm{diag}(id_{\mathbb{T}},n)} = M U_n$$

if $M = M_{\widetilde{\Theta}}$ for a function $\widetilde{\Theta} := \Psi_{m\times n}(\Theta) \in L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n})$.
Regarding sufficiency, we define $T := \Psi_m^{-1} \circ M \circ \Psi_n$. The operator T is well defined since M maps $L^2_+(\mathbb{T};\mathbb{C}^n)$ into $L^2_+(\mathbb{T};\mathbb{C}^m)$, so $T \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)\big)$. Obviously, $S_i = \Psi_i^{-1} \circ U_i \circ \Psi_i$. Hence, using $U_m M = M U_n$ yields

$$\begin{aligned} S_m \circ T &= \big(\Psi_m^{-1} \circ U_m \circ \Psi_m\big) \circ \big(\Psi_m^{-1} \circ M \circ \Psi_n\big) = \\ &= \Psi_m^{-1} \circ U_m \circ M \circ \Psi_n = \\ &= \Psi_m^{-1} \circ M \circ U_n \circ \Psi_n = \\ &= \big(\Psi_m^{-1} \circ M \circ \Psi_n\big) \circ \big(\Psi_n^{-1} \circ U_n \circ \Psi_n\big) = \\ &= T \circ S_n. \end{aligned}$$

Therefore, T satisfies all conditions of Theorem 3.2.14. Thus, there exists a function $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ such that $\Psi_m^{-1}\circ M\circ\Psi_n = T = T_\Theta$. We set $\widetilde{\Theta} := \Psi_{m\times n}(\Theta) \in L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n})$.By Lemma 3.3.10 we also have $T = T_\Theta = \Psi_m^{-1}\circ M_{\widetilde{\Theta}}\circ\Psi_n$. In particular, $M_{\widetilde{\Theta}}\circ\Psi_n = M\circ\Psi_n$, which means that $M_{\widetilde{\Theta}} \restriction L^2_+(\mathbb{T};\mathbb{C}^n) = M \restriction L^2_+(\mathbb{T};\mathbb{C}^n)$.
The property $MU_n = U_mM$ obviously extends to $MU_n^j = U_m^jM$ for $j\geq 1$ by induction. Because of $U_n^{-1} = M_{\mathrm{diag}(\zeta\mapsto\bar{\zeta},n)}$ it follows that $M = MU_nU_n^{-1} = U_mMU_n^{-1}$. Applying U_m^{-1} from the left gets us $U_m^{-1}M = MU_n^{-1}$ and this again extends to $U_m^{-j}M = MU_n^{-j}$ for $j\geq 1$. Similarly, $U_m^jM_{\widetilde{\Theta}} = M_{\widetilde{\Theta}}U_n^j$ for $j\in\mathbb{Z}$.
Consider the ring of trigonometric polynomials

$$\mathcal{T}_n := \left\{ \sum_{k=-N}^{N} \zeta^k a_k \;\middle|\; N\in\mathbb{N}_0, a_{-N},\ldots,a_N\in\mathbb{C}^n \right\} \subseteq L^2(\mathbb{T};\mathbb{C}^n)$$

and let $p\in\mathcal{T}_n$. Obviously, $U^Np \in L^2_+(\mathbb{T};\mathbb{C}^n)$ for sufficiently large $N\in\mathbb{N}_0$. Hence, $MU_n^Np = M_{\widetilde{\Theta}}U_n^Np$. Applying U_m^{-N} gives

$$M_{\widetilde{\Theta}}p = U_m^{-N}M_{\widetilde{\Theta}}U_n^Np = U_m^{-N}MU_n^Np = Mp.$$

Thus, $M = M_{\widetilde{\Theta}}$ holds on $\mathcal{T}_n$. Since $\mathcal{T}_n$ is densely contained in $L^2(\mathbb{T};\mathbb{C}^n)$ and because M and $M_{\widetilde{\Theta}}$ are both continuous, we conclude that M is an $(m\times n)$-matrix valued multiplier operator with $(m\times n)$-matrix valued multiplier function $\widetilde{\Theta}\in L^2_+(\mathbb{T};\mathbb{C}^{m\times n})$ on the whole space $L^2(\mathbb{T};\mathbb{C}^n)$, i.e. $M = M_{\widetilde{\Theta}}$ everywhere.
The uniqueness of Θ is a consequence of Theorem 3.2.14 and Equation (3.8). □

LEMMA 3.3.13. *For every $\widetilde{\Theta}\in L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n})$ there exists a function $\Theta\in H^\infty_{m\times n}(\mathbb{D})$ satisfying $\Psi_{m\times n}(\Theta) = \widetilde{\Theta}$.*

Proof. Due to Lemma 3.3.5 there exists $M_{\widetilde{\Theta}} \in \mathcal{B}\big(L^2(\mathbb{T};\mathbb{C}^n), L^2(\mathbb{T};\mathbb{C}^m)\big)$ connected to $\widetilde{\Theta}\in L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n})$ and with $\|\widetilde{\Theta}\|_{L^\infty(\mathbb{T};\mathbb{C}^{m\times n})} = \|M_{\widetilde{\Theta}}\|$. Clearly, $U_mM_{\widetilde{\Theta}} = M_{\widetilde{\Theta}}U_n$ holds. Furthermore, let $\widetilde{\Theta}$ have Fourier coefficients $(\Theta_k)_{k\in\mathbb{Z}}$ — take into account that for negative indices k we have that Θ_k is the zero-operator — and take a polynomial $p\in\Psi_n\big(\mathbb{C}[z;\mathbb{C}^n]\big)\subseteq L^2_+(\mathbb{T};\mathbb{C}^n)$ of the form $p(\zeta) = \sum_{k=0}^N \zeta^k b_k$. For technical reasons, we set $b_k := 0$ for $k\in\mathbb{Z}\backslash\{0,\ldots,N\}$. Then

$$M_{\widetilde{\Theta}}p = \widetilde{\Theta}\cdot p = \left(\zeta\mapsto\Big(\sum_{k=0}^\infty \zeta^k\Theta_k\Big)\cdot\Big(\sum_{k=0}^N\zeta^kb_k\Big)\right) = \left(\zeta\mapsto\sum_{k=0}^\infty\zeta^k\Big(\sum_{j=0}^k\Theta_jb_{k-j}\Big)\right).$$

Since $M_{\widetilde{h}}$ maps into $L^2(\mathbb{T};\mathbb{C}^m)$, the sequence $\big(\sum_{j=0}^k\Theta_jb_{k-j}\big)_{k\in\mathbb{Z}}$ must be square-summable. Therefore, $M_{\widetilde{\Theta}}$ maps the norm dense subset $\Psi_n\big(\mathbb{C}[z;\mathbb{C}^n]\big)$ of $L^2_+(\mathbb{T};\mathbb{C}^n)$ into $L^2_+(\mathbb{T};\mathbb{C}^m)$. For $f\in L^2_+(\mathbb{T};\mathbb{C}^n)$ choose a sequence of polynomials $(p_N)_{N\in\mathbb{N}}$ from the space $\Psi_n\big(\mathbb{C}[z;\mathbb{C}^n]\big)$ such that $(p_N)_{N\in\mathbb{N}}$ converges to f in norm. Since $M_{\widetilde{\Theta}}$ is continuous and because $L^2_+(\mathbb{T};\mathbb{C}^m)$ is closed, the calculation

$$M_{\widetilde{\Theta}}f = M_{\widetilde{\Theta}}\left(\lim_{N\to\infty}p_N\right) = \lim_{N\to\infty}M_{\widetilde{\Theta}}p_N$$

shows that $M_{\widetilde{\Theta}}$ maps $L^2_+(\mathbb{T};\mathbb{C}^n)$ into $L^2_+(\mathbb{T};\mathbb{C}^m)$.
Hence, Theorem 3.3.12 is applicable and there exists a function $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ such that $\Psi_{m\times n}(\Theta) = \widetilde{\Theta}$. □

We collect the statements of Lemmata 3.3.10, 3.3.11 and 3.3.13 in the following

THEOREM 3.3.14. *The mapping*

$$\Psi_{m\times n} \upharpoonright H^\infty_{m\times n}(\mathbb{D}) : H^\infty_{m\times n}(\mathbb{D}) \to L^\infty_+(\mathbb{T};\mathbb{C}^{m\times n})$$

is linear, bijective and isometric, i.e. $\|\Psi_{m\times n}(.)\|_{L^\infty(\mathbb{T};\mathbb{C}^{m\times n})} = \|.\|_\infty$. *Furthermore, if* $\Theta_1 \in H^\infty_{m\times n}(\mathbb{D})$, $\Theta_2 \in H^\infty_{k\times m}(\mathbb{D})$ *and* $h \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, *we have the following multiplicativity property:*

$$\begin{aligned}\Psi_{k\times n}(\Theta_2\Theta_1) &= \Psi_{k\times m}(\Theta_2)\Psi_{m\times n}(\Theta_1)\\ \Psi_m(\Theta_1 h) &= \Psi_{m\times n}(\Theta_1)\Psi_n(h)\end{aligned}.$$

3.4 The Structure of Higher-Dimensional Multipliers

We follow [Nag10], V, in this section. In our source the following results are proved in the more general case of analytic functions that take values in arbitrary separable Hilbert spaces $\mathfrak{H}$ or $\mathfrak{G}$. This also involves working with $\mathcal{H}^2(\mathbb{D};\mathfrak{H})$ and considering operator valued functions mapping $\mathbb{D}$ to $\mathcal{B}(\mathfrak{H},\mathfrak{G})$. However, we restrict ourselves to the finite dimensional case.

DEFINITION 3.4.1. Let $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ be a bounded analytic function. Then we define $\Theta^\# : \mathbb{D} \to \mathbb{C}^{n\times m}$, where $\Theta^\#(z) := \Theta(\overline{z})^H = \overline{\Theta(\overline{z})}^T$, and call it the pointwise conjugate adjoint of Θ.

Note that $\Theta^\#$ is NOT the adjoint of Θ, i.e. in general we have

$$\big(\Theta f, g\big)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} \neq \big(f, \Theta^\# g\big)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}$$

for $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $g \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$.

LEMMA 3.4.2. *Let* $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ *be a bounded analytic function. Then* $\Theta^\#$ *is analytic and bounded as well, i.e.* $\Theta^\# \in H^\infty_{n\times m}(\mathbb{D})$. *Furthermore, we have* $\|\Theta\|_\infty = \|\Theta^\#\|_\infty$.

Proof. Because of

$$\Theta^\#(z) = \Theta(\overline{z})^H = \left(\sum_{n=0}^\infty \overline{z}^n\Theta_n\right)^H = \sum_{n=0}^\infty (\overline{z}^n\Theta_n)^H = \sum_{n=0}^\infty z^n\Theta_n^H$$

it easily follows that the pointwise conjugate adjoint of a bounded analytic function, i.e. $(z \mapsto \sum_{n=0}^\infty z^n\Theta_n^H)$, is analytic in $\mathbb{D}$ as well. Furthermore, we have

$$\sup_{z\in\mathbb{D}} \|\Theta^\#(z)\| = \sup_{z\in\mathbb{D}} \|\Theta(\overline{z})^H\| = \sup_{z\in\mathbb{D}} \|\Theta(\overline{z})\| = \sup_{z\in\mathbb{D}} \|\Theta(z)\|.$$

So it is bounded by the same bound as Θ . □

DEFINITION 3.4.3. Let $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ be a bounded analytic function. If $\|\Theta\|_\infty \le 1$, then it is called a contractive analytic function. If Θ additionally satisfies $\|\Theta(0)x\| < \|x\|$ for all $x \in \mathbb{C}^n\backslash\{0\}$, then it is called a purely contractive analytic function.

A direct consequence of Lemmata 3.2.13 and 3.3.10 is the following

COROLLARY 3.4.4. *If Θ is contractive, then so are T_Θ and $M_{\widetilde{\Theta}}$.*

DEFINITION 3.4.5. If $\Theta \in H^\infty_{n\times n}(\mathbb{D})$ satisfies that $\Theta(0)$ is a unitary operator, i.e. $\Theta(0)^{-1} = \Theta(0)^H$, it is called a unitary constant.

The next proposition casts some light on the nomenclature of this definition. Note that for $n = 1$ this is just the maximum modulus principle in complex analysis.

LEMMA 3.4.6. *Let $\Theta \in H^\infty_{n\times n}(\mathbb{D})$ be a contractive analytic function. If $\Theta(0) \in \mathbb{C}^{n\times n}$ is a unitary operator, then $\Theta(z) = \Theta(0)$ for all $z \in \mathbb{D}$. Hence, Θ is a unitary constant.*

Proof. Take $x \in \mathbb{C}^n$ and define $f_x := (z \mapsto \Theta(z)x)$. Clearly, $(z \mapsto x) \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. By Lemma 3.2.13 we have $f_x = T_\Theta(z \mapsto x) \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, and by the Cauchy Integral Formula we have

$$f_x(0) = \frac{1}{2\pi i}\int_{\partial U_r(0)} \frac{f_x(\zeta)}{\zeta}\, d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f_x\big(re^{i\theta}\big)\, d\theta$$

for $0 < r < 1$. Additionally, since $\|\Theta(z)\| \le 1$ on $\mathbb{D}$, Lemma 3.2.13 yields

$$\|f_x\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \|T_\Theta(z \mapsto x)\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} \le \|T_\Theta\| \cdot \|z \mapsto x\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} \le \|x\|_{\mathbb{C}^n}.$$

Clearly, the constant function $(z \mapsto f_x(0))$ is an element of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ as well. Thus,

$$\begin{aligned}(f_x, f_x(0))_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} &= \lim_{r\nearrow 1}\frac{1}{2\pi}\int_0^{2\pi}\Big(f_x(re^{i\theta}), f_x(0)\Big)_{\mathbb{C}^n}\, d\theta\\ &= \lim_{r\nearrow 1}\left(\frac{1}{2\pi}\int_0^{2\pi} f_x(re^{i\theta})\, d\theta, f_x(0)\right)_{\mathbb{C}^n}\\ &= (\Theta(0)x, \Theta(0)x)_{\mathbb{C}^n} = \|x\|^2_{\mathbb{C}^n} = \|x\|_{\mathbb{C}^n}\cdot\|\Theta(0)x\|_{\mathbb{C}^n}\\ &\ge \|f_x\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}\cdot\|f_x(0)\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}.\end{aligned}$$

The Cauchy-Schwarz inequality assures us of the opposite estimate. Hence, f_x and $(z \mapsto f_x(0))$ are linearly dependent, meaning $\Theta(z)x = \kappa\cdot\Theta(0)x$ for every $z \in \mathbb{D}$. Setting in particular $z = 0$ shows that the scalar κ must be 1. Therefore, $\Theta(z)x = \Theta(0)x$ for every $x \in \mathbb{C}^n$. Hence, the range of Θ in $\mathbb{C}^{n\times n}$ consists of one single unitary operator. □

Contractive analytic functions can be decomposed in a particular way.

PROPOSITION 3.4.7. *Let $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ be a contractive analytic function. Then there exist unique decompositions $\mathbb{C}^n = A_< \oplus A_=$ and $\mathbb{C}^m = B_< \oplus B_=$ with*

$$k := \dim(A_=) = \dim(B_=) \le \min(n, m),$$

such that for every fixed $z \in \mathbb{D}$:

- *The range of* $\Theta_<(z) := \Theta(z) \restriction A_<$ *is contained in* $B_<$*, and*
- *The range of* $\Theta_=(z) := \Theta(z) \restriction A_=$ *is contained in* $B_=$*.*

Hereby,

- $\Theta_< : \mathbb{D} \to \mathcal{B}(A_<, B_<)$ *is a purely contractive analytic function, and*
- $\Theta_= : \mathbb{D} \to \mathcal{B}(A_=, B_=)$ *is a unitary constant.*

We can therefore write Θ *as a block operator*

$$\Theta = \begin{pmatrix} \Theta_< & 0 \\ & \\ 0 & \Theta_= \end{pmatrix} : \mathbb{D} \to \mathcal{B}\left(\begin{pmatrix} A_< \\ \oplus \\ A_= \end{pmatrix}, \begin{pmatrix} B_< \\ \oplus \\ B_= \end{pmatrix}\right)$$

with purely contractive part $\Theta_< \in H^\infty_{(m-k)\times(n-k)}(\mathbb{D})$ *and unitary part* $\Theta_= \in H^\infty_{k\times k}(\mathbb{D})$*.*

Proof. To start, we define

$$A_= := \left\{x \in \mathbb{C}^n \;\middle|\; x = \Theta(0)^H\Theta(0)x\right\} \text{ and } B_= := \left\{y \in \mathbb{C}^m \;\middle|\; y = \Theta(0)\Theta(0)^H y\right\}.$$

Evidently, $A_=$ and $B_=$ are linear subspaces. For $\Theta_= := (z \mapsto \Theta(z) \restriction A_=)$ we will show that $\Theta_=(0)$ maps $A_=$ onto $B_=$. First, take $x \in A_=$, apply $\Theta(0)$ and remember that we have $\Theta(0)x = \Theta(0)\Theta(0)^H\Theta(0)x$, hence $\Theta(0)x \in B_=$. For the other inclusion, the same argument for $y \in B_=$ and $\Theta(0)^H$ shows that $\Theta(0)^H$ maps $B_=$ into $A_=$, i.e. $\Theta(0)^H B_= \subseteq A_=$. Applying $\Theta(0)$ to this relation and using that it is the inverse of $\Theta(0)^H$ on $B_=$ yields the desired result.
Moreover, $\Theta_=(0) : A_= \to B_=$ is an isometry, since for $x \in A_=$ we have

$$\|\Theta_=(0)x\|^2_{\mathbb{C}^m} = (\Theta_=(0)x, \Theta_=(0)x)_{\mathbb{C}^m} = \left(\Theta_=(0)^H\Theta_=(0)x, x\right)_{\mathbb{C}^n} = (x,x)_{\mathbb{C}^n} = \|x\|^2_{\mathbb{C}^n}.$$

Thus, it is also injective and therefore a unitary operator. In particular, the two spaces $A_=$ and $B_=$ have the same dimension k. Applying Lemma 3.4.6 to $z \mapsto P_=\Theta_=(z)$, where $P_=$ is the orthogonal projection onto $B_=$, shows that $P_=\Theta_=(z) = \Theta_=(0)$ for all $z \in \mathbb{D}$. From

$$\|x\|^2 \geq \|\Theta_=(z)x\|^2 = \|P_=\Theta_=(z)x\|^2 + \|(I - P_=)\Theta_=(z)\|^2 = \|x\|^2 + \|(I - P_=)\Theta_=(z)\|^2$$

for all $x \in A_=$ we infer that $P_=\Theta_=(z) = \Theta_=(z)$ on $\mathbb{D}$. Hence, $\Theta_=(z) = \Theta_=(0)$ for every $z \in \mathbb{D}$.
Next, we rewrite $A_=$ and $B_=$ as

$$A_= = \left\{x \in \mathbb{C}^n \;\middle|\; x = \Theta^\#(0)\Theta^\#(0)^H x\right\} \text{ and } B_= = \left\{y \in \mathbb{C}^m \;\middle|\; y = \Theta^\#(0)^H\Theta^\#(0)y\right\}$$

and repeat the same arguments with $\Theta^\#$ instead of Θ. Then we get $\Theta^\#(z)(B_=) = \Theta^\#(0)(B_=) = A_=$.

Let us now define $A_< := \mathbb{C}^n \ominus A_=$ and $B_< := \mathbb{C}^m \ominus B_=$. For $x \in A_<$ and $y \in B_=$, we have $\Theta^\#(\overline{z})y \in A_=$ and, therefore,

$$(\Theta(z)x, y)_{\mathbb{C}^m} = (x, \Theta(z)^H y)_{\mathbb{C}^n} = (x, \Theta^\#(\overline{z})y)_{\mathbb{C}^n} = 0.$$

Thus, the range of $\Theta_<(z)$ is contained in $B_<$.
Finally, we show that $\Theta_< : A_< \to B_<$ is purely contractive. If there existed an $x \in A_<$ such that $\|\Theta_<(0)x\|_{\mathbb{C}^m} = \|x\|_{\mathbb{C}^n}$, then we would have

$$\begin{aligned} 0 &= \|x\|^2_{\mathbb{C}^n} - \|\Theta_<(0)x\|^2_{\mathbb{C}^m} = (x,x)_{\mathbb{C}^n} - (\Theta(0)x, \Theta(0)x)_{\mathbb{C}^m} \\ &= \big((I - \Theta(0)^H\Theta(0))x, x\big)_{\mathbb{C}^n}, \end{aligned}$$

and in turn $x = \Theta(0)^H\Theta(0)x$, i.e. $x \in A_=$. Since the constructed decomposition of $\mathbb{C}^n$ is orthogonal, this forces $x = 0$.
So we have shown that there are in fact decompositions as postulated in the proposition. To show uniqueness suppose that there is another pair of decompositions with the same properties, that is $\mathbb{C}^n = C_< \oplus C_=$ and $\mathbb{C}^m = D_< \oplus D_=$. Since $\Theta(0) : C_= \to D_=$ is a unitary transformation, we have $\|c\|_{\mathbb{C}^n} = \|\Theta(0)c\|_{\mathbb{C}^m}$ for every $c \in C_=$, which infers $C_= \subseteq A_=$. The inclusion cannot be proper, though, since if it were, there would be a nonzero element $c \in A_= \cap C_<$ satisfying simultaneously $\|\Theta(0)c\|_{\mathbb{C}^m} = \|c\|_{\mathbb{C}^n}$ because of $c \in A_=$ and $\|\Theta(0)c\|_{\mathbb{C}^m} < \|c\|_{\mathbb{C}^n}$ because of $c \in C_<$. Hence, $C_= = A_=$ and this implies

$$D_= = \Theta(0)C_= = \Theta(0)A_= = B_=$$

as well as

$$C_< = \mathbb{C}^n \ominus C_= = \mathbb{C}^n \ominus A_= = A_<$$

and

$$D_< = \mathbb{C}^m \ominus D_= = \mathbb{C}^m \ominus B_= = B_<.$$

□

The operator T_Θ connected to a contractive analytic function Θ affords us a way to generalise some notions from Hardy space theory to our setting of vector valued analytic functions.

DEFINITION 3.4.8. A contractive analytic function Θ is said to be

1. inner, if T_Θ is an isometry from $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ into $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$.
2. outer, if $\overline{T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$.

There is an alternative characterisation available for inner functions.

PROPOSITION 3.4.9. *A contractive analytic function $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ is inner if and only if $\widetilde{\Theta}(\zeta)$ is an isometry from $\mathbb{C}^n$ into $\mathbb{C}^m$ for almost every $\zeta \in \mathbb{T}$, where $\widetilde{\Theta} = \Psi_{m\times n}(\Theta)$.*

Proof. Suppose that $\widetilde{\Theta}(\zeta)$ is an isometry almost everywhere on $\mathbb{T}$. This clearly makes $M_{\widetilde{\Theta}}$ an isometry from $L^2(\mathbb{T};\mathbb{C}^n)$ into $L^2(\mathbb{T};\mathbb{C}^m)$. Take any $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, then Lemma 3.3.10 assures us of

$$\|T_\Theta f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} = \left\|\left(\Psi_m^{-1} \circ M_{\widetilde{\Theta}} \circ \Psi_n\right) f\right\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} = \left\|\left(M_{\widetilde{\Theta}} \circ \Psi_n\right) f\right\|_{L^2(\mathbb{T};\mathbb{C}^m)} = \\ = \|\Psi_n f\|_{L^2(\mathbb{T};\mathbb{C}^n)} = \|f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)},$$

so T_Θ is an isometry, i.e. Θ is inner.
Conversely, suppose that $\|T_\Theta f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} = \|f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)}$ for all $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Since the shift operators U_m and U_n are unitary transformations, we use Theorem 3.3.12 to show

$$\begin{aligned}\left\|\left(M_{\widetilde{\Theta}} \circ U_n^{-k} \circ \Psi_n\right) f\right\|_{L^2(\mathbb{T};\mathbb{C}^m)} &= \left\|\left(U_m^{-k} \circ M_{\widetilde{\Theta}} \circ \Psi_n\right) f\right\|_{L^2(\mathbb{T};\mathbb{C}^m)} = \\ &= \left\|\left(\Psi_m \circ T_\Theta\right) f\right\|_{L^2(\mathbb{T};\mathbb{C}^m)} = \|T_\Theta f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} = \\ &= \|f\|_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \|\Psi_n f\|_{L^2(\mathbb{T};\mathbb{C}^n)} = \\ &= \left\|\left(U_n^{-k} \circ \Psi_n\right) f\right\|_{L^2(\mathbb{T};\mathbb{C}^n)}\end{aligned}$$

for $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $k \in \mathbb{N}_0$. Since $\left\{\left(U_n^{-k} \circ \Psi_n\right) f \;\middle|\; f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n),\ k \in \mathbb{N}_0\right\}$ is dense in $L^2(\mathbb{T};\mathbb{C}^n)$, this implies that $M_{\widetilde{\Theta}}$ is an isometry, i.e. $M_{\widetilde{\Theta}}^* M_{\widetilde{\Theta}} = M_{\widetilde{I}}$, where $I \in \mathbb{C}^{n\times n}$ is the identity matrix, obviously satisfying $\widetilde{I} = \Psi_{n\times n}(I) = I$. As a result, we get for all $g, h \in L^2(\mathbb{T};\mathbb{C}^n)$ that $\left([M_{\widetilde{I}} - M_{\widetilde{\Theta}}^* M_{\widetilde{\Theta}}]g, h\right)_{L^2(\mathbb{T};\mathbb{C}^n)} = 0$. As $M_{\widetilde{\Theta}}^* = M_{\widetilde{\Theta}^*}$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\left[I - \widetilde{\Theta}(e^{i\theta})^* \widetilde{\Theta}(e^{i\theta})\right] g(e^{i\theta}), h(e^{i\theta})\right)_{\mathbb{C}^n} d\theta = 0.$$

Taking in particular $g = \chi_A x$ and $h = \chi_A y$, where x, y runs over the canonical basis of $\mathbb{C}^n$ and A runs over the Borel sets of $\mathbb{T}$, we conclude that $I - \widetilde{\Theta}(e^{i\theta})^* \widetilde{\Theta}(e^{i\theta}) = 0$ almost everywhere. Hence, $\widetilde{\Theta}(\zeta)$ is an isometry almost everywhere on $\mathbb{T}$. □

COROLLARY 3.4.10. *Let $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ be a contractive analytic function. If it is inner, then $n \leq m$.*

Proof. By Proposition 3.4.9, $\widetilde{\Theta}(\zeta)$ is an isometry from $\mathbb{C}^n$ into $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$. We fix one such $\zeta' \in \mathbb{T}$ and get that $\widetilde{\Theta}(\zeta')$ is injective. Thus, $\widetilde{\Theta}(\zeta')\mathbb{C}^n$ is an n-dimensional subspace of $\mathbb{C}^m$. Consequently, we have $n \leq m$. □

For outer functions it is possible to make a statement about the dimensions of the involved spaces, too.

PROPOSITION 3.4.11. *Let $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ be an outer function and $\widetilde{\Theta} = \Psi_{m\times n}(\Theta)$. Then $m \leq n$ and*

(i) $\dim \operatorname{ran} \Theta(z) = m$ *for every* $z \in \mathbb{D}$ *and*

(ii) $\dim \operatorname{ran} \widetilde{\Theta}(\zeta) = m$ *for almost every* $\zeta \in \mathbb{T}$.

Proof. As Θ is outer, we know that T_Θ has dense range in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. Regarding (i), it therefore follows that the orthogonal complement of $T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ contains only the zero function. We remember the function $K_{m,w} : \mathbb{C}^m \to \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ as defined in Lemma 3.2.7, which satisfies $\big(T_\Theta f, K_{m,w}(\alpha)\big)_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} = \big(\Theta(w)f(w), \alpha\big)_{\mathbb{C}^m}$ for any $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$.
We proceed indirectly. If there existed $w \in \mathbb{D}$ such that $\dim \operatorname{ran} \Theta(w) < m$, then $\Theta(w)\mathbb{C}^n \subsetneq \mathbb{C}^m$. Hence, there would be a nonzero element $\alpha \in \mathbb{C}^m$ orthogonal to $\Theta(w)\mathbb{C}^n$. But then the nonzero element $K_{m,w}(\alpha)$ is perpendicular to $T_\Theta f$ for every $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and this is a contradiction. Consequently, we also get $m \leq n$.
Concerning (ii), let $B = \{e_i \mid i = 1, \ldots, m\}$ be the canonical basis of $\mathbb{C}^m$ and take $x \in B$. We notice that T_Θ having dense image in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ means that for every constant function $g_x := (z \mapsto x)$ there exists a sequence of functions $(f_k) \subset \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ such that $T_\Theta f_k$ converges to g_x in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. By way of the mapping Ψ_m from (3.5) we have the identiy $\Psi_m\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)\big) = L^2_+(\mathbb{T};\mathbb{C}^m)$ and this implies

$$\lim_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left\| \widetilde{\Theta}(e^{i\theta})\widetilde{f}_k(e^{i\theta}) - \widetilde{g_x}(e^{i\theta}) \right\|^2_{\mathbb{C}^m} \, d\theta = 0.$$

Convergence in the space $L^2_+(\mathbb{T};\mathbb{C}^m)$ implies the existence of a subsequence $(\widetilde{\Theta}\widetilde{f}_{k_j})$ that converges pointwise almost everywhere on $\mathbb{T}$, i.e. $\widetilde{\Theta}(e^{i\theta})\widetilde{f}_{k_j}(e^{i\theta}) \to \widetilde{g_x}$ on $\mathbb{T}$ with the exception of a set E_x of zero measure. Letting x run over the set B and taking the union of all the respective exceptional subsets of $\mathbb{T}$, we end up with $E \subset \mathbb{T}$ of zero measure. For $\zeta \notin E$, we get $B \subseteq \widetilde{\Theta}(\zeta)\mathbb{C}^n$, which must therefore coincide with the whole space $\mathbb{C}^m$, i.e. $\dim \operatorname{ran} \widetilde{\Theta}(\zeta) = m$ almost everywhere on $\mathbb{T}$. □

PROPOSITION 3.4.12. *Let* $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ *be a contractive analytic function. Then for* Θ *to be simultaneously inner and outer it is necessary and sufficient that it is a unitary constant. In particular,* $m = n$.

Proof. If we assume Θ to be a unitary constant, then $m = n$ must hold and $\Theta(0) \in \mathbb{C}^{n\times n}$ is a unitary matrix. By Lemma 3.4.6 we have $\Theta(z) \equiv \Theta(0)$ as a function on $\mathbb{D}$. Hence T_Θ is a unitary transformation on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. In particular, it is an isometry, i.e. Θ has the property inner. As T_Θ is also a bijection, it trivially has dense range, so it is outer as well.
To show sufficiency, we notice that Θ being simultaneously inner and outer infers $n \leq m$ and $m \leq n$ due to Corollary 3.4.10 and Proposition 3.4.11. Moreover, T_Θ is a unitary transformation of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, because on the one hand Θ being inner means that T_Θ is isometric and on the other hand Θ being outer means that $T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ is dense in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. Since the isometric image of the Banach space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ must be closed, this can only be true if T_Θ is bijective. Thus, it is unitary. Due to Theorem 3.2.14 we have $T_\Theta S_n \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = S_n T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = S_n \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$, where S_n is the shift operator on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Therefore,

$$T_\Theta[\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \ominus S_n\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)] = \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \ominus S_n\mathcal{H}^2(\mathbb{D};\mathbb{C}^n). \tag{3.12}$$

But the right hand side of (3.12) contains precisely the constant functions on $\mathbb{D}$, because

$$S_n\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \cong \left\{(a_k) \in \ell^2(\mathbb{N}_0;\mathbb{C}^n) \;\middle|\; a_0 = 0\right\}$$

and consequently

$$\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \ominus S_n\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \cong \left\{(a_k) \in \ell^2(\mathbb{N}_0;\mathbb{C}^n) \;\middle|\; a_k = 0 \text{ for all } k > 0\right\}.$$

Thus, $\big(z \mapsto \Theta(z)h(z)\big)$ is constant for all constant functions $h \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Hence, Θ is a constant function itself. Equation (3.12) furthermore assures us of the validity of $\operatorname{ran}\Theta(z) = \mathbb{C}^n$ for $z \in \mathbb{D}$, so $\Theta(z)$ is a bijective operator on $\mathbb{C}^n$ for every $z \in \mathbb{D}$. Finally, since $\widetilde{\Theta}(\zeta)$ is an isometry for almost every $\zeta \in \mathbb{T}$, this means that the constant value of Θ must have this property as well. In particular, $\Theta(0) \in \mathbb{C}^{n\times n}$ is a unitary operator, so Θ is indeed a unitary constant. □

We will conclude this section with some simple examples.

Example 3.4.13. Let the range of the following bounded analytic functions always be $\mathbb{C}^{2\times 2}$.

1. It is clear that $\Theta_1 \equiv I$ is a unitary constant and therefore both inner and outer.
2. Consider

$$\Theta_2(z) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

   Clearly, the functions $\lambda_1(z) \equiv 1$ and $\lambda_2(z) = z$ describe the eigenvalues of Θ_2. It is also not difficult to see, that Θ_2 can be extended onto $\mathbb{T}$ and that

$$\widetilde{\Theta_2}(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}.$$

   Θ_2 is also contractive, although not purely contractive, because $\Theta_2(0) = I$ on $\mathbb{C} \times \{0\}$. Since $\dim\operatorname{ran}\Theta_2(0) = 1 < 2$, it cannot be outer due to Proposition 3.4.11. However, since

$$\widetilde{\Theta_2}(\zeta)^H\widetilde{\Theta_2}(\zeta) = \widetilde{\Theta_2}(\zeta)\widetilde{\Theta_2}(\zeta)^H = \begin{pmatrix} 1 & 0 \\ 0 & |\zeta|^2 \end{pmatrix} = I,$$

   it is inner as per Proposition 3.4.9.
3. Define

$$\Theta_3(z) := I - S_2 = I - zI = \begin{pmatrix} 1-z & 0 \\ 0 & 1-z \end{pmatrix}.$$

   It can, just like Θ_2, be extended onto $\mathbb{T}$ by changing z to ζ. This, however, implies

$$\widetilde{\Theta_3}(\zeta)^H\widetilde{\Theta_3}(\zeta) = |1-\zeta|^2 I = (2 - 2\operatorname{Re}\zeta)I \neq I$$

on $\mathbb{T}$ with the exception of the points $\zeta_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so Θ_3 is not inner.

We will now, however, show that Θ_3 is outer. As per definition, $T_{\Theta_3}\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ must have dense image in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$, so we will proof that the orthogonal complement of

$$T_{\Theta_3}\mathcal{H}^2(\mathbb{D};\mathbb{C}^2) = \left\{ \begin{pmatrix} f_1 - Sf_1 \\ f_2 - Sf_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \;\middle|\; \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \right\}$$

contains only the zero function. We proceed indirectly and suppose that there is a nonzero function $g = (g_1, g_2)^\top \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ such that

$$(f_1 - Sf_1, g_1)_{\mathcal{H}^2(\mathbb{D})} + (f_2 - Sf_2, g_2)_{\mathcal{H}^2(\mathbb{D})} = 0$$

for all $(f_1, f_2)^\top \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$. Let us only consider the case where $g_1 \not\equiv 0$ as the second coordinate can be dealt with analoguously, and keep in mind that $f_1 - Sf_1 = a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \dots$. Then we have the following distinction of cases:

Case 1 If $g_1 = \left(z \mapsto \sum_{n=0}^N b_n z^n\right)$ with $b_N \neq 0$ then $f_1 := \left(z \mapsto z^N\right)$ and $f_2 := 0$ produces $f_1 - Sf_1 = \left(z \mapsto z^N - z^{N+1}\right)$ and by the original definition of the scalar product we get

$$(f_1 - Sf_1, g_1) + (f_2 - Sf_2, g_2) = (f_1 - Sf_1, g_1) = \overline{b_N} \neq 0.$$

Case 2 If $g_1 = \left(z \mapsto \sum_{n=0}^\infty b_n z^n\right)$ and there exists a pair of coefficients $b_k \neq b_{k+1}$ then $f_1 := \left(z \mapsto z^k\right)$ and $f_2 := 0$ yields $f_1 - Sf_1 = \left(z \mapsto z^k - z^{k+1}\right)$ and

$$(f_1 - Sf_1, g_1) + (f_2 - Sf_2, g_2) = (f_1 - Sf_1, g_1) = \overline{b_k} - \overline{b_{k+1}} \neq 0.$$

Case 3 If g_1 is as in the second case but all b_n are the same, then $g_1 \notin \mathcal{H}^2(\mathbb{D})$ or $g_1 \equiv 0$

Either way we get a contradiction, so $g = (g_1, g_2)^\top \equiv 0$. Thus, Θ_3 is outer.

3.5 A Theorem of Beurling for Higher Dimensions

THEOREM 3.5.1 (Beurling)**.** *Let $\mathcal{L} \leq \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ be a closed subspace with $\mathcal{L} \neq \{0\}$. Then $\mathcal{L}$ is S_m-invariant if and only if it can be represented as*

$$\mathcal{L} = T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n),$$

where $n \leq m$ and $T_\Theta \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n), \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)\big)$ is the $(m \times n)$-matrix valued multiplier operator connected to an inner $(m \times n)$-matrix valued multiplier function $\Theta \in H^\infty_{m\times n}(\mathbb{D})$.

Proof. Regarding necessity, we note that by definition the operator T_Θ is an isometry if $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ is inner. Hence, $\mathcal{L} := T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ is a closed subspace of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. With the help of Theorem 3.2.14 we easily see that

$$S_m\mathcal{L} = S_m T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = T_\Theta S_n \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \subseteq T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = \mathcal{L}.$$

So $\mathcal{L}$ is necessarily left invariant by S_m.
Conversely, we need to show that if $S_m\mathcal{L} \subseteq \mathcal{L}$, then $\mathcal{L}$ can be represented with the help of an inner function. First, by identifying $x \in \mathbb{C}^m$ with the constant function $(z \mapsto x) \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ we can embed $\mathbb{C}^m$ in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$ as a subspace. Clearly, S_m is isometric and $S_m^k\mathbb{C}^m \perp_{\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)} \mathbb{C}^m$ for every $k \in \mathbb{N}$. So $\mathbb{C}^m$ is wandering with respect to S_m. Since the monomials form an orthonormal basis of the Hardy-Hilbert space, we can write

$$\mathcal{H}^2(\mathbb{D};\mathbb{C}^m) = M_+(\mathbb{C}^m) = \bigoplus_{k=0}^{\infty} S_m^k\mathbb{C}^m.$$

Hence, S_m is a unilateral shift with generating subspace $\mathbb{C}^m$ in the space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. Now set $V := S_m \restriction \mathcal{L}$. Of course, V is isometric. Since the subspace $\mathcal{L}$ is closed, it is a Hilbert space and Theorem 1.2.11 on the Wold decomposition is applicable. But the calculation

$$\bigcap_{k=0}^{\infty} V^k\mathcal{L} \subseteq \bigcap_{k=0}^{\infty} S_m^k\mathcal{H}^2(\mathbb{D};\mathbb{C}^m) = \{0\}$$

shows that V lacks a unitary part. Setting $\mathfrak{L} := \mathcal{L} \ominus V\mathcal{L}$, the Wold decomposition reduces to $\mathcal{L} = \bigoplus_{k=0}^{\infty} V^k\mathfrak{L}$. Notice that $n := \dim \mathfrak{L} \le \dim \mathbb{C}^m = m$ due to Proposition 1.2.13. Let $B = \{e_j \in \mathbb{C}^n \mid j = 1,\dots,n\}$ be the canonical basis of $\mathbb{C}^n$ in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and let

$$E := \{\Theta_j \in \mathcal{L} \subseteq \mathcal{H}^2(\mathbb{D};\mathbb{C}^m) \mid j = 1,\dots,n\}$$

be an orthonormal basis of $\mathfrak{L}$ in $\mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$. We define a function $\Theta : \mathbb{D} \to \mathbb{C}^{m\times n}$ by setting $\Theta := (\Theta_1,\dots,\Theta_n)$. Clearly,

$$G := \left\{S_n^k e_j \mid k \in \mathbb{N}_0, j = 1,\dots,n\right\} \quad \text{and} \quad H := \left\{V^k\Theta_j \mid k \in \mathbb{N}_0, j = 1,\dots,n\right\}$$

are then orthonormal bases of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and $\mathcal{L}$, respectively. If we define a mapping

$$T : \begin{cases} G & \to & H \\ S_n^k e_j & \mapsto & V^k\Theta_j \end{cases}$$

we can extend T uniquely to a unitary transformation, also called T, with domain $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ and range $\mathcal{L} \subseteq \mathcal{H}^2(\mathbb{D};\mathbb{C}^m)$, i.e. $T\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = \mathcal{L}$.
We need to show that $\Theta \in H^\infty_{m\times n}(\mathbb{D})$ and $T = T_\Theta$ and that Θ is inner. The operator T clearly satisfies

$$TS_nS_n^k e_j = TS_n^{k+1}e_j = V^{k+1}\Theta_j = VV^k\Theta_j = VTS_n^k e_j = S_mTS_n^k e_j$$

for arbitrary $j \in \{1, \dots, n\}$ and $k \in \mathbb{N}_0$, i.e. we have $TS_n = S_m T$. According to Theorem 3.2.14 there exists a uniquely determined $\Theta' \in H^\infty_{m\times n}(\mathbb{D})$ such that $T = T_{\Theta'}$. Because of

$$(z \mapsto \Theta'(z)e_j) = T_{\Theta'}(z \mapsto e_j) = T(z \mapsto e_j) = (z \mapsto \Theta_j)$$

we conclude that $\Theta = \Theta'$. Since T is isometric, this means that Θ must be inner. □

As in the one-dimensional case, the inner function Θ in the multi-dimensional Beurling Theorem is unique up to multiplication with a unitary matrix.

PROPOSITION 3.5.2. *Let $\Theta_1, \Theta_2 \in H^\infty_{m\times n}(\mathbb{D})$ be inner functions and suppose that $T_{\Theta_1}\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = T_{\Theta_2}\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Then there exists a unitary matrix $C \in \mathrm{U}(n)$ such that $\Theta_2 = \Theta_1 C$.*

Proof. We set $\mathcal{L} := T_{\Theta_1}\mathcal{H}^2(\mathbb{D};\mathbb{C}^n) = T_{\Theta_2}\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. By definition, T_{Θ_1} and T_{Θ_2} are isometric due to Θ_1 and Θ_2 being inner. As the isometric image of a Banach space, $\mathcal{L}$ is closed. Furthermore, $T_{\Theta_1} : \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \to \mathcal{L}$ is bijective and bounded. By the Bounded Inverse Theorem 1.2.4, the operator $T_{\Theta_1}^{-1} : \mathcal{L} \to \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$ is bounded, too. Consequently, we have $T_{\Theta_1}^{-1} \circ T_{\Theta_2} \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)\big)$.
Now we infer from $S_m T_{\Theta_1} = T_{\Theta_1} S_n$ that

$$S_n \circ T_{\Theta_1}^{-1} = T_{\Theta_1}^{-1} \circ T_{\Theta_1} \circ S_n \circ T_{\Theta_1}^{-1} = T_{\Theta_1}^{-1} \circ S_m \circ T_{\Theta_1} \circ T_{\Theta_1}^{-1} = T_{\Theta_1}^{-1} \circ S_m$$

holds on $\mathcal{L}$. This implies that $T_{\Theta_1}^{-1} \circ T_{\Theta_2}$ commutes with S_n because

$$S_n \circ T_{\Theta_1}^{-1} \circ T_{\Theta_2} = T_{\Theta_1}^{-1} \circ S_m \circ T_{\Theta_2} = T_{\Theta_1}^{-1} \circ T_{\Theta_2} \circ S_n$$

holds on $\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. So all requirements of Theorem 3.2.14 are satisfied for $T_{\Theta_1}^{-1} \circ T_{\Theta_2}$. Thus, there exists a function $\Theta_3 \in H^\infty_{n\times n}(\mathbb{D})$ such that $T_{\Theta_1}^{-1} \circ T_{\Theta_2} = T_{\Theta_3}$, or alternatively $T_{\Theta_2} = T_{\Theta_1} \circ T_{\Theta_3}$. As $T_{\Theta_1} : \mathcal{H}^2(\mathbb{D};\mathbb{C}^n) \to \mathcal{L}$ is an isometry, so is $T_{\Theta_1}^{-1} : \mathcal{L} \to \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$. Therefore, since T_{Θ_3} is a composition of isometric operators, it is itself isometric. By Lemma 3.2.13 we have $\|\Theta_3\|_\infty = \|T_{\Theta_3}\| = 1$, so Θ_3 is a contractive analytic function and, furthermore, it is inner. Moreover,

$$\overline{T_{\Theta_3}\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = \overline{T_{\Theta_1}^{-1} \circ T_{\Theta_2}\mathcal{H}^2(\mathbb{D})} = T_{\Theta_1}^{-1}\overline{T_{\Theta_2}\mathcal{H}^2(\mathbb{D};\mathbb{C}^n)} = T_{\Theta_1}^{-1}\mathcal{L} = \mathcal{H}^2(\mathbb{D};\mathbb{C}^n)$$

proves that Θ_3 is also outer. According to Proposition 3.4.12, Θ_3 is a unitary constant. Therefore, $\Theta_2 = \Theta_1 C$ is fulfilled for $C := \Theta_3(0) \in \mathrm{U}(n)$. □

Chapter 4

Generalisation to Linear Relations

Since linear relations expand on the consept of linear operators, it seems natural to ask whether a version of Theorem 2.1.14 can be formulated to hold also in this more comprehensive case. We have already shown that subsets of $\mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ can be understood as subsets of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$. Therefore, it is possible to employ the theory of vector-valued functions developed in the previous chapter. It remains to find a suitable notion to replace the condition of commutating with the shift operator.

4.1 Shift-Invariant Linear Relations

We begin with the following conception.

DEFINITION 4.1.1. Let $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ be a closed linear relation. Then we call it S_2-stable, if R, considered as a linear subspace of $\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$, is invariant unter $S_2 \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)\big)$.

Since we are going to deal with the interplay of the operators S and S_2, we formulate the next result for our convenience.

LEMMA 4.1.2. *Consider the shift operators $S \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D})\big)$ and $S_2 \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)\big)$. Let τ be the mapping that identifies the Cartesian product $\mathcal{H}^2(\mathbb{D})\times\mathcal{H}^2(\mathbb{D})$ with $\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ from Proposition 3.2.4. Then the diagram*

$$\begin{array}{ccc} \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) & \xrightarrow{\ \tau\ } & \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \\ {\scriptstyle S\times S}\downarrow & & \downarrow{\scriptstyle S_2} \\ \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) & \xrightarrow{\ \tau\ } & \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \end{array}$$

commutes.

Proof. By definition, an ordered pair $[f,g] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ is mapped to the vector-valued function $h := (f,g)^\top \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ by τ. Since S_2 does nothing else but

$$S_2 \begin{pmatrix} f \\ g \end{pmatrix} = S_2 \left(z \mapsto \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \right) = \left(z \mapsto z \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \right) = \left(z \mapsto \begin{pmatrix} zf(z) \\ zg(z) \end{pmatrix} \right) = \begin{pmatrix} Sf \\ Sg \end{pmatrix},$$

the assertion follows. □

Some operators and linear relations inherit being S_2-stable through another of their properties.

LEMMA 4.1.3. *Consider the operators $S \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D})\big)$ and $S_2 \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)\big)$ and let $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ be a closed linear relation with* $\operatorname{dom} R = \mathcal{H}^2(\mathbb{D})$*. If S and R commute as linear relations, then R is S_2-stable.*

Proof. Assume that $SR = RS$ as linear relations. If we apply S^{-1} from the right, we get $SRS^{-1} = RSS^{-1}$, which is well-defined on $\operatorname{dom} S^{-1} = \operatorname{ran} S = \big\{ f \in \mathcal{H}^2(\mathbb{D}) \;\big|\; f(0) = 0 \big\}$. On the one hand, it is easy to check that $SS^{-1} = I_{\operatorname{ran} S \times \operatorname{ran} S}$. Consequently, the condition $[f,g] \in RI_{\operatorname{ran} S \times \operatorname{ran} S}$ is equivalent to $[f,g] \in R$ and $f(0) = 0$. We therefore get $SRS^{-1} = RSS^{-1} \subseteq R$.
On the other hand, we notice that $[f,g] \in SRS^{-1}$ is equivalent to $[f,p] \in S^{-1}$, that is $[p,f] \in S$, $[p,q] \in R$ and $[q,g] \in S$, for some $p,q \in \mathcal{H}^2(\mathbb{D})$. Using that S is an operator, we conclude $f = Sp$ and $g = Sq$. In essence this shows that $[p,q] \in R$ if and only if $[Sp,Sq] \in SRS^{-1}$. By Lemma 4.1.2, $[Sp,Sq] = S_2([p,q])$. Hence,

$$SRS^{-1} = \big\{ [Sp,Sq] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\big|\; [p,q] \in R \big\} = S_2(R).$$

Therefore, we have $S_2(R) \subseteq R$. □

COROLLARY 4.1.4. *If an operator $T \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$ commutes with the shift operator S, then its graph is S_2-stable.*

LEMMA 4.1.5. *Let $R, T \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ be two S_2-stable linear relations. Then we have:*

(i) $\operatorname{dom} R$, $\operatorname{ran} R$, $\ker R$ *and* $\operatorname{mul} R$ *are all invariant under S.*

(ii) The sets

$$\overline{\operatorname{dom} R} \times \{0\}, \quad \ker R \times \{0\}, \quad \{0\} \times \overline{\operatorname{ran} R} \quad \textit{and} \quad \{0\} \times \operatorname{mul} R$$

are all S_2-stable linear relations.

(iii) R^{-1} is S_2-stable.

(iv) αR is S_2-stable for $\alpha \in \mathbb{C}\backslash\{0\}$.

(v) $\overline{R+T}$ is S_2-stable.

Proof. Suppose that $[f, g] \in R$, then Lemma 4.1.2 assures us of $S_2 = S \times S$ and we have

$$[Sf, Sg] = S_2\big([f, g]\big) \in S_2(R) \subseteq R \tag{4.1}$$

Regarding (i): for $f \in \text{dom } R$ there exists $g \in \mathcal{H}^2(\mathbb{D})$ such that $[f, g] \in R$. By (4.1), we have $[Sf, Sg] \in R$, and thus, $Sf \in \text{dom } R$. This shows $S(\text{dom } R) \subseteq \text{dom } R$. The cases for the range, kernel and multi-valued part are completely analoguous.
Concerning (ii), we notice that the linear subspaces $\overline{\text{dom } R} \times \{0\}$ and $\{0\} \times \overline{\text{ran } R}$ are both clearly closed. The continuity of S then shows that

$$S(\overline{\text{dom } R}) \subseteq \overline{S(\text{dom } R)} \subseteq \overline{\text{dom } R}.$$

Obviously, we have $S\{0\} = \{0\}$, so $\overline{\text{dom } R} \times \{0\}$ is S_2-stable. For $\{0\} \times \overline{\text{ran } R}$ we proceed analoguously. Furthermore, $\ker R$ and $\text{mul } R$ are both closed, due to Corollary 1.3.6, and invariant under S, so $\ker R \times \{0\}$ and $\{0\} \times \text{mul } R$ are S_2-stable as well.
For (iii), we remember Corollary 1.3.8 and see that $R^{-1}$is closed. Since by definition $[f, g] \in R^{-1}$ if and only if $[g, f] \in R$, equation (4.1) shows that $[Sf, Sg] \in R^{-1}$ if and only if $[Sg, Sf] \in R$, so R^{-1} is S_2-stable.
In (iv), we also use Corollary 1.3.8 to convince ourselves that αR is closed for $\alpha \in \mathbb{C}\backslash\{0\}$. Again by definition, $[f, g] \in \alpha R$ if and only if $[f, \frac{1}{\alpha} g] \in R$. Equation (4.1) combined with the fact that $S\frac{1}{\alpha}g = \frac{1}{\alpha}Sg$ proves that $[Sf, Sg]$ also belongs to αR, and thus, αR is S_2-stable.
Finally, to show (v) we notice that $[f, g] \in R + T$ is equivalent to there being functions $h, k \in \mathcal{H}^2(\mathbb{D})$ such that $g = h + k$ with $[f, h] \in R$ and $[f, k] \in T$. Clearly, if we have such a decomposition of g then due to linearity of S we get a decomposition of Sg, namely $Sg = Sh + Sk$, with $[Sf, Sh] \in S_2(R)$ and $[Sf, Sk] \in S_2(T)$. Hence, we have shown the inclusion marked with an asterisk in

$$S_2(\overline{R + T}) \subseteq \overline{S_2(R + T)} \overset{*}{\subseteq} \overline{S_2(R) + S_2(T)} \subseteq \overline{R + T}$$

and the first inclusion is due to S_2 being continuous. Thus, $\overline{R + T}$ is S_2-stable. □

We remember Theorem 2.1.14, which stated that a continuous operator T on the Hardy-Hilbert space commutes with the shift operator S if and only if T is a multiplier operator. We can formulate a similar result for linear relations, using the property of S_2-stability.

THEOREM 4.1.6. *Let $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ be a closed linear relation with $R \supsetneq \{[0, 0]\}$. Then R is S_2-stable if and only if there exists an inner function $\Theta \in H^\infty_{2\times n}(\mathbb{D})$ such that*

$$R = T_\Theta \mathcal{H}^2(\mathbb{D}; \mathbb{C}^n)$$

with $n \leq 2$.

Proof. By Proposition 3.2.4, we can consider $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ as a closed linear subspace of $\mathcal{H}^2(\mathbb{D}; \mathbb{C}^2)$. Due to Theorem 3.5.1 there exists an inner function $\Theta \in H^\infty_{2\times n}(\mathbb{D})$, where $n \leq 2$, such that $R = T_\Theta \mathcal{H}^2(\mathbb{D}; \mathbb{C}^n)$ if and only if $S_2(R) \subseteq R$. □

We direct our attention to the fact that Theorem 4.1.6 permits two cases, i.e. an S_2-stable linear relation can either be described as the transform of the Hardy-space $\mathcal{H}^2(\mathbb{D})$ or of the vector-valued function space $\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$.
The reason for this differentiation stems from Beurling's Theorem 3.5.1. Setting, as in its proof, $V := S_2 \restriction R$ and $\mathfrak{L} := R \ominus VR$, we have a Wold decomposition of R of the form $R = \bigoplus_{k=0}^{\infty} V^k \mathfrak{L}$ and because of Proposition 1.2.13 we know that $\dim \mathfrak{L} \leq 2$. In other words, the number n takes the value 2 if and only if we can find two linearly independent functions belonging to R, i.e. two vector-valued functions, that span $\mathfrak{L}$. The fact that $\mathfrak{L}$ is spanned by a single function is therefore equivalent to n being equal to 1. We have therefore proven

PROPOSITION 4.1.7 (Dimensional Condition)**.** *Let $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ be a non-trivial S_2-stable linear relation. Then R is the transform of*

- $\mathcal{H}^2(\mathbb{D})$ *if and only if $R \ominus S_2(R)$ is one-dimensional.*
- $\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ *if and only if $R \ominus S_2(R)$ is two-dimensional.*

We will now discuss the two cases separately.

The case $n = 1$

Let $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ be an S_2-stable linear relation. In this section, suppose that the orthogonal complement of $S_2(R)$ in R is one-dimensional. By Theorem 4.1.6 and the Dimensional Condition, Proposition 4.1.7, there exists $\Theta \in H^{\infty}_{2\times 1}(\mathbb{D})$ with $R = T_\Theta \mathcal{H}^2(\mathbb{D})$. Therefore, Θ has the form

$$\Theta = \begin{pmatrix} a \\ b \end{pmatrix}$$

for $a, b \in H^{\infty}(\mathbb{D})$. Notice that due to Proposition 3.5.2 Θ is uniquely determined up to multiplication by a unimodular constant. The functions a and b therefore share this uniqueness property.

PROPOSITION 4.1.8. *Let $R = T_\Theta \mathcal{H}^2(\mathbb{D})$ and $\Theta = (a, b)^\top \in H^{\infty}_{2\times 1}(\mathbb{D})$ with the coordinate functions $a, b \in H^{\infty}(\mathbb{D})$. Then*

$$R = \left\{ [a \cdot f, b \cdot f] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; f \in \mathcal{H}^2(\mathbb{D}) \right\} \tag{4.2}$$

with

$$\operatorname{dom} R = T_a \mathcal{H}^2(\mathbb{D}) \qquad \text{and} \qquad \operatorname{ran} R = T_b \mathcal{H}^2(\mathbb{D}).$$

Furthermore, $|\widetilde{a}(\zeta)|^2 + |\widetilde{b}(\zeta)|^2 = 1$ must hold almost everywhere on $\mathbb{T}$. In particular, a and b cannot both be the zero function.

Proof. First, we notice that

$$\left\{\Theta f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \;\middle|\; f \in \mathcal{H}^2(\mathbb{D})\right\} = \left\{\begin{pmatrix} a \cdot f \\ b \cdot f \end{pmatrix} \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \;\middle|\; f \in \mathcal{H}^2(\mathbb{D})\right\}.$$

So with the help of Proposition 3.2.4 we see that R has the form given in Equation (4.2). Secondly, we obtain from (4.2)

$$\begin{aligned}\operatorname{dom} R &= \left\{g \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; \exists h \in \mathcal{H}^2(\mathbb{D}) : [g,h] \in R\right\} = \\ &= \left\{a \cdot f \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; f \in \mathcal{H}^2(\mathbb{D})\right\} = \\ &= T_a \mathcal{H}^2(\mathbb{D}).\end{aligned}$$

A symmetric argument yields $\operatorname{ran} R = T_b \mathcal{H}^2(\mathbb{D})$.
Finally, since Θ must be inner, $\widetilde{\Theta}(\zeta)$ is an isometric mapping from $\mathbb{C}$ into $\mathbb{C}^2$ for almost all $\zeta \in \mathbb{T}$ as by Proposition 3.4.9. This directly implies that $|\widetilde{a}(\zeta)|^2 + |\widetilde{b}(\zeta)|^2 = 1$ must hold almost everywhere on $\mathbb{T}$ and that a and b cannot both vanish. □

A direct consequence of Proposition 4.1.8 is the following

COROLLARY 4.1.9. *Let R and Θ be as above. Then:*

(i) R is densely defined if and only if the coordinate function $a \in H^\infty(\mathbb{D})$ is outer.

(ii) R has dense range if and only if the coordinate function $b \in H^\infty(\mathbb{D})$ is outer.

PROPOSITION 4.1.10. *Let R and Θ be as above. Then:*

(i) R is an operator if and only if the coordinate function $a \in H^\infty(\mathbb{D})$ is not the zero function.

(ii) R is a continuous linear operator $R : \operatorname{dom} R \to \mathcal{H}^2(\mathbb{D})$ if and only if $T_a \mathcal{H}^2(\mathbb{D})$ is nontrivial and closed, i.e. if and only if $\operatorname{ess\,inf} |\widetilde{a}| > 0$.

(iii) R has trivial kernel if and only if the coordinate function $b \in H^\infty(\mathbb{D})$ is not the zero function.

Proof. A linear relation is an operator if and only if its multi-valued part only contains the element zero. In particular, by (4.2) we have

$$\operatorname{mul} R = \left\{g \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; [0,g] \in R\right\} = \left\{b \cdot f \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; a \cdot f = 0\right\}.$$

Whenever $a \not\equiv 0$ the first coordinate only vanishes for $f \equiv 0$, in which case the second coordinate automatically amounts to zero as well. Only for $a \equiv 0$ can — and will — $\operatorname{mul} R$ contain elements different from zero. This proves the first claim.
The second claim is a consequence of the first claim and Lemma 2.2.14.
Finally, since

$$\ker R = \left\{g \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; [g,0] \in R\right\} = \left\{a \cdot f \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; b \cdot f = 0\right\},$$

the third claim is symmetric to the first one. □

COROLLARY 4.1.11. *Let R and Θ be as above and suppose that R is an operator. In this case we have in fact*

$$R = \left(T_{\frac{b}{a}} \restriction T_a\mathcal{H}^2(\mathbb{D})\right), \tag{4.3}$$

where

$$R : \begin{cases} T_a\mathcal{H}^2(\mathbb{D}) & \to & T_b\mathcal{H}^2(\mathbb{D}) \\ a\cdot f & \mapsto & b\cdot f \end{cases} \qquad \text{and} \qquad T_{\frac{b}{a}} : \begin{cases} \operatorname{dom} T_{\frac{b}{a}} & \to & \mathcal{H}^2(\mathbb{D}) \\ f & \mapsto & \frac{b}{a}\cdot f \end{cases},$$

with $\operatorname{dom} T_{\frac{b}{a}} = \left\{ f \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; \frac{b}{a}\cdot f \in \mathcal{H}^2(\mathbb{D})\right\}$ *as the domain of the multiplication operator* $T_{\frac{b}{a}}$*.*

Proof. Clearly, every function $g = a\cdot f \in T_a\mathcal{H}^2(\mathbb{D}) = \operatorname{dom} R$ satisfies $\frac{b}{a}\cdot g = \frac{b}{a}\cdot a\cdot f = b\cdot f \in \mathcal{H}^2(\mathbb{D})$. □

In general, $\operatorname{dom} T_{\frac{b}{a}}$ might very well be a proper superset of $T_a\mathcal{H}^2(\mathbb{D})$. So $T_{\frac{b}{a}}$ could potentially be a proper extension of the operator R. We give an example to show how this can happen.

Example 4.1.12. Let

$$\Theta(z) := \frac{1}{\sqrt{2}}\begin{pmatrix} z \\ z \end{pmatrix}$$

on $\mathbb{D}$. Clearly, Θ belongs to $H^\infty_{2\times 1}(\mathbb{D})$ and it is inner. Now set $R := T_\Theta\mathcal{H}^2(\mathbb{D})$. Since we have that $a(z) = b(z) = \frac{1}{\sqrt{2}}z$ for $z \in \mathbb{D}$ we get

$$T_a\mathcal{H}^2(\mathbb{D}) = \operatorname{ran} S = \left\{ f \in \mathcal{H}^2(\mathbb{D}) \;\middle|\; f(0) = 0 \right\} \subsetneq \mathcal{H}^2(\mathbb{D}).$$

However,

$$T_{\frac{b}{a}} = T_1 = id_{\mathcal{H}^2(\mathbb{D})},$$

which is obviously defined everywhere. Hence, $\operatorname{dom} T_{\frac{b}{a}} = \mathcal{H}^2(\mathbb{D}) \supsetneq T_a\mathcal{H}^2(\mathbb{D})$. Consequently, $T_{\frac{b}{a}}$ is a multiplier operator with multiplier function 1 whereas R is an operator that is only defined on a closed proper subspace of $\mathcal{H}^2(\mathbb{D})$.

We also remark that we find ourselves in a similar situation whenever b is a multiple of a, i.e. $\frac{b}{a} \in H^\infty(\mathbb{D})$, implying that $T_{\frac{b}{a}}$ is a multiplier, and $T_a\mathcal{H}^2(\mathbb{D}) \subsetneq \mathcal{H}^2(\mathbb{D})$.
The following result is an obvious consequence of Equation (4.2).

COROLLARY 4.1.13. *Let R and Θ be as above. Then:*

(i) If $a \equiv 0$, then $R = \overline{\{0\} \times T_b\mathcal{H}^2(\mathbb{D})}$ *and* $\operatorname{mul} R = \overline{T_b\mathcal{H}^2(\mathbb{D})}$.

(ii) If $b \equiv 0$, then $R = \overline{T_a \mathcal{H}^2(\mathbb{D}) \times \{0\}}$ and $\ker R = \overline{T_a \mathcal{H}^2(\mathbb{D})}$.

PROPOSITION 4.1.14. *Let R and Θ be as above. Then:*

(i) The inverse R^{-1} can be represented as $R^{-1} = T_{\Theta_\updownarrow} \mathcal{H}^2(\mathbb{D})$.

(ii) For $\alpha \in \mathbb{T}$ we have $\alpha R = T_{\Theta[\alpha]} \mathcal{H}^2(\mathbb{D})$.

Thereby we set

$$\Theta_\updownarrow = \begin{pmatrix} b \\ a \end{pmatrix} \qquad \text{and} \qquad \Theta[\alpha] = \begin{pmatrix} a \\ \alpha b \end{pmatrix}.$$

Proof. According to Lemma 4.1.5 *(iii)* and *(iv)*, R^{-1} and αR are both S_2-stable if R has this property. It is clear that the Dimensional Condition, Proposition 4.1.7, is unaffected by forming R^{-1} or αR. Furthermore, $\Theta_\updownarrow$ and $\Theta[\alpha]$ are both inner. So a short calculation involving the mappings Φ_{inv} and Φ_α from Lemma 1.3.7 yields

$$\begin{aligned} R^{-1} &= \Phi_{inv}(R) \\ &= \Phi_{inv}\Big(\big\{ [a \cdot f, b \cdot f] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\big|\; f \in \mathcal{H}^2(\mathbb{D}) \big\} \Big) \\ &= \big\{ [b \cdot f, a \cdot f] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\big|\; f \in \mathcal{H}^2(\mathbb{D}) \big\} \\ &= T_{\Theta_\updownarrow} \mathcal{H}^2(\mathbb{D}) \end{aligned}$$

and

$$\begin{aligned} \alpha R &= \Phi_\alpha(R) \\ &= \Phi_\alpha\Big(\big\{ [a \cdot f, b \cdot f] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\big|\; f \in \mathcal{H}^2(\mathbb{D}) \big\} \Big) \\ &= \big\{ [a \cdot f, \alpha b \cdot f] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\big|\; f \in \mathcal{H}^2(\mathbb{D}) \big\} \\ &= T_{\Theta[\alpha]} \mathcal{H}^2(\mathbb{D}). \end{aligned}$$

□

Notice that it would be possible to define $\Theta[\alpha]$ also for complex numbers of modulus different from 1. However, this then creates the problem of assuring that $\Theta[\alpha]$ is inner, which can very quickly become too complicated to solve, at least if we have to rely on the tools we have developed in this work. A similar problem is discussed in Example 4.2.3.

The case $n = 2$

Let $R \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ again be S_2-stable, but now suppose that there exist two linearly independet functions spanning the orthogonal complement of $S_2(R)$ in R. Then there exists a representation such that $R = T_\Theta \mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ with $\Theta \in H^\infty_{2\times 2}(\mathbb{D})$ and Θ can be represented by

$$\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.4}$$

for functions $a, b, c, d \in H^\infty(\mathbb{D})$.

PROPOSITION 4.1.15. *Let $R = T_\Theta \mathcal{H}^2(\mathbb{D})$ and $\Theta \in H^\infty_{2\times 2}(\mathbb{D})$ of the form (4.4) with the coordinate functions $a, b, c, d \in H^\infty(\mathbb{D})$. Then*

$$R = \left\{ [a \cdot f_1 + b \cdot f_2, c \cdot f_1 + d \cdot f_2] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2) \right\} \tag{4.5}$$

with

$$\operatorname{dom} R = T_a \mathcal{H}^2(\mathbb{D}) + T_b \mathcal{H}^2(\mathbb{D}) \qquad \textit{and} \qquad \operatorname{ran} R = T_c \mathcal{H}^2(\mathbb{D}) + T_d \mathcal{H}^2(\mathbb{D}).$$

Furthermore, $\widetilde{\Theta}(\zeta)$ must belong to the unitary group $\mathrm{U}(2)$ for almost every $\zeta \in \mathbb{T}$. In particular, there cannot be a row or column in Θ made up entirely of zero functions. Finally,

$$\det \Theta : \begin{cases} \mathbb{D} & \to \quad \mathbb{C} \\ z & \mapsto \quad \det \Theta(z) \end{cases}$$

belongs to $H^\infty(\mathbb{D})$ and is not the zero function on $\mathbb{D}$.

Proof. Since

$$\Theta f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a \cdot f_1 + b \cdot f_2 \\ c \cdot f_1 + d \cdot f_2 \end{pmatrix}$$

for any $f \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2)$ with coordinate functions $f_1, f_2 \in \mathcal{H}^2(\mathbb{D})$, the representation (4.5) follows with the help of Proposition 3.2.4.
It is clear that $\operatorname{dom} R$ is contained in $T_a \mathcal{H}^2(\mathbb{D}) + T_b \mathcal{H}^2(\mathbb{D})$, so we need to show the opposite inclusion. By the definition of the domain of a linear relation, we must find for every $f \in T_a \mathcal{H}^2(\mathbb{D}) + T_b \mathcal{H}^2(\mathbb{D})$ a function $g \in \mathcal{H}^2(\mathbb{D})$ such that $[f, g] \in R$. Due to our choice of f there exist functions $k \in T_a \mathcal{H}^2(\mathbb{D})$ and $\ell \in T_b \mathcal{H}^2(\mathbb{D})$ such that $f = k + \ell$. If we set $f_1 := \frac{k}{a}$ and $f_2 := \frac{\ell}{b}$, then they are both clearly well-defined and belong to $\mathcal{H}^2(\mathbb{D})$. Furthermore, $f = a \cdot f_1 + b \cdot f_2$ is obviously satisfied. With $g := c \cdot f_1 + d \cdot f_2$ we get $g \in \mathcal{H}^2(\mathbb{D})$ and $[f, g] \in R$ due to (4.5). A symmetric argument for $\operatorname{ran} R$ runs completely analoguously.
Since Θ is guaranteed to be inner by Theorem 4.1.6, $\widetilde{\Theta}(\zeta)$ must be a unitary matrix almost everywhere on $\mathbb{T}$, which would be impossible to achieve if a column or row were to consist just of zero functions. To prove the second claim regarding $\det \Theta$, we proceed indirectly. Assume that $\det \Theta \equiv 0$, then

$$\Theta \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} a \cdot (-b) + b \cdot a \\ c \cdot (-b) + d \cdot a \end{pmatrix} = \begin{pmatrix} 0 \\ \det \Theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

shows that

$$\ker T_\Theta \supseteq \operatorname{ls}\left\{\begin{pmatrix}-b\\ a\end{pmatrix} \in \mathcal{H}^2(\mathbb{D};\mathbb{C}^2)\right\} \supsetneq \{0\}.$$

Thus, $T_\Theta \in \mathcal{B}\big(\mathcal{H}^2(\mathbb{D};\mathbb{C}^2)\big)$ cannot be an isometry and Θ is not inner. We arrive at a contradiction. Finally, we also see $\det\Theta = a\cdot d - b\cdot c \in H^\infty(\mathbb{D})$ since $H^\infty(\mathbb{D})$ is closed under addition and multiplication. □

We can immediately derive the following

COROLLARY 4.1.16. *Let R and Θ be as above. Then:*

(i) R is densely defined if and only if $T_a\mathcal{H}^2(\mathbb{D}) + T_b\mathcal{H}^2(\mathbb{D})$ is dense in $\mathcal{H}^2(\mathbb{D})$.

(ii) R has dense range if and only if $T_c\mathcal{H}^2(\mathbb{D}) + T_d\mathcal{H}^2(\mathbb{D})$ is dense in $\mathcal{H}^2(\mathbb{D})$.

In the previous section, where $n = 1$, we have seen that R is always an operator, disregarding one pathological case. The behaviour of R for $n = 2$ is quite different, however.
To explain this properly, we remember a certain concept of classical Hardy space theory: Let $g, h \in H^\infty(\mathbb{D})$ and suppose that h is inner. Then we call g divisible by h if there exists a function $f \in H^\infty(\mathbb{D})$ such that $h\cdot f = g$ holds. Furthermore, given any two functions $a, b \in H^\infty(\mathbb{D})$, there exists a unique — within multiplication by unimodular constant — inner function $k \in H^\infty(\mathbb{D})$ such that both a and b are divisble by k and k is maximal in the sense that any other inner function $k' \in H^\infty(\mathbb{D})$ that is a divisor of both a and b is also a divisor of k. This particular function k is the greatest common divisor of a and b, denoted by $\gcd(a,b)$.

PROPOSITION 4.1.17. *Let R and Θ be as above. Then R is never an operator. In fact,*

$$\operatorname{mul} R = \begin{cases} T_c\mathcal{H}^2(\mathbb{D}) & \text{for} \quad a = 0, b \neq 0 \\ T_d\mathcal{H}^2(\mathbb{D}) & \text{for} \quad a \neq 0, b = 0 \\ T_{\frac{\det\Theta}{\gcd(a,b)}}\mathcal{H}^2(\mathbb{D}) & \text{for} \quad a, b \neq 0 \end{cases}$$

and each of the three spaces is a proper superset of $\{0\}$.

Proof. Since $\operatorname{mul} R$ is defined as the set of all functions $g \in \mathcal{H}^2(\mathbb{D})$ such that $[0, g]$ belongs to the linear relation, we are faced with solving

$$a\cdot f_1 + b\cdot f_2 = 0 \tag{4.6}$$
$$c\cdot f_1 + d\cdot f_2 = g \tag{4.7}$$

for $f_1, f_2 \in \mathcal{H}^2(\mathbb{D})$. We proceed by distinguishing four cases.

Case 1 $a = b = 0$. This is impossible due to Proposition 4.1.15.

Case 2 $a = 0, b \neq 0$. We first notice that (4.6) implies $f_2 \equiv 0$. Accordingly, (4.7) now reads as $c \cdot f_1 = g$ and $c \not\equiv 0$ due to Proposition 4.1.15. In fact, if a vanishes, the coordinate function $\widetilde{c}$ of $\widetilde{\Theta}$ must have modulus 1 almost everywhere on $\mathbb{T}$ or else $\widetilde{\Theta}$ would not belong to the unitary group almost everywhere. This means that c is an inner function. Consequently, T_c is an isometry and isometries automatically have closed range. The above problem is therefore solvable if and only if $g \in T_c\mathcal{H}^2(\mathbb{D})$ because then $f_1 := \frac{g}{c}$ will belong to $\mathcal{H}^2(\mathbb{D})$. This shows $\operatorname{mul} R = T_c\mathcal{H}^2(\mathbb{D}) \supsetneq \{0\}$.

Case 3 $a \neq 0, b = 0$. This case can be dealt with analoguously to Case 2 and yields $\operatorname{mul} R = T_d\mathcal{H}^2(\mathbb{D}) \supsetneq \{0\}$.

Case 4 $a, b \neq 0$. We will first look at the operator $T_{(a,b)} : \mathcal{H}^2(\mathbb{D}; \mathbb{C}^2) \to \mathcal{H}^2(\mathbb{D})$ and determine $\ker T_{(a,b)}$ to solve condition (4.6).

Lemma 4.1.5 implies that $\ker T_{(a,b)}$ is invariant under S. Furthermore, it contains at least the vector function $(-b, a)$, so it is not just the space $\{0\}$. Since the kernel is always closed, all requirements of Theorem 3.5.1 are satisfied. Consequently, we have one of the representations

$$\ker T_{(a,b)} = T_{\Theta'}\mathcal{H}^2(\mathbb{D}; \mathbb{C}^2) \qquad \text{or} \qquad \ker T_{(a,b)} = T_{(p,q)^\top}\mathcal{H}^2(\mathbb{D}),$$

where $\Theta' \in H^\infty_{2\times 2}(\mathbb{D})$ and $p, q \in H^\infty(\mathbb{D})$ and both Θ' and $(p, q)^\top$ are inner. However, if $\ker T_{(a,b)}$ were the transform of $\mathcal{H}^2(\mathbb{D}; \mathbb{C}^2)$, this would imply $\big(\widetilde{a}(\zeta), \widetilde{b}(\zeta)\big) \cdot \widetilde{\Theta'}(\zeta) = 0$ for almost every $\zeta \in \mathbb{T}$. But as $\widetilde{\Theta'}$ is unitary almost everywhere on $\mathbb{T}$, its kernel must be trivial, so we have a contridiction. Consequently, $\ker T_{(a,b)} = T_{(p,q)^\top}\mathcal{H}^2(\mathbb{D})$ holds.

Next, notice that

$$T_{(-b,a)^\top}\mathcal{H}^2(\mathbb{D}) \subseteq \ker T_{(a,b)} = T_{(p,q)^\top}\mathcal{H}^2(\mathbb{D})$$

holds. Therefore, we can define the operator $Q := T^{-1}_{(p,q)^\top} \circ T_{(-b,a)^\top}$, which maps $\mathcal{H}^2(\mathbb{D})$ into itself and is bounded. Because of

$$\begin{aligned} S \circ T^{-1}_{(p,q)^\top} &= T^{-1}_{(p,q)^\top} \circ T_{(p,q)^\top} \circ S \circ T^{-1}_{(p,q)^\top} = \\ &= T^{-1}_{(p,q)^\top} \circ S_2 \circ T_{(p,q)^\top} \circ T^{-1}_{(p,q)^\top} = \\ &= T^{-1}_{(p,q)^\top} \circ S_2 \end{aligned}$$

we arrive at

$$\begin{aligned} Q \circ S &= T^{-1}_{(p,q)^\top} \circ T_{(-b,a)^\top} \circ S = \\ &= T^{-1}_{(p,q)^\top} \circ S_2 \circ T_{(-b,a)^\top} = \\ &= S \circ T^{-1}_{(p,q)^\top} \circ T_{(-b,a)^\top} = \\ &= S \circ Q \end{aligned}$$

i.e. Q commutes with S. By Theorem 2.1.14 there exists a function $k \in H^\infty(\mathbb{D})$ such that $Q = T_k$. Hence, we get $T_{(-b,a)^\top} = T_{(p,q)^\top} \circ T_k$. Applying this operator equality to the constant function with value 1 we get

$$k(z)\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \begin{pmatrix} -b(z) \\ a(z) \end{pmatrix}$$

for $z \in \mathbb{D}$ and a similar relation for the boundary functions on $\mathbb{T}$. We remember that the vector $(-b, a)^\top$ is in essence the first row of the inner function Θ, satisfying $|\widetilde{a}(\zeta)|^2 + |\widetilde{b}(\zeta)|^2 = 1$ almost everywhere on $\mathbb{T}$. Consequently, the calculation

$$|\widetilde{k}(\zeta)|^2 = |\widetilde{k}(\zeta)|^2 \cdot \big(|\widetilde{p}(\zeta)|^2 + |\widetilde{q}(\zeta)|^2\big) = |\widetilde{b}(\zeta)|^2 + |\widetilde{a}(\zeta)|^2 = 1$$

shows that k is inner. Additionally, we have that k is a divisor of both a and b.

We will show that k is in fact the greatest common divisor. Suppose that the inner function $\kappa \in H^\infty(\mathbb{D})$ also divides both a and b, i.e. there exist functions $f_a, f_b \in H^\infty(\mathbb{D})$ such that $a = \kappa \cdot f_a$ and $b = \kappa \cdot f_b$. We immediately get that $(-f_b, f_a)^\top \in H^\infty_{2\times 1}(\mathbb{D})$ is inner. Because of

$$\kappa \cdot \big(-a \cdot f_b + b \cdot f_a \big) = -a \cdot b + b \cdot a \equiv 0$$

we get that $-a \cdot f_b + b \cdot f_a \equiv 0$, implying

$$T_{(-f_b, f_a)^\top} \mathcal{H}^2(\mathbb{D}) \subseteq \ker T_{(a,b)}.$$

We follow the arguments already used for the operator Q above to conclude that there exists a function $\lambda \in H^\infty(\mathbb{D})$ such that

$$T^{-1}_{(p,q)^\top} \circ T_{(-f_b, f_a)^\top} = T_\lambda.$$

Applying $T_{(p,q)^\top}$ from the left and looking at the image of the constant function with value 1 lets us conclude that λ is inner and that we have

$$\begin{aligned} (\kappa \cdot \lambda) \cdot p &= -b = k \cdot p \\ (\kappa \cdot \lambda) \cdot q &= \quad a = k \cdot q \end{aligned} \; .$$

At least one line is nontrivial and allows us to cancel either p or q. This shows $\kappa \cdot \lambda = k$, and thus, κ is a divisor of k. We have therefore proven $k = \gcd(a, b)$. Furthermore, the kernel of $T_{(a,b)}$ satisfies $\ker T_{(a,b)} = T_{(-\frac{b}{k}, \frac{a}{k})^\top} \mathcal{H}^2(\mathbb{D})$.

For (4.6) to be satisfied, we consequently have to set $f_1 := -h \cdot \frac{b}{k}$ and $f_2 := h \cdot \frac{a}{k}$ with some nonzero function $h \in \mathcal{H}^2(\mathbb{D})$. However, this transforms Equation (4.7) into $c \cdot (-h \cdot \frac{b}{k}) + d \cdot h \cdot \frac{a}{k} = h \cdot \frac{\det \Theta}{k} = g$, inferring $\operatorname{mul} R = T_{\frac{\det \Theta}{\gcd(a,b)}} \mathcal{H}^2(\mathbb{D}) \supsetneq \{0\}$. Clearly, $\frac{\det \Theta}{\gcd(a,b)} \not\equiv 0$, so the last inclusion is due to Proposition 4.1.15. Additionally, we notice that $\frac{\det \Theta}{\gcd(a,b)}$ belongs to $H^\infty(\mathbb{D})$ and that it is inner, since the determinant of a unitary matrix has modulus 1 and because $\gcd(a, b)$ was defined to be inner. Consequently, $T_{\frac{\det \Theta}{\gcd(a,b)}}$ is an isometry and $\operatorname{ran} T_{\frac{\det \Theta}{\gcd(a,b)}}$ is automatically closed.

This shows that the multi-valued part of R is never trivial and therefore, R cannot be an operator for any choice of coordinate functions. □

By a symmetric argument we easily get the next

COROLLARY 4.1.18. *Let R and Θ be as above. Then R never has trivial kernel. In fact,*

$$\ker R = \begin{cases} T_a \mathcal{H}^2(\mathbb{D}) & \text{for} \quad c = 0, d \neq 0 \\ T_b \mathcal{H}^2(\mathbb{D}) & \text{for} \quad c \neq 0, d = 0 \\ T_{\frac{\det \Theta}{\gcd(c,d)}} \mathcal{H}^2(\mathbb{D}) & \text{for} \quad c, d \neq 0 \end{cases}$$

and each of these three spaces is a proper superset of $\{0\}$.

Furthermore, with the help of Equation (4.5) the next result is immediate. We remind ourselves of the fact that there can only be one zero function per column and row due to Proposition 4.1.15.

COROLLARY 4.1.19. *Let R and Θ be as above. Then:*

(i) If $a = d \equiv 0$ then $R = \overline{(\{0\} \times T_b \mathcal{H}^2(\mathbb{D})) \dot{\boxplus} (T_c \mathcal{H}^2(\mathbb{D}) \times \{0\})}$.

(ii) If $b = c \equiv 0$ then $R = \overline{(T_a \mathcal{H}^2(\mathbb{D}) \times \{0\}) \dot{\boxplus} (\{0\} \times T_d \mathcal{H}^2(\mathbb{D}))}$.

Some easy operations that we can perform on linear relations are again reflected in the structure of the inner function Θ.

PROPOSITION 4.1.20. *Let R and Θ be as above. Then:*

(i) The inverse R^{-1} can be represented as $R^{-1} = T_{\Theta_\updownarrow} \mathcal{H}^2(\mathbb{D})$.

(ii) For $\alpha \in \mathbb{T}$ we have $\alpha R = T_{\Theta[\alpha]} \mathcal{H}^2(\mathbb{D})$.

Thereby we set

$$\Theta_\updownarrow = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \qquad \text{and} \qquad \Theta[\alpha] = \begin{pmatrix} a & b \\ \alpha c & \alpha d \end{pmatrix}.$$

Proof. Both R^{-1} and αR are S_2-stable if R has this property due to to Lemma 4.1.5 *(iii)* and *(iv)*. Furthermore, the Dimensional Condition, Proposition 4.1.7, is invariant with respect to forming R^{-1} or αR and $\Theta_\updownarrow$ and $\Theta[\alpha]$ are both inner. As in the case where $n = 1$ we can make use of the mappings Φ_{inv} and Φ_α from Lemma 1.3.7 to calculate

$$\begin{aligned} R^{-1} &= \Phi_{inv}(R) \\ &= \Phi_{inv}\left(\left\{[a \cdot f_1 + b \cdot f_2, c \cdot f_1 + d \cdot f_2] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D})\right\}\right) \end{aligned}$$

$$= \left\{ [c \cdot f_1 + d \cdot f_2, a \cdot f_1 + b \cdot f_2] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D}) \right\}$$
$$= T_{\Theta_\updownarrow} \mathcal{H}^2(\mathbb{D})$$

and

$$\begin{aligned} \alpha R &= \Phi_\alpha(R) \\ &= \Phi_\alpha \left(\left\{ [a \cdot f_1 + b \cdot f_2, c \cdot f_1 + d \cdot f_2] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D}) \right\} \right) \\ &= \left\{ [a \cdot f_1 + b \cdot f_2, \alpha c \cdot f_1 + \alpha d \cdot f_2] \in \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D}) \;\middle|\; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}^2(\mathbb{D}) \right\} \\ &= T_{\Theta[\alpha]} \mathcal{H}^2(\mathbb{D}). \end{aligned}$$

□

4.2 Some Examples

Example 4.2.1. We will look at some simple linear relations and determine how they can represented in the spirit of Theorem 4.1.6.

1. The linear relation $R = \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ is obviously equal to $\mathcal{H}^2(\mathbb{D}; \mathbb{C}^2)$ and thus S_2-stable. It can therefore be represented using the function $\Theta \equiv I$, where I is the (2×2)-identity matrix, i.e.
$$R = T_\Theta \mathcal{H}^2(\mathbb{D}; \mathbb{C}^2).$$
It goes without saying that Θ is inner. It should be noted that this choice is not unique since for any $\alpha \in \mathbb{T}$ we could also use $\Theta \equiv \alpha I$ to represent R.

2. Consider the linear relation $R = \{[0, 0]\} \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$. Setting
$$\Theta_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \Theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
we can clearly write R in two different ways, namely
$$R = T_{\Theta_1} \mathcal{H}^2(\mathbb{D}) \qquad \text{and} \qquad R = T_{\Theta_2} \mathcal{H}^2(\mathbb{D}; \mathbb{C}^2).$$
Although R is S_2-stable, neither of these representations arise from Theorem 4.1.6. The reason for this is that Beurling's Theorem 3.5.1 excludes trivial subspaces and, by extension, so does Theorem 4.1.6. Additionally, Θ_1 and Θ_2 both lack the property inner.

3. Both the spaces $R_1 = \mathcal{H}^2(\mathbb{D}) \times \{0\}$ and $R_2 = \{0\} \times \mathcal{H}^2(\mathbb{D})$ satisfy the requirements of Theorem 4.1.6. It is immediately clear that
$$\Theta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \Theta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are both inner and fulfil

$$R_1 = T_{\Theta_1}\mathcal{H}^2(\mathbb{D}) \qquad \text{and} \qquad R_2 = T_{\Theta_2}\mathcal{H}^2(\mathbb{D}).$$

As in the first case, using $\alpha\Theta_i$ for $\alpha \in \mathbb{T}$ yields the same linear relations R_i for $i \in \{1,2\}$.

The above examples are straightforward since the linear relations in questions are rather easy to describe. The situation gets a more interesting, albeit more difficult, if the linear relations arise from the graph of an operator.

Example 4.2.2. Let us first discuss two special classes of operators. As usual, the choices we make for inner functions are unique up to multiplication by a constant $\beta \in \mathbb{T}$.

1. The operator

$$T_\alpha : \begin{cases} \mathcal{H}^2(\mathbb{D}) & \to & \mathcal{H}^2(\mathbb{D}) \\ f & \mapsto & \alpha f \end{cases}$$

i.e. multiplication by a scalar $\alpha \neq 0$, clearly satisfies $T_\alpha = \alpha I$ as a linear relation, where $I \leq \mathcal{H}^2(\mathbb{D}) \times \mathcal{H}^2(\mathbb{D})$ is the identity relation. Take note that $\alpha = 0$ describes the linear relation R_1 in Example 4.2.1.3. It is easy to see that T_α is S_2-stable and thus we have $T_\alpha = T_{\Theta[\alpha]}\mathcal{H}^2(\mathbb{D})$. The function $\Theta[\alpha]$ can be chosen as

$$\Theta[\alpha] = \frac{1}{\sqrt{1+|\alpha|^2}}\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

and therefore is clearly inner.

2. The shift operator S is trivially S_2-stable. By induction, the same is then true for S^k as k runs over $\mathbb{N}$. The representation $S^k = T_{\Theta[k]}\mathcal{H}^2(\mathbb{D})$ can then be achieved through

$$\Theta[k] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ z^k \end{pmatrix}$$

for $k \in \mathbb{N}$.

Finally, let us examine how Theorems 2.1.14 and 4.1.6 interact.

Example 4.2.3. Suppose that $T \in \mathcal{B}(\mathcal{H}^2(\mathbb{D}))$ commutes with the shift operator S. On the one hand, there must exist a function $h \in H^\infty(\mathbb{D})$ such that $T = T_h$ as we have shown at the very beginning of this work in Theorem 2.1.14.
On the other hand, by identifying T with its graph and considering it as a linear relation, it is closed, according to Lemma 2.1.11, and S_2-stable, due to Corollary 4.1.4. Hence, Theorem 4.1.6 is applicable, so there exists $\Theta \in H^\infty_{2\times 1}(\mathbb{D})$ such that $T = T_\Theta\mathcal{H}^2(\mathbb{D})$.

In the light of the previous examples it seems natural to suspect that up to multiplication by a unimodular constant we will arrive at

$$\Theta = \frac{1}{\kappa}\begin{pmatrix}1\\h\end{pmatrix}.$$

The difficulty now lies in the fact of finding a suitable function κ such that $\frac{1}{\kappa}$ and $\frac{h}{\kappa}$ both belong to $H^\infty(\mathbb{D})$ and such that on $\mathbb{T}$ we have

$$\Big|\frac{1}{\widetilde{\kappa}(\zeta)}\Big|^2 + \Big|\frac{\widetilde{h}(\zeta)}{\widetilde{\kappa}(\zeta)}\Big|^2 = 1$$

almost everywhere. Unless h is the scalar multiple of an inner function, which implies that $|\widetilde{h}| \equiv c \in \mathbb{C}$ on $\mathbb{T}$ and enables $\kappa := \sqrt{1+|c|^2}$, we cannot make any more refined statements about the particular appearance of Θ. In fact, we would need other tools from classical Hardy space theory to advance this further.

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