TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

The Liouville Transformation in Sturm-Liouville Theory

Ausgeführt am Institut für

Analysis und Scientific Computing
der Technischen Universität Wien

unter der Anleitung von

Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Harald Woracek

durch

Matthäus Kleindeßner, B.Sc.
Matr. Nr. 0625345

Märzstraße 122
1150 Wien

Wien, am 30. August 2012

Contents

Introduction

This thesis deals with Sturm-Liouville Theory, the vast area of mathematical research based on the examination of second-order linear ordinary differential equations of the kind

$$-(py')' + qy = \lambda ry \quad \text{on } (a,b) \subseteq \mathbb{R} \tag{0.1}$$

and named after J. Sturm[1] and J. Liouville[2]. Their work in the 1830s was the beginning of a systematic approach to equations of the form (0.1) and the question of finding the values of the (complex-valued) spectral parameter λ for which there exists a non-trivial solution satisfying certain boundary conditions (Sturm-Liouville problem).
Today Sturm-Liouville Theory plays an important part in applied mathematics and especially in mathematical physics, where equations of the form (0.1) (called Sturm-Liouville equations) occur very often, in particular after separation of the variables in linear partial differential equations. We just cite the wave equation or the Schrödinger equation as examples, additional ones can be found in [Wei87].

In 1837 Liouville introduced a transformation (today known as Liouville transformation) which enables reducing the general Sturm-Liouville equation (0.1) to the more special and simpler one

$$-y'' + \tilde{q}y = \lambda y \quad \text{on } (\tilde{a},\tilde{b}) \subseteq \mathbb{R}. \tag{0.2}$$

It is the main task of this thesis to systematically discuss Liouville´s transformation in a rather general way, i.e. we will not only consider the Liouville transformation which transforms (0.1) into (0.2), but consider related transformations (which we also call a Liouville transformation) transforming (0.1) into another equation of this kind (i.e. p, q, r and (a,b) replaced by $\tilde{p}, \tilde{q}, \tilde{r}$ and $(\tilde{a},\tilde{b})$). Liouville´s original transformation is then obtained as a special case.
In this thesis we examine invariance properties of the equation (0.1) under a Liouville transformation and deal with the question to which extent a Liouville transformation can be used to transform a given equation into a simpler one. We will see that a Liouville transformation gives rise to a unitary mapping between Hilbert spaces and that certain operators associated with Sturm-Liouville equations are unitarily equivalent via this mapping. A main result of this thesis is an inverse theorem stating sufficient conditions for the existence of a Liouville transformation such that two considered operators are unitarily equivalent via it (considered as a mapping between the associated Hilbert spaces).

It should be mentioned that - clearly - depending on the underlying situation one can expect the functions p, q an r to satisfy different conditions. Many attempts have been

[1]Jacques Sturm (1803-1855)
[2]Joseph Liouville (1809-1882)

made to keep necessary conditions for mathematical treatment as general as possible and going so far that the functions p, q and r can be replaced by abstract Borel measures (see, e.g., [ET] and the references therein). On the other hand, Liouville´s original transformation requires considerable restrictions on the coefficients p, q and r for its feasibility. In this thesis we consider the classical right-definite case of Sturm-Liouville Theory (see Section 1.1 for a definition) and then tempt to keep additional restrictions for working with the concept of a Liouville transformation as lean as possible.

The contents of this thesis are as follows: After giving some notations and commenting some basic concepts in the next section, we give an introduction into Sturm-Liouville Theory in Chapter 1. This includes Sturm-Liouville differential expressions and their realizations in Hilbert spaces, endpoint classification and a description of selfadjoint realizations via boundary conditions. We cite some results needed later on, in particular concerning spectral theory of selfadjoint realizations of Sturm-Liouville differential expressions (Section 1.5). The following Chapter 2 is rather short. There we establish a connection between the generalized Fourier transform of Section 1.5 and an increasing chain of de Branges' spaces, which is of course interesting on its own, but for us mainly serves as a preparation for the proof of the inverse result in the following chapter. However, since the assertions there can be shown under the weak assumptions of Chapter 1 (other than the inverse result itself), this is an own chapter.
In Chapter 3 we introduce the concept of a Liouville transformation and describe its basic properties. We see that certain operators introduced in Chapter 1 are unitarily equivalent via a Liouville transformation and formulate and prove the above mentioned inverse Theorem 3.2.1. Chapter 4 is then concerned with the task of transforming Sturm-Liouville equations by a Liouville transformation.
Finally, we collect some important assertions concerning absolutely continuous functions which we essentially use in Chapter 3 and give a brief review of de Branges' theory of Hilbert spaces of entire functions in the appendix.

Acknowledgements. At this point I want to thank my thesis adviser Harald Woracek for his constant, helpful and kind support as well as Jonathan Eckhardt for some helpful hints.

Notations and Basic Concepts

The real and complex fields are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively; $\mathbb{R}^+$ ($\mathbb{R}^-$) denotes the positive (negative) reals without zero, $\mathbb{C}^+$ ($\mathbb{C}^-$) denotes the open upper (lower) complex half plane. $\mathbb{N}$ denotes the natural numbers without zero, $\mathbb{N}_0$ the natural numbers with zero included. An open interval $I \subseteq \mathbb{R}$ is denoted by (a, b), with $-\infty \le a < b \le \infty$; $[a, b]$ denotes the closed interval $I \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ which includes the left endpoint a and the right endpoint b, regardless of whether these are finite or infinite. There is a corresponding notation for half open intervals.

For $z \in \mathbb{C}$ its conjugate complex number is denoted by $\overline{z}$. For any complex-valued function f we denote the pointwise conjugate function by $\overline{f}$. Whenever we write $\sqrt{z}$ for some $z \in \mathbb{C}$ we mean the principal branch of the complex root (i.e. if $z = re^{i\phi}$ with $r \ge 0$ and $\phi \in (-\pi, \pi]$, then $\sqrt{z} = r^{\frac{1}{2}} e^{i\frac{\phi}{2}}$), thus for $z \in \mathbb{R}^+ \cup \{0\}$ this coincides with the common definition of the square root.

Let $U \subseteq \mathbb{C}$ be open. By $\mathcal{H}(U)$ we denote the class of complex-valued functions defined and holomorphic on U.

By $\mathcal{M}((a, b); \mathbb{C})$ we denote the set of complex-valued Borel measurable functions defined on (a, b).

For $I = (a, b)$ and $f \in \mathcal{M}((a, b); \mathbb{C})$ the Lebesgue integral of f - if it exists - is written in one of the following equivalent forms

$$\int_I f d\lambda = \int_I f(x) d\lambda(x) = \int_I f(x) dx = \int_a^b f(x) dx = \int_a^b f dx.$$

More general, if μ is a Borel measure on I, we write

$$\int_I f d\mu \quad \text{or} \quad \int_I f(x) d\mu(x)$$

for the integral of f with respect to the measure μ - again, this assumes that the integral exists. For $J = (c, d) \subseteq (a, b) = I$ and $f \in \mathcal{M}((a, b); \mathbb{C})$ we write - as customary -

$$\int_c^d f(x) dx \quad \text{instead of} \quad \int_c^d f\big|_{(c,d)}(x) dx,$$

similarly for the other representations.

For $p \in [1, \infty)$ and a Borel measure on $I = (a, b)$ by $L^p(I; \mu)$ we denote the Banach space (of equivalence classes) of functions $f \in \mathcal{M}((a, b); \mathbb{C})$ whose absolute value raised to the p-th power has finite integral, i.e.

$$\int_I |f|^p d\mu < \infty.$$

We write $\|\cdot\|_{L^p(I;\mu)}$ for the norm in $L^p(I;\mu)$. The inner product of the Hilbert space $L^2(I;\mu)$ is denoted by $(\cdot,\cdot)_{L^2(I;\mu)}$. Recall that

$$\|f\|_{L^p(I;\mu)} = \left(\int_I |f|^p d\mu\right)^{\frac{1}{p}} \quad \text{and} \quad (f,g)_{L^2(I;\mu)} = \int_I f\overline{g}d\mu.$$

A weight function w on I is a Borel measurable function $w : I \to \mathbb{R}$ satisfying $w(x) > 0$ for almost all $x \in I$. We additionally assume that w is integrable on all compact subintervals $[c,d] \subseteq I$ (i.e. $w \in L^1_{loc}(I)$, see below). If μ is the Borel measure $w\lambda$, where $w\lambda$ is given by

$$w\lambda(B) = \int_B w d\lambda,$$

we write $L^p(I;w)$ instead of $L^p(I;w\lambda)$ and if $w \equiv 1$, i.e. $\mu = \lambda$, we only write $L^p(I)$ or[3] $L^p(a,b)$.

By $L^p_{loc}(I;\mu)$ we denote the vector space of functions $f \in \mathcal{M}((a,b);\mathbb{C})$ which satisfy $f|_J \in L^p(J;\mu|_J)$ for all compact sub-intervals $J \subseteq I$; again, if $\mu = w\lambda$ or $\mu = \lambda$ we write $L^p_{loc}(I;w)$ and $L^p_{loc}(I)$, respectively.

$L^\infty(a,b)$ denotes the Banach space (of equivalence classes) of functions $f \in \mathcal{M}((a,b);\mathbb{C})$ which are essentially bounded with respect to the Lebesgue measure, i.e. bounded up to a set of Lebesgue measure zero.

Let $(a,b) \subseteq \mathbb{R}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. By $C^k(a,b)$ ($C^k((a,b);\mathbb{C})$) we denote the class of real-valued (complex-valued) functions which are k-times continuously differentiable on (a,b). In particular $C^0(a,b)$ ($C^0((a,b);\mathbb{C})$) denotes the class of continuous functions on (a,b), $C^\infty(a,b)$ ($C^\infty((a,b);\mathbb{C})$) denotes the class of smooth functions on (a,b). $C^k_0(a,b)$ ($C^k_0((a,b);\mathbb{C})$) denotes the collection of functions $f \in C^k(a,b)$ ($f \in C^k((a,b);\mathbb{C})$) with compact support in (a,b).
If a,b are finite, i.e. $[a,b] \subseteq \mathbb{R}$ is a compact interval, by $C^k[a,b]$ ($C^k([a,b];\mathbb{C})$) we mean the class of functions k-times continuously differentiable on $[a,b]$, where continuity and differentiability at an endpoint have to be understood as one-sided continuity and differentiability, respectively.

Let $[a,b] \subseteq \mathbb{R}$ be a compact interval. Then $AC[a,b]$ ($Lip[a,b]$) denotes the vector space of all complex-valued and absolutely continuous (Lipschitz continuous) functions on $[a,b]$. However, in this thesis more often we encounter the vector space of locally absolutely continuous functions $AC_{loc}(a,b)$ for some interval $(a,b) \subseteq \mathbb{R}$, i.e. the collection of functions f which satisfy $f|_{[c,d]} \in AC[c,d]$ for all compact subintervals $[c,d] \subseteq (a,b)$. If $f \in AC_{loc}(a,b)$ then, of course, $f \in C^0((a,b);\mathbb{C})$. Furthermore, it has a unique derivative almost everywhere, which we denote by f'. We have $f' \in L^1_{loc}(a,b)$ and, for $c \in (a,b)$,

$$f(x) = f(c) + \int_c^x f(t)dt, \quad x \in (a,b).$$

[3] We should mention that in general: If F is any space of functions defined on an open interval $I = (a,b)$, we write $F(a,b)$ instead of $F((a,b))$ (which is $F(I)$).

As for the L^p spaces, in any normed vector space X the norm is denoted by $\|\cdot\|_X$, however, in a general Hilbert space H (which is not a L^2 space) the inner product is denoted by $\langle\cdot,\cdot\rangle_H$.

Let T be a linear operator. By $\mathcal{D}(T)$ we denote the domain of T, by $\ker T$ its kernel and by $\operatorname{ran} T$ its range. If T is a densely defined operator acting on a Hilbert space, by T^* we denote its adjoint. For a closed operator T acting on a Banach space X we write $\sigma(T)$ to denote the spectrum of T.

If S is a set of vectors in a vector space, we write $\operatorname{span}(S)$ in order to denote the linear span of S. If S is finite, $S = \{v_1, \ldots, v_n\}$, we also write $\operatorname{span}(v_1, \ldots, v_n)$.

Let μ be a measure on $(\mathbb{R}, \mathfrak{B}(\mathcal{E}))$, where $\mathcal{E}$ denotes the Euclidean topology and $\mathfrak{B}(\mathcal{E})$ the associated Borel σ-algebra on $\mathbb{R}$. By $\mathfrak{U}(x)$ we denote the neighborhood filter of $x \in \mathbb{R}$ (equipped with $\mathcal{E}$). Then we define the support of μ as the set

$$\operatorname{supp}\mu = \{x \in \mathbb{R} : \forall U \in \mathfrak{U}(x) \cap \mathfrak{B}(\mathcal{E}) \Rightarrow \mu(U) > 0\}.$$

One can show that $\operatorname{supp}\mu$ is always a closed subset of $\mathbb{R}$. If μ is a Borel measure, i.e. $\mu(K) < \infty$ for all compact $K \subseteq \mathbb{R}$, we have, since $(\mathbb{R}, \mathcal{E})$ is a locally compact Hausdorff space satisfying the second axiom of countability, $\mu(\mathbb{R} \setminus \operatorname{supp}\mu) = 0$.

Let $f \in \mathcal{M}((a,b);\mathbb{C})$. By $\mathfrak{U}(x)$ we denote the neighborhood filter of $x \in (a,b)$ (equipped with the induced topology from $(\mathbb{R}, \mathcal{E})$). We define the support of f as the set

$$\operatorname{supp} f = \{x \in (a,b) : \forall U \in \mathfrak{U}(x) \cap \mathfrak{B}(\mathcal{E}) \Rightarrow \lambda(\{x \in U : f(x) \neq 0\}) > 0\}.$$

One can show that $\operatorname{supp} f$ is always closed in (a,b). Since (a,b) is a locally compact Hausdorff space satisfying the second axiom of countability, we have $f = 0$ in $(a,b) \setminus \operatorname{supp} f$ almost everywhere. Clearly, if $f = g$ a.e., then $\operatorname{supp} f = \operatorname{supp} g$ and hence the definition makes sense for $f \in L^2((a,b);w)$. If f is continuous on (a,b), this definition coincides with the classical one,

$$\operatorname{supp} f = \overline{\{x \in (a,b) : f(x) \neq 0\}},$$

where the closure is taken in (a,b). In general, $\operatorname{supp} f$ is nothing else than $(a,b) \cap \operatorname{supp}\mu_f$, where u_f is the measure given by

$$\mu_f(A) = \int\limits_{A\cap(a,b)} |f| d\lambda, \quad A \in \mathfrak{B}(\mathcal{E}).$$

Chapter 1

Preliminaries

This chapter is an introduction into the basic notions and concepts of Sturm-Liouville Theory and collects some results needed later on. We mainly cite [Zet05], [Tes09] and [Wei03] as the sources of the results given in the first four sections; however, actually these sections are based on [Eck09b], where most of the proofs omitted here can be found too. Section 1.5 is based on [BE05].

1.1 Sturm-Liouville Differential Expressions and Equations

Let $(a,b) \subseteq \mathbb{R}$ be any open interval, finite or infinite, and (p,q,r) be a tuple of functions defined on (a,b) and satisfying

$$\begin{cases} (i) & p,q,r : (a,b) \to \mathbb{R} \\ (ii) & p^{-1}, q, r \in L^1_{loc}(a,b) \\ (iii) & r(x) > 0 \text{ for almost all } x \in (a,b). \end{cases} \tag{1.1.1}$$

We then call (p,q,r) a tuple of Sturm-Liouville coefficients (SL coefficients).

We define the associated Sturm-Liouville differential expression (SL differential expression) $\tau = \tau(p,q,r)$ by

$$\mathcal{D}(\tau) = \{u \in AC_{loc}(a,b) : pu' \in AC_{loc}(a,b)\},$$
$$\tau u = \frac{1}{r}\big[-(pu')' + qu\big], \quad u \in \mathcal{D}(\tau).$$

Note that $\mathcal{D}(\tau)$ is the maximal function space such that this differential expression is well-defined on $\mathcal{D}(\tau)$. Furthermore, note that $r\tau u \in L^1_{loc}(a,b)$ for all $u \in \mathcal{D}(\tau)$.

We will consider equations of the kind

$$(\tau - \lambda)u = \tau u - \lambda u = f, \tag{1.1.2}$$

where f is a complex-valued function on (a,b) and $\lambda \in \mathbb{C}$. By a solution of (1.1.2) we mean a function $u \in \mathcal{D}(\tau)$ fulfilling (1.1.2) a.e. on (a,b). Since $r > 0$ a.e., this is equivalent to

$$-(pu')'(x) + (q(x) - \lambda r(x))\, u(x) = r(x) f(x) \quad \text{for almost all } x \in (a,b).$$

In particular, for $f \equiv 0$ we obtain the equation

$$(\tau - \lambda)u = 0 \quad \text{or} \quad \tau u = \lambda u,$$

which is equivalent to

$$-(pu')' + qu = \lambda r u \quad \text{on } (a, b). \tag{1.1.3}$$

We call equation (1.1.3) the associated[1] Sturm-Liouville equation (SL equation) with spectral parameter $\lambda \in \mathbb{C}$.

Let us say some words on the conditions (1.1.1). The second condition guarantees that, for each $\lambda \in \mathbb{C}$, the equation (1.1.3) together with some initial conditions has a unique solution - see Theorem 1.1.1 and Corollary 1.1.2, respectively. In fact, this condition is necessary for Theorem 1.1.1 or Corollary 1.1.2 to hold (see [Zet05], Theorem 2.2.2). The third condition is crucial since it allows applying Hilbert space theory in the Hilbert space $L^2((a, b); r)$ to study Sturm-Liouville problems, i.e. the question of finding the values of λ for which there exists a non-trivial solution of (1.1.3) satisfying certain boundary conditions. Now, due to the first condition, the linear operators appearing there are self-adjoint (see below).
Hence, conditions (1.1.1) are usually considered as the standard minimal conditions on (p, q, r), leading to standard Sturm-Liouville Theory of so called right-definite problems. In this thesis we do not consider the "non-standard" case where r is allowed to change sign, leading to so called left-definite problems (see [Zet05], Chapter 5 and Chapter 12, for an introduction) and always assume a tuple of SL coefficients to satisfy (1.1.1). In Chapter 3 and Chapter 4 we will tighten these conditions.

Obviously, τ is a linear differential expression. Since its coefficients are real-valued, it is a real differential expression, by which we mean that

$$u \in \mathcal{D}(\tau) \Rightarrow \overline{u} \in \mathcal{D}(\tau) \quad \text{and then} \quad \tau\overline{u} = \overline{\tau u}.$$

In general, the first derivative u' of a function $u \in \mathcal{D}(\tau)$ only satisfies $u' \in L^1_{loc}(a, b)$, whereas pu' satisfies $pu' \in AC_{loc}(a, b)$ - we refer to pu' as the first quasi-derivative of u. Note that therefore equation (1.1.3) cannot be written as

$$-pu'' - p'u' + qu = \lambda r u.$$

Nevertheless, we want to speak of (1.1.3) as a second-order linear ordinary differential equation.

Concerning existence and uniqueness of solutions of (1.1.2) and (1.1.3) we have the following results.

THEOREM 1.1.1 ([Zet05], Theorem 2.2.1 together with Theorem 2.5.2, [Tes09], Theorem 9.1, [Wei03], Corollary 13.3). *Let $f : (a, b) \to \mathbb{C}$ such that $rf \in L^1_{loc}(a, b)$. Then for arbitrary $\lambda \in \mathbb{C}$, $c \in (a, b)$ and $d_1, d_2 \in \mathbb{C}$ the initial value problem*

$$(\tau - \lambda)u = f \quad \textit{together with} \quad u(c) = d_1, (pu')(c) = d_2$$

[1] associated with (p, q, r)

has a unique solution $u(\cdot,\lambda)$*. For each* $x \in (a,b)$ *we have that* $\lambda \mapsto u(x,\lambda)$ *and* $\lambda \mapsto (pu')(x,\lambda)$ *are entire functions, i.e.* $u(x,\cdot),(pu')(x,\cdot) \in \mathcal{H}(\mathbb{C})$*. Moreover, if* f *is real-valued and* $d_1, d_2 \in \mathbb{R}$*, then* $u(x,\lambda) \in \mathbb{R}$ *for all* $x \in (a,b)$ *and* $\lambda \in \mathbb{R}$*.*

Note that in Theorem 1.1.1 the assumption $rf \in L^1_{loc}(a,b)$ is necessary since we have $r(\tau-\lambda)u \in L^1_{loc}(a,b)$ for all $u \in \mathcal{D}(\tau)$.

COROLLARY 1.1.2. *For arbitrary* $\lambda \in \mathbb{C}$*,* $c \in (a,b)$ *and* $d_1, d_2 \in \mathbb{C}$ *there is a unique solution* $u(\cdot,\lambda)$ *of the initial value problem*

$$-(pu')' + qu = \lambda r u \quad \textit{together with} \quad u(c) = d_1, (pu')(c) = d_2.$$

For each $x \in (a,b)$ *we have* $u(x,\cdot),(pu')(x,\cdot) \in \mathcal{H}(\mathbb{C})$*. If* $d_1, d_2 \in \mathbb{R}$*, then* $u(x,\lambda) \in \mathbb{R}$ *for all* $x \in (a,b)$ *and* $\lambda \in \mathbb{R}$*.*

COROLLARY 1.1.3. *For each* $\lambda \in \mathbb{C}$ *the solution space of* $(\tau-\lambda)u = 0$ *is two-dimensional.*

Suppose that in addition to (1.1.1) we have $p^{-1}|_{(a,c)}, q|_{(a,c)}, r|_{(a,c)} \in L^1(a,c)$ for some (and hence for all) $c \in (a,b)$. Then we say that τ is regular at a or that a is a regular endpoint (for τ). Otherwise, we say that τ is singular at a or that a is a singular endpoint (for τ). Analogously we define regular/singular for the endpoint b and say that τ is regular if it is regular at a and regular at b. The SL differential expression τ is said to be singular whenever it is not regular.

REMARK 1.1.4. *Note that this classification of an endpoint as a regular or singular endpoint does not depend on whether it is a finite or infinite endpoint - unlike in much of the literature (see [Zet05], Remark 2.3.1).*

Solutions and their first quasi-derivatives can be continuously extended to a regular endpoint.

PROPOSITION 1.1.5 ([Zet05], Theorem 2.3.1). *Let* a *be a regular endpoint and* $f : (a,b) \to \mathbb{C}$ *such that* $(rf)|_{(a,c)} \in L^1(a,c)$ *for some* $c \in (a,b)$*. Then for any solution* $u(\cdot,\lambda)$ *of* (1.1.2) *the limits*

$$u(a,\lambda) := \lim_{x\to a^+} u(x,\lambda) \quad \textit{and} \quad (pu')(a,\lambda) := \lim_{x\to a^+} (pu')(x,\lambda) \tag{1.1.4}$$

both exist and are finite. Corresponding assertions hold for the endpoint b*.*

Sometimes it is quite useful to consider τ only on a subinterval of (a,b). For $\alpha,\beta \in [a,b]$, $\alpha < \beta$, by $\tau|_{(\alpha,\beta)}$ we mean the SL differential expression which is associated with $(p|_{(\alpha,\beta)}, q|_{(\alpha,\beta)}, r|_{(\alpha,\beta)})$. If $\alpha,\beta \in (a,b)$, then, of course, $\tau|_{(\alpha,b)}$ is regular at α, $\tau|_{(a,\beta)}$ is regular at β and $\tau|_{(\alpha,\beta)}$ is regular.

For $u, v \in \mathcal{D}(\tau)$ we define the (modified) Wronskian $W(u,v) : (a,b) \to \mathbb{C}$ by

$$W(u,v)(x) = \begin{vmatrix} u(x) & v(x) \\ (pu')(x) & (pv')(x) \end{vmatrix} = u(x)(pv')(x) - (pu')(x)v(x).$$

The following properties are easy to prove.

PROPOSITION 1.1.6.

- *Let $u, v \in \mathcal{D}(\tau)$. Then $W(u,v) \in AC_{loc}(a,b)$ and*
$$W(u,v)' = r(v\tau u - u\tau v). \tag{1.1.5}$$
- *$W(\cdot,\cdot)$ is linear in both arguments.*
- *$W(\cdot,\cdot)$ is skew-symmetric, i.e. $W(u,v) = -W(v,u)$ for $u, v \in \mathcal{D}(\tau)$.*
- *$W(\overline{u}, \overline{v}) = \overline{W(u,v)}$ for all $u, v \in \mathcal{D}(\tau)$.*
- *Let $\lambda \in \mathbb{C}$. If both u and v are solutions of $(\tau - \lambda)u = 0$, then $W(u,v)$ is constant and one has*
$$W(u,v) \equiv 0 \Leftrightarrow u \text{ and } v \text{ are linearly dependent.}$$

Let $\lambda \in \mathbb{C}$ and u, v be two linearly independent solutions of $(\tau - \lambda)u = 0$. We then call $\{u, v\}$ a fundamental system of the equation $(\tau - \lambda)u = 0$. By Corollary 1.1.3 any fundamental system of $(\tau - \lambda)u = 0$ is a basis of the solution space of $(\tau - \lambda)u = 0$. From Theorem 1.1.1 and Proposition 1.1.6 we have the following corollary.

COROLLARY 1.1.7. *For arbitrary $\lambda \in \mathbb{C}$ and $c \in (a,b)$ there exists a fundamental system $\{u, v\}$ of the equation $(\tau - \lambda)u = 0$ satisfying $W(u,v) \equiv 1$. If $\lambda \in \mathbb{R}$, then this fundamental system can be chosen to be real, i.e. u and v are real-valued.*

Fundamental systems can be used to gain a representation of solutions of the inhomogeneous equation $(\tau - \lambda)u = f$. The following proposition is the key for computing resolvents of selfadjoint realizations of τ.

PROPOSITION 1.1.8 ([Tes09], Lemma 9.2, [Wei03], Section 13.1). *Let $\lambda \in \mathbb{C}$ and $\{u, v\}$ be a fundamental system of $(\tau - \lambda)u = 0$. Let $c \in (a,b)$, $d_1, d_2 \in \mathbb{C}$ and $f : (a,b) \to \mathbb{C}$ such that $rf \in L^1_{loc}(a,b)$. Then there exist $c_1, c_2 \in \mathbb{C}$ such that the solution y of*

$$(\tau - \lambda)u = f \quad \text{together with} \quad u(c) = d_1, (pu')(c) = d_2$$

is given by

$$y(x) = c_1 u(x) + c_2 v(x) + \int_c^x \frac{u(x)v(t) - v(x)u(t)}{W(u,v)(c)} f(t)r(t)dt.$$

Finally, we want to show that the initial value problem at a regular endpoint can be solved.

Theorem 1.1.9. *Assume that a is a regular endpoint and let $f : (a,b) \to \mathbb{C}$ such that $rf \in L^1(a,c)$ for all $c \in (a,b)$. Then for arbitrary $\lambda, d_1, d_2 \in \mathbb{C}$ the initial value problem*

$$(\tau - \lambda)u = f \quad \text{together with} \quad u(a) = d_1, (pu')(a) = d_2,$$

where $u(a)$ and $(pu')(a)$ have to be understood as in (1.1.4)*, has a unique solution $u(\cdot, \lambda)$. For each $x \in [a,b)$ we have that $\lambda \mapsto u(x,\lambda)$ and $\lambda \mapsto (pu')(x,\lambda)$ are entire functions, i.e. $u(x,\cdot), (pu')(x,\cdot) \in \mathcal{H}(\mathbb{C})$. Again, if f is real-valued and $d_1, d_2 \in \mathbb{R}$, then $u(x,\lambda) \in \mathbb{R}$ for all $x \in [a,b)$ and $\lambda \in \mathbb{R}$. Corresponding assertions hold for the endpoint b.*

Proof. Let, according to Corollary 1.1.7, $\{v,w\}$ be a fundamental system of $(\tau - \lambda)u = 0$ with $W(v,w) \equiv 1$ and which is real if $\lambda \in \mathbb{R}$. By Proposition 1.1.5 the limits $v(a)$, $(pv')(a)$, $w(a)$ and $(pv')(a)$ all exist and are finite, where the $\mathbb{C}^2$-vectors $(v(a), (pv')(a))$ and $(w(a), (pw')(a))$ are linearly independent since otherwise we had

$$0 = v(a)(pw')(a) - (pv')(a)w(a) = \lim_{x \to a^+} W(v,w)(x) = 1.$$

Thus, given any solution $\hat{u}$ of $(\tau - \lambda)u = f$ (which we assume to be real-valued if $\lambda \in \mathbb{R}$ and f is real-valued), a solution u satisfying the initial conditions is given by $u = c_1 v + c_2 w + \hat{u}$ for certain $c_1, c_2 \in \mathbb{C}$. In order to see that u is unique, assume that u_1, u_2 are two solutions of $(\tau - \lambda)u = f$ satisfying the same initial conditions at a. Then $u_1 - u_2$ is a solution of the homogenous equation $(\tau - \lambda)u = 0$ satisfying $(u_1 - u_2)(a) = (pu_1' - pu_2')(a) = 0$. By Proposition 1.1.6 we infer that $u_1 - u_2$ and any other solution of $(\tau - \lambda)u = 0$ are linearly dependent and hence that $u_1 - u_2 \equiv 0$. The other assertions are clear (compare Theorem 1.1.1). □

Corollary 1.1.10. *Assume that a is a regular endpoint. For arbitrary $\lambda, d_1, d_2 \in \mathbb{C}$ there is a unique solution $u(\cdot, \lambda)$ of the initial value problem*

$$-(pu')' + qu = \lambda r u \quad \text{together with} \quad u(a) = d_1, (pu')(a) = d_2.$$

For each $x \in [a,b)$ we have $u(x,\cdot), (pu')(x,\cdot) \in \mathcal{H}(\mathbb{C})$. If $d_1, d_2 \in \mathbb{R}$, then $u(x,\lambda) \in \mathbb{R}$ for all $x \in [a,b)$ and $\lambda \in \mathbb{R}$. Corresponding assertions hold for the endpoint b.

Remark 1.1.11. *It is easy to see that all assertions of Corollary 1.1.10 hold true if the initial values $d_1, d_2 \in \mathbb{C}$ are replaced by entire functions with argument λ, i.e. one considers the initial value problem $-(pu')' + qu = \lambda r u$ together with $u(a,\lambda) = d_1(\lambda)$ and $(pu')(a,\lambda) = d_2(\lambda)$ for $d_1, d_2 \in \mathcal{H}(\mathbb{C})$.*

1.2 Operator Theory

We want to consider SL differential expressions as operators in appropriate Hilbert spaces. To this end let (p,q,r) be a tuple of SL coefficients defined on $(a,b) \subseteq \mathbb{R}$. The Hilbert space $L^2((a,b);r)$ turns out to be appropriate.

We define the associated[2] maximal operator
$T_{max} = T_{max}(p,q,r) : \mathcal{D}(T_{max}) \subseteq L^2((a,b);r) \to L^2((a,b);r)$ by

$$\begin{aligned}\mathcal{D}(T_{max}) &= \{u \in \mathcal{D}(\tau(p,q,r)) : u, \tau u \in L^2((a,b);r)\},\\ T_{max}u &= \tau(p,q,r)u, \quad u \in \mathcal{D}(T_{max}),\end{aligned}$$

and the associated preminimal operator $T_0 = T_0(p,q,r) : \mathcal{D}(T_0) \subseteq L^2((a,b);r) \to L^2((a,b);r)$ by

$$\begin{aligned}\mathcal{D}(T_0) &= \{u \in \mathcal{D}(T_{max}(p,q,r)) : \operatorname{supp} u \text{ is compact in } (a,b)\},\\ T_0 u &= \tau(p,q,r)u, \quad u \in \mathcal{D}(T_0).\end{aligned}$$

Clearly, we have

$$u \in \mathcal{D}(T_{max}) \Rightarrow \overline{u} \in \mathcal{D}(T_{max}) \quad \text{and then} \quad T_{max}\overline{u} = \overline{T_{max}u}.$$

A corresponding assertion holds for T_0.

We want to use the following convenient notation: A function $u \in \mathcal{M}((a,b);\mathbb{C})$ is said to lie in $L^2((a,b);r)$ at a if for every $c \in (a,b)$ the restriction of u to (a,c), $u|_{(a,c)}$, is in $L^2((a,c);r|_{(a,c)})$. A function $u \in \mathcal{D}(\tau(p,q,r))$ is said to lie in $\mathcal{D}(T_{max})$ at a if u and $\tau(p,q,r)u$ lie in $L^2((a,b);r)$ at a. There is a corresponding notation for the endpoint b. Clearly, we have $u \in L^2((a,b);r)$ if and only if u lies in $L^2((a,b);r)$ at a and at b and $u \in \mathcal{D}(T_{max})$ if and only if u lies in $\mathcal{D}(T_{max})$ at a and at b.

Lemma 1.2.1.

- *For any $u, v \in \mathcal{D}(\tau(p,q,r))$ and $\alpha, \beta \in (a,b)$, $\alpha < \beta$, we have the Lagrange identity*

$$\int_\alpha^\beta (\overline{v}\tau u - u\overline{\tau v})\, r dx = W(u,\overline{v})(\beta) - W(u,\overline{v})(\alpha). \tag{1.2.1}$$

- *If u and v lie in $\mathcal{D}(T_{max})$ at a, then the limit*

$$W(u,\overline{v})(a) := \lim_{\alpha \to a^+} W(u,\overline{v})(\alpha)$$

exists and is finite. A corresponding assertion holds for the endpoint b.

- *If $u, v \in \mathcal{D}(T_{max})$, then*

$$(T_{max}u, v)_{L^2((a,b);r)} - (u, T_{max}v)_{L^2((a,b);r)} = W(u,\overline{v})(b) - W(u,\overline{v})(a).$$

Proof.

- This is due to (1.1.5).

[2] associated with (p,q,r) or with $\tau(p,q,r)$

- If u and v lie in $\mathcal{D}(T_{max})$ at a, then in (1.2.1) the limit for $\alpha \to a^+$ exists and is finite. Hence, the limit of the assertion exists and is finite.
- This can be concluded from the previous points.

□

Now we may define the associated minimal operator $T_{min} = T_{min}(p,q,r) : \mathcal{D}(T_{min}) \subseteq L^2((a,b);r) \to L^2((a,b);r)$ by

$$\mathcal{D}(T_{min}) = \{u \in \mathcal{D}(T_{max}(p,q,r)) : W(u,v)(a) = W(u,v)(b) = 0,\ v \in \mathcal{D}(T_{max}(p,q,r))\}$$

and

$$T_{min}u = \tau(p,q,r)u \quad u \in \mathcal{D}(T_{min}).$$

We have the following theorem linking T_0, T_{min} and T_{max}.

THEOREM 1.2.2 ([Tes09], Lemma 9.4). *The preminimal operator T_0 is densely defined, its adjoint is T_{max} and its closure is given by T_{min}.*

Note that it is by no means clear that T_0 is densely defined (which implies that T_{min} and T_{max} are densely defined) since in general we do not have $C_0^\infty((a,b);\mathbb{C}) \subseteq \mathcal{D}(T_0)$ (compare Section 4.2).
Also note that, since T_{min} is the closure of T_0, we have

$$u \in \mathcal{D}(T_{min}) \Rightarrow \overline{u} \in \mathcal{D}(T_{min}) \quad \text{and then} \quad T_{min}\overline{u} = \overline{T_{min}u}.$$

We want to summarize:

COROLLARY 1.2.3.

- *The preminimal operator T_0 is densely defined and symmetric.*
- *The minimal operator T_{min} is densely defined, closed and symmetric.*
- *The maximal operator T_{max} is densely defined and closed.*
- *We have*

$$T_0 \subseteq T_{min} \subseteq T_{max}, \quad T_{min} = \overline{T_0}, \quad T_0^* = T_{min}^* = T_{max}, \quad T_{max}^* = T_{min}.$$

The following theorem yields the existence of selfadjoint extensions of T_{min}.

THEOREM 1.2.4 ([Zet05], Section 10.4, [Wei03], Theorem 13.10). *The minimal operator T_{min} has equal deficiency indices*[3] $n := n_+ = n_-$ *with* $0 \leq n \leq 2$.

Clearly, any selfadjoint extension S of T_{min} satisfies $T_{min} \subseteq S \subseteq T_{max}$.

[3] $(n_+, n_-) = (\dim\ker(T_{min}^* + \mathrm{i}), \dim\ker(T_{min}^* - \mathrm{i})) = (\dim\ker(T_{max} + \mathrm{i}), \dim\ker(T_{max} - \mathrm{i}))$

Later on we will need the following assertions.

Lemma 1.2.5 ([Zet05], Lemma 10.4.3). *Assume that a is a regular endpoint. Then, for any $f \in \mathcal{D}(T_{max})$, the limits*

$$f(a) := \lim_{x \to a^+} f(x) \quad \text{and} \quad (pf')(a) := \lim_{x \to a^+} (pf')(x) \tag{1.2.2}$$

both exist and are finite. A corresponding assertion holds for the endpoint b.

Remark 1.2.6.

- *The assertion of Lemma 1.2.5 holds true if instead of $f \in \mathcal{D}(T_{max})$ we only assume that f lies in $\mathcal{D}(T_{max})$ at a.*
- *Lemma 1.2.5 implies that for $f, g \in \mathcal{D}(T_{max})$ (for f and g which lie in $\mathcal{D}(T_{max})$ at a) the limit $W(f,g)(a)$ is given by $f(a)(pg')(a) - (pf')(a)g(a)$.*
- *If, in addition, a is a finite endpoint, we even have[4] $f, pf' \in AC[a,c]$ for every $c \in (a,b)$: Since $f \in \mathcal{D}(T_{max})$ (f lies in $\mathcal{D}(T_{max})$ at a), we have $\tau f|_{(a,c)} \in L^2((a,c); r|_{(a,c)})$. The measure $r\lambda$ is finite on (a,c), thus we also have $\tau f|_{(a,c)} \in L^1((a,c); r|_{(a,c)})$. However, this just means $(-(pf')' + qf)|_{(a,c)} \in L^1(a,c)$. We have $q|_{(a,c)} \in L^1(a,c)$ and $|f(x)| < C$, $x \in (a,c)$, for some $C \in \mathbb{R}^+$, implying that $(pf')'|_{(a,c)} \in L^1(a,c)$ and hence $pf' \in AC[a,c]$. We also have $p^{-1}|_{(a,c)} \in L^1(a,c)$ and $|(pf')(x)| < C$, $x \in (a,c)$, for some $C \in \mathbb{R}^+$, implying that $f'|_{(a,c)} = p^{-1}pf'|_{(a,c)} \in L^1(a,c)$ and hence $f \in AC[a,c]$.*
  Clearly, if a and b are both regular and finite, these arguments can be adapted to yield $f, pf' \in AC[a,b]$.

Corresponding assertions hold for the endpoint b.

Lemma 1.2.7 ([Zet05], Lemma 10.4.4). *Assume that a and b are both regular endpoints, i.e. τ is regular. Then for any d_1, d_2, e_1, e_2 there exists a function $g \in \mathcal{D}(T_{max})$ such that*

$$g(a) = d_1, \quad (pg')(a) = d_2, \quad g(b) = e_1, \quad (pg')(b) = e_2,$$

whereas this has to be understood as in (1.2.2). Furthermore, for $f \in \mathcal{D}(T_{max})$ we have $f \in \mathcal{D}(T_{min})$ if and only if $f(a) = (pf')(a) = f(b) = (pf')(b) = 0$.

Lemma 1.2.8. *Let $f_a \in \mathcal{D}(\tau)$ be lying in $\mathcal{D}(T_{max})$ at a and $f_b \in \mathcal{D}(\tau)$ be lying in $\mathcal{D}(T_{max})$ at b. Then there exists $f \in \mathcal{D}(T_{max})$ equaling f_a near a and f_b near b.*

Proof. Let $\alpha, \beta \in (a,b)$, $\alpha < \beta$, such that

$$f_a\big|_{(a,\alpha)}, \tau f_a\big|_{(a,\alpha)} \in L^2((a,\alpha); r|_{(a,\alpha)})$$

[4] To be precise: With this we mean that f and pf', respectively, restricted to $(a,c]$ and extended to $[a,c]$ by (1.2.2) is in $AC[a,c]$.

and

$$f_b\big|_{(\beta,b)}, \tau f_b\big|_{(\beta,b)} \in L^2((\beta,b); r|_{(\beta,b)}).$$

According to Lemma 1.2.8, there exists a function $g \in \mathcal{D}(T_{max}(p|_{(\alpha,\beta)}, q|_{(\alpha,\beta)}, r|_{(\alpha,\beta)}))$ satisfying[5]

$$g(\alpha) = f_a(\alpha), \quad (pg')(\alpha) = (pf_a')(\alpha), \quad g(\beta) = f_b(\beta), \quad (pg')(\beta) = (pf_b')(\beta).$$

Clearly, $g, \tau|_{(\alpha,\beta)} g \in L^2((\alpha,\beta); r|_{(\alpha,\beta)})$ and by Remark 1.2.6 we have $g, pg' \in AC[\alpha,\beta]$. Define $f : (a,b) \to \mathbb{C}$ by

$$f(x) = \begin{cases} f_a(x), & x \le \alpha, \\ g(x), & \alpha < x < \beta, \\ f_b(x), & \beta \le x. \end{cases}$$

Then it is easy to check that $f, pf' \in AC_{loc}(a,b)$ and $f, \tau f \in L^2((a,b); r)$ meaning that $f \in \mathcal{D}(T_{max})$. □

1.3 Endpoint Classification

Theorem 1.3.1 (Weyl's alternative) ([Zet05], Theorem 7.2.2, [Wei03], Theorem 13.18). *For any SL differential expression $\tau = \tau(p,q,r)$ defined on $(a,b) \subseteq \mathbb{R}$ it holds that either*

(i) for every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda)u = 0$ lie in $L^2((a,b); r)$ at a

or

(ii) for every $\lambda \in \mathbb{C}$ there exists at least one solution of $(\tau - \lambda)u = 0$ which does not lie in $L^2((a,b); r)$ at a.

A corresponding assertion holds for the endpoint b.

If the case *(i)* is accurate in Theorem 1.3.1, we say that a is in the limit-circle case or that a is limit-circle (for τ). Otherwise, i.e. if the case *(ii)* is accurate, we say that a is in the limit-point case or that a is limit-point (for τ). There is a corresponding notation for the endpoint b.

Proposition 1.3.2. *If a is a regular endpoint for τ, then a is limit-circle for τ. A corresponding assertion holds for the endpoint b.*

Proof. If a is a regular endpoint for τ, $\lambda \in \mathbb{C}$ and u a solution of $(\tau - \lambda)u = 0$, then by Proposition 1.1.5 the limit $u(x)$ as $x \to a^+$ exists and is finite. In particular, for $c \in (a,b)$ we have

$$|u(x)| < C, \quad x \in (a,c),$$

[5] pg' should rather be written as $p|_{(\alpha,\beta)} g'$.

for some constant $C \in \mathbb{R}^+$ (depending on c) and hence

$$\int_a^c |u|^2 r dx < C^2 \|r|_{(a,c)}\|_{L^1(a,c)} < \infty.$$

□

PROPOSITION 1.3.3 ([Zet05], Lemma 10.4.1). *The endpoint a is limit-point for τ if and only if $W(u,v)(a) = 0$ for all $u, v \in \mathcal{D}(T_{max}(p,q,r))$. Hence, the endpoint a is limit-circle if and only if there exists $u \in \mathcal{D}(T_{max}(p,q,r))$ such that $W(u,v)(a) \neq 0$ for some $v \in \mathcal{D}(T_{max}(p,q,r))$. A corresponding assertion holds for the endpoint b.*

Note that if $W(u,v)(a) \neq 0$, then $W(\operatorname{Re} u, v)(a) \neq 0$ or $W(\operatorname{Im} u, v)(a) \neq 0$. Clearly, we have $\operatorname{Re} u, \operatorname{Im} u \in \mathcal{D}(T_{max})$ and $W(\operatorname{Re} u, \overline{\operatorname{Re} u}) = W(\operatorname{Re} u, \operatorname{Re} u) = 0$ and $W(\operatorname{Im} u, \overline{\operatorname{Im} u}) = W(\operatorname{Im} u, \operatorname{Im} u) = 0$. Hence, we can state Proposition 1.3.3 as follows:

COROLLARY 1.3.4. *The endpoint a is limit-circle for τ if and only if there exists $u \in \mathcal{D}(T_{max}(p,q,r))$ such that $W(u,\overline{u})(a) = 0$ and $W(u,v)(a) \neq 0$ for some $v \in \mathcal{D}(T_{max}(p,q,r))$. A corresponding assertion holds for the endpoint b.*

1.4 Selfadjoint Realizations

We have already seen in Section 1.2 that there exist selfadjoint extensions of T_{min} or, equivalently, selfadjoint restrictions of T_{max}. These are precisely the n-dimensional, symmetric extensions of T_{min} or the n-dimensional, symmetric restrictions of T_{max} if T_{min} has equal deficiency indices (n,n) (compare Theorem 1.2.4 and see [Wei00], Theorem 10.10). We want to call such a selfadjoint extension and restriction, respectively, a selfadjoint realization (of $\tau = \tau(p,q,r)$).

We need the following notation: We say that $v \in \mathcal{D}(T_{max})$ satisfies (1.4.1) if

$$W(v,\overline{v})(a) = 0 \quad \text{and} \quad W(h,\overline{v})(a) \neq 0 \ \text{ for some } h \in \mathcal{D}(T_{max}). \tag{1.4.1}$$

Correspondingly, we say that $v \in \mathcal{D}(T_{max})$ satisfies (1.4.2) if

$$W(v,\overline{v})(b) = 0 \quad \text{and} \quad W(h,\overline{v})(b) \neq 0 \ \text{ for some } h \in \mathcal{D}(T_{max}). \tag{1.4.2}$$

By Corollary 1.3.4 we see that the endpoint a is in the limit-circle case for τ if and only if there exists $v \in \mathcal{D}(T_{max})$ satisfying (1.4.1) - similarly for the endpoint b.

PROPOSITION 1.4.1.

- *For all $f_1, f_2, f_3, f_4 \in \mathcal{D}(\tau)$ we have the Plücker identity*

$$W(f_1,f_2)W(f_3,f_4) + W(f_1,f_3)W(f_4,f_2) + W(f_1,f_4)W(f_2,f_3) \equiv 0 \tag{1.4.3}$$

on (a,b).

- *Let $v \in \mathcal{D}(T_{max})$ be satisfying (1.4.1). For f lying in $\mathcal{D}(T_{max})$ at a we have*

$$W(f,\overline{v})(a) = 0 \quad \Leftrightarrow \quad W(\overline{f},\overline{v})(a) = 0 \tag{1.4.4}$$

and for f and g lying in $\mathcal{D}(T_{max})$ at a we have

$$W(f,\overline{v})(a) = W(g,\overline{v})(a) = 0 \quad \Rightarrow \quad W(f,g)(a) = 0. \tag{1.4.5}$$

Corresponding assertions hold for the endpoint b.

PROOF.

- A calculation shows that the left side of (1.4.3) equals the determinant

$$\frac{1}{2}\begin{vmatrix} f_1 & f_2 & f_3 & f_4 \\ pf_1' & pf_2' & pf_3' & pf_4' \\ f_1 & f_2 & f_3 & f_4 \\ pf_1' & pf_2' & pf_3' & pf_4' \end{vmatrix},$$

which is obviously vanishing.

- Let $v \in \mathcal{D}(T_{max})$ be satisfying (1.4.1). Then there exists $h \in \mathcal{D}(T_{max})$ such that $W(h,\overline{v})(a) \neq 0$. Choosing $f_1 = v$, $f_2 = \overline{v}$, $f_3 = h$ and $f_4 = \overline{h}$ in (1.4.3), one sees that also $W(h,v)(a) \neq 0$. Now choosing $f_1 = f$, $f_2 = v$, $f_3 = \overline{v}$, $f_4 = h$ in (1.4.3) yields (1.4.4), choosing $f_1 = f$, $f_2 = g$, $f_3 = \overline{v}$, $f_4 = h$ yields (1.4.5).

□

The next theorem gives a characterization of all selfadjoint realizations depending on endpoint classification.

THEOREM 1.4.2 ([Zet05], Theorem 10.4.1, [Wei03], Theorem 13.19 together with Theorem 13.20).

- *Neither endpoint is limit-circle, i.e. both endpoints are limit-point: This is the case if and only if $n = 0$. The minimal operator $T_{min} = T_{max}$ itself is self-adjoint and hence the only selfadjoint realization of τ.*

- *One endpoint is limit-circle and the other one limit-point: This is the case if and only if $n = 1$.*

  Let a be limit-circle and b be limit-point. An operator S is a selfadjoint realization of τ if and only if there exists $v \in \mathcal{D}(T_{max})$ satisfying (1.4.1) such that

$$\begin{aligned} \mathcal{D}(S) &= \{f \in \mathcal{D}(T_{max}) : W(f,\overline{v})(a) = 0\}, \\ Sf &= \tau f, \quad f \in \mathcal{D}(S). \end{aligned}$$

  Let a be limit-point and b be limit-circle. An operator S is a selfadjoint realization of τ if and only if there exists $v \in \mathcal{D}(T_{max})$ satisfying (1.4.2) such that

$$\begin{aligned} \mathcal{D}(S) &= \{f \in \mathcal{D}(T_{max}) : W(f,\overline{v})(b) = 0\}, \\ Sf &= \tau f, \quad f \in \mathcal{D}(S). \end{aligned}$$

- *Both endpoints are limit-circle: This is the case if and only if $n = 2$.*

  An operator S is a selfadjoint realization of τ if and only if there exist $v, w \in \mathcal{D}(T_{max})$ satisfying

  (i) v and w are linearly independent modulo $\mathcal{D}(T_{min})$, i.e. no nontrivial linear combination of v and w is in $\mathcal{D}(T_{min})$

  and[6]

  (ii) $W_a^b(v,\overline{v}) = W_a^b(w,\overline{w}) = W_a^b(v,\overline{w}) = 0$

  such that

$$\mathcal{D}(S) = \left\{ f \in \mathcal{D}(T_{max}) : W_a^b(f,\overline{v}) = W_a^b(f,\overline{w}) = 0 \right\}, \qquad (1.4.6)$$
$$Sf = \tau f, \quad f \in \mathcal{D}(S).$$

In this thesis we mainly consider selfadjoint realizations with separated boundary conditions. With these we mean all selfadjoint realizations of τ if precisely one endpoint is limit-circle or, if both endpoints are limit-circle, those selfadjoint realizations S whose domain is given by

$$\mathcal{D}(S) = \{ f \in \mathcal{D}(T_{max}) : W(f,\overline{v})(a) = W(f,\overline{w})(b) = 0 \} \qquad (1.4.7)$$

for some $v \in \mathcal{D}(T_{max})$ satisfying (1.4.1) and some $w \in \mathcal{D}(T_{max})$ satisfying (1.4.2). In order to check that each operator S with domain (1.4.7) and acting as τ is indeed selfadjoint, one may simply show that (1.4.7) can be written in the form of (1.4.6) (or see [Wei03], Theorem 13.21): According to Lemma 1.2.8, there exist v_0 equaling v near a and the zero function near b and w_0 equaling the zero function near a and w near b. Since v and w satisfy (1.4.1) and (1.4.2), respectively, it is easy to prove that v_0 and w_0 are linearly independent modulo $\mathcal{D}(T_{min})$ and that these functions satisfy $W_a^b(v_0,\overline{v_0}) = W_a^b(w_0,\overline{w_0}) = W_a^b(v_0,\overline{w_0}) = 0$. Clearly, we have

$$\{ f \in \mathcal{D}(T_{max}) : W(f,\overline{v})(a) = W(f,\overline{w})(b) = 0 \} = \left\{ f \in \mathcal{D}(T_{max}) : W_a^b(f,\overline{v_0}) = W_a^b(f,\overline{w_0}) = 0 \right\}.$$

By Proposition 1.4.1 one sees that a selfadjoint realization S with separated boundary conditions has the property

$$f \in \mathcal{D}(S) \Rightarrow \overline{f} \in \mathcal{D}(S) \quad \text{and then} \quad S\overline{f} = \overline{Sf}.$$

This does not hold true for all selfadjoint realisations[7].

We want to describe boundary conditions of the form $W(f,\overline{v})(a) = 0$ and $W(f,\overline{w})(b) = 0$, respectively, where $v, w \in \mathcal{D}(T_{max})$ satisfy (1.4.1) or (1.4.2), also in an alternative way. This is the content of Proposition 1.4.3 and Corollary 1.4.5.

[6] We use $W_a^b(\cdot,\cdot)$ as an abbreviation for $W(\cdot,\cdot)(b) - W(\cdot,\cdot)(a)$.

[7] Selfadjoint realizations of τ (where both endpoints are limit-circle) whose domain (1.4.6) cannot be written in the form of (1.4.7) are called selfadjoint realizations with coupled boundary conditions.

Proposition 1.4.3. *Assume that the endpoint a is in the limit-circle case for τ and let $\lambda \in \mathbb{R}$ and $\{\gamma, \delta\}$ be a real fundamental system of $(\tau - \lambda)u = 0$ satisfying $W(\gamma, \delta) \equiv 1$.*

- *Let $v \in \mathcal{D}(T_{max})$ be satisfying* (1.4.1)*. Then there exists a unique $\alpha \in [0, \pi)$ such that we have*[8]

$$W(f, \overline{v})(a) = 0 \Leftrightarrow W(f, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = 0, \quad f \in \mathcal{D}(T_{max}). \tag{1.4.8}$$

- *Conversely, given a linear combination $\cos\alpha\ \gamma + \sin\alpha\ \delta$ for some $\alpha \in [0, \pi)$, there exists $v \in \mathcal{D}(T_{max})$ (which is surely not unique) satisfying* (1.4.1) *and such that we have* (1.4.8)*.*

Corresponding assertions hold for the endpoint b.

Proof.

- Let $h \in \mathcal{D}(T_{max})$ such that $W(h, \overline{v})(a) \neq 0$. Then the Plücker identity (1.4.3)

$$W(h, \overline{v}) \underbrace{W(\gamma, \delta)}_{\equiv 1} + W(h, \gamma)W(\delta, \overline{v}) + W(h, \delta)W(\overline{v}, \gamma) \equiv 0$$

yields

$$0 \neq W(h, \overline{v})(a) = W(h, \gamma)(a)W(\overline{v}, \delta)(a) - W(h, \delta)(a)W(\overline{v}, \gamma)(a), \tag{1.4.9}$$

implying that

$$\big(W(\overline{v}, \delta)(a), W(\overline{v}, \gamma)(a)\big) = \big(\overline{W(v, \delta)(a)}, \overline{W(v, \gamma)(a)}\big) \neq (0, 0).$$

The Plücker identity also yields (replace h by v in the right equation of (1.4.9))

$$\begin{aligned} 0 = W(v, \overline{v})(a) &= W(v, \gamma)(a)W(\overline{v}, \delta)(a) - W(v, \delta)(a)W(\overline{v}, \gamma)(a) \\ &= W(v, \gamma)(a)W(\overline{v}, \delta)(a) - \overline{W(v, \gamma)(a)W(\overline{v}, \delta)(a)} \\ &= 2\mathrm{i}\,\mathrm{Im}(W(v, \gamma)(a)W(\overline{v}, \delta)(a)), \end{aligned}$$

implying that $W(v, \gamma)(a)\overline{W(v, \delta)(a)} \in \mathbb{R}$.

Hence, we have

$$D \begin{pmatrix} W(v, \gamma)(a) \\ W(v, \delta)(a) \end{pmatrix} \in \mathbb{R}^2 \backslash \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

where $D = \overline{W(v, \delta)(a)}$ if $W(v, \delta)(a) \neq 0$ and $D = \overline{W(v, \gamma)(a)}$ else, and there exist a unique $\alpha \in [0, \pi)$ and a unique $B \in \mathbb{R}\backslash\{0\}$ such that

$$D \begin{pmatrix} W(v, \gamma)(a) \\ W(v, \delta)(a) \end{pmatrix} = B \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}$$

[8] Note that the assumption of a being limit-circle implies that γ and δ (and hence also every linear combination) lie in $\mathcal{D}(T_{max})$ at a.

and

$$\begin{pmatrix} W(v,\gamma)(a) \\ W(v,\delta)(a) \end{pmatrix} = \frac{B}{D} \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}.$$

Now, for $f \in \mathcal{D}(T_{max})$ (replace h by f in the right equation of (1.4.9))

$$\begin{aligned} W(f,\overline{v})(a) &= W(f,\gamma)(a)W(\overline{v},\delta)(a) - W(f,\delta)(a)W(\overline{v},\gamma)(a) \\ &= W(f,\gamma)(a)\overline{W(v,\delta)(a)} - W(f,\delta)(a)\overline{W(v,\gamma)(a)} \\ &= -W(f,\gamma)(a)\frac{B}{\overline{D}}\cos\alpha - W(f,\delta)(a)\frac{B}{\overline{D}}\sin\alpha \end{aligned}$$

and hence (1.4.8).

It remains to show that α is unique in (1.4.8). Suppose that there is $\tilde{\alpha} \in [0,\pi)$ such that, for $f \in \mathcal{D}(T_{max})$,

$$W(f, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = 0 \Leftrightarrow W(f, \cos\tilde{\alpha}\ \gamma + \sin\tilde{\alpha}\ \delta)(a) = 0.$$

By Lemma 1.2.8 there exists $f_0 \in \mathcal{D}(T_{max})$ equaling $\cos\alpha\ \gamma + \sin\alpha\ \delta$ near a. Clearly,

$$W(f_0, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = W(\cos\alpha\ \gamma + \sin\alpha\ \delta, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = 0$$

and thus ($W(\gamma,\delta) \equiv 1$, $W(\gamma,\gamma) \equiv W(\delta,\delta) \equiv 0$)

$$\begin{aligned} 0 &= W(f_0, \cos\tilde{\alpha}\ \gamma + \sin\tilde{\alpha}\ \delta)(a) = W(\cos\alpha\ \gamma + \sin\alpha\ \delta, \cos\tilde{\alpha}\ \gamma + \sin\tilde{\alpha}\ \delta)(a) \\ &= \cos\alpha\sin\tilde{\alpha} - \sin\alpha\cos\tilde{\alpha}. \end{aligned}$$

We conclude that

$$\begin{pmatrix} \cos\tilde{\alpha} \\ \sin\tilde{\alpha} \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

and hence $\tilde{\alpha} = \alpha$.

- Choose $v \in \mathcal{D}(T_{max})$ equaling $\cos\alpha\ \gamma + \sin\alpha\ \delta$ near a, which is possible due to Lemma 1.2.8. Then, clearly, we have (1.4.8) and $W(v,\overline{v})(a) = 0$. Furthermore, for $h_1 \in \mathcal{D}(T_{max})$ equaling γ near a we have $W(h_1,\overline{v})(a) = \sin\alpha$ and for $h_2 \in \mathcal{D}(T_{max})$ equaling δ near a we have $W(h_2,\overline{v})(a) = -\cos\alpha$, thus there certainly exists $h \in \mathcal{D}(T_{max})$ with $W(h,\overline{v})(a) \neq 0$, and v indeed satisfies (1.4.1).

□

PROPOSITION 1.4.4. *Assume that the endpoint a is regular. Then for any $\lambda \in \mathbb{R}$ there exists a real fundamental system $\{\gamma,\delta\}$ of $(\tau-\lambda)u = 0$ satisfying $W(\gamma,\delta) \equiv 1$ and*

$$W(f,\gamma)(a) = f(a), \quad W(f,\delta)(a) = (pf')(a)$$

for all f lying in $\mathcal{D}(T_{max})$ at a. A corresponding assertion holds for the endpoint b.

Proof. Let γ be the solution of $(\tau-\lambda)u = 0$ together with $u(a) = 0$ and $(pu')(a) = 1$ and δ be the solution of $(\tau - \lambda)u = 0$ together with $u(a) = -1$ and $(pu')(a) = 0$, see Corollary 1.1.10. Then $\{\gamma, \delta\}$ has the claimed properties. □

Corollary 1.4.5. *Assume that the endpoint a is regular.*

- *Let $v \in \mathcal{D}(T_{max})$ be satisfying* (1.4.1). *Then there exists a unique $\alpha \in [0, \pi)$ such that we have*

$$W(f, \overline{v})(a) = 0 \Leftrightarrow f(a)\cos\alpha + (pf')(a)\sin\alpha = 0, \quad f \in \mathcal{D}(T_{max}). \tag{1.4.10}$$

- *Conversely, for $\alpha \in [0, \pi)$ there exists $v \in \mathcal{D}(T_{max})$ (which is surely not unique) satisfying* (1.4.1) *and such that we have* (1.4.10).

Corresponding assertions hold for the endpoint b.

In particular, choosing $\alpha = 0$ in $f(a)\cos\alpha + (pf')(a)\sin\alpha = 0$ leads to the Dirichlet boundary condition $f(a) = 0$; choosing $\alpha = \pi/2$ leads to $(pf')(a) = 0$, which is called the Neumann boundary condition.

1.5 Spectral Theory

Throughout this section let $\tau = \tau(p, q, r)$ be a SL differential expression defined on $(a, b) \subseteq \mathbb{R}$.

Boundary conditions as described in the previous section lead to an extension of the initial value problem at a regular endpoint (see Corollary 1.1.10) to the case of an endpoint in the limit-circle case.

Theorem 1.5.1 ([BE05], Theorem 5.1, [ESSZ97], Theorem 2). *Assume that the endpoint a is in the limit-circle case for τ, let, for some $\lambda \in \mathbb{R}$, $\{\gamma, \delta\}$ be a real fundamental system of $(\tau - \lambda)u = 0$ satisfying $W(\gamma, \delta) \equiv 1$ and let $\eta, \xi \in \mathcal{H}(\mathbb{C})$. Then there exists a unique mapping $\psi : (a, b) \times \mathbb{C} \to \mathbb{C}$ with the properties*

- *$\psi(\cdot, \lambda)$ is a solution of $(\tau - \lambda)u = 0$ for every $\lambda \in \mathbb{C}$*
- *$W(\psi(\cdot, \lambda), \gamma)(a) = \eta(\lambda)$ and $W(\psi(\cdot, \lambda), \delta)(a) = \xi(\lambda)$ for all $\lambda \in \mathbb{C}$*
- *$\psi(x, \cdot), (p\psi')(x, \cdot) \in \mathcal{H}(\mathbb{C})$ for all $x \in (a, b)$.*

A corresponding assertion holds for the endpoint b.

Remark 1.5.2. *In the case of a regular endpoint a and a choice $\{\gamma, \delta\}$ as in Proposition 1.4.4, Theorem 1.5.1 reduces to Corollary 1.1.10 together with Remark 1.1.11.*

Proposition 1.5.3 ([BE05], Corollary 5.1). *If in the hypotheses of Theorem 1.5.1 the initial value functions η and ξ are replaced by real numbers, i.e. $\eta(\cdot) \equiv \eta \in \mathbb{R}$ and*

$\xi(\cdot) \equiv \xi \in \mathbb{R}$, *then*

$$\overline{\psi}(\cdot,\lambda) = \psi\big(\cdot,\overline{\lambda}\big) \quad \text{and} \quad \big(p\overline{\psi}'\big)(\cdot,\lambda) = (p\psi')\big(\cdot,\overline{\lambda}\big), \quad \lambda \in \mathbb{C}.$$

From now on always assume that the endpoint a is in the limit-circle case for τ and that S is a selfadjoint realization of τ with separated boundary conditions. Fix some real fundamental system $\{\gamma, \delta\}$ of $(\tau - \lambda)u = 0$ for some $\lambda \in \mathbb{R}$ satisfying $W(\gamma, \delta) \equiv 1$. By Theorem 1.4.2 and Proposition 1.4.3 there is a unique $\alpha \in [0, \pi)$ such that the domain of S is given by

$$\mathcal{D}(S) = \{f \in \mathcal{D}(T_{max}) : W(f, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = 0\} \tag{1.5.1}$$

if b is limit-point, or by

$$\mathcal{D}(S) = \{f \in \mathcal{D}(T_{max}) : W(f, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = W(f, \overline{w})(b) = 0\} \tag{1.5.2}$$

if b is in the limit-circle case too, where $w \in \mathcal{D}(T_{max})$ satisfies (1.4.2).

Let, according to Theorem 1.5.1, $\theta = \theta(\cdot,\cdot)$ and $\varphi = \varphi(\cdot,\cdot)$ be the solutions of $(\tau-\lambda)u = 0$, $\lambda \in \mathbb{C}$, satisfying

$$\begin{aligned} W(\theta(\cdot,\lambda),\gamma)(a) &= \cos\alpha, & W(\theta(\cdot,\lambda),\delta)(a) &= \sin\alpha, \\ W(\varphi(\cdot,\lambda),\gamma)(a) &= -\sin\alpha, & W(\varphi(\cdot,\lambda),\delta)(a) &= \cos\alpha. \end{aligned} \tag{1.5.3}$$

We refer to θ and φ as the basic solutions. Obviously, $\varphi(\cdot,\lambda)$ satisfies the boundary condition of S at a, i.e. $W(\varphi(\cdot,\lambda), \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = 0$, for every $\lambda \in \mathbb{C}$.

Proposition 1.5.4. *The basic solutions θ and φ have the following properties:*

- *$\theta(\cdot,\lambda)$ and $\varphi(\cdot,\lambda)$ lie in $\mathcal{D}(T_{max})$ at a for each $\lambda \in \mathbb{C}$.*
- *For all $x \in (a,b)$ and $\lambda \in \mathbb{C}$ we have*

$$\overline{\theta}(x,\lambda) = \theta\big(x,\overline{\lambda}\big) \quad \text{and} \quad \overline{\varphi}(x,\lambda) = \varphi\big(x,\overline{\lambda}\big). \tag{1.5.4}$$

- *For all $\lambda \in \mathbb{R}$ it holds that $\theta(\cdot,\lambda)$ and $\varphi(\cdot,\lambda)$ are real-valued.*
- *For all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ it holds that $\theta(\cdot,\lambda), (p\theta')(\cdot,\lambda), \varphi(\cdot,\lambda)$ and $(p\varphi')(\cdot,\lambda)$ do not have any zeros.*
- *For each $x \in (a,b)$ we have $\theta(x,\cdot), (p\theta')(x,\cdot), \varphi(x,\cdot), (p\varphi')(x,\cdot) \in \mathcal{H}(\mathbb{C})$.*
- *For all $\lambda \in \mathbb{C}$ we have*

$$W(\theta(\cdot,\lambda), \varphi(\cdot,\lambda)) \equiv 1. \tag{1.5.5}$$

- *For all $\lambda_1, \lambda_2 \in \mathbb{C}$ we have*

$$W(\varphi(\cdot,\lambda_1), \varphi(\cdot,\lambda_2))(a) = 0. \tag{1.5.6}$$

Proof.

- This is clear since a is assumed to be in the limit-circle case.
- This immediately follows from Proposition 1.5.3.
- This follows from Corollary 1.1.2 on examination of the proof of Theorem 1.5.1 given in [ESSZ97].
- Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $\varphi(\cdot,\lambda)$ or $(p\varphi')(\cdot,\lambda)$ had any zero, let us say $c \in (a,b)$, then $\varphi(\cdot,\lambda)|_{(a,c)}$ would be an eigenfunction (with eigenvalue λ) of the selfadjoint realization of $\tau|_{(a,c)}$ provided with the boundary condition of S at a and the Dirichlet or the Neumann boundary condition at c. However, the spectrum of any selfadjoint operator is real (see, e.g., [Wei00], Theorem 5.14).
  It may be seen similarly that $\theta(\cdot,\lambda)$ and $(p\theta')(\cdot,\lambda)$ do not have any zeros.
- This immediately follows from Theorem 1.5.1.
- This is (6.14) in [BE05], however, simply follows from the Plücker identity (1.4.3), Proposition 1.1.6 and the initial conditions (1.5.3).
- This is (6.15) in [BE05].

□

Theorem 1.5.5 ([BE05], Theorem 8.1 together with Remark 8.1). *There is a unique function $m : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ with the properties*

- *m is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, i.e. $m \in \mathcal{H}(\mathbb{C} \setminus \mathbb{R})$*
- $m(\overline{\lambda}) = \overline{m}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$
- *The solution $\psi = \psi(\cdot,\cdot)$ of $(\tau - \lambda)u = 0$ defined by*

$$\psi(\cdot,\lambda) = \theta(\cdot,\lambda) + m(\lambda)\varphi(\cdot,\lambda) \quad \textit{for } \lambda \in \mathbb{C} \setminus \mathbb{R}$$

  satisfies

$$\psi(\cdot,\lambda) \in L^2((a,b);r), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

  and in addition, if b is in the limit-circle case for τ too,

$$W(\psi(\cdot,\lambda),\overline{w})(b) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

  i.e. $\psi(\cdot,\lambda)$ satisfies the boundary condition of S at b for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

We refer to the function m as the Titchmarsh-Weyl m-function and to ψ as the Weyl solution (for S).

Remark 1.5.6. *The Titchmarsh-Weyl m-function is uniquely determined by the last property in Theorem 1.5.5 and hence so is the Weyl solution ψ (assuming that it is of the form $\theta(\cdot,\lambda) + m(\lambda)\varphi(\cdot,\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$): Assume that, for arbitrary $\lambda \in \mathbb{C} \setminus \mathbb{R}$, there is another solution $\theta(\cdot,\lambda) + \tilde{m}(\lambda)\varphi(\cdot,\lambda)$ with this property. Since*

$$W(\theta(\cdot,\lambda) + m(\lambda)\varphi(\cdot,\lambda), \theta(\cdot,\lambda) + \tilde{m}(\lambda)\varphi(\cdot,\lambda)) \equiv \tilde{m}(\lambda) - m(\lambda),$$

we see (compare Proposition 1.1.6) that $m(\lambda) \neq \tilde{m}(\lambda)$ *if and only if* $\theta(\cdot,\lambda)+m(\lambda)\varphi(\cdot,\lambda)$ *and* $\theta(\cdot,\lambda)+\tilde{m}(\lambda)\varphi(\cdot,\lambda)$ *are linearly independent. However, if b is in the limit-point case, due to Corollary 1.1.3 and Theorem 1.3.1, the two solutions have to be linearly dependent according to the definition of an endpoint in the limit-point case. If b is in the limit-circle case, the two solutions are linearly dependent by* (1.4.5) *(and Proposition 1.1.6).*

Proposition 1.5.7 ([BE05], Corollary 8.1). *We have*

$$0 < \int_a^b |\psi(x,\lambda)|^2 r(x)dx = \frac{\operatorname{Im} m(\lambda)}{\operatorname{Im} \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proposition 1.5.7 implies that m is a Nevanlinna function, with what we mean a function $m : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ satisfying[9]

$$\begin{cases} (i) & m \text{ is holomorphic} \\ (ii) & m\big|_{\mathbb{C}^+} : \mathbb{C}^+ \to \mathbb{C}^+ \quad \text{and} \quad m\big|_{\mathbb{C}^-} : \mathbb{C}^- \to \mathbb{C}^- \\ (iii) & m(\overline{\lambda}) = \overline{m}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

As a Nevanlinna function, m has an integral representation of the following form (compare [BE05], Section 10, or [Eck09a], Appendix A.1, where also several references to literature concerning Nevanlinna functions can be found), where $A, B \in \mathbb{R}$ with $B \geq 0$,

$$m(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\rho(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.5.7}$$

Herein ρ is a uniquely determined Borel measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\rho(t) < \infty. \tag{1.5.8}$$

We want to call ρ the spectral measure of the Titchmarsh-Weyl m-function or (due to the next theorem) the spectral measure for S.

Theorem 1.5.8 ([BE05], Theorem 11.1 together with Corollary 11.2, Lemma 17.1). *Let* $f \in L^2((a,b);r)$ *be a function with compact support* $[\alpha,\beta] \subseteq (a,b)$*. Define a function* F *by*

$$F(t) = \int_\alpha^\beta f(x)\varphi(x,t)r(x)dx, \quad t \in \mathbb{R}. \tag{1.5.9}$$

Then $F \in L^2(\mathbb{R};\rho)$ *with* $\|F\|_{L^2(\mathbb{R};\rho)} = \|f\|_{L^2((a,b);r)}$ *and the mapping* $U_0 : D \to L^2(\mathbb{R};\rho) : f \mapsto F$*, where* $D = \{f \in L^2((a,b);r) : f$ *has compact support in* $(a,b)\}$*, uniquely extends to a unitary mapping* $U : L^2((a,b);r) \to L^2(\mathbb{R};\rho)$.

[9] Often one defines a Nevanlinna function as a holomorphic function $m : \mathbb{C}^+ \to \mathbb{C}^+$. However, by setting $m(\lambda) = \overline{m}(\overline{\lambda})$ for $\lambda \in \mathbb{C}^-$ one obtains a Nevanlinna function according to our notation.

Let $G \in L^2(\mathbb{R};\rho)$ be a function with compact support $[\alpha', \beta'] \subseteq \mathbb{R}$. Define a function g by

$$g(x) = \int_{\alpha'}^{\beta'} G(t)\varphi(x,t)d\rho(t), \quad x \in (a,b).$$

Then $g \in L^2((a,b);r)$ with $\|g\|_{L^2((a,b);r)} = \|G\|_{L^2(\mathbb{R};\rho)}$ and the mapping $V_0 : D' \to L^2((a,b);r) : G \mapsto g$, where $D' = \{G \in L^2(\mathbb{R};\rho) : G$ has compact support in $\mathbb{R}\}$, uniquely extends to a unitary mapping $V : L^2(\mathbb{R};\rho) \to L^2((a,b);r)$.

The mapping V is the inverse of the mapping U (and vice versa) and the operator S is unitarily equivalent to the operator of multiplication[10] M_{id} in $L^2(\mathbb{R};\rho)$ via U, i.e. $U(L^2((a,b);r)) = \mathcal{D}(M_{\mathrm{id}})$ and

$$S = VM_{\mathrm{id}}U.$$

In particular, we have

$$\sigma(S) = \operatorname{supp}\rho. \tag{1.5.10}$$

We refer to the mapping U as the generalized Fourier transform and for $f \in L^2((a,b);r)$ to Uf as the generalized Fourier transform of f. The mapping V is referred to as the generalized inverse Fourier transform.

Remark 1.5.9.

- *Note that in the above construction the basic solutions θ and φ and consequently the Titchmarsh-Weyl m-function m, the Weyl solution ψ, the spectral measure ρ and the generalized Fourier transform U depend on the choice of the real fundamental system $\{\gamma, \delta\}$.*

- *One would like to have, once a SL differential expression τ with a limit-circle endpoint a and some selfadjoint realization S with separated boundary conditions are fixed, a unique construction, i.e. to be able to uniquely determine these objects. There seems to be no (canonical) way of generally adapting the construction in order to achieve this, however, we can at least adapt it in such a manner that ρ and U (which are the objects of ultimate interest) are determined up to a real constant:*

  Given τ and S, choose $\{\gamma, \delta\}$ as real fundamental system of $(\tau - \lambda_0)u = 0$ satisfying $W(\gamma, \delta) \equiv 1$, where $\lambda_0 \in \mathbb{R}$ is a priori fixed, e.g. always assume $\lambda_0 = 0$, and such that $\alpha = 0$ in (1.5.1) or (1.5.2). This is always possible[11] and it is straightforward to see that any other fundamental system $\{\tilde{\gamma}, \tilde{\delta}\}$ of $(\tau - \lambda_0)u = 0$ has the same properties if and only if it is of the form

  $$\tilde{\gamma} = e\gamma, \quad \tilde{\delta} = \frac{1}{e}\delta + f\gamma, \quad e, f \in \mathbb{R},\ e \neq 0.$$

[10] M_{id} denotes the operator of multiplication with the function id : $t \mapsto t$, i.e. $M_{\mathrm{id}} : D(M_{\mathrm{id}}) \to L^2(\mathbb{R};\rho) : G \mapsto (t \mapsto tG(t))$, where $D(M_{\mathrm{id}}) = \{G \in L^2(\mathbb{R};\rho) : (t \mapsto tG(t)) \in L^2(\mathbb{R};\rho)\}$.

[11] Given any real fundamental system $\{\gamma, \delta\}$ of $(\tau - \lambda_0)u = 0$ with $W(\gamma, \delta) \equiv 1$ and $\alpha \in [0, \pi)$ such that the boundary condition of S at a is given by $W(f, \cos\alpha\ \gamma + \sin\alpha\ \delta)(a) = 0$, the real fundamental

Now, if the construction starting from $\{\gamma,\delta\}$ *yields the basic solutions* θ *and* φ, *the Titchmarsh-Weyl* m*-function* m, *the Weyl solution* ψ, *the spectral measure* ρ *and the generalized Fourier transform* U, *it is easy to check that the corresponding objects obtained from the construction starting from* $\{\tilde\gamma,\tilde\delta\}$ *are given by*

$$\tilde\theta = \frac{1}{e}\theta - f\varphi, \quad \tilde\varphi = e\varphi, \quad \tilde m(\cdot) = \frac{1}{e^2}m(\cdot) + \frac{1}{e}f, \quad \tilde\rho = \frac{1}{e^2}\rho \quad \textit{and} \quad \tilde U = eU.$$

- *If we additionally assume the endpoint* a *to be regular, we are able to eliminate the dependence on the choice of the real fundamental system* $\{\gamma,\delta\}$ *and thus to uniquely determine* θ, φ, m, ψ, ρ *and* U*:*

  By Corollary 1.4.5 there is a unique $\alpha \in [0,\pi)$ *such that the boundary condition of* S *at* a *is given by*

$$f(a)\cos\alpha + (pf')(a)\sin\alpha = 0.$$

  Now we may uniquely determine the basic solutions θ *and* φ *by the initial conditions (compare Corollary 1.1.10)*

$$\begin{aligned} \theta(a,\lambda) &= \cos\alpha, & (p\theta')(a,\lambda) &= \sin\alpha, \\ \varphi(a,\lambda) &= -\sin\alpha, & (p\varphi')(a,\lambda) &= \cos\alpha, \end{aligned} \tag{1.5.11}$$

  which in fact means nothing else than choosing $\{\gamma,\delta\}$ *as in Proposition 1.4.4.*

  The case of a regular endpoint a *is the classical one for the above construction, which is most often dealt with in the literature. There the construction of* m, ψ, ρ *and* U *always starts from the basic solutions determined by* (1.5.11) *(possibly with some minor alterations concerning signs). Also in this thesis, whenever we encounter a SL differential expression which is regular at* a *and speak of the basic solutions or the Titchmarsh-Weyl* m*-function, the Weyl solution, etc. of a selfadjoint realization with separated boundary conditions, we mean the basic solutions determined by* (1.5.11) *or the objects constructed from these. However, note that in most of the literature the endpoint* a *is even assumed to be finite (compare Remark 1.1.4) - an assumption that we certainly do not need.*

Remark 1.5.10. *The only assumption on the SL differential expression* τ *for the feasibility of the above construction is that the endpoint* a *is in the limit-circle case. Unsurprisingly, the construction works analogously if the endpoint* b *is assumed to be in the limit-circle case for* τ *instead. However, in this case one has to make the ansatz*

$$\psi(\cdot,\lambda) = \theta(\cdot,\lambda) - m(\lambda)\varphi(\cdot,\lambda)$$

for the Weyl solution in order to obtain m *being a Nevanlinna function (compare [Tes09], Section 9.6, p. 211).*

---

system $\{\hat\gamma,\hat\delta\}$ given by

$$\{\hat\gamma,\hat\delta\} = \begin{cases} \{\cos\alpha\ \gamma + \sin\alpha\ \delta, \frac{1}{\cos\alpha}\ \delta\} & \text{if } \alpha \neq \frac{\pi}{2} \\ \{\delta, -\gamma\} & \text{if } \alpha = \frac{\pi}{2} \end{cases}$$

has the claimed properties.

Given a SL differential expression τ with a limit-circle endpoint a and some selfadjoint realization S with separated boundary conditions, the following assertions hold for the basic solutions, the Titchmarsh-Weyl m-function, etc. as constructed above, independently of the choice of real fundamental system $\{\gamma, \delta\}$ the construction has started from.

The following two lemmas will be crucial for Chapter 2.

LEMMA 1.5.11. *Let $c \in (a,b)$ and $f \in L^2((a,b);r)$ be a function vanishing outside of $(a,c]$ almost everywhere. Then the generalized Fourier transform Uf of f is given by*

$$Uf(t) = \int_a^c \varphi(x,t) f(x) r(x) dx, \quad t \in \mathbb{R}. \tag{1.5.12}$$

PROOF. First of all note that the integral is absolutely convergent for every $t \in \mathbb{R}$ since $f \in L^2((a,b);r)$ and $\varphi(\cdot,t)$ lies in $\mathcal{D}(T_{max})$ at a (even for all $t \in \mathbb{C}$). Now consider the sequence of functions $f_n = \mathbb{1}_{[\alpha_n,c]} f$, $n \in \mathbb{N}$, where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of reals satisfying $\alpha_n \in (a,b)$, $n \in \mathbb{N}$, and $\lim_{n\in\mathbb{N}} \alpha_n = a$. Then, of course, $\lim_{n\in\mathbb{N}} f_n = f$ in $L^2((a,b);r)$ and hence $\lim_{n\in\mathbb{N}} Uf_n = Uf$ in $L^2(\mathbb{R};\rho)$. However, since

$$Uf_n(t) = \int_{\alpha_n}^c \varphi(x,t) f(x) r(x) dx, \quad t \in \mathbb{R},$$

we see that, for every $t \in \mathbb{R}$, $Uf_n(t)$ converges to $\int_a^c \varphi(x,t) f(x) r(x) dx$ as $n \to \infty$ and infer that Uf is given by (1.5.12). □

LEMMA 1.5.12 ([BE05], Lemma 9.3). *Let $c \in (a,b)$ and $f \in L^2((a,b);r)$ be arbitrary. Then*

$$z \mapsto \int_a^c \varphi(x,z) f(x) r(x) dx, \quad z \in \mathbb{C},$$

is an entire function, i.e. in $\mathcal{H}(\mathbb{C})$. A corresponding assertion holds for the basic solution θ.

For the proof of the inverse Theorem 3.2.1 in Section 3.2 we will need the following lemma.

LEMMA 1.5.13 ([BE05], Lemma 18.1). *The generalized Fourier transform of the Weyl solution $\psi(\cdot,\lambda)$ is given, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, by*

$$U\psi(\cdot,\lambda)(t) = \frac{1}{t-\lambda}, \quad t \in \mathbb{R}.$$

Furthermore, we will need asymptotic formulas for the basic solution φ.

PROPOSITION 1.5.14 ([Ben89], Corollary 6.2). *Let, for each $\lambda \in \mathbb{C}$, $u(\cdot,\lambda)$ be a solution of*

$$-(py')' + qy = \lambda r y \quad \text{on } (a,b) \subseteq \mathbb{R},$$

where, as before, p, q and r satisfy (1.1.1)*, and, additionally, a is a finite and regular endpoint. Assume that $u(a,\cdot)/(pu')(a,\cdot)$ (restricted to $\mathbb{C}\setminus\mathbb{R}$) is a Nevanlinna function.*

Then, for any constants $A, B \in \mathbb{C}$ and any $x \in (a,b)$ for which $Au(x,\lambda) + B(pu')(x,\lambda)$ does not vanish identically in λ, we have

$$Au(x,\lambda) + B(pu')(x,\lambda) = u(a,\lambda) \exp\left\{ \int_a^x \sqrt{-\lambda/(pr)}\; r dt + o\left(\sqrt{|\lambda|}\right) \right\},$$

unless $u(a,\lambda)$ vanishes for all λ, in which case the first factor is replaced by $(pu')(a,\lambda)$. The estimate holds as $\lambda \to \infty$ in any non-real sector, and the error in the exponent is locally uniform in $x \in (a,b)$.

COROLLARY 1.5.15. *For each $\hat{x}, \tilde{x} \in (a,b)$ the basic solution φ satisfies*

$$\lim_{t\to\infty} \sqrt{\frac{2}{t}} \ln \left| \frac{\varphi(\tilde{x}, \mathrm{i}t)}{\varphi(\hat{x}, \mathrm{i}t)} \right| = \int_{\hat{x}}^{\tilde{x}} \sqrt{\frac{r}{|p|}} dx. \tag{1.5.13}$$

PROOF. There is nothing to show when $\hat{x} = \tilde{x}$. W.l.o.g. let $\tilde{x} > \hat{x}$ (after proving the assertion for this case, the assertion for the case $\tilde{x} < \hat{x}$ immediately follows by reversing the roles of $\hat{x}$ and $\tilde{x}$).

In order to apply Proposition 1.5.14, we have to show that $\varphi(\hat{x},\cdot)/(p\varphi')(\hat{x},\cdot)$ (restricted to $\mathbb{C}\setminus\mathbb{R}$) is a Nevanlinna function. To this end consider the function

$$\hat{\psi}(\cdot,\lambda) : x \mapsto \frac{\varphi(x,\lambda)}{(p\varphi')(\hat{x},\lambda)}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

on $(a,\hat{x})$. Note that $\hat{\psi}$ is well-defined, i.e. the denominator in the fraction never equals zero, due to Proposition 1.5.4. Since $\hat{\psi}$ satisfies the boundary condition of S at a and $(p\hat{\psi}')(\hat{x},\lambda) = 1$, it equals the Weyl solution of the selfadjoint realization (which will be referred to as $\hat{S}$) of $\tau|_{(a,\hat{x})}$ with the same boundary condition as S at a and the Neumann boundary condition at $\hat{x}$, where we consider $\hat{x}$ as the initial point (compare Remark 1.5.10). Hence,

$$\hat{\psi}(x,\lambda) = \hat{\theta}(x,\lambda) - \hat{m}(\lambda)\hat{\varphi}(x,\lambda),$$

where $\hat{m}$ is the Titchmarsh-Weyl m-function for $\hat{S}$ and $\hat{\varphi}$ and $\hat{\theta}$ are the basic solutions of $\hat{S}$, satisfying the initial conditions

$$\begin{aligned} \hat{\theta}(\hat{x},\lambda) &= 0, & (p\hat{\theta}')(\hat{x},\lambda) &= 1, \\ \hat{\varphi}(\hat{x},\lambda) &= -1, & (p\hat{\varphi}')(\hat{x},\lambda) &= 0, \end{aligned}$$

compare Remark 1.5.9.

Now, obviously, $\hat{\psi}(\hat{x},\lambda) = \hat{m}(\lambda)$ and hence

$$\hat{\psi}(\hat{x},\cdot) = \left.\frac{\varphi(\hat{x},\cdot)}{(p\varphi')(\hat{x},\cdot)}\right|_{\mathbb{C}\setminus\mathbb{R}}$$

is indeed a Nevanlinna function.

Clearly, $\varphi(\hat{x},\cdot)$ and $\varphi(\tilde{x},\cdot)$ do not vanish identically (by Proposition 1.5.4 they only have real zeros) and thus by Proposition 1.5.14 one gets (with $a = \hat{x}$, $x = \tilde{x}$, $A = 1$, $B = 0$ and $\lambda = \mathrm{i}t$, $t > 0$)

$$\varphi(\tilde{x},\mathrm{i}t) = \varphi(\hat{x},\mathrm{i}t)\exp\left\{\int_{\hat{x}}^{\tilde{x}} \sqrt{-\mathrm{i}t/(pr)}\; r dx + o\left(\sqrt{t}\right)\right\} \quad \text{as } t \to \infty.$$

This leads to

$$\ln\left|\frac{\varphi(\tilde{x},\mathrm{i}t)}{\varphi(\hat{x},\mathrm{i}t)}\right| = \sqrt{\frac{t}{2}}\int_{\hat{x}}^{\tilde{x}} \sqrt{\frac{r}{|p|}} dx + o\left(\sqrt{t}\right) \quad \text{as } t \to \infty,$$

which shows the assertion. □

We conclude this section with a proposition which will be essential in Section 3.2 too, but is also interesting on its own.

Proposition 1.5.16. *For each $x \in (a,b)$ the entire functions*

$$\theta(x,\cdot), (p\theta')(x,\cdot), \varphi(x,\cdot) \text{ and } (p\varphi')(x,\cdot) \tag{1.5.14}$$

belong to the Cartwright class[12].

Proof. We already know from Proposition 1.5.4 that these functions are entire and prove the claim only for the function $\varphi(x,\cdot)$ where $x \in (a,b)$ is arbitrary. The claim for the other functions may be proved similarly.

Because of (1.5.5), i.e. $W(\theta(\cdot,\lambda),\varphi(\cdot,\lambda)) \equiv 1$, $\lambda \in \mathbb{C}$, we have

$$\frac{1}{\varphi(x,\lambda)^2} = \frac{\theta(x,\lambda)}{\varphi(x,\lambda)}\left(\frac{(p\varphi')(x,\lambda)}{\varphi(x,\lambda)} - \frac{(p\theta')(x,\lambda)}{\theta(x,\lambda)}\right), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{1.5.15}$$

where all fractions are well-defined due to Proposition 1.5.4. In the proof of Corollary 1.5.15 we have seen that

$$\lambda \mapsto \frac{\varphi(x,\lambda)}{(p\varphi')(x,\lambda)}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

[12] An entire function f belongs to the Cartwright class $\mathcal{C}$ (see [Lev96], Section 16.1) if it is of exponential type and satisfies

$$\int_{\mathbb{R}} \frac{\ln^+|f(\lambda)|}{1+\lambda^2} d\lambda < \infty,$$

where $\ln^+$ is the positive part of the natural logarithm, i.e. $\ln^+(x) = \max\{\ln x, 0\}$. In particular, the class $\mathcal{C}$ contains all entire functions of exponential order less than one.

is a Nevanlinna function. Similarly, one can show that

$$\lambda \mapsto \frac{\theta(x,\lambda)}{(p\theta')(x,\lambda)}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

is a Nevanlinna function. Since the reciprocal of a Nevanlinna function multiplied by -1 is a Nevanlinna function too, we infer that the second and the third fraction on the right-hand side of (1.5.15) are Nevanlinna functions up to sign.

Now, the function

$$\hat{\psi}(\cdot,\lambda) : t \mapsto \theta(t,\lambda) + \left(-\frac{\theta(x,\lambda)}{\varphi(x,\lambda)}\right)\varphi(t,\lambda), \quad t \in (a,x),\ \lambda \in \mathbb{C}\setminus\mathbb{R},$$

equals the Weyl solution of the selfadjoint realization of $\tau|_{(a,x)}$ with the boundary condition of S at a and the Dirichlet boundary condition at x (note that other than in the proof of Corollary 1.5.15 we consider a as the initial point for the construction of the basic solutions), and we infer that the first fraction on the right-hand side of (1.5.15) is a Nevanlinna function up to sign too.

We claim that any Nevanlinna function m satisfies an estimate of the kind

$$|\pm m(\lambda)| \leq C\frac{1+|\lambda|^2}{|\operatorname{Im}\lambda|} \leq \exp\left(M\frac{1+\sqrt{|\lambda|}}{\sqrt[4]{|\operatorname{Im}\lambda|}}\right), \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

for some constants $C, M \in \mathbb{R}^+$. The first inequality follows from the representation (1.5.7): Because of, for $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$,

$$\left|\frac{1+\lambda t}{t-\lambda}\right| \leq \frac{1}{|t-\lambda|} + |\lambda|\frac{|t|}{|t-\lambda|} \leq \frac{1}{|\operatorname{Im}\lambda|} + |\lambda|\left(1+\frac{|\lambda|}{|t-\lambda|}\right) \leq |\lambda| + \frac{1+|\lambda|^2}{|\operatorname{Im}\lambda|},$$

we have

$$\begin{aligned} |m(\lambda)| &= \left|A + B\lambda + \int_{\mathbb{R}}\left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)d\rho(t)\right| \\ &\leq |A| + B|\lambda| + \int_{\mathbb{R}}\left|\frac{1+\lambda t}{(t-\lambda)(1+t^2)}\right| d\rho(t) \\ &\leq |A| + B|\lambda| + \left(|\lambda| + \frac{1+|\lambda|^2}{|\operatorname{Im}\lambda|}\right)\int_{\mathbb{R}}\frac{1}{1+t^2}d\rho(t) \\ &\leq C\frac{1+|\lambda|^2}{|\operatorname{Im}\lambda|}, \end{aligned}$$

where $C = |A| + B + 2\tilde{C}$ with $\tilde{C} = \int_{\mathbb{R}} 1/(1+t^2)d\rho(t) < \infty$ due to (1.5.8).
The second one simply follows from the elementary inequality $x \leq \exp(2\sqrt[4]{x})$ for $x \geq 0$:

$$C\frac{1+|\lambda|^2}{|\operatorname{Im}\lambda|} \leq \exp\left(2\sqrt[4]{C}\frac{\sqrt[4]{1+|\lambda|^2}}{\sqrt[4]{|\operatorname{Im}\lambda|}}\right) \leq \exp\left(2\sqrt[4]{C}\frac{1+\sqrt{|\lambda|}}{\sqrt[4]{|\operatorname{Im}\lambda|}}\right), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Consequently, $\varphi(x,\cdot)$ satisfies

$$\left|\frac{1}{\varphi(x,\lambda)^2}\right| \leq 2\exp\left(2\tilde{M}\frac{1+\sqrt{|\lambda|}}{\sqrt[4]{|\operatorname{Im}\lambda|}}\right), \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

and

$$|\varphi(x,\lambda)| \geq \frac{1}{\sqrt{2}} \exp\left(-\tilde{M}\frac{1+\sqrt{|\lambda|}}{\sqrt[4]{|\operatorname{Im}\lambda|}}\right), \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

for some $\tilde{M} \in \mathbb{R}^+$, and a theorem by Matsaev (see [Lev96], Section 26.4, Theorem 4) implies that $\varphi(x,\cdot)$ belongs to the Cartwright class. □

Remark 1.5.17. *In the case of a regular endpoint a (where the basic solutions θ and φ are given as the solutions of the classical initial value problem $(\tau - \lambda)u = 0$ together with* (1.5.11)*, compare Remark 1.5.9), there is an elementary proof even showing that the entire functions in* (1.5.14) *are of exponential order at most* 1/2 *(see, e.g., [Zet05], Theorem 2.5.3).*

Chapter 2

Associated de Branges Spaces

In this chapter we retain the notation of Section 1.5: $\tau = \tau(p, q, r)$ is a SL differential expression defined on $(a, b) \subseteq \mathbb{R}$ where a is in the limit-circle case and S a selfadjoint realization with separated boundary conditions. By θ and φ we denote the basic solutions, by ρ the spectral measure and by $U : L^2((a, b); r) \to L^2(\mathbb{R}; \rho)$ the corresponding generalized Fourier transform constructed as in Section 1.5.

2.1 Generalized Fourier Transform and its Connection to de Branges Spaces

For $c \in (a, b)$ by $L^2((a, c); r)$ we denote the linear subspace of $L^2((a, b); r)$ consisting of all functions vanishing outside of $(a, c]$ almost everywhere and by $(\cdot, \cdot)_{L^2((a,c);r)}$ the induced inner product (equaling $(\cdot, \cdot)_{L^2((a,b);r)}$). Clearly, this is a closed subspace which is isometrically isomorph to $L^2((a, c); r|_{(a,c)})$. For $f \in L^2((a, c); r)$ the generalized Fourier transform Uf of f is given by

$$Uf(t) = \int_a^c \varphi(x, t) f(x) r(x) dx, \quad t \in \mathbb{R},$$

considered as an element of $L^2(\mathbb{R}; \rho)$, compare Lemma 1.5.11.

Now, due to Lemma 1.5.12, we may also consider - for each fixed $c \in (a, b)$ - the linear transformation $\widehat{U}_c : L^2((a, b); r) \to \mathcal{H}(\mathbb{C}) : f \mapsto \widehat{U}_c f$ with $\widehat{U}_c f$ given by

$$\widehat{U}_c f(z) = \int_a^c \varphi(x, z) f(x) r(x) dx, \quad z \in \mathbb{C}. \tag{2.1.1}$$

Obviously, $\widehat{U}_c f$ is an (actually natural) entire continuation of Uf, i.e. $\widehat{U}_c f\big|_{\mathbb{R}} = Uf$, for $f \in L^2((a, c); r)$. Therefore, we have

$$U\big|_{L^2((a,c);r)} = (\varrho \circ \widehat{U}_c)\big|_{L^2((a,c);r)}, \tag{2.1.2}$$

where ϱ is restricting to $\mathbb{R}$. Clearly, we have $\widehat{U}_c f = \widehat{U}_{\tilde{c}} f$ for $f \in L^2((a, c); r)$, $c, \tilde{c} \in (a, b)$ and $c < \tilde{c}$.

Note that $\widehat{U}_c f$ is not necessarily the only entire continuation of Uf (considered as an element of $L^2(\mathbb{R}; \rho)$) for $f \in L^2((a, c); r)$. For example, if the spectrum of S is purely discrete (as it is the case when both endpoints are limit-circle, see, e.g., Theorem 9.10 in [Tes09]) so that $\operatorname{supp} \rho$ consists of isolated points (compare (1.5.10)), according to the Weierstrass theorem, there exist entire functions vanishing precisely on $\operatorname{supp}\rho$, and therefore an entire continuation never can be unique in this case.

Using the transformation $\widehat{U}_c$, we now want to link the sub-Hilbert space $U(L^2((a,c);r)) \subseteq L^2(\mathbb{R};\rho)$ to a de Branges space - see Appendix A.2 for a short introduction into de Branges' theory of Hilbert spaces of entire functions as far as we will need it. The ideas presented here are taken from [Eck], where the author himself refers to [Rem02], with the slight difference that we consider general right-definite SL differential equations, i.e. we do not assume that $p = r \equiv 1$.

For $c \in (a,b)$ consider the entire function

$$E_c(\lambda) = \varphi(c,\lambda) + \mathrm{i}(p\varphi')(c,\lambda), \quad \lambda \in \mathbb{C},$$

compare Proposition 1.5.4. Note that E_c does not have any real zero λ_0 because otherwise both $\varphi(c,\lambda_0)$ and $(p\varphi')(c,\lambda_0)$ would vanish (since these numbers are both real, compare Proposition 1.5.4), implying that φ would equal zero. E_c is actually a de Branges function.

Lemma 2.1.1. *E_c satisfies the inequality*

$$|E_c(\lambda)| > |E_c(\overline{\lambda})|, \quad \lambda \in \mathbb{C}^+, \tag{2.1.3}$$

and hence gives rise to a de Branges space $B(c)$. The reproducing kernel $K_c(\cdot,\cdot)$ of $B(c)$ is given by

$$K_c(z,\lambda) = \int_a^c \varphi(\cdot,\overline{z})\varphi(\cdot,\lambda)r(\cdot)dx = \int_a^c \overline{\varphi}(\cdot,z)\varphi(\cdot,\lambda)r(\cdot)dx, \quad z,\lambda \in \mathbb{C}. \tag{2.1.4}$$

Proof. We claim that

$$\begin{aligned} \frac{E_c(\lambda)E_c^{\#}(\overline{z}) - E_c(\overline{z})E_c^{\#}(\lambda)}{2\mathrm{i}(\overline{z}-\lambda)} &= \int_a^c \varphi(\cdot,\overline{z})\varphi(\cdot,\lambda)r(\cdot)dx \\ &= \int_a^c \overline{\varphi}(\cdot,z)\varphi(\cdot,\lambda)r(\cdot)dx, \quad \lambda, z \in \mathbb{C}. \end{aligned} \tag{2.1.5}$$

Taking $z = \lambda \in \mathbb{C}^+$, this shows

$$\frac{|E_c(\lambda)| - |E_c(\overline{\lambda})|}{4\operatorname{Im}\lambda} = \int_a^c \overline{\varphi(\cdot,\lambda)}\varphi(\cdot,\lambda)r(\cdot)dx > 0$$

and hence inequality (2.1.3). Because of (A.2.1) the reproducing kernel is then given by (2.1.4).

Let us show now (2.1.5). Since $\varphi(c,\overline{z}) = \overline{\varphi(c,z)}$ and $\varphi(c,\overline{\lambda}) = \overline{\varphi(c,\lambda)}$ by (1.5.4), one immediately gets

$$\begin{aligned} \frac{E_c(\lambda)E_c^{\#}(\overline{z}) - E_c(\overline{z})E_c^{\#}(\lambda)}{2\mathrm{i}(\overline{z}-\lambda)} &= \frac{1}{\overline{z}-\lambda}\big(\varphi(c,\overline{z})(p\varphi')(c,\lambda) - \varphi(c,\lambda)(p\varphi')(c,\overline{z})\big) \\ &= \frac{1}{\overline{z}-\lambda} W(\varphi(\cdot,\overline{z}),\varphi(\cdot,\lambda))(c). \end{aligned}$$

Using the Lagrange identity (1.2.1) and $\tau\varphi(\cdot,\zeta)=\zeta\varphi(\cdot,\zeta)$, $\zeta\in\mathbb{C}$, we obtain

$$\frac{E_c(\lambda)E_c^{\#}(\overline{z})-E_c(\overline{z})E_c^{\#}(\lambda)}{2\mathrm{i}(\overline{z}-\lambda)}=\int_a^c\varphi(\cdot,\overline{z})\varphi(\cdot,\lambda)r(\cdot)dx+\frac{1}{\overline{z}-\lambda}W(\varphi(\cdot,\overline{z}),\varphi(\cdot,\lambda))(a),$$

where the second summand vanishes because of (1.5.6). □

The assertion of the next theorem is the main result of this chapter. Together with (2.1.2) it shows the connection between the generalized Fourier transform and a family of de Branges spaces.

THEOREM 2.1.2. *For every $c\in(a,b)$ the transformation $\widehat{U}_c$ is unitary from $L^2((a,c);r)$ onto $B(c)$. In particular, we have*

$$B(c)=\left\{\widehat{U}_cf : f\in L^2((a,c);r)\right\}.$$

PROOF. For each $\lambda\in\mathbb{R}$ we consider the function $f_\lambda\in L^2((a,c);r)$ given by

$$f_\lambda(x)=\begin{cases}\varphi(x,\lambda), & x\in(a,c],\\ 0, & x\in(c,b).\end{cases}$$

Due to (2.1.1) and (2.1.4), we have, for $z\in\mathbb{C}$,

$$\widehat{U}_cf_\lambda(z)=\int_a^c\varphi(x,z)f_\lambda(x)r(x)dx=\int_a^c\varphi(x,z)\varphi(x,\lambda)r(x)dx=K_c\big(\overline{\lambda},z\big)=K_c(\lambda,z),$$

and hence $\widehat{U}_cf_\lambda=K_c(\lambda,\cdot)\in B(c)$ for $\lambda\in\mathbb{R}$. Furthermore, we have for all λ_1, $\lambda_2\in\mathbb{R}$

$$\begin{aligned}(f_{\lambda_1},f_{\lambda_2})_{L^2((a,c);r)}&=\int_a^c f_{\lambda_1}(x)\overline{f_{\lambda_2}}(x)r(x)dx=\int_a^c\varphi(x,\lambda_1)\varphi(x,\lambda_2)r(x)dx\\&=K_c(\lambda_1,\lambda_2)=\langle K_c(\lambda_1,\cdot),K_c(\lambda_2,\cdot)\rangle_{B(c)},\end{aligned}$$

which shows that $\widehat{U}_c$ is an isometry on the linear span D of all functions f_λ, $\lambda\in\mathbb{R}$. However, in order to show that D is dense in $L^2((a,c);r)$, consider the restriction of S to (a,c), which we refer to as S_c, with a Dirichlet boundary condition at c (i.e. S_c is the selfadjoint realization of $\tau|_{(a,c)}$ with the boundary condition of S at a and a Dirichlet boundary condition at c). Its eigenfunctions are precisely (the scalar multiples of) the functions $\varphi(\cdot,\lambda)|_{(a,c)}=f_\lambda|_{(a,c)}$, $\lambda\in\mathbb{R}$, which satisfy $\varphi(c,\lambda)=0$. Since a and c are both limit-circle for $\tau|_{(a,c)}$, the span of the eigenfunctions of S_c is dense in $L^2((a,c);r|_{(a,c)})$ (there is an orthonormal basis of eigenfunctions of S_c, see, e.g., Theorem 9.10 in [Tes09]), and by applying the isometric isomorphism between $L^2((a,c);r|_{(a,c)})$ and $L^2((a,c);r)$ we see that D is indeed dense in $L^2((a,c);r)$.

Moreover, the linear span of all transforms $\widehat{U}_cf_\lambda=K_c(\lambda,\cdot)$, $\lambda\in\mathbb{R}$, is dense in $B(c)$ because if $h\in B(c)$ such that

$$0=\langle h,K_c(\lambda,\cdot)\rangle_{B(c)}=h(\lambda),\quad \lambda\in\mathbb{R},$$

h must vanish identically because of the identity theorem for holomorphic functions. Thus, $\widehat{U}_c$ restricted to D uniquely extends to a unitary map V from $L^2((a,c);r)$ onto $B(c)$.

We have to show that $\widehat{U}_c|_{L^2((a,c);r)}$ equals V. Note that for each fixed $z \in \mathbb{C}$ both $f \mapsto \widehat{U}_c f(z)$ and $f \mapsto Vf(z)$ are continuous on $L^2((a,c);r)$. This is clear for the second map since V is continuous and $B(c)$ is a reproducing kernel Hilbert space. For the first one this follows from

$$\begin{aligned}\left|\widehat{U}_c f_1(z) - \widehat{U}_c f_2(z)\right| &= \left|\int_a^c \varphi(x,z)(f_1(x) - f_2(x))r(x)dx\right| \\ &\leq \left\|\varphi(\cdot,z)\big|_{(a,c)}\right\|_{L^2((a,c);r|_{(a,c)})} \|f_1 - f_2\|_{L^2((a,c);r)}.\end{aligned}$$

Hence, for all $f \in L^2((a,c);r)$ and $z \in \mathbb{C}$ we have

$$\widehat{U}_c f(z) = \lim_{n\to\infty} \widehat{U}_c f_n(z) = \lim_{n\to\infty} V f_n(z) = Vf(z),$$

where f_n is a sequence of functions in D converging to f. □

We have the following two corollaries.

Corollary 2.1.3. *For each $c \in (a,b)$ the de Branges space $B(c)$ is isometrically embedded in $L^2(\mathbb{R};\rho)$, where the embedding is just restricting to $\mathbb{R}$, i.e.*

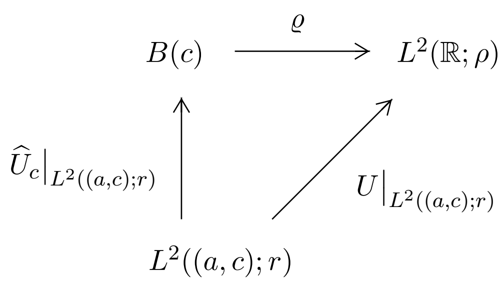
$$\int_{\mathbb{R}} |h(t)|^2 d\rho(t) = \|h\|^2_{B(c)}, \quad h \in B(c).$$

Moreover, the union of the spaces $B(c)$, $c \in (a,b)$, is dense in $L^2(\mathbb{R};\rho)$, i.e.

$$\overline{\bigcup_{c\in(a,b)} B(c)} = L^2(\mathbb{R};\rho), \tag{2.1.6}$$

where we suppress the embedding.

Proof. For each $c \in (a,b)$ we have the following commutative diagram, where $U|_{L^2((a,c);r)}$ is an isometry and $\widehat{U}_c|_{L^2((a,c);r)}$ is unitary:



This immediately shows the embedding. Since

$$\overline{\bigcup_{c\in(a,b)} L^2((a,c);r)} = L^2((a,b);r)$$

and $U : L^2((a,b);r) \to L^2(\mathbb{R};\rho)$ is unitary, we have (2.1.6). □

Corollary 2.1.4. *If $c_1, c_2 \in (a,b)$ with $c_1 < c_2$, then $B(c_1) \subsetneq B(c_2)$ (including inner product).*
Moreover, for each $c \in (a,b)$ we have

$$\overline{\bigcup_{x\in(a,c)} B(x)} = B(c) = \bigcap_{x\in(c,b)} B(x). \tag{2.1.7}$$

Proof.

$$B(c_1) = \widehat{U}_{c_1}(L^2((a,c_1);r)) = \widehat{U}_{c_2}(L^2((a,c_1);r)) \subsetneq \widehat{U}_{c_2}(L^2((a,c_2);r)) = B(c_2)$$

and

$$\begin{aligned}
\langle g,h\rangle_{B(c_1)} &= \Big(\big(\widehat{U}_{c_1}\big|_{L^2((a,c_1);r)}\big)^{-1} g, \big(\widehat{U}_{c_1}\big|_{L^2((a,c_1);r)}\big)^{-1} h\Big)_{L^2((a,c_1);r)} \\
&= \Big(\big(\widehat{U}_{c_1}\big|_{L^2((a,c_1);r)}\big)^{-1} g, \big(\widehat{U}_{c_1}\big|_{L^2((a,c_1);r)}\big)^{-1} h\Big)_{L^2((a,c_2);r)} \\
&= \Big\langle \widehat{U}_{c_2}\big(\widehat{U}_{c_1}\big|_{L^2((a,c_1);r)}\big)^{-1} g, \widehat{U}_{c_2}\big(\widehat{U}_{c_1}\big|_{L^2((a,c_1);r)}\big)^{-1} h\Big\rangle_{B(c_2)} \\
&= \langle g,h\rangle_{B(c_2)}, \quad g,h \in B(c_1),
\end{aligned}$$

which shows the first claim.

The second claim follows from

$$\overline{\bigcup_{x\in(a,c)} L^2((a,x);r)} = L^2((a,c);r) = \bigcap_{x\in(c,b)} L^2((a,x);r).$$

□

Chapter 3

The Liouville Transformation

We consider the SL equation

$$-(p_1 y')' + q_1 y = \lambda r_1 y \quad \text{on } (a_1, b_1) \subseteq \mathbb{R}, \tag{3.0.1}$$

where compared to the two previous chapters (see Section 1.1) the SL coefficients[1] (p_1, q_1, r_1) comply with the stronger requirements

$$\begin{cases} (i) & p_1, q_1, r_1 : (a_1, b_1) \to \mathbb{R} \\ (ii) & p_1, r_1 \in AC_{loc}(a_1, b_1), q_1 \in L^1_{loc}(a_1, b_1) \\ (iii) & p_1, r_1 > 0 \text{ on } (a_1, b_1). \end{cases} \tag{3.0.2}$$

For the treatment of the Liouville transformation (as we will introduce it below) these conditions seem to be appropriate minimal conditions, which turn out to be crucial repeatedly. Hence we make the following convention:

Till the end of this thesis the minimal conditions (1.1.1) *are replaced by the conditions* (3.0.2)*, i.e. every tuple of SL coefficients* (p_1, q_1, r_1) *is assumed to satisfy* (3.0.2).

Note that in this case the domain of an associated SL differential expression $\tau_1 = \tau(p_1, q_1, r_1)$ is given by $\mathcal{D}(\tau_1) = \{y \in AC_{loc}(a_1, b_1) : y' \in AC_{loc}(a_1, b_1)\}$ since one has $y' \in AC_{loc}(a_1, b_1)$ if and only if $p_1 y' \in AC_{loc}(a_1, b_1)$ according to Theorem A.1.1 and that $(p_1 y')'$ equals $p_1 y'' + p_1' y'$. In particular, note that the domain of any SL differential expression depends on its coefficients only by means of its interval of definition.

One can do the transformation

$$y(z) = v(z) w(u(z)), \tag{3.0.3}$$

where $u, v : (a_1, b_1) \to \mathbb{R}$ are appropriate functions. As we will see, this leads to a SL equation

$$-(p_2 w')' + q_2 w = \lambda r_2 w \quad \text{on } (a_2, b_2) \subseteq \mathbb{R} \tag{3.0.4}$$

for the unknown function w such that y is a solution of equation (3.0.1) if and only if w is a solution of equation (3.0.4). In this chapter we want to systematically discuss the transformation (3.0.3).

[1] In this chapter we will relate equation (3.0.1) to another SL equation and hence the subscript.

3.1 The Liouville Transformation

DEFINITION 3.1.1. *Let $(a,b) \subseteq \mathbb{R}$ and v and u be two functions defined on (a,b) and satisfying*

$$\begin{cases} (i) & v,u : (a,b) \to \mathbb{R} \\ (ii) & v,u \in C^1(a,b) \text{ and } v',u' \in AC_{loc}(a,b) \\ (iii) & v \neq 0 \text{ and } u' > 0 \text{ in } (a,b). \end{cases} \tag{3.1.1}$$

Given any complex-valued function f defined on $I \supseteq u((a,b))$, by the Liouville transform of f we mean the function g defined by

$$g(x) = v(x)f(u(x)), \quad x \in (a,b).$$

For any linear function space $\mathcal{I}$ of complex-valued functions defined on $I \supseteq u((a,b))$ we refer to the linear map

$$L : \mathcal{I} \ni f \mapsto g$$

as the Liouville transformation (belonging to v and u).

REMARK 3.1.2.

- *Since $u' > 0$ on (a,b), there exists a unique continuous extension of u (to which we also may refer as u since we can identify u with this extension) to $u : [-\infty,\infty] \supseteq [a,b] \to [-\infty,\infty]$. Then u is a strictly increasing bijection between (a,b) and $(u(a),u(b))$ (or between $[a,b]$ and $[u(a),u(b)]$, respectively) and has an also strictly increasing and continuously differentiable inverse function $u^{-1} : (u(a),u(b)) \to (a,b)$ with derivative $(u^{-1})'(x) = \frac{1}{u'(u^{-1}(x))}$. Because of Theorem A.1.1 it is $(u^{-1})' \in AC_{loc}(u(a),u(b))$.*
- *It is easy to see that L is a bijection between the complex-valued functions defined on $(u(a),u(b))$ and those defined on (a,b). Theorem A.1.1 shows that its inverse L^{-1} is a Liouville transformation (belonging to $\frac{1}{v} \circ u^{-1}$ and u^{-1}) as well. The same theorem shows that the composition of two Liouville transformations is a Liouville transformation too.*
- *Since v is assumed to be real-valued, we have $L\overline{f} = \overline{Lf}$.*
- *If $\hat{v} : (a,b) \to \mathbb{R}$, $\hat{v} \in C^1(a,b)$, $\hat{v}' \in AC_{loc}(a,b)$, $\hat{v} \neq 0$ in (a,b), then (since $(\hat{v}v)' \in AC_{loc}(a,b)$ by Theorem A.1.1) $M_{\hat{v}}L$, where $M_{\hat{v}}$ is multiplication by $\hat{v}$, is a Liouville transformation (belonging to $\hat{v}v$ and u). In particular, cL is a Liouville transformation for $c \in \mathbb{R} \setminus \{0\}$.*

Now we have the following theorem linking (3.0.1) to (3.0.4).

THEOREM 3.1.3. *Let (p_1,q_1,r_1) be a tuple of SL coefficients defined on $(a_1,b_1) \subseteq \mathbb{R}$ and subject to* (3.0.2) *and let $\tau_1 = \tau(p_1,q_1,r_1)$ be the associated SL differential* *expression. Let $v, u : (a_1, b_1) \to \mathbb{R}$ be two functions satisfying* (3.1.1) *and L be the Liouville transformation belonging to $\frac{1}{v} \circ u^{-1}$ and u^{-1}, i.e. the inverse of the Liouville transformation belonging to v and u (see Definition 3.1.1 and Remark 3.1.2).*

Set $a_2 = u(a_1), b_2 = u(b_1)$ and define the functions $p_2, q_2, r_2 : (a_2, b_2) \to \mathbb{R}$ by

$$p_2 = (v^2 p_1 u') \circ u^{-1}, \quad q_2 = \left[\frac{v}{u'}(q_1 v - (p_1 v')')\right] \circ u^{-1}, \quad r_2 = \frac{v^2 r_1}{u'} \circ u^{-1}. \tag{3.1.2}$$

Then:

- (p_2, q_2, r_2) *is a tuple of SL coefficients defined on $(a_2, b_2) \subseteq \mathbb{R}$ satisfying conditions* (3.0.2).
- *Set $\tau_2 = \tau(p_2, q_2, r_2)$. Then we have $y \in \mathcal{D}(\tau_1)$ if and only if $Ly \in \mathcal{D}(\tau_2)$ and*

$$L\tau_1 y = \tau_2 L y, \quad y \in \mathcal{D}(\tau_1).$$

- *A function y is a solution of $-(p_1 y')' + q_1 y = \lambda r_1 y$ on (a_1, b_1) if and only if the function $w = Ly$ is a solution of $-(p_2 w')' + q_2 w = \lambda r_2 w$ on (a_2, b_2).*
- *The map L is a unitary mapping of $L^2((a_1, b_1); r_1)$ onto $L^2((a_2, b_2); r_2)$.*
- *We have $y \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ if and only if $Ly \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$ and*

$$LT_{max}(p_1, q_1, r_1)y = T_{max}(p_2, q_2, r_2)Ly, \quad y \in \mathcal{D}(T_{max}(p_1, q_1, r_1)),$$

  i.e. $T_{max}(p_1, q_1, r_1)$ and $T_{max}(p_2, q_2, r_2)$ are unitarily equivalent operators. The same assertions hold for the preminimal operators $T_0(p_1, q_1, r_1)$ and $T_0(p_2, q_2, r_2)$, respectively.

PROOF.

- Under the assumptions for p_1, q_1, r_1, v and u, for p_2 and r_2 this is an immediate consequence of Theorem A.1.1, and for q_2 this immediately follows from the substitution rule and the fact that u maps compact intervals in (a_1, b_1) onto compact intervals in (a_2, b_2).

For the following let y always be a function defined on (a_1, b_1) and set $w = Ly$. By z we always mean a variable in (a_1, b_1), whereas x denotes a variable in (a_2, b_2) and let z and x be related by $u(z) = x$. The following relations hold:

$$\begin{aligned} w(x) &= Ly(x) = \frac{1}{v(u^{-1}(x))} y(u^{-1}(x)), \\ y(z) &= L^{-1}w(z) = v(z)w(u(z)), \\ u(z) &= x \quad \text{and} \quad z = u^{-1}(x). \end{aligned}$$

- Let $y \in \mathcal{D}(\tau_1) = \{y \in AC_{loc}(a_1, b_1) : y' \in AC_{loc}(a_1, b_1)\}$. By Theorem A.1.1 we have $w \in AC_{loc}(a_2, b_2)$. Now calculate

$$\begin{aligned} y'(z) &= v'(z)w(u(z)) + v(z)w'(u(z))u'(z), \\ y'(u^{-1}(x)) &= v'(u^{-1}(x))w(x) + v(u^{-1}(x))w'(x)u'(u^{-1}(x)). \end{aligned}$$

Rearranging (and again Theorem A.1.1) shows that $w' \in AC_{loc}(a_2, b_2)$, and hence we have $w \in \mathcal{D}(\tau_2)$. If given $w \in \mathcal{D}(\tau_2)$, then an analogous argument gives $y \in \mathcal{D}(\tau_1)$.

Now calculate

$$\begin{aligned}(p_1y')'(z) &= (p_1v'(w\circ u) + p_1v(w'\circ u)u')'(z)\\ &= (p_1v')'(z)w(u(z)) + (p_1v')(z)w'(u(z))u'(z) + (p_1u')'(z)v(z)w'(u(z))\\ &\quad + (p_1u')(z)v'(z)w'(u(z)) + (p_1u')(z)v(z)w''(u(z))u'(z).\end{aligned}$$

Hence,

$$\begin{aligned}\tau_1y(z) &= \frac{1}{r_1(z)}(-(p_1y')' + q_1y)(z)\\ &= \frac{1}{r_1(z)}\Big[-(p_1v(u')^2)(z)w''(u(z)) + (-2p_1v'u' - v(p_1u')')(z)w'(u(z))\\ &\quad + (-(p_1v')' + q_1v)(z)w(u(z))\Big]\end{aligned}$$

and

$$\begin{aligned}\frac{1}{v(u^{-1}(x))}(\tau_1y)(u^{-1}(x)) = \frac{1}{vr_1}\circ u^{-1}(x)\Big[&-(p_1v(u')^2)\circ u^{-1}(x)\cdot w''(x)\\ &+(-2p_1v'u' - v(p_1u')')\circ u^{-1}(x)\cdot w'(x)\\ &+(-(p_1v')' + q_1v)\circ u^{-1}(x)\cdot w(x)\Big],\end{aligned}$$

where the expression on the left-hand side is just $L\tau_1y(x)$. On the right-hand side we have a general second-order linear differential expression

$$a(x)w''(x) + b(x)w'(x) + c(x)w(x),$$

which we have to rearrange as a SL differential expression.

Setting

$$\widetilde{p_2} = e^{\int \frac{b}{a}dx}, \quad \widetilde{r_2} = -\frac{p_2}{a}, \quad \widetilde{q_2} = -\frac{cp_2}{a}, \tag{3.1.3}$$

where $\int \frac{b}{a}dx$ denotes an arbitrary antiderivative of $\frac{b}{a}$, one gets

$$a(x)w''(x) + b(x)w'(x) + c(x)w(x) = \frac{1}{\widetilde{r_2}(x)}\big[-(\widetilde{p_2}(x)w'(x))' + \widetilde{q_2}(x)w(x)\big]. \tag{3.1.4}$$

Note that the choice (3.1.3) is necessary for (3.1.4) to hold for any $w \in L(\mathcal{D}(\tau_1))$, but (since $\int \frac{b}{a}dx$ is unique only up to an additive constant $C \in \mathbb{R}$) is unique only up to a multiplicative constant $e^C \in \mathbb{R}^+$ - see Remark 3.1.5 after the proof of this theorem.

In our case we have

$$a = \Big(-\frac{p_1}{r_1}(u')^2 \Big) \circ u^{-1},$$
$$b = \Big(-\frac{2p_1 v' u'}{v r_1} - \frac{(p_1 u')'}{r_1} \Big) \circ u^{-1},$$
$$c = \Big(\frac{q_1}{r_1} - \frac{(p_1 v')'}{v r_1} \Big) \circ u^{-1}$$

and hence obtain

$$\begin{aligned} \int \frac{b}{a} dx &= \int \frac{2p_1 v' u' + v(p_1 u')'}{v p_1 (u')^2} \circ u^{-1}(x) dx = \begin{vmatrix} x & = & u(z) \\ dx & = & u'(z)dz \end{vmatrix} \\ &= \int \frac{2p_1 v' u' + v(p_1 u')'}{v p_1 u'}(z) dz = \int \frac{2v'}{v}(z)dz + \int \frac{(p_1 u')'}{p_1 u'}(z)dz \\ &= \ln v^2(z) + \ln(p_1 u')(z) = \ln(v^2 p_1 u')(u^{-1}(x)) \end{aligned}$$

as one particular antiderivative of $\frac{b}{a}$. Note that in here $\ln v^2$ and $\ln p_1 u'$ are indeed well-defined and elements of $AC_{loc}(a_1, b_1)$ according to Theorem A.1.1.

It follows that

$$\widetilde{p_2} = (v^2 p_1 u') \circ u^{-1} = p_2,$$
$$\widetilde{q_2} = \Big[\frac{v}{u'}(q_1 v - (p_1 v')') \Big] \circ u^{-1} = q_2,$$
$$\widetilde{r_2} = \frac{v^2 r_1}{u'} \circ u^{-1} = r_2$$

and

$$L\tau_1 y = \frac{1}{r_2}\big[-(p_2 w')' + q_2 w \big] = \tau_2 L y.$$

- This is an immediate consequence of the previous assertion.
- Let $y \in L^2((a_1, b_1); r_1)$. Then

$$\begin{aligned} \int_{a_2}^{b_2} |w(x)|^2 r_2(x) dx &= \int_{a_2}^{b_2} \Big| \frac{1}{v(u^{-1}(x))} y(u^{-1}(x)) \Big|^2 \cdot \frac{v^2 r_1}{u'} \circ u^{-1}(x) dx \\ &= \begin{vmatrix} z & = & u^{-1}(x) \\ dz & = & (u^{-1})'(x)dx \end{vmatrix} \\ &= \int_{a_1}^{b_1} |y(z)|^2 r_1(z) dz \\ &= \|y\|^2_{L^2((a_1,b_1);r_1)}, \end{aligned}$$

where we have used that $(u^{-1})'(x) = \frac{1}{u'(z)}$. Hence, $w \in L^2((a_2, b_2); r_2)$ and $\|w\|_{L^2((a_2,b_2);r_2)} = \|y\|_{L^2((a_1,b_1);r_1)}$.

If given $w \in L^2((a_2, b_2); r_2)$, then the argument may be reversed to give $y \in L^2((a_1, b_1); r_1)$ and hence L is unitary from $L^2((a_1, b_1); r_1)$ onto $L^2((a_2, b_2); r_2)$.

- The assertion for the maximal operators is a consequence of the previous points as $\mathcal{D}(T_{max}(p_1,q_1,r_1)) = \{y \in L^2((a_1,b_1);r_1) : y \in \mathcal{D}(\tau_1), \tau_1 y \in L^2((a_1,b_1);r_1)\}$ and $T_{max}(p_1,q_1,r_1)y = \tau_1 y$ for $y \in \mathcal{D}(T_{max}(p_1,q_1,r_1))$ and the same relations hold for the maximal operator and the SL differential expression associated with $\{p_2,q_2,r_2\}$.

  The assertions for the preminimal operators follow from this by the observation that $\operatorname{supp} y$ is compact in (a_1,b_1) if and only if $\operatorname{supp} w$ is compact in (a_2,b_2).

□

Theorem 3.1.3 motivates the following definition.

Definition 3.1.4. *We say that the SL equation*

$$-(p_1y')' + q_1y = \lambda r_1 y \quad \text{on } (a_1,b_1) \subseteq \mathbb{R}$$

can be transformed into

$$-(p_2w')' + q_2w = \lambda r_2 w \quad \text{on } (a_2,b_2) \subseteq \mathbb{R}$$

by a Liouville transformation (or by the Liouville transformation $y = v\cdot(w\circ u)$) if the SL coefficients (p_1,q_1,r_1) and (p_2,q_2,r_2) are related as in (3.1.2) for some functions $v,u:(a_1,b_1)\to\mathbb{R}$ satisfying (3.1.1). In this case we write $(p_1,q_1,r_1)\sim_{\mathcal{K}}(p_2,q_2,r_2)$. We define $\mathcal{K}$ as the set of all tuples of SL coefficients satisfying conditions (3.0.2),

$$\begin{aligned}\mathcal{K} = \{(p,q,r) : &(p,q,r) \text{ is a tuple of SL coefficients defined on}\\ &\text{some interval } (a,b)\subseteq\mathbb{R} \text{ and satisfying conditions (3.0.2)}\},\end{aligned}$$

and consider $\sim_{\mathcal{K}}$ as a relation on $\mathcal{K}$.

Remark 3.1.5.

- *In the proof of Theorem 3.1.3 we have seen that we have*

$$L\tau(p_1,q_1,r_1)y = \tau(p_2,q_2,r_2)Ly$$

  and hence

$$-(p_1y')' + q_1y = \lambda r_1 y \Leftrightarrow -(p_2w')' + q_2w = \lambda r_2 w$$

  ($y\in\mathcal{D}(\tau_1)$, $w = Ly$) not only for the SL coefficients (p_2,q_2,r_2) given by (3.1.2), but precisely for those SL coefficients differing from these by a positive multiplicative constant. However, only the choice (3.1.2) leads to a unitary mapping $L : L^2((a_1,b_1);r_1)\to L^2((a_2,b_2);r_2)$.

  Note that if $(\widetilde{p_2},\widetilde{q_2},\widetilde{r_2}) = (Cp_2,Cq_2,Cr_2)$ for some $C\in\mathbb{R}^+$, then $\tau(\widetilde{p_2},\widetilde{q_2},\widetilde{r_2}) = \tau(p_2,q_2,r_2)$, but the associated SL equations

$$\begin{aligned}-(\widetilde{p_2}\omega')' + \widetilde{q_2}\omega &= \lambda\widetilde{r_2}\omega,\\ -(Cp_2\omega')' + Cq_2\omega &= \lambda C r_2\omega\end{aligned} \tag{3.1.5}$$

and

$$-(p_2w')' + q_2w = \lambda r_2 w \tag{3.1.6}$$

differ by multiplication with C. However, (3.1.5) *can be transformed into* (3.1.6) *by the Liouville transformation $\omega = \frac{1}{\sqrt{C}}w$ in terms of Definition 3.1.4 and therefore a SL equation can be transformed into* (3.1.6) *by a Liouville transformation if and only if it can be transformed into* (3.1.5) *by a Liouville transformation - this is due to the fact that transformability is an equivalence relation, see the next item of this remark.*

- *Using Theorem A.1.1, it is easy to check that $\sim_{\mathcal{K}}$ is actually an equivalence relation on $\mathcal{K}$.*

For the following we retain the notation of Theorem 3.1.3.

PROPOSITION 3.1.6. *Let W^1 be the Wronskian associated with τ_1 and W^2 be the Wronskian associated with τ_2, respectively. Let $y, s \in \mathcal{D}(\tau_1)$. Then*

$$W^2(w,t)(x) = W^1(y,s)(u^{-1}(x)), \quad x \in (a_2, b_2),$$

if $w = Ly$ and $t = Ls$.
If $y, s \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ (and hence $w, t \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$) we also have

$$W^2(w,t)(a_2) = W^1(y,s)(a_1) \quad \text{and} \quad W^2(w,t)(b_2) = W^1(y,s)(b_1).$$

PROOF. $W^2(w,t)(x) = W^1(y,s)(u^{-1}(x))$ for $x \in (a_2, b_2)$ is shown by a simple calculation. For $y, s \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ we then have

$$\begin{aligned} W^2(w,t)(a_2) &= \lim_{x \to a_2^+} W^2(w,t)(x) = \lim_{x \to a_2^+} W^1(y,s)(u^{-1}(x)) \\ &= \lim_{z \to a_1^+} W^1(y,s)(z) = W^1(y,s)(a_1) \end{aligned}$$

and

$$\begin{aligned} W^2(w,t)(b_2) &= \lim_{x \to b_2^-} W^2(w,t)(x) = \lim_{x \to b_2^-} W^1(y,s)(u^{-1}(x)) \\ &= \lim_{z \to b_1^-} W^1(y,s)(z) = W^1(y,s)(b_1). \end{aligned}$$

□

COROLLARY 3.1.7. *The endpoint a_1 is limit-point (limit-circle) for τ_1 if and only if a_2 is limit-point (limit-circle) for τ_2. Corresponding assertions hold for the right endpoint b_1.*

Proof. We use the criterion of Proposition 1.3.3. From Proposition 3.1.6 and Theorem 3.1.3 it follows that $W^1(y,s)(a_1) = 0$ for all $y, s \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ is equivalent to $W^2(w,t)(a_2) = 0$ for all $w, t \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$. Since each endpoint is either limit-point or limit-circle, this shows the assertion. □

Corollary 3.1.7 states an important invariance property of the Liouville transformation: Endpoints which are limit-point (limit-circle) are transformed into endpoints which are limit-point (limit-circle). Considering the definition of an endpoint being limit-point (limit-circle) this is not all that surprising since we know that the Liouville transformation maps solutions of $\tau_1 y = \lambda y$ to solutions of $\tau_2 w = \lambda w$ and that it is an isometric isomorphism between $L^2((a_1, b_1); r_1)$ and $L^2((a_2, b_2); r_2)$.

However, there is obviously no invariance of endpoints concerning their nature of being finite or infinite as, e.g., the Liouville transformation belonging to $v \equiv 1$ and $u(z) = \tan(-\frac{\pi}{2} + \frac{z-a_1}{b_1-a_1}\pi)$ (in the case that both a_1 and b_1 are finite) shows. Furthermore, there is no invariance of endpoints concerning their nature of being regular or singular as we will see in the following example.

Example 3.1.8. *Consider the Bessel equation*

$$(-z^{\frac{1}{2}} y'(z))' = \lambda z^{\frac{1}{2}} y(z) \quad on\ (0,1),$$

where $a_1 = 0$ and $b_1 = 1$ are both regular endpoints since the functions $\frac{1}{p_1} : z \mapsto z^{-\frac{1}{2}}$ and $r_1 : z \mapsto z^{\frac{1}{2}}$ are integrable on $[0,1]$.

Applying the Liouville transformation $w(x) = x^{\frac{1}{4}} y(x)$, i.e. $u(z) = z, v(z) = z^{-\frac{1}{4}}$, one obtains the transformed equation

$$-w''(x) - \frac{3}{16} x^{-2} w(x) = \lambda w(x) \quad on\ (0,1),$$

where $b_2 = 1$ is still a regular endpoint, but $a_2 = 0$ is now a singular endpoint since $q_2 : x \mapsto -\frac{3}{16} x^{-2}$ is not integrable near 0.

Remark 3.1.9. *Under additional assumptions one can guarantee that a regular endpoint is transformed into a regular endpoint: Assume that $0 < C_1 < v(z) < C_2 < \infty, z \in (a_1, c)$, and $(p_1 v')' \in L^1(a_1, c)$ for some $c \in (a_1, b_1)$ and $C_1, C_2 \in \mathbb{R}$. Then one can show that, provided a_1 is a regular endpoint for τ_1, then a_2 is a regular endpoint for τ_2. Corresponding assertions hold for the right endpoint.*
Clearly, in Example 3.1.8 both of these assumptions are not satisfied.

We collect further properties of the Liouville transformation: The minimal operators associated with τ_1 and τ_2, respectively, are unitarily equivalent via L too. Selfadjoint realizations of τ_1 (with separated boundary conditions) correspond to selfadjoint realizations of τ_2 (with separated boundary conditions) such that - loosely speaking - associated spectral measures coincide.

THEOREM 3.1.10. *Let (p_1, q_1, r_1) be a tuple of SL coefficients defined on $(a_1, b_1) \subseteq \mathbb{R}$ and subject to* (3.0.2) *and let $\tau_1 = \tau(p_1, q_1, r_1)$ be the associated SL differential expression. Let $v, u : (a_1, b_1) \to \mathbb{R}$ be two functions satisfying* (3.1.1) *and L be the Liouville transformation belonging to $\frac{1}{v} \circ u^{-1}$ and u^{-1}, i.e. the inverse of the Liouville transformation belonging to v and u. Let $a_2 = u(a_1)$ and $b_2 = u(b_1)$, (p_2, q_2, r_2) be the tuple of SL coefficients defined on (a_2, b_2) given by* (3.1.2) *and $\tau_2 = \tau(p_2, q_2, r_2)$ (compare Theorem 3.1.3).*

Then:

- *We have $y \in \mathcal{D}(T_{min}(p_1, q_1, r_1))$ if and only if $Ly \in \mathcal{D}(T_{min}(p_2, q_2, r_2))$ and hence $T_{min}(p_1, q_1, r_1)$ and $T_{min}(p_2, q_2, r_2)$ are unitarily equivalent via L too.*
- *Let S_1 be a selfadjoint realization of τ_1. Then $L(\mathcal{D}(S_1))$ is the domain of some selfadjoint realization S_2 of τ_2 such that S_1 and S_2 are unitarily equivalent via L. If S_1 is a selfadjoint realization with separated boundary conditions, then S_2 is a selfadjoint realization with separated boundary conditions too.*
- *Assume that the endpoint a_1 is limit-circle for τ_1 (and hence so is a_2 for τ_2), let S_1 be a selfadjoint realization of τ_1 with separated boundary conditions and let S_2 be the selfadjoint realization of τ_2 of the previous point. If we start with the real-valued fundamental system $\{\gamma, \delta\}$ in the construction of the spectral measure for S_1 (compare Section 1.5 and Remark 1.5.9) and with $\{L\gamma, L\delta\}$ in the construction of the spectral measure for S_2, then we obtain the same Titchmarsh-Weyl m-function and hence also the same spectral measure ρ for S_1 and S_2.*

PROOF.

- Recall the definition of the minimal operator in Section 1.2.

  Let $y \in \mathcal{D}(T_{min}(p_1, q_1, r_1))$, then $W^1(y, f)(a_1) = 0$ and $W^1(y, f)(b_1) = 0$ for all $f \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$.
  Set $w = Ly$, then $w \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$ according to Theorem 3.1.3. Let now be $t \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$ arbitrary, then $t = Ls$ for some $s \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ again according to Theorem 3.1.3. Now we have (see Proposition 3.1.6)

  $$W^2(w,t)(a_2) = W^1(y,s)(a_1) = 0 \quad \text{and} \quad W^2(w,t)(b_2) = W^2(y,s)(b_1) = 0$$

  and hence $w \in \mathcal{D}(T_{min}(p_2, q_2, r_2))$. Given $w = Ly \in \mathcal{D}(T_{min}(p_2, q_2, r_2))$, the argument may be reversed to get $y \in \mathcal{D}(T_{min}(p_1, q_1, r_1))$.
- We have $S_1 = T_{max}(p_1, q_1, r_1)|_{\mathcal{D}(S_1)}$ and $LT_{max}(p_1, q_1, r_1)L^{-1} = T_{max}(p_2, q_2, r_2)$ and hence

  $$\begin{aligned} LS_1L^{-1} &= LT_{max}(p_1,q_1,r_1)\big|_{\mathcal{D}(S_1)}L^{-1} \\ &= LT_{max}(p_1,q_1,r_1)L^{-1}\big|_{L(\mathcal{D}(S_1))} = T_{max}(p_2,q_2,r_2)\big|_{L(\mathcal{D}(S_1))} \end{aligned}$$

  and

  $$\begin{aligned} T_{max}(p_2,q_2,r_2)\big|_{L(\mathcal{D}(S_1))}{}^{*} &= \left(LS_1L^{-1}\right)^{*} = LS_1^{*}L^{-1} \\ &= LS_1L^{-1} = T_{max}(p_2,q_2,r_2)\big|_{L(\mathcal{D}(S_1))} \end{aligned} \tag{3.1.7}$$

in terms of linear relations, where we have used in (3.1.7) that L is a unitary operator and S_1 a densely defined, selfadjoint operator (see [Wei00], Theorem 2.43). This shows the first assertion with $S_2 = T_{max}(p_2, q_2, r_2)|_{L(\mathcal{D}(S_1))}$.

Now assume S_1 to be a selfadjoint realization of τ_1 with separated boundary conditions so that $\mathcal{D}(S_1) = \{f \in \mathcal{D}(T_{max}(p_1, q_1, r_1)) : W^1(f, \overline{v})(a_1) = 0\}$ if a_1 is limit-circle and b_1 is limit-point ($\mathcal{D}(S_1) = \{f \in \mathcal{D}(T_{max}(p_1, q_1, r_1)) : W^1(f, \overline{u})(b_1) = 0\}$ if a_1 is limit-point and b_1 is limit-circle) or $\mathcal{D}(S_1) = \{f \in \mathcal{D}(T_{max}(p_1, q_1, r_1)) : W^1(f, \overline{v})(a_1) = W^1(f, \overline{u})(b_1) = 0\}$ if a_1 and b_1 are both limit-circle, for some $v, u \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ fulfilling

$$W^1(v, \overline{v})(a_1) = 0 \quad \text{and} \quad W^1(h_1, \overline{v})(a_1) \neq 0, \tag{3.1.8}$$

$$W^1(u, \overline{u})(b_1) = 0 \quad \text{and} \quad W^1(h_2, \overline{u})(b_1) \neq 0 \tag{3.1.9}$$

for some $h_1, h_2 \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ - recall Section 1.4. Note that according to Corollary 3.1.7 the endpoint $a_1(b_1)$ is limit-point if and only if $a_2(b_2)$ is limit-point.

Now, by Theorem 3.1.3 and Proposition 3.1.6 we have

$$Lv, Lu, Lh_1, Lh_2 \in \mathcal{D}(T_{max}(p_2, q_2, r_2)),$$

$$\begin{aligned} W^2(Lv, \overline{Lv})(a_2) &= W^2(Lv, L\overline{v})(a_2) = W^1(v, \overline{v})(a_1) = 0, \\ W^2(Lh_1, \overline{Lv})(a_2) &= W^2(Lh_1, L\overline{v})(a_2) = W^1(h_1, \overline{v})(a_1) \neq 0, \end{aligned} \tag{3.1.10}$$

$$\begin{aligned} W^2(Lu, \overline{Lu})(b_2) &= W^2(Lu, L\overline{u})(b_2) = W^1(u, \overline{u})(b_1) = 0, \\ W^2(Lh_2, \overline{Lu})(b_2) &= W^2(Lh_2, L\overline{u})(b_2) = W^1(h_2, \overline{u})(b_1) \neq 0, \end{aligned} \tag{3.1.11}$$

$$f \in \mathcal{D}(T_{max}(p_1, q_1, r_1)) \Leftrightarrow Lf \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$$

and

$$\begin{aligned} W^1(f, \overline{v})(a_1) = 0 &\Leftrightarrow W^2(Lf, \overline{Lv})(a_2) = 0, \quad f \in \mathcal{D}(T_{max}(p_1, q_1, r_1)), \\ W^1(f, \overline{u})(b_1) = 0 &\Leftrightarrow W^2(Lf, \overline{Lu})(b_2) = 0, \quad f \in \mathcal{D}(T_{max}(p_1, q_1, r_1)). \end{aligned}$$

This shows that $\mathcal{D}(S_2) = L(\mathcal{D}(S_1)) = \{g \in \mathcal{D}(T_{max}(p_2, q_2, r_2)) :$ $W^2\big(g, \overline{Lv}\big)(a_2) = 0\}$ if a_1 is limit-circle and b_1 is limit-point ($\mathcal{D}(S_2) = L(\mathcal{D}(S_1)) = \{g \in \mathcal{D}(T_{max}(p_2, q_2, r_2)) : W^2\big(g, \overline{Lu}\big)(b_2) = 0\}$ if a_1 is limit-point and b_1 is limit-circle) or $\mathcal{D}(S_2) = L(\mathcal{D}(S_1)) = \{g \in \mathcal{D}(T_{max}(p_2, q_2, r_2)) : W^2\big(g, \overline{Lv}\big)(a_2) = W^2\big(g, \overline{Lu}\big)(b_2) = 0\}$ if a_1 and b_1 are both limit-circle, where $Lv, Lu \in \mathcal{D}(T_{max}(p_2, q_2, r_2))$ fulfill (3.1.10) and (3.1.11), respectively.

- Suppose that a_1 is limit-circle for τ_1 and let S_1 be a selfadjoint realization of τ_1 with separated boundary conditions. We use the characterization of $\mathcal{D}(S_1)$ as described in Proposition 1.4.3:

$$f \in \mathcal{D}(S_1) \Leftrightarrow \cos\alpha_1 \; W^1(f, \gamma)(a_1) + \sin\alpha_1 \; W^1(f, \delta)(a_1) = 0$$

and, in addition, $W^1(f,\overline{u})(b_1) = 0$, if b_1 is limit-circle too, for some $\alpha_1 \in [0,\pi)$ and some $u \in \mathcal{D}(T_{max}(p_1,q_1,r_1))$ fulfilling (3.1.9), where $\{\gamma,\delta\}$ is some fixed, real-valued fundamental system of $(\tau_1 - z)y = 0$ (for some $z \in \mathbb{R}$) with $W^1(\gamma,\delta) \equiv 1$.

Similarly to the proof of the previous point one concludes that functions $g \in \mathcal{D}(S_2)$ are characterized by

$$\cos\alpha_1 \; W^2(g, L\gamma)(a_2) + \sin\alpha_1 \; W^2(g, L\delta)(a_2) = 0$$

and, in addition, $W^2\big(g, \overline{Lu}\big)(b_2) = 0$ if b_1 is limit-circle. Note that indeed (Theorem 3.1.3 and Proposition 3.1.6) $\{L\gamma, L\delta\}$ is a real-valued fundamental system of $(\tau_2 - z)w = 0$ with $W^2(L\gamma, L\delta) \equiv 1$.

Now let $\theta_1(\cdot,\lambda)$ and $\phi_1(\cdot,\lambda)$ be the basic solutions for S_1 fulfilling equation

$$-(p_1 y')' + q_1 y = \lambda r_1 y$$

together with

$$\begin{aligned} W^1(\theta_1(\cdot,\lambda),\gamma)(a_1) &= \cos\alpha_1, & W^1(\theta_1(\cdot,\lambda),\delta)(a_1) &= \sin\alpha_1 \\ W^1(\phi_1(\cdot,\lambda),\gamma)(a_1) &= -\sin\alpha_1, & W^1(\phi_1(\cdot,\lambda),\delta)(a_1) &= \cos\alpha_1. \end{aligned}$$

Let $m_1 : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be the Titchmarsh-Weyl m-function for S_1, i.e. $m_1(\lambda)$ is the unique complex number such that the Weyl solution

$$\psi_1(\cdot,\lambda) = \theta_1(\cdot,\lambda) + m_1(\lambda)\phi_1(\cdot,\lambda)$$

satisfies $\psi_1(\cdot,\lambda) \in L^2((a_1,b_1);r_1)$ and, if b_1 is limit-circle, $W^1(\psi_1(\cdot,\lambda),\overline{u})(b_1) = 0$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

From Theorem 3.1.3 it follows that $L\theta_1(\cdot,\lambda)$ and $L\phi_1(\cdot,\lambda)$ are solutions of

$$-(p_2 w')' + q_2 w = \lambda r_2 w.$$

From Proposition 3.1.6 it follows that

$$\begin{aligned} W^2(L\theta_1(\cdot,\lambda),L\gamma)(a_2) &= \cos\alpha_1, & W^2(L\theta_1(\cdot,\lambda),L\delta)(a_2) &= \sin\alpha_1 \\ W^2(L\phi_1(\cdot,\lambda),L\gamma)(a_2) &= -\sin\alpha_1, & W^2(L\phi_1(\cdot,\lambda),L\delta)(a_2) &= \cos\alpha_1, \end{aligned}$$

and hence $L\theta_1(\cdot,\lambda)$ and $L\phi_1(\cdot,\lambda)$ are the basic solutions for S_2.

Finally, by the linearity of L and by Theorem 3.1.3 we have

$$L\theta_1(\cdot,\lambda) + m_1(\lambda)L\phi_1(\cdot,\lambda) = L\psi_1(\cdot,\lambda) \in L^2((a_2,b_2);r_2),$$

and, if b_1 is limit-circle, according to Proposition 3.1.6 it holds that $W^2\big(L\psi_1(\cdot,\lambda),\overline{Lu}\big)(b_2) = 0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. This shows that $L\psi_1(\cdot,\lambda)$ is the Weyl solution and m_1 is the Titchmarsh-Weyl m-function for S_2.

□

Remark 3.1.11. *The assertion concerning the minimal operators could have also been proved by using*

$$\begin{aligned} T_{min}(p_1, q_1, r_1) &= \overline{T_0(p_1, q_1, r_1)} \\ &= \overline{L^{-1} T_0(p_2, q_2, r_2) L} \\ &= L^{-1} \overline{T_0(p_2, q_2, r_2)} L = L^{-1} T_{min}(p_2, q_2, r_2) L \end{aligned}$$

in terms of linear relations.

3.2 An Inverse Result

In this section we want to prove the converse of Theorem 3.1.10. We show that, if two selfadjoint realizations with separated boundary conditions of some SL differential expressions τ_1 and τ_2 have the same spectral measure, there is a Liouville transformation transforming τ_1 into τ_2. Under several different hypotheses this result is known from the literature. For the case of finite and regular left endpoints and $r \equiv 1$ it can be found in [Ben03] or for the left-definite case in [BBW09] or [Eck12]. See also [Eck], which is the source the ideas presented here are mainly taken from (parts come from [Ben03]), but as mentioned above this paper only deals with the case $p = r \equiv 1$ and hence the concept of a Liouville transformation reduces to a simple shift of the underlying interval.

In this section let $\tau_1 = \tau(p_1, q_1, r_1)$, $\tau_2 = \tau(p_2, q_2, r_2)$ be two SL differential expressions on intervals (a_1, b_1) respectively (a_2, b_2) such that a_1 is limit-circle for τ_1 and a_2 is limit-circle for τ_2 and let S_1, S_2 be two associated self-adjoint realizations with separated boundary conditions. We provide every associated quantity and object of Chapter 1 and Chapter 2 with a subscript or superscript to indicate its affiliation, e.g. by ρ_1 we denote a spectral measure constructed as in Section 1.5 associated with S_1, whereas ρ_2 denotes the spectral measure associated with S_2.

Theorem 3.2.1. *Let (p_1, q_1, r_1) and (p_2, q_2, r_2) be two tuples of SL coefficients defined on (a_1, b_1) respectively (a_2, b_2) and both subject to* (3.0.2)*. Let $\tau_1 = \tau(p_1, q_1, r_1)$ and $\tau_2 = \tau(p_2, q_2, r_2)$ be the associated SL differential expressions and assume that a_1 is limit-circle for τ_1 and that a_2 is limit-circle for τ_2. Let S_1 and S_2 be two self-adjoint realizations of τ_1 respectively τ_2 with separated boundary conditions and let ρ_1 and ρ_2 be spectral measures constructed as in Section 1.5 associated with S_1 respectively S_2.*

Suppose that $\rho_1 = \rho_2$. Then there exists a Liouville transformation L such that S_1 and S_2 are unitarily equivalent via L, i.e. $S_2 = LS_1L^{-1}$. The Liouville transformation L is the inverse of a Liouville transformation belonging to functions v and u, where v and u actually satisfy (3.1.1)*. Furthermore, the SL coefficients (p_1, q_1, r_1) and (p_2, q_2, r_2) are related as in* (3.1.2) *and hence we even have $L\tau_1 y = \tau_2 L y$ for all $y \in \mathcal{D}(\tau_1)$ (compare Theorem 3.1.3). In particular, the SL equation associated with (p_1, q_1, r_1) can be transformed into the one associated with (p_2, q_2, r_2) by a Liouville transformation.*

We state the first part of the proof as a separate lemma.

Lemma 3.2.2. *Let the hypotheses of Theorem 3.2.1 hold. Then there exists a function u mapping (a_1, b_1) bijectively onto (a_2, b_2) such that the de Branges spaces (compare Chapter 2) associated with S_1 and S_2, respectively, satisfy $B_1(x_1) = B_2(u(x_1))$, $x_1 \in (a_1, b_1)$. We have $u \in C^1(a_1, b_1)$, $u' \in AC_{loc}(a_1, b_1)$ and $u' > 0$ on (a_1, b_1).*

Proof. We claim that for each $x_1 \in (a_1, b_1)$ and each $x_2 \in (a_2, b_2)$ the quotient of the de Branges functions $E^1_{x_1}$ and $E^2_{x_2}$ is of bounded type in $\mathbb{C}^+$: According to Proposition 1.5.16, for each $x_1 \in (a_1, b_1)$ the functions $\varphi_1(x_1, \cdot)$ and $(p\varphi_1')(x_1, \cdot)$ belong to the Cartwright class $\mathcal{C}$ and so does

$$E^1_{x_1} = \varphi_1(x_1, \cdot) + \mathrm{i}(p\varphi_1')(x_1, \cdot).$$

The same holds for the de Branges function $E^2_{x_2}$, $x_2 \in (a_2, b_2)$. By a theorem of Krein (see [Lev96], Section 16.1, Theorem 1 or [BS01], Theorem A) the class $\mathcal{C}$ consists of all entire functions f for which the function $\ln^+ |f|$ has (positive) harmonic majorants in both the upper and the lower complex half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$. These functions, in turn, are precisely the entire functions which are of bounded type in $\mathbb{C}^+$ and $\mathbb{C}^-$ (see, e.g., [Wor04], Section 3.1). Actually, this equivalence is often used for an alternative definition of being of bounded type. However, we see that $E^1_{x_1}$ and $E^2_{x_2}$ are of bounded type in $\mathbb{C}^+$ for each $x_1 \in (a_1, b_1)$ and each $x_2 \in (a_2, b_2)$ and so is their quotient $E^1_{x_1}/E^2_{x_2}$ because $E^2_{x_2}$ does not have any zeros in $\mathbb{C}^+$.

Now fix some arbitrary $x_1 \in (a_1, b_1)$. By Corollary 2.1.3, for each $x_2 \in (a_2, b_2)$, both $B_1(x_1)$ and $B_2(x_2)$ are isometrically embedded in $L^2(\mathbb{R}; \rho)$ ($\rho := \rho_1 = \rho_2$), and we infer from Theorem A.2.1 that $B_1(x_1)$ is contained in $B_2(x_2)$ or $B_2(x_2)$ is contained in $B_1(x_1)$. Note that $E^1_{x_1}/E^2_{x_2}$ has indeed no real zeros or singularities since $E^1_{x_1}$ and $E^2_{x_2}$ do not have real zeros.

Set (at the moment x_1 is still fixed)

$$u(x_1) = \inf\{x_2 \in (a_2, b_2) : B_1(x_1) \subseteq B_2(x_2)\}.$$

First, note that the set on the right-hand side is not empty because otherwise we had $B_2(x_2) \subsetneq B_1(x_1)$ for all $x_2 \in (a_2, b_2)$. By Corollary 2.1.3 this would mean that $B_1(x_1)$ is dense in $L^2(\mathbb{R}; \rho)$, contradicting Corollary 2.1.4. Hence, we always have $u(x_1) \in [a_2, b_2)$. However, $u(x_1) = a_2$ if and only if $B_1(x_1) \subseteq B_2(x_2)$ for all $x_2 \in (a_2, b_2)$. This would imply that for every function $h \in B_1(x_1)$ and $\zeta \in \mathbb{C}$ we had

$$\begin{aligned} |h(\zeta)| = |\langle h, K^2_{x_2}(\zeta, \cdot)\rangle_{B_2(x_2)}| &\leq \|h\|_{B_2(x_2)} \langle K^2_{x_2}(\zeta, \cdot), K^2_{x_2}(\zeta, \cdot)\rangle^{\frac{1}{2}}_{B_2(x_2)} \\ &= \|h\|_{B_1(x_1)} K^2_{x_2}(\zeta, \zeta)^{\frac{1}{2}} \end{aligned}$$

for each $x_2 \in (a_2, b_2)$. Since $K^2_{x_2}(\zeta, \zeta) \xrightarrow{x_2 \to a_2} 0$ by (2.1.4), we then had $B_1(x_1) = \{0\}$, contradicting Theorem 2.1.2. So we have $u(x_1) \in (a_2, b_2)$.

Now, from (2.1.7) we infer that

$$B_2(u(x_1)) = \overline{\bigcup_{x_2 < u(x_1)} B_2(x_2)} \subseteq B_1(x_1) \subseteq \bigcap_{u(x_1) < x_2} B_2(x_2) = B_2(u(x_1))$$

and hence even

$$B_1(x_1) = B_2(u(x_1)), \tag{3.2.1}$$

including the inner product.

Now we may vary with x_1 and consider u as a function $u : (a_1, b_1) \to (a_2, b_2) : x_1 \mapsto u(x_1)$. Note that u is uniquely defined by (3.2.1). It remains to show that u has the claimed properties. By Corollary 2.1.4 it is clear that u is strictly increasing. Therefore, to show that u is continuous let $x, x_n \in (a_1, b_1)$, $n \in \mathbb{N}$, and $x_n \uparrow x$. By Corollary 2.1.4 we have

$$B_1(x) = \overline{\bigcup_{n\in\mathbb{N}} B_1(x_n)} = \overline{\bigcup_{n\in\mathbb{N}} B_2(u(x_n))} = B_2\left(\sup_{n\in\mathbb{N}} u(x_n)\right) = B_2\left(\lim_{n\in\mathbb{N}} u(x_n)\right)$$

and hence $u(x) = \lim_{n\in\mathbb{N}} u(x_n)$. Similarly, if $x_n \downarrow x$, we obtain

$$B_1(x) = \bigcap_{n\in\mathbb{N}} B_1(x_n) = \bigcap_{n\in\mathbb{N}} B_2(u(x_n)) = B_2\left(\inf_{n\in\mathbb{N}} u(x_n)\right) = B_2\left(\lim_{n\in\mathbb{N}} u(x_n)\right),$$

meaning that $u(x) = \lim_{n\in\mathbb{N}} u(x_n)$.

To show that u is actually a bijection it is sufficient to show that $u(x_1) \to a_2$ as $x_1 \downarrow a_1$ and that $u(x_1) \to b_2$ as $x_1 \uparrow b_1$. However, the first claim follows from (compare (2.1.4))

$$K^2_{u(x_1)}(\zeta, \zeta) = K^1_{x_1}(\zeta, \zeta) \xrightarrow{x_1 \to a_1} 0, \quad \zeta \in \mathbb{C},$$

the second one is again a simple consequence of (2.1.6) and (2.1.7). Indeed, assume that $u(x_1) \nrightarrow b_2$ as $x_1 \uparrow b_1$. Then we had $\sup_{x_1 \in (a_1, b_1)} u(x_1) = c < b_2$ and

$$L^2(\mathbb{R}; \rho) = \overline{\bigcup_{x_1\in(a_1,b_1)} B_1(x_1)} = \overline{\bigcup_{x_1\in(a_1,b_1)} B_2(u(x_1))} = B_2(c) \subsetneq L^2(\mathbb{R}; \rho).$$

Now, because of $K^2_{u(x_1)}(z, z) = K^1_{x_1}(z, z)$, $z \in \mathbb{C}$, and using (2.1.4) we get

$$\int_{a_2}^{u(x_1)} |\varphi_2(\cdot, z)|^2 r_2(\cdot) dx = \int_{a_1}^{x_1} |\varphi_1(\cdot, z)|^2 r_1(\cdot) dx, \quad x_1 \in (a_1, b_1).$$

Herein, due to the conditions (3.0.2), both integrands are continuous. If $z \in \mathbb{C}\backslash\mathbb{R}$, both integrands, in particular the first one, do not vanish and hence the implicit function theorem yields that u is continuously differentiable and that

$$|\varphi_2(u(x_1), z)|^2 r_2(u(x_1)) \cdot u'(x_1) = |\varphi_1(x_1, z)|^2 r_1(x_1), \quad x_1 \in (a_1, b_1). \tag{3.2.2}$$

Note that this holds for all $z \in \mathbb{C}$. However, for $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$u'(x_1) = \frac{|\varphi_1(x_1, z)|^2 r_1(x_1)}{|\varphi_2(u(x_1), z)|^2 r_2(u(x_1))} > 0, \quad x_1 \in (a_1, b_1), \tag{3.2.3}$$

and Theorem A.1.1 shows that $u' \in AC_{loc}(a_1, b_1)$. □

For the proof of Theorem 3.2.1 we need another lemma.

LEMMA 3.2.3. *Suppose H is a unitary map on a weighted L^2-space $L^2((a,b);r)$ with the property that*

$$\sup(\operatorname{supp} Hy) = \sup(\operatorname{supp} y), \quad y \in L^2((a,b);r). \tag{3.2.4}$$

Then H is multiplication by a function $h \in L^\infty(a,b)$ of absolute value 1 almost everywhere.

PROOF. W.l.o.g. we may assume $r \equiv 1$. Otherwise, consider the map THT^{-1} where

$$T : L^2((a,b);r) \to L^2(a,b) : y \mapsto r^{\frac{1}{2}} y.$$

Obviously, T is unitary and preserves supports and hence $THT^{-1} : L^2(a,b) \to L^2(a,b)$ is unitary and has the assumed property. If then $THT^{-1} = M_h$ is multiplication by h, so is $H = T^{-1} M_h T = M_h$.

Let $H : L^2(a,b) \to L^2(a,b)$ be unitary and satisfy (3.2.4). Clearly, H^{-1} has the same property as H and thus $H(L^2(a,c)) = L^2(a,c)$ for every $c \in (a,b)$. Hence, if $y = 0$ in (a,c) a.e., then so is its image since it is orthogonal to $L^2(a,c)$, and H actually preserves convex hulls of supports.

Let $(c,d) \subseteq (a,b)$ be an arbitrary open subinterval of (a,b). For $y \in L^2(a,b)$ we have

$$y = \mathbb{1}_{(a,c)} y + \mathbb{1}_{(c,d)} y + \mathbb{1}_{(d,b)} y \quad \text{a.e.}$$

and

$$\mathbb{1}_{(a,c)} Hy + \mathbb{1}_{(c,d)} Hy + \mathbb{1}_{(d,b)} Hy = Hy = H \mathbb{1}_{(a,c)} y + H \mathbb{1}_{(c,d)} y + H \mathbb{1}_{(d,b)} y \quad \text{a.e..}$$

Because of the previous observation we conclude that

$$H \mathbb{1}_{(c,d)} y = \mathbb{1}_{(c,d)} Hy, \tag{3.2.5}$$

and hence we have

$$\int_c^d |Hy|^2 dx = \int_c^d |y|^2 dx, \quad a \le c < d \le b, y \in L^2(a,b). \tag{3.2.6}$$

Now let $\big((c_n,d_n)\big)_{n\in\mathbb{N}}$ be an increasing sequence of subintervals of (a,b) such that every (c_n,d_n) is finite and

$$\bigcup_{n\in\mathbb{N}} (c_n,d_n) = (a,b).$$

We then may define a function $h \in L^\infty(a,b)$ by defining it on every (c_n,d_n) as follows: Let $y_n = \mathbb{1}_{(c_n,d_n)}$. Since (c_n,d_n) is finite, we have $y_n \in L^2(a,b)$ and we can set $h = Hy_n$ on (c_n,d_n). Because of (3.2.6) we obtain from the Lebesgue differentiation Theorem that h has absolute value 1 a.e. in (c_n,d_n), and (3.2.5) shows that h is indeed well-defined on all of (a,b). We even have $\mathbb{1}_{(c,d)} h = H \mathbb{1}_{(c,d)}$ for every finite subinterval

(c,d), however, this just means $M_h \mathbb{1}_{(c,d)} = H\mathbb{1}_{(c,d)}$, where M_h is multiplication by h. By linearity it follows that $M_h y = Hy$ for every integrable step function y. Since the set of integrable step functions is dense in $L^2(a,b)$ and both M_h and H are continuous on $L^2(a,b)$, we conclude that $M_h = H$.

□

Proof of Theorem 3.2.1. Let u be the function from Lemma 3.2.2. Define $v : (a_1, b_1) \to \mathbb{R}$ by

$$v(x) = \sqrt{u'(x)\frac{r_2(u(x))}{r_1(x)}}, \quad x \in (a_1, b_1). \tag{3.2.7}$$

Note that $v(x) > 0$, $x \in (a_1, b_1)$. By (3.2.3) we have, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$v(x) = \sqrt{\frac{|\varphi_1(x,\lambda)|^2}{|\varphi_2(u(x),\lambda)|^2}} = \frac{|\varphi_1(x,\lambda)|}{|\varphi_2(u(x),\lambda)|}, \quad x \in (a_1, b_1). \tag{3.2.8}$$

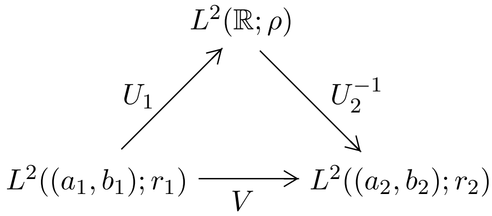
Since the radicand is continuously differentiable with absolutely continuous derivative (by Theorem A.1.1) and greater than zero for all $x \in (a_1, b_1)$, we see that $v \in C^1(a_1, b_1)$ with $v' \in AC_{loc}(a_1, b_1)$ (by Theorem A.1.1).

So the functions v and u indeed satisfy (3.1.1) and hence give rise to a Liouville transformation whose inverse we want to denote by L, i.e. L is the Liouville transformation belonging to $\frac{1}{v} \circ u^{-1}$ and u^{-1}. From (3.2.7) we obtain the relation

$$r_2 = \frac{v^2 r_1}{u'} \circ u^{-1}, \tag{3.2.9}$$

and hence using Theorem 3.1.3 we see that L is a unitary mapping of $L^2((a_1, b_1); r_1)$ onto $L^2((a_2, b_2); r_2)$.

Now consider the generalized Fourier transforms U_1 and U_2 belonging to S_1 and S_2, respectively. The assumption $\rho_1 = \rho_2$ (in the following we write $\rho = \rho_1 = \rho_2$) implies that U_1 and U_2 have the same target space, so we may consider the composition $V = U_2^{-1} U_1$ and obtain a unitary map $V : L^2((a_1, b_1); r_1) \to L^2((a_2, b_2); r_2)$.



We will show that $V = \pm L$. Since

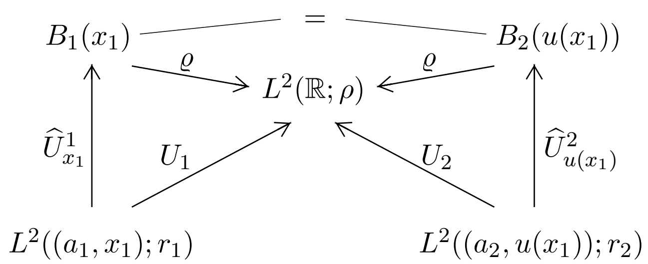
$$S_1 = U_1^{-1} M_{\mathrm{id}} U_1, \quad S_2 = U_2^{-1} M_{\mathrm{id}} U_2$$

(remember, M_{id} is multiplication with $\mathrm{id} : t \mapsto t$ in $L^2(\mathbb{R};\rho)$), we have

$$S_2 = U_2^{-1} U_1 S_1 U_1^{-1} U_2 = V S_1 V^{-1} = L S_1 L^{-1}.$$

The main step in proving $V = \pm L$ is to note that V maps $L^2((a_1, x_1); r_1)$ onto $L^2((a_2, u(x_1); r_2)$ for each $x \in (a_1, b_1)$. However, this follows from (see the diagram below and compare with the proof of Corollary 2.1.3)

$$U_1(L^2((a_1, x_1); r_1)) = \varrho(B_1(x_1)) = \varrho(B_2(u(x_1)) = U_2(L^2((a_2, u(x_1)); r_2)). \quad (3.2.10)$$



In particular, (3.2.10) implies that[2]

$$\sup(\operatorname{supp} V f) = u(\sup(\operatorname{supp} f)), \quad f \in L^2((a_1, b_1); r_1).$$

Consider the Liouville transformation $L^{-1} : L^2((a_2, b_2); r_2) \to L^2((a_1, b_1); r_1) : f \mapsto v \cdot (f \circ u)$. Since $v \neq 0$ in (a_1, b_1) and $u : (a_1, b_1) \to (a_2, b_2)$ is a diffeomorphism, we have

$$\operatorname{supp} L^{-1} f = u^{-1}(\operatorname{supp} f), \quad f \in L^2((a_2, b_2); r_2).$$

It immediately follows that the unitary map $L^{-1}V : L^2((a_1, b_1); r_1) \to L^2((a_1, b_1); r_1)$ satisfies the assumption of Lemma 3.2.3, i.e.

$$\sup(\operatorname{supp} L^{-1} V f) = \sup(\operatorname{supp} f), \quad f \in L^2((a_1, b_1); r_1),$$

and we infer that $L^{-1}V = M_h$ is multiplication by a function $h \in L^\infty(a_1, b_1)$ of absolute value 1 almost everywhere.

Now, by (1.5.9) and Proposition 1.5.4 it is clear that U_1 maps real-valued functions $f \in L^2((a_1, b_1); r_1)$ with compact support to real-valued functions in $L^2(\mathbb{R};\rho)$. Since the set of real-valued functions in $L^2(\mathbb{R};\rho)$ is a closed subspace of $L^2(\mathbb{R};\rho)$ and $U_1 f = \lim_{n\to\infty} U_1 f_n$ for every $f \in L^2((a_1, b_1); r_1)$, where $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $L^2((a_1, b_1); r_1)$ with compact support converging to f, we see that U_1 maps real-valued functions to real-valued functions. The same holds for U_2 and so it does for V and even for $L^{-1}V$ since L^{-1} obviously maps real-valued functions to real-valued functions. We infer that h must be real-valued and hence $h = \pm 1$ almost everywhere.

[2] If $\sup(\operatorname{supp} f) = b_1$, this equation has to be understood in terms of Remark 3.1.2.

From Lemma 1.5.13 it follows that $V\psi_1(\cdot,\lambda) = \psi_2(\cdot,\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and hence that

$$L^{-1}\psi_2(\cdot,\lambda) = h\cdot\psi_1(\cdot,\lambda).$$

In here the function on the left-hand side is absolutely continuous and so is the function on the right-hand side. Because $\psi_1(x,\lambda) \neq 0$, $x \in (a_1,b_1)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, h cannot have any points of discontinuity, meaning $h \equiv 1$ or $h \equiv -1$ a.e., thus $L^{-1}V = \pm\,\mathrm{Id}$ and $V = \pm L$ indeed.

We have to show that (p_1,q_1,r_1) and (p_2,q_2,r_2) are related as in (3.1.2). For r_1 and r_2 this is (3.2.9). In order to show the claim for p_1 and p_2, we use Corollary 1.5.15. By (1.5.13) we have

$$\begin{aligned}\lim_{t\to\infty}\sqrt{\frac{2}{t}}\ln\left|\frac{\varphi_1(\tilde{x},\mathrm{i}t)}{\varphi_1(\hat{x},\mathrm{i}t)}\right| &= \int_{\hat{x}}^{\tilde{x}}\sqrt{\frac{r_1}{p_1}}dx, \quad \tilde{x},\hat{x}\in(a_1,b_1),\\ \lim_{t\to\infty}\sqrt{\frac{2}{t}}\ln\left|\frac{\varphi_2(\tilde{y},\mathrm{i}t)}{\varphi_2(\hat{y},\mathrm{i}t)}\right| &= \int_{\hat{y}}^{\tilde{y}}\sqrt{\frac{r_2}{p_2}}dy, \quad \tilde{y},\hat{y}\in(a_2,b_2).\end{aligned} \tag{3.2.11}$$

From (3.2.2) and (3.2.7) it follows that we have

$$|\varphi_1(x_1,\lambda)|^2 = v(x_1)^2|\varphi_2(u(x_1),\lambda)|^2, \quad x_1\in(a_1,b_1),\lambda\in\mathbb{C}, \tag{3.2.12}$$

and hence

$$\frac{|\varphi_1(\tilde{x},\mathrm{i}t)|}{|\varphi_1(\hat{x},\mathrm{i}t)|} = \frac{v(\tilde{x})}{v(\hat{x})}\frac{|\varphi_2(u(\tilde{x}),\mathrm{i}t)|}{|\varphi_2(u(\hat{x}),\mathrm{i}t)|}, \quad \tilde{x},\hat{x}\in(a_1,b_1), t>0.$$

From (3.2.11) we infer that

$$\int_{\hat{x}}^{\tilde{x}}\sqrt{\frac{r_1}{p_1}}dx = \int_{u(\hat{x})}^{u(\tilde{x})}\sqrt{\frac{r_2}{p_2}}dy, \quad \tilde{x},\hat{x}\in(a_1,b_1).$$

Differentiating with respect to $\tilde{x}$ yields

$$\sqrt{\frac{r_1}{p_1}}(\tilde{x}) = \sqrt{\frac{r_2}{p_2}}\circ u(\tilde{x})\cdot u'(\tilde{x}), \quad \tilde{x}\in(a_1,b_1),$$

and using (3.2.9) we finally get after a simple calculation

$$p_2 = (v^2p_1u')\circ u^{-1}. \tag{3.2.13}$$

It remains to show that q_1 and q_2 are related as in (3.1.2).

By (3.2.12) we have

$$\varphi_1(x_1,\lambda)^2 = (v(x_1)\varphi_2(u(x_1),\lambda))^2, \quad x_1\in(a_1,b_1),\lambda\in\mathbb{R}.$$

Since the zeros of every nontrivial solution of a SL equation with real spectral parameter are isolated in the interior of the underlying interval, compare Theorem 2.6.1 in [Zet05], we may take logarithmic derivatives and obtain

$$\frac{\varphi_1'(x_1,\lambda)}{\varphi_1(x_1,\lambda)} = \frac{(v(x_1)\varphi_2(u(x_1),\lambda))'}{v(x_1)\varphi_2(u(x_1),\lambda)} \quad \text{for almost all } x_1\in(a_1,b_1),\lambda\in\mathbb{R}, \tag{3.2.14}$$

and hence

$$\begin{aligned}\frac{p_1(x_1)\varphi_1'(x_1,\lambda)}{\varphi_1(x_1,\lambda)} &= \frac{p_1(x_1)v'(x_1)\varphi_2(u(x_1),\lambda)}{v(x_1)\varphi_2(u(x_1),\lambda)} + \frac{p_1(x_1)v(x_1)\varphi_2'(u(x_1),\lambda)u'(x_1)}{v(x_1)\varphi_2(u(x_1),\lambda)}\\ &= \frac{p_1(x_1)v'(x_1)}{v(x_1)} + \frac{p_2(u(x_1))\varphi_2'(u(x_1),\lambda)}{v^2(x_1)\varphi_2(u(x_1),\lambda)} \quad \text{a.e.}, \lambda\in\mathbb{R},\end{aligned}$$

where the second equality is due to (3.2.13).

Differentiating and using that

$$\begin{aligned}-(p_1\varphi_1(\cdot,\lambda)')' + q_1\varphi_1(\cdot,\lambda) &= \lambda r_1\varphi_1(\cdot,\lambda) \quad \text{a.e. in } (a_1,b_1),\\ -(p_2\varphi_2(\cdot,\lambda)')' + q_2\varphi_2(\cdot,\lambda) &= \lambda r_2\varphi_2(\cdot,\lambda) \quad \text{a.e. in } (a_2,b_2),\end{aligned}$$

we obtain

$$\begin{aligned}q_1(x_1) - \lambda r_1(x_1) - p_1(x_1)\left(\frac{\varphi_1'(x_1,\lambda)}{\varphi_1(x_1,\lambda)}\right)^2 &= \frac{(p_1v')'(x_1)}{v(x_1)} - p_1(x_1)\left(\frac{v'(x_1)}{v(x_1)}\right)^2\\ &+\frac{u'(x_1)}{v(x_1)^2}q_2(u(x_1)) - \lambda\frac{u'(x_1)r_2(u(x_1))}{v(x_1)^2} - 2\,\frac{p_2(u(x_1))v'(x_1)}{v(x_1)^3}\frac{\varphi_2'(u(x_1),\lambda)}{\varphi_2(u(x_1),\lambda)}\\ &-\frac{p_2(u(x_1))u'(x_1)}{v(x_1)^2}\left(\frac{\varphi_2'(u(x_1),\lambda)}{\varphi_2(u(x_1),\lambda)}\right)^2\end{aligned} \tag{3.2.15}$$

for almost all $x_1\in(a_1,b_1)$ and all $\lambda\in\mathbb{R}$.

By (3.2.9) we have, for $\lambda\in\mathbb{C}$,

$$-\lambda r_1(x_1) = -\lambda\frac{u'(x_1)r_2(u(x_1))}{v(x_1)^2}, \quad x_1\in(a_1,b_1),$$

and using (3.2.13) it is easy to verify that

$$\begin{aligned}&-p_1(x_1)\left(\frac{v'(x_1)}{v(x_1)}\right)^2 - 2\,\frac{p_2(u(x_1))v'(x_1)}{v(x_1)^3}\frac{\varphi_2'(u(x_1),\lambda)}{\varphi_2(u(x_1),\lambda)}\\ &\quad-\frac{p_2(u(x_1))u'(x_1)}{v(x_1)^2}\left(\frac{\varphi_2'(u(x_1),\lambda)}{\varphi_2(u(x_1),\lambda)}\right)^2 = -p_1(x_1)\left(\frac{(v(x_1)\varphi_2(u(x_1),\lambda))'}{v(x_1)\varphi_2(u(x_1),\lambda)}\right)^2\end{aligned}$$

for almost all $x_1\in(a_1,b_1)$.

Hence, using (3.2.14), we obtain from (3.2.15)

$$q_1(x_1) = \frac{(p_1v')'(x_1)}{v(x_1)} + \frac{u'(x_1)}{v(x_1)^2}q_2(u(x_1)) \quad \text{a.e. in } (a_1,b_1),$$

which is equivalent to

$$q_2 = \left[\frac{v}{u'}(q_1v - (p_1v')')\right]\circ u^{-1} \quad \text{a.e. in } (a_1,b_1),$$

so that the proof is complete. □

Chapter 4

Transforming Sturm-Liouville Equations by a Liouville Transformation

Origin of our considerations of the Liouville transformation at the beginning of Chapter 3 was the transformation

$$y(z) = v(z)w(u(z))$$

in

$$-(p_1y')' + q_1y = \lambda r_1y \quad \text{on } (a_1, b_1) \subseteq \mathbb{R}, \tag{4.0.1}$$

leading to the SL equation

$$-(p_2w')' + q_2w = \lambda r_2w \quad \text{on } (a_2, b_2) \subseteq \mathbb{R}, \tag{4.0.2}$$

where the SL coefficients (p_1, q_1, r_1) and (p_2, q_2, r_2) are related as in (3.1.2).

Solving equation (4.0.1) is then equivalent to solving equation (4.0.2), and hence one may tempt to choose u and v in such a way that the transformed equation (4.0.2) is easier to deal with than the original equation (4.0.1). For example, if we could choose u and v such that (p_2, q_2, r_2) is a tuple of constant SL coefficients, (4.0.2) is always explicit solvable and hence so is (4.0.1).

Furthermore, there are several special forms of the general SL equation (like, e.g., the potential form, see the following section), and some assertions of Sturm-Liouville Theory may only (or at least easier) be shown for equations in one (or some) of these special forms. Using an appropriate Liouville transformation (and having the knowledge of the previous chapter about it), one can try to transfer such an assertion to the case of a general SL equation.

In the following we describe transformability by the equivalence relation $\sim_{\mathcal{K}}$ on the set of SL coefficients $\mathcal{K}$ as introduced in Definition 3.1.4. We present some special forms of SL equations and deal with the question whether a general SL equation can be transformed into them[1] and - if so - how this can be done. We tempt to identify "canonical representative systems" (see Definition 4.1.5) and give necessary and sufficient conditions for a SL equation in order to can be transformed into an equation with constant coefficients. Finally, we show in Section 4.2 that for a restricted class of SL coefficients the equivalence relation $\sim_{\mathcal{K}}$ corresponds to an equivalence relation on the set of associated maximal operators.

[1] In this chapter, if we speak of transforming a SL equation, we always mean transforming by a Liouville transformation.

4.1 Special Forms of Sturm-Liouville Equations

In this section we present some special forms of SL equations and deal with the question whether a general SL equation can be transformed into them. To this end let[2] $(p_1, q_1, r_1) \in \mathcal{K}$ (defined on (a_1, b_1)) always be a tuple of general SL coefficients (i.e. satisfying only conditions (3.0.2)) - however, sometimes additional requirements will be necessary. By the notation $(p, q, 1)$ (as it appears in the next subsection) we mean a tuple $(p, q, r) \in \mathcal{K}$ where $r \equiv 1$ - the meaning of similar notations should be clear then.

4.1.1 $(p, q, 1)$-Form

In order to obtain $(p_1, q_1, r_1) \sim_{\mathcal{K}} (p, q, 1)$ for some p and q, we have to solve

$$1 = \frac{v^2 r_1}{u'} \circ u^{-1}$$

or equivalently

$$1 = \frac{v^2 r_1}{u'}.$$

This can always be done, e.g., by choosing $v \equiv 1$ on (a_1, b_1) and

$$u(x) = \begin{cases} \int_c^x r_1(t)dt + d, & c \le x < b_1, \\ -\int_x^c r_1(t)dt + d, & a_1 < x < c, \end{cases}$$

for some $c \in (a_1, b_1)$ and $d \in \mathbb{R}$. Since $r_1 \in AC_{loc}(a_1, b_1)$ and $r_1 > 0$ on (a_1, b_1), we indeed have $u \in C^1(a_1, b_1), u' = r_1 \in AC_{loc}(a_1, b_1)$ and $u' > 0$ on (a_1, b_1).

Hence, we then have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (p_1 r_1 \circ u^{-1}, \frac{q_1}{r_1} \circ u^{-1}, 1)$.

However, if r_1 is not only out of $AC_{loc}(a_1, b_1)$ but instead $r_1 \in C^1(a_1, b_1)$ with $r_1' \in AC_{loc}(a_1, b_1)$, we also can choose u to be the identity on (a_1, b_1), $u = \mathrm{id}_{(a_1,b_1)}$, and $v = \sqrt{\frac{1}{r_1}}$. We then have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (p, q, 1)$ with

$$p = \frac{p_1}{r_1} \quad \text{and} \quad q = \frac{q_1}{r_1} + \frac{1}{2}\sqrt{\frac{1}{r_1}}\left(p_1 r_1^{-\frac{3}{2}} r_1'\right)'.$$

Finally, it is easy to check that $(1, 1, 1) \sim_{\mathcal{K}} (x \mapsto 4x, x \mapsto 1 + \frac{\sqrt{2}}{4} x^{-\frac{3}{4}}, 1)$, where both tuples are defined on $(0, 1)$ ($v(z) = \sqrt{2z}$, $u(z) = z^2$), and so we see that there are tuples $(p_1, q_1, 1)$ and $(p_2, q_2, 1)$ both defined on the same interval with $(p_1, q_1, 1) \sim_{\mathcal{K}} (p_2, q_2, 1)$ but $(p_1, q_1) \neq (p_2, q_2)$.

[2]For the following it is essential to recall Definition 3.1.4 and the second item of Remark 3.1.5.

4.1.2 $(1, q, r)$-Form

Similarly to above we see that $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q, r)$ for some q and r if we choose $v \equiv 1$ on (a_1, b_1) and

$$u(x) = \begin{cases} \int_c^x \frac{1}{p_1}(t)dt + d, & c \leq x < b_1, \\ -\int_x^c \frac{1}{p_1}(t)dt + d, & a_1 < x < c, \end{cases}$$

for some $c \in (a_1, b_1)$ and $d \in \mathbb{R}$ and that we then have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q_1 p_1 \circ u^{-1}, r_1 p_1 \circ u^{-1})$.

Like above, if p_1 is not only out of $AC_{loc}(a_1, b_1)$ but instead $p_1 \in C^1(a_1, b_1)$ with $p_1' \in AC_{loc}(a_1, b_1)$, we can choose u to be the identity on (a_1, b_1), $u = \mathrm{id}_{(a_1,b_1)}$, and $v = \sqrt{\frac{1}{p_1}}$. We then have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q, r)$ with

$$q = \frac{q_1}{p_1} + \frac{1}{\sqrt{p_1}}(\sqrt{p_1})'' \quad \text{and} \quad r = \frac{r_1}{p_1}.$$

Again, one can easily find an example of tuples $(1, q_1, r_1)$ and $(1, q_2, r_2)$ both defined on the same (finite) interval with $(1, q_1, r_1) \sim_{\mathcal{K}} (1, q_2, r_2)$ but $(q_1, r_1) \neq (q_2, r_2)$.

4.1.3 Potential Form

A SL equation of the form

$$-y'' + qy = \lambda y,$$

i.e. $p \equiv r \equiv 1$, is usually called a potential equation or a SL equation in potential form or Schrödinger form with potential q - due to its application in physics.

In order to obtain $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q, 1)$ for some q, we have to solve

$$\begin{aligned} 1 &= v^2 p_1 u', \\ 1 &= \frac{v^2 r_1}{u'}. \end{aligned} \tag{4.1.1}$$

For u and v to solve (4.1.1) it is necessary that $u' = \sqrt{\frac{r_1}{p_1}}$ and that $v = \pm(p_1 r_1)^{-\frac{1}{4}}$. However, in order to have $v \in C^1(a_1, b_1)$ with $v' \in AC_{loc}(a_1, b_1)$, this in general requires $p_1, r_1 \in C^1(a_1, b_1)$ with $p_1', r_1' \in AC_{loc}(a_1, b_1)$. If this is the case, we may choose

$$v = (p_1 r_1)^{-\frac{1}{4}} \tag{4.1.2}$$

and

$$u(x) = \begin{cases} \int_c^x \sqrt{\frac{r_1}{p_1}}(t)dt + d, & c \leq x < b_1, \\ -\int_x^c \sqrt{\frac{r_1}{p_1}}(t)dt + d, & a_1 < x < c, \end{cases} \tag{4.1.3}$$

for some $c \in (a_1, b_1)$ and $d \in \mathbb{R}$.

We then have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q, 1)$ with

$$q = \left(\frac{q_1}{r_1} + \frac{1}{4} \sqrt[4]{\frac{p_1}{r_1^3}} \left(\sqrt[4]{\frac{1}{p_1 r_1^5}} \, (p_1 r_1)' \right)' \right) \circ u^{-1}. \tag{4.1.4}$$

Once again, we want to point out that this only works for a restricted class of tuples of SL coefficients $(p_1, q_1, r_1) \in \mathcal{K}$, namely those satisfying $p_1, r_1 \in C^1(a_1, b_1)$ with $p_1', r_1' \in AC_{loc}(a_1, b_1)$ (in fact, it is necessary and sufficient that $(p_1 r_1) \in C^1(a_1, b_1)$ with $(p_1 r_1)' \in AC_{loc}(a_1, b_1)$).

The Liouville transformation belonging to v and u from (4.1.2) and (4.1.3), respectively, or rather its inverse is the one originally invented by Liouville and often referred to as *the* Liouville transformation, and in this context a potential equation is also called an equation in Liouville normal form. In the literature dealing with *the* Liouville transformation one finds rather different assumptions on the endpoints a_1, b_1 and the SL coefficients (p_1, q_1, r_1). Some authors assume a_1 and b_1 to be finite, $q_1 \in C^0[a_1, b_1]$, $p_1, r_1 \in C^2[a_1, b_1]$ and $p_1, r_1 > 0$ on $[a_1, b_1]$ (e.g. [Tesed], Problem 5.13; similarly [She05]) or a_1, b_1 to be finite, $q_1, r_1 \in C^0[a_1, b_1]$, $p_1 \in C^1[a_1, b_1]$ with $(p_1 r_1) \in C^2[a_1, b_1]$ and $p_1, r_1 > 0$ on $[a_1, b_1]$ (e.g. [Yos60]). Under these assumptions a_2, b_2 clearly are finite too and, according to Remark 3.1.9, *the* Liouville transformation transforms regular endpoints into regular endpoints. Furthermore, note that under these assumptions one can choose $c = a_1$ in (4.1.3), which is usually done then. In general, choosing $c = a_1$ in (4.1.3) works whenever $\sqrt{r_1/p_1}$ is integrable near a_1 - similarly for b_1.

In [DKS76] the authors' assumptions are a_1, b_1 being finite, $q_1 \in C^0[a_1, b_1]$ and $p_1, r_1 \in C^2[a_1, b_1]$ with $p_1, r_1 > 0$ on (a_1, b_1) and the slight difference of assuming $p_1, r_1 > 0$ only on (a_1, b_1) (compared to $p_1, r_1 > 0$ on $[a_1, b_1]$) brings that the properties stated in the paragraph above do not hold anymore. In [Eve82] the assumptions on the endpoints a_1, b_1 and the SL coefficients (p_1, q_1, r_1) are the same as ours, i.e. arbitrary $a_1, b_1 \in \mathbb{R} \cup \{-\infty, \infty\}$ (of course $a_1 < b_1$), (p_1, q_1, r_1) satisfying (3.0.2) and additionally $p_1, r_1 \in C^1(a_1, b_1)$ with $p_1', r_1' \in AC_{loc}(a_1, b_1)$.

Finally, let $(1, q_1, 1) \sim_{\mathcal{K}} (1, q_2, 1)$. Then it immediately follows from (4.1.1) that $v^2 \equiv 1$ and $u' \equiv 1$, and we infer that u is a shift mapping (a_1, b_1) onto (a_2, b_2).

4.1.4 String Form

An equation of the form

$$-y'' = \lambda r y, \tag{4.1.5}$$

i.e. $p \equiv 1$ and $q \equiv 0$, is usually called a string equation or a SL equation in string form, and the weight function r is then called the density function of (4.1.5).

The question of whether a general SL equation can be transformed into an equation in string form is a bit more subtle to deal with than in the previous cases. In order to obtain $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, 0, r)$ for some $(1, 0, r) \in \mathcal{K}$, we have to find functions v and u satisfying (3.1.1) and solving

$$\begin{aligned} 1 &= v^2 p_1 u', \\ 0 &= \frac{v}{u'}(q_1 v - (p_1 v')'), \end{aligned} \tag{4.1.6}$$

and then we had $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, 0, \frac{v^2 r_1}{u'} \circ u^{-1})$ or, using the first equation of (4.1.6), $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, 0, v^4 p_1 r_1 \circ u^{-1})$. Since v has to satisfy $v \neq 0$ in (a_1, b_1), the second equation of (4.1.6) is equivalent to $0 = q_1 v - (p_1 v')'$.

We want to note two things: Finding an appropriate v requires finding a real-valued, zero free solution of the given SL equation with spectral parameter $\lambda = 0$ - in general, if ever, this is not explicitly feasible. Having found an appropriate v, finding an appropriate u is no further difficulty - one can (and in fact has to) choose

$$u(x) = \begin{cases} \int_c^x \frac{1}{v^2 p_1}(t)dt + d, & c \leq x < b_1, \\ -\int_x^c \frac{1}{v^2 p_1}(t)dt + d, & a_1 < x < c, \end{cases}$$

for some $c \in (a_1, b_1)$ and $d \in \mathbb{R}$.

However, we are able to state a sufficient condition for having $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, 0, r)$ for some $(1, 0, r) \in \mathcal{K}$.

THEOREM 4.1.1. *Assume that $q_1 \geq 0$ a.e. on (a_1, b_1). Then we have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, 0, r)$ for some $(1, 0, r) \in \mathcal{K}$.*

PROOF. Since we always have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q_1 p_1 \circ u^{-1}, r_1 p_1 \circ u^{-1})$ (compare Subsection 4.1.2), where $q_1 \geq 0$ a.e. if and only if $q_1 p_1 \circ u^{-1} \geq 0$ a.e., we may assume that $p_1 \equiv 1$. Now, as we have seen above, all we have to show is that there is a real-valued, zero free solution of

$$-v'' + q_1 v = 0 \quad \text{on } (a_1, b_1). \tag{4.1.7}$$

To this end let v be the solution of the initial value problem

$$-v'' + q_1 v = 0, \quad v(c) = 1, v'(c) = 0$$

for some $c \in (a_1, b_1)$. This solution uniquely exists and is real-valued, compare Corollary 1.1.2. Suppose that $v(z) = 0$ for some $z \in (a_1, b_1)$. Clearly, then there is $z_0 \in (c, b_1)$ such that $v(z_0) = 0$ and $v(z) > 0$ for $z \in [c, z_0)$ or $z_0 \in (a_1, c)$ such that $v(z_0) = 0$ and $v(z) > 0$ for $z \in (z_0, c]$ (of course, both could be true - for different z_0). We suppose $z_0 > c$, the other case can be treated in a similar way. Then we have

$$\begin{aligned} v'(z) &= \underbrace{v'(c)}_{=0} + \int_c^z v''(t)dt = \int_c^z \underbrace{q_1 v}_{\geq 0} dt \geq 0, \quad z \in [c, z_0], \\ v(z) &= \underbrace{v(c)}_{=1} + \int_c^z \underbrace{v'(t)}_{\geq 0} dt > 0, \quad z \in [c, z_0], \end{aligned}$$

contradicting[3] $v(z_0) = 0$. □

As an example consider the tuple $(1, q, 1)$ defined on (a, b) where $q \equiv c$ for some $c < 0$. In this case the real-valued solutions of (4.1.7) are precisely the functions

$$v(z) = d_1 \cos(\sqrt{|c|}\ z) + d_2 \sin(\sqrt{|c|}\ z), \quad d_1, d_2 \in \mathbb{R}. \tag{4.1.8}$$

Now, if $\sqrt{|c|}b - \sqrt{|c|}a > \pi$ or equivalently $c < -\frac{\pi^2}{(b-a)^2}$, we have

$$\bigcup_{z \in (a,b)} \operatorname{span}\left(\begin{pmatrix} \cos(\sqrt{|c|}\ z) \\ \sin(\sqrt{|c|}\ z) \end{pmatrix}\right) = \mathbb{R}^2,$$

and hence for all $d_1, d_2 \in \mathbb{R}$ there is some $z \in (a, b)$ such that

$$\left\langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \begin{pmatrix} \cos(\sqrt{|c|}\ z) \\ \sin(\sqrt{|c|}\ z) \end{pmatrix} \right\rangle_{\mathbb{R}^2} = d_1 \cos(\sqrt{|c|}\ z) + d_2 \sin(\sqrt{|c|}\ z) = 0,$$

meaning that there is not any real-valued, zero free solution of (4.1.7). Thus a potential equation with constant potential $q \equiv c$ cannot be transformed into a string equation if $c < -\frac{\pi^2}{(b-a)^2}$.

Conversely, let $0 > c \geq -\frac{\pi^2}{(b-a)^2}$. Then, since

$$\begin{aligned} \sin x\ \sin y &= \frac{1}{2}(\cos(x-y) - \cos(x+y)), \\ \cos x\ \cos y &= \frac{1}{2}(\cos(x-y) + \cos(x+y)), \quad x, y \in \mathbb{R}, \end{aligned}$$

we have, for $z \in (a, b)$,

$$\cos\left(\sqrt{|c|}\ \frac{a+b}{2}\right)\cos(\sqrt{|c|}\ z) + \sin\left(\sqrt{|c|}\ \frac{a+b}{2}\right)\sin(\sqrt{|c|}\ z) = \\ \cos\Big(\underbrace{\sqrt{|c|}}_{\leq \frac{\pi}{b-a}}\Big(\underbrace{\frac{a+b}{2} - z}_{|\cdot| < \frac{b-a}{2}}\Big)\Big) \neq 0,$$

meaning that there is a real-valued, zero free solution of (4.1.7). Consequently, a potential equation with constant potential $q \equiv c$, $0 > c \geq -\frac{\pi^2}{(b-a)^2}$, can be transformed into a string equation. Because of Theorem 4.1.1 this also holds for $c \geq 0$.

In particular, the example of a potential equation with constant potential shows two interesting things which we want to state as a remark.

Remark 4.1.2.

- *The assumption of Theorem 4.1.1 that $q_1 \geq 0$ a.e. on (a_1, b_1) is not necessary in order to have $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, 0, r)$ for some $(1, 0, r) \in \mathcal{K}$.*

[3] In fact, one sees that v is convex on (a_1, b_1) with global minimum at $z = c$.

- *There are SL equations which cannot be transformed into a string equation.*

At this point let us conclude with an example of tuples $(1,0,r_1)$ and $(1,0,r_2)$ both defined on the same interval with $(1,0,r_1) \sim_{\mathcal{K}} (1,0,r_2)$ but $r_1 \neq r_2$. Let $(a_1,b_1) = (a_2,b_2) = (1,2)$, $r_1 \equiv 1$ and $r_2(x) = 4x^{-4}$. Then it is easy to check that $(1,0,r_1) \sim_{\mathcal{K}} (1,0,r_2)$ ($v(z) = \frac{1}{\sqrt{2}}z - \frac{3}{\sqrt{2}}$ and $u(z) = -(\frac{1}{2}z - \frac{3}{2})^{-1}$).

4.1.5 Impedance Form

A SL equation associated with coefficients of the form $(p,0,p)$, i.e $p = r$, is called an equation in impedance form.

We claim that $(p_1,q_1,r_1) \sim_{\mathcal{K}} (p,0,p)$ for some p if and only if $(p_1,q_1,r_1) \sim_{\mathcal{K}} (1,0,r)$ for some r, i.e. a SL equation can be transformed into impedance form if and only if it can be transformed into string form (see Subsection 4.1.4). To this end it suffices to show that each tuple $(1,0,r_1) \in \mathcal{K}$ satisfies $(1,0,r_1) \sim_{\mathcal{K}} (p,0,p)$ for some p and - conversely - each tuple $(p_1,0,p_1) \in \mathcal{K}$ satisfies $(p_1,0,p_1) \sim_{\mathcal{K}} (1,0,r)$ for some r.

Let $(1,0,r_1) \in \mathcal{K}$. Choosing $v \equiv 1$ on (a_1,b_1) and

$$u(x) = \begin{cases} \int_c^x \sqrt{r_1}(t)dt + d, & c \leq x < b_1, \\ -\int_x^c \sqrt{r_1}(t)dt + d, & a_1 < x < c, \end{cases}$$

for some $c \in (a_1,b_1)$ and $d \in \mathbb{R}$, one sees that $(1,0,r_1) \sim_{\mathcal{K}} (\sqrt{r_1} \circ u^{-1}, 0, \sqrt{r_1} \circ u^{-1})$. If $(p_1,0,p_1) \in \mathcal{K}$, choosing $v \equiv 1$ on (a_1,b_1) and

$$u(x) = \begin{cases} \int_c^x \frac{1}{p_1}(t)dt + d, & c \leq x < b_1, \\ -\int_x^c \frac{1}{p_1}(t)dt + d, & a_1 < x < c, \end{cases}$$

for some $c \in (a_1,b_1)$ and $d \in \mathbb{R}$ leads to $(p_1,0,p_1) \sim_{\mathcal{K}} (1,0,p_1^2 \circ u^{-1})$.

In order to see that there are tuples $(p_1,0,p_1)$ and $(p_2,0,p_2)$ both defined on the same interval with $(p_1,0,p_1) \sim_{\mathcal{K}} (p_2,0,p_2)$ but $p_1 \neq p_2$, consider, e.g., $p_1(z) = z^2$ and $p_2 \equiv 1$ on $(1,2)$ ($v(z) = \frac{1}{z}$ and $u = \mathrm{id}_{(1,2)}$).

4.1.6 Canonical Representative Systems and Equations with Constant Coefficients

For the following it is essential to notice that each shift $u(z) = z + d$, $d \in \mathbb{R}$, gives rise to a Liouville transformation belonging to $v \equiv 1$ and u, transforming

$$-(p_1 y')' + q_1 y = \lambda r_1 y$$

via

$$y(z) = w(u(z))$$

into

$$-((p_1 \circ u^{-1})w')' + (q_1 \circ u^{-1})w = \lambda(r_1 \circ u^{-1})w,$$

i.e. $(p_1, q_1, r_1) \sim_{\mathcal{K}} (p_1 \circ u^{-1}, q_1 \circ u^{-1}, r_1 \circ u^{-1})$.

DEFINITION 4.1.3. *Let $(p_1, q_1, r_1), (p_2, q_2, r_2) \in \mathcal{K}$. We say that (p_1, q_1, r_1) and (p_2, q_2, r_2) are equal up to some shift if $(p_2, q_2, r_2) = (p_1 \circ u^{-1}, q_1 \circ u^{-1}, r_1 \circ u^{-1})$ for some shift u.*

REMARK 4.1.4. *Clearly, if (p_1, q_1, r_1) is defined on (a_1, b_1), (p_2, q_2, r_2) is defined on (a_2, b_2), where $a_1 = a_2$ is finite and (p_1, q_1, r_1) and (p_2, q_2, r_2) are equal up to some shift, then $(p_1, q_1, r_1) = (p_2, q_2, r_2)$. Note that in general this does not hold true if $a_1 = a_2 = -\infty$.*

DEFINITION 4.1.5. *Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{K}$. We say that $\mathcal{A}$ is a canonical representative system (c.r.s.) for the tuples of SL coefficients in $\mathcal{B}$ (in short, $\mathcal{A}$ is a c.r.s. for $\mathcal{B}$) if*

(i) each $t_{\mathcal{B}} \in \mathcal{B}$ satisfies $t_{\mathcal{B}} \sim_{\mathcal{K}} t_{\mathcal{A}}$ for at least one $t_{\mathcal{A}} \in \mathcal{A}$

and

(ii) if $t_{\mathcal{A}} \sim_{\mathcal{K}} \tilde{t}_{\mathcal{A}}$ for $t_{\mathcal{A}}, \tilde{t}_{\mathcal{A}} \in \mathcal{A}$, then $t_{\mathcal{A}}$ and $\tilde{t}_{\mathcal{A}}$ are equal up to some shift.

EXAMPLE 4.1.6. *Let $\mathcal{A} = \{(1, q, 1) \in \mathcal{K}\}$, $\mathcal{B} = \{(p, q, r) \in \mathcal{K} : p, r \in C^1$ with $p', r' \in AC_{loc}\}$. Then the considerations above (Subsection 4.1.3) show that $\mathcal{A}$ is a c.r.s. for $\mathcal{B}$. However, by using Remark 4.1.4, we see that none of the other described special forms gives rise to a canonical representative system.*

EXAMPLE 4.1.7. *Let (p_1, q_1, r_1) be a tuple of constant SL coefficients on $(a_1, b_1) \subseteq \mathbb{R}$. Clearly, then $p_1, r_1 \in C^1(a_1, b_1)$ with $p', r' \in AC_{loc}(a_1, b_1)$ and hence $(p_1, q_1, r_1) \sim_{\mathcal{K}} (1, q, 1)$ for some $(1, q, 1) \in \mathcal{K}$. By (4.1.4) q is constant. Consequently, $\mathcal{A} = \{(1, q, 1) \in \mathcal{K} : q \equiv c$ for some $c \in \mathbb{R}\}$ is a c. r. s. for $\mathcal{B} = \{(p, q, r) \in \mathcal{K} : (p, q, r)$ is a tuple of constant SL coefficients$\}$.*

From Example 4.1.7 we infer that a SL equation with coefficients $(p_1, q_1, r_1) \in \mathcal{K}$ defined on (a_1, b_1) can be transformed into a SL equation with constant coefficients (by means of a Liouville transformation) if and only if it can be transformed into Liouville normal form with constant potential. This is equivalent to $(p_1 r_1) \in C^1(a_1, b_1)$ with $(p_1 r_1)' \in AC_{loc}(a_1, b_1)$ and (see (4.1.4))

$$\frac{q_1}{r_1} + \frac{1}{4}\sqrt[4]{\frac{p_1}{r_1^3}}\left(\sqrt[4]{\frac{1}{p_1 r_1^5}}\,(p_1 r_1)'\right)' \equiv c$$

for some $c \in \mathbb{R}$. In particular, every SL equation with coefficients $(p_1, q_1, r_1) \in \mathcal{K}$ where $p_1 r_1 \equiv c_1$ for some $c_1 \in \mathbb{R}^+$ and $q_1 / r_1 \equiv c_2$ for some $c_2 \in \mathbb{R}$ can be transformed into an equation with constant coefficients.

Example 4.1.8. *Consider the Cauchy-Euler equation*

$$z^2y''(z) + zy'(z) + \lambda y(z) = 0 \quad on\ (0, b_1) \tag{4.1.9}$$

or - equivalently - in SL form

$$-(zy'(z))' = \lambda \frac{1}{z} y(z) \quad on\ (0, b_1). \tag{4.1.10}$$

We have $q_1 \equiv 0$ *and* $p_1 r_1 \equiv 1$ *(*$p_1 : z \mapsto z$, $r_1 : z \mapsto \frac{1}{z}$*) and infer that* (4.1.9) *or* (4.1.10)*, respectively, can be transformed into a SL equation with constant coefficients. Indeed, applying the Liouville transformation* $w(x) = y(\mathrm{e}^x)$*, i.e.* $u(z) = \ln z, v \equiv 1$*, yields the transformed equation*

$$-w''(x) = \lambda w(x) \quad on\ (-\infty, \ln b_1).$$

4.2 Equivalence Relation on the Set of Maximal Operators

We may consider the set $\mathcal{T}$ of all maximal operators associated with SL coefficients $(p, q, r) \in \mathcal{K}$,

$$\mathcal{T} = \{T_{max}(p, q, r) : (p, q, r) \in \mathcal{K}\},$$

and declare an equivalence relation $\sim_\mathcal{T}$ on $\mathcal{T}$ by

$$T_{max}(p_1, q_1, r_1) \sim_\mathcal{T} T_{max}(p_2, q_2, r_2) \quad \Leftrightarrow \quad \begin{array}{l} \exists \text{ Liouville transformation } L \text{ such that} \\ T_{max}(p_1, q_1, r_1) \text{ and } T_{max}(p_2, q_2, r_2) \\ \text{are unitarily equivalent via } L. \end{array}$$

We want to make a connection between the relation $\sim_\mathcal{K}$ on the set of SL coefficients and the relation $\sim_\mathcal{T}$ on the set of associated maximal operators. For a restricted class $\mathcal{K}_r$ of SL coefficients this is possible:

Let $\mathcal{K}_r$ be the subset of $\mathcal{K}$ which consists of all tuples of SL coefficients satisfying not only conditions (3.0.2) but additionally $p', q \in L^2_{loc}(a, b)$,

$$\mathcal{K}_r = \{(p, q, r) \in \mathcal{K} : (p, q, r) \text{ is defined on } (a, b) \subseteq \mathbb{R} \text{ and } p', q \in L^2_{loc}(a, b)\}.$$

The additional assumptions on the SL coefficients guarantee that smooth functions with compact support are in the domain of associated maximal operators (in fact, of course, even in the domain of associated preminimal operators): Let $(p, q, r) \in \mathcal{K}_r$ be defined on (a, b) and $y \in C_0^\infty((a, b); \mathbb{C})$ with $\operatorname{supp} y = [c, d] \subseteq (a, b)$. Clearly, y and y'

are in $AC_{loc}(a,b)$ and hence $y \in \mathcal{D}(\tau(p,q,r))$, and $y \in L^2((a,b);r)$. Furthermore,

$$\begin{aligned}\int_a^b |\tau(p,q,r)y|^2 r dx &= \int_a^b \left|\frac{1}{r}\big[-(py')' + qy\big]\right|^2 r dx = \int_c^d \frac{1}{r}\left|-py'' - p'y' + qy\right|^2 dx \\ &\leq 3 \max_{x\in[c,d]} \frac{1}{r(x)} \left(\int_c^d |p|^2 |y''|^2 dx + \int_c^d |p'|^2 |y'|^2 dx + \int_c^d |q|^2|y|^2 dx\right) \\ &\leq 3 \max_{x\in[c,d]} \frac{1}{r(x)} \max_{x\in[c,d]} \left(|y(x)|^2 + |y'(x)|^2 + |y''(x)|^2\right) \\ &\quad \left((d-c)\max_{x\in[c,d]} |p(x)|^2 + \int_c^d |p'|^2 dx + \int_c^d |q|^2 dx\right) < \infty.\end{aligned}$$

This property, i.e. $C_0^\infty((a,b);\mathbb{C}) \subseteq \mathcal{D}(T_{max}(p,q,r))$, is crucial for the proof of Theorem 4.2.1.

Note that other than $\mathcal{K}$ the set $\mathcal{K}_r$ is not closed under applying a Liouville transformation, meaning that if we start with $(p_1,q_1,r_1) \in \mathcal{K}_r$ and define (p_2,q_2,r_2) by (3.1.2) for v and u satisfying (3.1.1), then in general $(p_2,q_2,r_2) \notin \mathcal{K}_r$. To this end one had to assume that $v'', u'' \in L^2_{loc}(a,b)$.

Theorem 4.2.1. *Let $(p_1,q_1,r_1), (p_2,q_2,r_2) \in \mathcal{K}_r$. Then*

$$(p_1,q_1,r_1) \sim_{\mathcal{K}} (p_2,q_2,r_2) \Leftrightarrow T_{max}(p_1,q_1,r_1) \sim_{\mathcal{T}} T_{max}(p_2,q_2,r_2).$$

Consequently, we have

$$\mathcal{K}_r/_{\sim_{\mathcal{K}}} \widehat{=} \mathcal{T}_r/_{\sim_{\mathcal{T}}},$$

where $\mathcal{T}_r \subseteq \mathcal{T}$ is the set of maximal operators associated with SL coefficients $(p,q,r) \in \mathcal{K}_r$,

$$\mathcal{T}_r = \{T_{max}(p,q,r) : (p,q,r) \in \mathcal{K}_r\}.$$

Proof. If $(p_1,q_1,r_1) \sim_{\mathcal{K}} (p_2,q_2,r_2)$, it follows from Theorem 3.1.3 that $T_{max}(p_1,q_1,r_1) \sim_{\mathcal{T}} T_{max}(p_2,q_2,r_2)$. This even holds if $(p_1,q_1,r_1), (p_2,q_2,r_2) \in \mathcal{K}$ instead of $(p_1,q_1,r_1), (p_2,q_2,r_2) \in \mathcal{K}_r$.

Conversely, assume that $T_{max}(p_1,q_1,r_1) \sim_{\mathcal{T}} T_{max}(p_2,q_2,r_2)$. Assume that (p_1,q_1,r_1) is defined on (a_1,b_1) and (p_2,q_2,r_2) is defined on (a_2,b_2) and let L be a Liouville transformation (belonging to some η and ξ) such that $L : L^2((a_1,b_1);r_1) \to L^2((a_2,b_2);r_2)$ is unitary, $L : \mathcal{D}(T_{max}(p_1,q_1,r_1)) \to \mathcal{D}(T_{max}(p_2,q_2,r_2))$ is bijective and

$$LT_{max}(p_1,q_1,r_1)y = T_{max}(p_2,q_2,r_2)Ly, \quad y \in \mathcal{D}(T_{max}(p_1,q_1,r_1)).$$

We write $\eta = \frac{1}{v} \circ u^{-1}$ and $\xi = u^{-1}$ (with $u = \xi^{-1}$ and $v = \frac{1}{\eta} \circ u$). Since $L : \mathcal{D}(T_{max}(p_1,q_1,r_1)) \to \mathcal{D}(T_{max}(p_2,q_2,r_2))$ is bijective, it is clear that u maps bijectively (a_1,b_1) onto (a_2,b_2). Consequently, v is defined on (a_1,b_1). By definition of a Liouville transformation (Definition 3.1.1) u and v satisfy $u,v \in C^1(a_1,b_1), u',v' \in AC_{loc}(a_1,b_1), u' > 0$ and $v \neq 0$ in (a_1,b_1).

We use that $L : L^2((a_1, b_1); r_1) \to L^2((a_2, b_2); r_2)$ is unitary. On the one hand we have

$$\begin{aligned}\|f\|^2_{L^2((a_1,b_1);r_1)} &= \int_{a_1}^{b_1} |f(z)|^2 r_1(z) dz = \begin{vmatrix} z &=& u^{-1}(x) \\ dz &=& (u^{-1})'(x)dx \end{vmatrix} \\ &= \int_{a_2}^{b_2} |f(u^{-1}(x))|^2 r_1(u^{-1}(x)) \frac{1}{u'(u^{-1}(x))} dx,\end{aligned}$$

on the other hand we have

$$Lf = \frac{1}{v \circ u^{-1}} \cdot (f \circ u^{-1}), \quad \|Lf\|^2_{L^2((a_2,b_2);r_2)} = \int_{a_2}^{b_2} \frac{1}{v^2(u^{-1}(x))} |f(u^{-1}(x))|^2 r_2(x) dx$$

and therefore

$$\int_{a_2}^{b_2} |f(u^{-1}(x))|^2 r_1(u^{-1}(x)) \frac{1}{u'(u^{-1}(x))} dx = \int_{a_2}^{b_2} \frac{1}{v^2(u^{-1}(x))} |f(u^{-1}(x))|^2 r_2(x) dx$$

for $f \in L^2((a_1, b_1); r_1)$.

In particular, for $f = \mathbb{1}_{(u^{-1}(c),u^{-1}(d))}$, $a_2 < c < d < b_2$, this yields

$$\int_c^d r_1(u^{-1}(x)) \frac{1}{u'(u^{-1}(x))} dx = \int_c^d \frac{1}{v^2(u^{-1}(x))} r_2(x) dx, \quad a_2 < c < d < b_2,$$

and hence we infer from the Lebesgue differentiation theorem that

$$r_1(u^{-1}(x)) \frac{1}{u'(u^{-1}(x))} = \frac{1}{v^2(u^{-1}(x))} r_2(x) \text{ for almost all } x \in (a_2, b_2).$$

Since in here both sides continuously depend on x, we even have

$$\frac{r_1}{u'} \circ u^{-1} = \frac{1}{v^2} \circ u^{-1} \cdot r_2$$

or equivalently

$$r_2 = \frac{v^2 r_1}{u'} \circ u^{-1}. \tag{4.2.1}$$

For the following let $\chi_{(c,d)}$, $a_1 < c < d < b_1$, be a function such that $\chi_{(c,d)} \in C_0^\infty(a_1, b_1)$, $0 \le \chi_{(c,d)} \le 1$ on (a_1, b_1) and $\chi_{(c,d)}(z) = 1$ for $z \in [c, d]$ (it is widely known that such a function exists, see, e.g., [Alt06], Section 2.18).

For $y \in \mathcal{D}(\tau(p_1, q_1, r_1))$ such that $\chi_{(c,d)} y \in \mathcal{D}(T_{max}(p_1, q_1, r_1))$ we then have

$$L\tau(p_1, q_1, r_1)\chi_{(c,d)} y = \tau(p_2, q_2, r_2) L\chi_{(c,d)} y$$

and also

$$\big(L\tau(p_1, q_1, r_1)\chi_{(c,d)} y\big)\big|_{(u(c),u(d))} = \big(\tau(p_2, q_2, r_2) L\chi_{(c,d)} y\big)\big|_{(u(c),u(d))}.$$

However, this is equivalent to

$$L\big|_{(u(c),u(d))}\tau(p_1,q_1,r_1)\big|_{(c,d)}y\big|_{(c,d)} = \tau(p_2,q_2,r_2)\big|_{(u(c),u(d))}L\big|_{(u(c),u(d))}y\big|_{(c,d)}, \tag{4.2.2}$$

where by $L|_{(u(c),u(d))}$ we mean L restricted to $(u(c),u(d))$, i.e. the Liouville transformation belonging to $\frac{1}{v}\circ u^{-1}|_{(u(c),u(d))}$ and $u^{-1}|_{(u(c),u(d))}$.

If y is the function which is constant to 1, i.e. $y \equiv 1$ on (a_1,b_1), then $\chi_{(c,d)}y \in C_0^\infty(a,b) \subseteq C_0^\infty((a,b);\mathbb{C}) \subseteq \mathcal{D}(T_{max}(p_1,q_1,r_1))$ and we obtain

$$L\big|_{(u(c),u(d))}\frac{q_1}{r_1}\Big|_{(c,d)} = \tau(p_2,q_2,r_2)\big|_{(u(c),u(d))}\left(\frac{1}{v}\circ u^{-1}\right)\Big|_{(u(c),u(d))}$$

and hence

$$\begin{aligned}\frac{1}{v(u^{-1}(x))}\frac{q(u^{-1}(x))}{r(u^{-1}(x))} &= \frac{1}{r_2(x)}\left[-\left(p_2\left(\frac{1}{v}\circ u^{-1}\right)'\right)'(x) + q_2(x)\frac{1}{v(u^{-1}(x))}\right]\\ &= \frac{1}{r_2(x)}\left[\left(p_2\cdot\frac{v'}{v^2u'}\circ u^{-1}\right)'(x) + q_2(x)\frac{1}{v(u^{-1}(x))}\right]\end{aligned} \tag{4.2.3}$$

for almost all $x \in (u(c),u(d))$. Since c,d ($a_1 < c < d < b_1$) were arbitrary, (4.2.3) even holds for almost all $x \in (a_2,b_2)$.

Similarly, for $y = \mathrm{id}_{(a_1,b_1)}$ we obtain from (4.2.2)

$$\begin{aligned}L\big|_{(u(c),u(d))}\left(\frac{1}{r_1}\big[-p_1' + q_1\,\mathrm{id}_{(a_1,b_1)}\big]\right)\Big|_{(c,d)} &=\\ \tau(p_2,q_2,r_2)\big|_{(u(c),u(d))}&\left(\frac{1}{v\circ u^{-1}}\cdot u^{-1}\right)\Big|_{(u(c),u(d))},\end{aligned}$$

leading to

$$\begin{aligned}&\frac{1}{v(u^{-1}(x))}\frac{1}{r_1(u^{-1}(x))}\big[-p_1'(u^{-1}(x)) + q_1(u^{-1}(x))u^{-1}(x)\big] =\\ &\quad\frac{1}{r_2(x)}\left[\left(p_2u^{-1}\cdot\frac{v'}{v^2u'}\circ u^{-1} - p_2\cdot\frac{1}{vu'}\circ u^{-1}\right)'(x) + q_2(x)\frac{1}{v(u^{-1}(x))}u^{-1}(x)\right]\end{aligned}$$

for almost all $x \in (a_2,b_2)$.

After a lengthy calculation using (4.2.1) and (4.2.3) we obtain from this

$$\left(p_2\cdot\frac{1}{v^2u'}\circ u^{-1}\right)' = (p_1\circ u^{-1})' \quad \text{a.e. in } (a_1,b_1)$$

and hence

$$p_2\cdot\frac{1}{v^2u'}\circ u^{-1} = p_1\circ u^{-1} + c$$

for some $c \in \mathbb{R}$.

Finally, we do the same as for $y \equiv 1$ and $y = \mathrm{id}_{(a_1,b_1)}$ for y with $y(z) = z^2$ yielding (again after a lengthy calculation) that $c = 0$, i.e.

$$p_2 = (v^2 p_1 u') \circ u^{-1}. \tag{4.2.4}$$

Putting (4.2.1) and (4.2.4) into (4.2.3) then yields

$$q_2 = \left[\frac{v}{u'}(q_1 v - (p_1 v')')\right] \circ u^{-1},$$

and together with (4.2.1) and (4.2.4) this shows $(p_1, q_1, r_1) \sim_{\mathcal{K}} (p_2, q_2, r_2)$. □

Appendix A

A.1 Absolutely Continuous Functions

The following theorem is essential throughout Chapter 3 and Chapter 4.

THEOREM A.1.1. *Let $f, g \in AC[a,b]$ and $\lambda \in \mathbb{C}$. Then $\lambda f, \overline{f}, |f|, f+g, f \cdot g \in AC[a,b]$. Furthermore, if $g \neq 0$ in $[a,b]$, then $f/g \in AC[a,b]$.*
Let $f \in AC[a,b]$ be real-valued and $h \in Lip[c,d]$ or $f \in AC[a,b]$ be real-valued and monotone and $h \in AC[c,d]$ such that $f([a,b]) \subseteq [c,d]$. Then $h \circ f \in AC[a,b]$.

PROOF. Clearly, λf and $\overline{f}$ are absolutely continuous. The function $|f|$ being absolutely continuous can be easily seen by using the ε-δ-definition of absolute continuity and the reverse triangle inequality. In the case of real-valued functions f, g and h, the remaining assertions can be found, e.g., in [App09] (Theorem 4.21 and Theorem 4.28) or [Bog07] (Lemma 5.3.2, Corollary 5.3.3 and Exercise 5.8.59). However, for complex-valued functions f, g and h the assertions follow from these since a complex-valued function is absolutely continuous (Lipschitz continuous) if and only if its real part and its imaginary part are absolutely continuous (Lipschitz continuous). □

A.2 Hilbert Spaces of Entire Functions

We want to briefly discuss the concept of de Branges[1] spaces and state some main results as far as we need them in Chapter 2 and Section 3.2. We refer to de Branges' book [dB68] for a detailed description.

We say a function f which is analytic in the upper complex half-plane is of bounded type in $\mathbb{C}^+$ if

$$f(z) = \frac{p(z)}{q(z)}, \quad z \in \mathbb{C}^+,$$

where p and q are both bounded and analytic in the upper complex half-plane and q is not identically zero[2]. Clearly, if $\lambda \in \mathbb{C}$ and f, g are of bounded type, so are $\lambda f, f+g, f \cdot g$ and (if f/g is analytic in $\mathbb{C}^+$) f/g.

If f is a function of bounded type in $\mathbb{C}^+$ and f does not vanish identically, the quantity

$$\limsup_{y\to\infty} \frac{\ln|f(\mathrm{i}y)|}{y} \in [-\infty, \infty]$$

[1] Louis de Branges de Bourcia (*1932)
[2] There is a corresponding notation for other simply connected complex domains.

is referred to as the mean type of f. It can be shown that this number in fact is finite and the mean type of $f \equiv 0$ is then taken to be $-\infty$.

A de Branges function is an entire function E which satisfies the inequality

$$|E(z)| > |E(\overline{z})|, \quad z \in \mathbb{C}^+.$$

Obviously, a de Branges function does not have any zeros in $\mathbb{C}^+$. Given such a function E, the associated de Branges space B is the vector space of all entire functions h such that

$$\int_{\mathbb{R}} \frac{|h(t)|^2}{|E(t)|^2} dt < \infty$$

and such that both h/E and $h^\#/E$ are of bounded type in $\mathbb{C}^+$ and of non-positive mean type, where $h^\#$ is the entire function given by

$$h^\#(z) = \overline{h(\overline{z})}, \quad z \in \mathbb{C}.$$

An inner product is defined on B by

$$\langle h, g \rangle_B = \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)\overline{g(t)}}{|E(t)|^2} dt, \quad h, g \in B,$$

and equipped with this inner product the de Branges space B can be proven to be a Hilbert space (see [dB68], Theorem 21, but note that the scale factor π^{-1} is missing there).

For each $\zeta \in \mathbb{C}$ point evaluation in ζ is a continuous linear functional on B, hence B is a reproducing kernel Hilbert space and point evaluation in ζ can be written as

$$h(\zeta) = \langle h, K(\zeta, \cdot) \rangle_B, \quad h \in B.$$

The reproducing kernel K is given by, compare [dB68] (Theorem 19),

$$K(\zeta, z) = \frac{E(z)E^\#(\overline{\zeta}) - E(\overline{\zeta})E^\#(z)}{2\mathrm{i}(\overline{\zeta} - z)}, \quad \zeta, z \in \mathbb{C}. \tag{A.2.1}$$

Note that though there is a multitude of de Branges functions giving rise to the same de Branges space (including the inner product), e.g. λE for $|\lambda| = 1$, the reproducing kernel K is of course independent of the actual de Branges function.

In Section 3.2 we need the following subspace ordering theorem.

Theorem A.2.1 ([dB68], Theorem 35). *Let E_1, E_2 be two de Branges functions and B_1, B_2 be the associated de Branges spaces. Suppose B_1, B_2 are isometrically embedded in $L^2(\mathbb{R}, \mu)$ for some Borel measure μ on $\mathbb{R}$. If E_1/E_2 is of bounded type in $\mathbb{C}^+$ and has no real zeros or singularities, then B_1 contains B_2 or B_2 contains B_1 (including the inner product).*

Furthermore, we need the following converse statement.

LEMMA A.2.2. *Let E_1, E_2 be two de Branges functions and B_1, B_2 be the associated de Branges spaces. If $B_1 \cap B_2 \neq \{0\}$ (which is surely the case if B_1 contains B_2 or B_2 contains B_1 since any de Branges space contains nonzero elements, compare [dB68], p. 50), then E_1/E_2 is of bounded type in $\mathbb{C}^+$.*

PROOF. For each $h \in B_1 \cap B_2 \setminus \{0\}$ the functions h/E_2 and h/E_1 are of bounded type by definition and hence so is their quotient, which equals E_1/E_2 (and is indeed analytic in $\mathbb{C}^+$ since E_2 does not have any zeros there). $\square$

Bibliography

[Alt06] H. W. Alt, *Lineare Funktionalanalysis - Eine anwendungsorientierte Einführung*, 5 ed., Springer, Providence, RI, 2006.

[App09] J. Appell, *Analysis in Beispielen und Gegenbeispielen - Eine Einführung in die Theorie reeller Funktionen*, Springer, Berlin, 2009.

[BBW09] C. Bennewitz, B. M. Brown, and R. Weikard, *Inverse spectral and scattering theory for the half-line left-definite Sturm-Liouville problem*, SIAM J. Math. Anal. **40** (2009), no. 5, 2105–2131.

[BE05] C. Bennewitz and W. N. Everitt, *The Titchmarsh-Weyl eigenfunction expansion theorem for Sturm-Liouville differential equations*, Sturm-Liouville Theory - Past and Present, Birkhäuser, Basel, 2005, pp. 137–171.

[Ben89] C. Bennewitz, *Spectral asymptotics for Sturm-Liouville equations*, Proc. London Math. Soc. (3) **59** (1989), no. 2, 294–338.

[Ben03] ———, *A Paley-Wiener theorem with applications to inverse spectral theory*, Advances in differential equations and mathematical physics, Contemp. Math., vol. 327, Amer. Math. Soc., Providence, RI, 2003, pp. 21–31.

[Bog07] V. I. Bogachev, *Measure Theory - Volume I*, Springer, Berlin, 2007.

[BS01] A. Borichev and M. Sodin, *Krein's entire functions and the Bernstein approximation problem*, Illinois Journal of Mathematics **45** (2001), 167–185.

[dB68] L. de Branges, *Hilbert spaces of entire functions*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968.

[DKS76] H. J. Dirschmid, W. Kummer, and M. Schweda, *Einführung in die mathematischen Methoden der Theoretischen Physik*, Vieweg, Braunschweig, 1976.

[Eck] J. Eckhardt, *Schrödinger operators with strongly singular potentials and de Branges spaces*, arXiv:1105.6355.

[Eck09a] ———, *Direkte und inverse Spektraltheorie von Sturm-Liouville Differentialoperatoren*, Master's thesis, Vienna University of Technology, 2009.

[Eck09b] ———, *Sturm-Liouville Differentialoperatoren*, project report, available at `http://asc.tuwien.ac.at/index.php?id=404`, 2009.

[Eck12] ———, *Direct and inverse spectral theory of singular left-definite Sturm-Liouville operators*, J. Differential Equations **253** (2012), no. 2, 604–634.

[ESSZ97] W. N. Everitt, C. Shubin, G. Stolz, and A. Zettl, *Sturm-Liouville problems with an infinite number of interior singularities*, Spectral theory and computational methods of Sturm-Liouville problems, Lecture Notes in Pure and

Applied Mathematics, vol. 191, Marcel Dekker, New York, 1997, pp. 211–249.

[ET] J. Eckhardt and G. Teschl, *Sturm-Liouville operators with measure-valued coefficients*, arXiv:1105.3755.

[Eve82] W. N. Everitt, *On the transformation theory of ordinary second-order linear symmetric differential expressions*, Czechoslovak Mathematical Journal **32** (1982), no. 2, 275–306.

[Lev96] B. Ya. Levin, *Lectures on Entire Functions*, Graduate Studies in Mathematics, vol. 150, American Math. Soc., Providence, RI, 1996.

[Rem02] C. Remling, *Schrödinger operators and de Branges spaces*, J. Funct. Anal **196** (2002), no. 2, 323–394.

[She05] Chao-Liang Shen, *On the Liouville transformation and some inverse spectral problems*, Inverse Problems **21** (2005), 591–614.

[Tes09] G. Teschl, *Mathematical Methods in Quantum Mechanics - With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, vol. 99, American Math. Soc., Providence, RI, 2009, online edition available at `http://www.mat.univie.ac.at/~gerald/ftp/book-schroe/index.html`.

[Tesed] ———, *Ordinary Differential Equations and Dynamical Systems*, Graduate Studies in Mathematics, vol. 140, American Math. Soc., Providence, RI, 2012 (expected), online edition available at `http://www.mat.univie.ac.at/~gerald/ftp/book-ode/index.html`.

[Wei87] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, vol. 1258, Springer, Berlin, 1987.

[Wei00] ———, *Lineare Operatoren in Hilberträumen - Teil 1: Grundlagen*, Teubner, Stuttgart, 2000.

[Wei03] ———, *Lineare Operatoren in Hilberträumen - Teil 2: Anwendungen*, Teubner, Stuttgart, 2003.

[Wor04] H. Woracek, *Komplexe Analysis im Einheitskreis*, lecture notes, available at `http://asc.tuwien.ac.at/funkana/`, 2004.

[Yos60] K. Yosida, *Lectures on Differential and Integral Equations*, Pure and Applied Mathematics, vol. 10, Interscience, New York, 1960.

[Zet05] A. Zettl, *Sturm-Liouville Theory*, American Math. Soc., Providence, RI, 2005.