DIPLOMARBEIT

Reproducing Kernel Spaces of Entire Functions

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Preface

A Hilbert space H which consists of entire functions is called a de Branges space, if it satisfies the axioms

$(H1)$ $\mathcal{H}$ is a reproducing kernel space.

$(H2)$ $\mathcal{S} : F(z) \mapsto z\,F(z)$ is a closed symmetric operator with deficiency index $(1,1)$.

$(H3)$ The mapping $F(z) \mapsto \overline{F(\overline{z})}$ is an antilinear isometry of $\mathcal{H}$ onto itself.

$(H4)$ For each nonreal $w \in \mathbb{C}$, there exist functions $f \in H$ such that $f(w) \neq 0$.

In his book [dB] L. de Branges developed a far reaching theory of such spaces. An application of this theory is concerned with the study of so-called canonical systems of differential equations. However, also in various other contexts (e.g. Hankel transforms, special functions of mathematical physics) Hilbert spaces of entire functions satisfying $(H1)$, $(H2)$,$(H3)$ and $(H4)$ appear. The methods employed by L. de Branges are function theoretic in their nature, however large parts seem to be motivated from an operator theoretic viewpoint.

Especially in connection with the study of canonical systems M.G.Krein used the theory of symmetric and selfadjoint operators. He investigated a model space for a symmetric operator with deficiency index $(1,1)$, which, in the case of entire operators, is isomorphic to a de Branges space of entire functions.

It is the aim of the present work to give proofs of some of L. de Branges results via an operator theoretic approach. In this way some results lying in the core of de Branges's theory are getting more structured and transparent, and the ideas and motivations behind are clarified.

We divide this thesis in two chapters. Chapter 1 treats representations of a symmetric operator S with deficiency index $(1,1)$ acting in a Hilbert space H with not necessarily dense domain, and can be seen as a preparation for our work on de Branges spaces. Two kinds of representations are discussed: The method of universal directing functionals, where we restrict ourselves to the special case of everywhere defined functionals. This setting is suitable for the considerations in Chapter 2. Secondly, the representation of S with respect to a generalized gauge. Moreover, in Chapter 1, we introduce and investigate regularized generalized u-resolvents and u-resolvent matrices. The results of this chapter are wellknown, see e.g. [KL], [GG] or [KW1].

In Chapter 2 we mainly deal with Hilbert spaces of entire functions which satisfy the properties $(H1)$ and $(H2)$ above. On the first sight this setting seems to be a bit more general than the setting of de Branges spaces where additionally $(H3)$ is required. However, it is basically the same. We only would like to point out that it is the validity of $(H1)$ and $(H2)$ which is responsible for most results.

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Contents

Chapter 1

Symmetric Operators with deficiency index $(1,1)$

1.1 Preliminaries

Let L be a linear space with inner product $[\cdot,\cdot]$, and denote by $(L^2,[\cdot,\cdot])$ the product space $L\times L$, endowed with the inner product

$$[\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}] = [f_1,f_2]+[g_1,g_2], \quad \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in L^2.$$

For a linear relation, i.e. subspace $T\subseteq L^2$ we define its adjoint T^* by

$$T^* = \{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : [f_1,g_2]-[f_2,g_1]=0 \text{ for all } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in T\}.$$

T is called symmetric if $T\subseteq T^*$ and selfadjoint if $T=T^*$. For $(f;g)\in L^2$, let J be the mapping defined by

$$J\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.1.1.1}$$

The operator iJ is selfadjoint and unitary. Define an inner product on L^2 by

$$\langle\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\rangle = [iJ\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}] = i([f_1,g_2]-[f_2,g_1]).$$

Obviously, the adjoint T^* of any linear relation $T\subseteq L^2$ is the orthogonal complement with respect to $\langle\cdot,\cdot\rangle$. Therefore, a relation T is symmetric if and only if T is a neutral subspace of

$(L^2, \langle\cdot,\cdot\rangle)$, and T is selfadjoint if and only T is a hypermaximal neutral subspace of $(L^2, \langle\cdot,\cdot\rangle)$. Any two vectors $a, b \in L^2$ are called skewly linked, if $\langle a, a\rangle = \langle b, b\rangle = 0$, and $\langle a, b\rangle = 1$.

Let S be a closed symmetric relation defined in the linear space $(L, [\cdot,\cdot])$. Define $S_\infty = \{(0; f) : (0; f) \in S)\}$ and $S_s = S_\infty^\perp \cap S$. By its definition, S_s is (the graph of) a closed operator which is also symmetric. We have

$$S = S_s \oplus S_\infty.$$

A point $z \in \mathbb{C}$ is called regular, if $\operatorname{ran}(S - z) = H$ and $(S - z)^{-1}$ is a bounded operator. We write $\rho(S)$ for the set of all regular points. A point $z \in \mathbb{C}$ is called a point of regular type of S if there is $c > 0$ such that

$$\|g - zf\| \geq c\,\|f\| \quad \text{for all } \begin{pmatrix} f \\ g \end{pmatrix} \in S. \tag{1.1.1.2}$$

The set of all points of regular type of S is denoted by $r(S)$. Clearly, all nonreal points are points of regular type. Hence, for nonreal z, the range

$$M_z = \operatorname{ran}(S - z) = \{g - zf : \begin{pmatrix} f \\ g \end{pmatrix} \in S\}$$

is a closed subspace of L. Denote by $N_{\overline{z}}$ its orthogonal companion. Note that $(S - z)^{-1}$ is an bounded operator with $\operatorname{ran}(S - z)$ as its domain and $S(0) = \{f : (0; f) \in S\}$ as its nullspace.

For symmetric relations, the Neumann formula reads as follows:

1.1.1. Lemma. *Suppose S is a closed symmetric relation defined in some linear space $(L, [\cdot,\cdot])$. For $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, define $N_{\overline{z}} = \operatorname{ran}(S - z)^\perp$, $N_z = \operatorname{ran}(S - \overline{z})^\perp$. Then*

$$S^* = S \dotplus \{\begin{pmatrix} n_{\overline{z}} \\ \overline{z} n_{\overline{z}} \end{pmatrix} : n_{\overline{z}} \in N_{\overline{z}}\} \dotplus \{\begin{pmatrix} n_z \\ z n_z \end{pmatrix} : n_z \in N_z\}. \tag{1.1.1.3}$$

The sum is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle$.

A proof (in the more general setting of Krein spaces) is given in [DS1]. The following theorem is taken from [DS2].

1.1.2. Theorem. *Suppose O is a subset of $\mathbb{C}$ such that $O \cap \overline{O} \neq \emptyset$. Assume that $R(a) : L \to L$, $a \in O$, is a family of operators acting on a liner space L with inner product $[\cdot,\cdot]$, which fulfills the following conditions: There exists $a_0 \in O \cap \overline{O}$ such that*

$$R(a_0)^* = R(\overline{a_0})$$

as subspaces of L^2, and $R(a)$ satisfies the resolvent identity

$$R(a) - R(b) = (a - b)\, R(b)R(a), \quad a, b \in O.$$

Then $R(a)$ is the resolvent operator of a selfadjoint linear relation $A \subseteq L^2$. In particular, $R(a)$ is a bounded operator for any $a \in \rho(A)$.

Proof. For any $a \in O$, define the relation

$$A(a) = R(a)^{-1} + a = \{(R(a)f; f + aR(a)f) \ : f \in L\}.$$

Since $R(b)(f + (a-b)R(a)f) = R(a)f$,

$$(A(a) - b)^{-1} = \{(f + (a-b)R(a)f; R(a)f) \ : f \in L\} \subseteq R(b),$$

and therefore $A(a) \subseteq R(b)^{-1} + b \subseteq A(b)$. For the same reason $A(b) \subseteq A(a)$, whence $A(a) = A(b) = A$ for $a, b \in O$. Moreover,

$$A^* = (R(a_0)^{-1} + a_0)^* = (R(a_0)^*)^{-1} + \overline{a_0} = R(\overline{a_0})^{-1} + \overline{a_0} = A.$$

The relation $R(a) = (A - a)^{-1}$ holds by definition, and the proof is complete. □

Suppose S is a symmetric relation defined in a Hilbert space $(H, (\, . \, , \, . \,))$. The Caley transform

$$C(S) = \{\begin{pmatrix} g - if \\ g + if \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in S\}$$

is an isometric mapping from $\operatorname{ran}(S - i)$ onto $\operatorname{ran}(S + i)$. It is unitary if and only if S is selfadjoint. If A is a selfadjoint linear relation, then its Caley transform $U = C(A)$ can be written as

$$U = \int_{|z|=1} z \, dE_U, \tag{1.1.1.4}$$

where $E_U(\, . \,)$ is the orthogonal resolution of the identity of U. Note that the vectors f which are invariant under the transformation U are exactly those elements satisfying $(0; f) \in A$. Thus $\operatorname{ran} E_U(\{ 1\}) = A(0)$. Note that $\tau : z \mapsto -i\frac{z+1}{z-1}$ maps the unit circle bijectively onto the real axis. For any Borel subset B of the real line we define $E(B) = E_U(\tau^{-1}((\, \cdot \,))$, and $E(\{\infty\}) := E_U(\{1\})$. We shall call $(E(\cdot), E(\{\infty\}))$ the orthogonal resolution of the identity of the selfadjoint relation A. It is easy to verify that

$$(A - z)^{-1} = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, dE_\lambda. \tag{1.1.1.5}$$

Different to the case where A is an selfadjoint operator, i.e $A(0) = \{0\}$, the range of $E(\mathbb{R})$ might not be the entire space H, in fact $\operatorname{ran} E(\mathbb{R}) = S(0)^\perp = \overline{\operatorname{dom} S}$. Observe that A_s consists of all pairs $(f; g)$ with $f \in \operatorname{dom} A$ and

$$g = A_s f = \int_{\mathbb{R}} \lambda \, dE_\lambda f.$$

Assume that S has deficiency index $(1,1)$ and choose a canonical selfadjoint extension A. Again write $M_z = \operatorname{ran}(S - z)$ and $N_{\overline{z}} = \operatorname{ran}(S - z)^\perp$, so that

$$H = M_z \dotplus N_{\overline{z}}.$$

For $z, w \in \rho(A)$, the mapping

$$U_{zw} = I + (z-w)(A-z)^{-1}$$

maps M_z bijectively onto M_w, thus $U_{zw}^* = U_{\overline{zw}}$ maps $N_{\overline{z}}$ bijectively onto $N_{\overline{w}}$. Fix $z_0 \in \mathbb{C}$ such that $\overline{z_0} \in \rho(A)$, and choose any $\varphi(z_0) \in N_{z_0}$, $\varphi(z_0) \neq 0$. Then

$$\varphi(z) = U_{z_0 z}\varphi(z_0) = \varphi(z_0) + (z-z_0)(A-z)^{-1}\varphi(z_0) \in N_z, \quad z \in \rho(A), \tag{1.1.1.6}$$

gives an analytic parametrization of the deficiency spaces N_z, $z \in \rho(A)$. We shall allways assume that the normalization condition $\|\varphi(i)\| = 1$ is satisfied and call the function $\varphi(z)$ a parametrization associated to S and A. A function Q is called Q-function of S and A, if

$$\frac{Q(z) - \overline{Q(w)}}{z - \overline{w}} = (\varphi(z), \varphi(w)) \quad z, w \in \rho(A). \tag{1.1.1.7}$$

Clearly, $\overline{Q(\overline{z})} = Q(z)$, i.e. Q is a real function, and

$$\operatorname{Im} Q(z) = (\varphi(z), \varphi(z)) \operatorname{Im} z,$$

thus $\operatorname{Im} Q(z) > 0$ for $z \in \mathbb{C}^+$. By $\mathcal{N}_0$ we denote the set of all functions $\tau(z)$ holomorphic on $\mathbb{C}\backslash\mathbb{R}$ such that $\overline{\tau(\overline{z})} = \tau(z)$ and $\operatorname{Im} \tau(z) \geq 0$ for $z \in \mathbb{C}^+$. With this notation, $Q \in \mathcal{N}_0$. By (1.1.1.7), the Q-function is uniquely determined up to a real constant. It is easy to verify that for any $z_0 \in \rho(A)$,

$$Q(z) = \operatorname{Im} z_0 + (z - z_0)(\varphi(z), \varphi(z_0))$$

is a Q-function of S and A.

If $\tilde{A}$ is a selfadjoint extension of S which acts in a possibly larger space $\tilde{H} \supseteq H$, and $\tilde{P}$ denotes the orthogonal projection of $\tilde{H}$ onto H, we shall call

$$R_z := \tilde{P}(\tilde{A} - z)^{-1}|_H, \quad z \in \rho(\tilde{A}),$$

the generalized (or compressed) resolvent of S generated by the extension $\tilde{A}$. Note that for different extensions their generalized resolvents may coincide. The following theorem of M.G.Krein gives an effective description of all generalized resolvents:

1.1.3. Theorem. *Assume $S \subseteq H^2$ is a symmetric relation with deficiency index $(1,1)$. Fix a canonical selfadjoint extension A of S, a parametrization $\varphi(z) \in N_z$ associated to S and A, and a Q-function $Q(z)$ of S and A. Then the formula*

$$R_z = (A-z)^{-1} + \frac{(\,.\,, \varphi(\overline{z}))}{\tau(z) + Q(z)}\,\varphi(z), \quad z \in \mathbb{C}\backslash\mathbb{R},$$

establishes a one-to-one correspondence between all functions $\tau \in \mathcal{N}_0 \cup \{\infty\}$ and all generalized resolvents of S. Thereby $\tilde{R}_z$ arises from a canonical extension if and only if τ is a real constant or $\tau \equiv \infty$.

For densely defined S, a proof is given in [AG2] or in [GG]. For the relational case, see [LT].

There is another characterization of points of regular type.

1.1.4. Lemma. *Suppose that the deficiency numbers of S are equal (not necesarily finite). Then $z \in r(S)$ if and only if there exists a canonical selfadjoint extension A such that $z \in \rho(A)$.*

Proof. If A is a canonical selfadjoint extension and if $z \in \rho(A)$, then (1.1.1.2) holds even for A instead of S, thus $z \in r(S)$. On the other hand, if $z \in r(S)$, there exists a canonical selfadjoint extension A such that $z \in \rho(A)$: This is clear for nonreal $z \in \mathbb{C}$. If $z \in \mathbb{R}$ then $(S - z)^{-1}$ is a bounded symmetric operator with domain $\operatorname{ran}(S - z)$. Hence there exists a bounded selfadjoint extension B defined on the entire space H (see [AG]). Then

$$A = B^{-1} + z$$

is a selfadjoint relation extending S which has the desired properties. □

We proceed with some results on reproducing kernel spaces. The presented facts are wellknown, see e.g. [Ar].

1.1.5. Definition. Suppose O is an open subset of the complex plane. A $n \times n$-matrix kernel $K(z,w)$ continous on O^2 is called positive definite, if for any choice $z_i \in O$, $\zeta_i \in \mathbb{C}^{n\times 1}$, $i = 1, \dots, m$, we have

$$\sum_{i,j=1}^{m} \zeta_i^* \, K(z_i, z_j) \, \zeta_j \geq 0.$$

Suppose $O \subseteq \mathbb{C}$ is open. A Hilbert space $(\mathfrak{H}, (.,.))$ whose elements are vector functions $\mathbf{f}(z) \in \mathbb{C}^{n\times 1}$ continuous on O is called reproducing kernel space, if for any $w \in O$ and $i = 1, \dots, n$, the linear functional

$$\mathbf{f} \mapsto \pi_i \mathbf{f}(w), \quad \mathbf{f} \in \mathfrak{H},$$

where π_i is the projection onto the i-th coordinate, is bounded. Since $\mathfrak{H}$ is a Hilbert space, there is an element $K_i(w) \in \mathfrak{H}$ such that

$$\pi_i \mathbf{f}(w) = (\mathbf{f}, K_i(w)), \quad f \in H.$$

The $n \times n$-matrix function $K(w,z) = (K_i(w)(z))_{i=1,\dots,n}$ is called the reproducing kernel of $\mathfrak{H}$. To any $z \in O$, $\zeta \in \mathbb{C}^{n\times 1}$, the function $K(z, \cdot\,)\zeta$ belongs to $\mathfrak{H}$, and

$$\zeta^* \, \mathbf{f}(z) = (\mathbf{f}, K(z, \cdot\,)\zeta), \quad \mathbf{f} \in \mathfrak{H}.$$

The kernel $K(w,z)$ is positive definite as one can easily verify.

1.1.6. Theorem. *To any $n \times n$-matrix kernel $K(w,z)$, $w,z \in O$, which is positive definite, there exists a unique reproducing Kernel space $(\mathfrak{K}(K),(.,.))$ of vector functions $\mathbf{f}(z)$ continuous on O, such that $K(w,z)$ is its reproducing kernel.*

Proof. We construct $\mathfrak{K}(K)$ as follows. On the linear set

$$\mathcal{L} = \operatorname{span}\{K(z_i,\,\cdot\,)\zeta_i \;: z_i \in O,\ \zeta_i \in \mathbb{C}^{n\times 1}\}$$

define a bilinear form $(\,\cdot\,,\,\cdot\,)$ by

$$(\sum_i K(z_i,\,\cdot\,)\zeta_i, \sum_j K(w_j,\,\cdot\,)\eta_j) := \sum_{i,j} \eta_j^* \, K(z_i,w_j)\zeta_i. \tag{1.1.1.8}$$

Since $K(w,z)$ is positive definite, we have $\|\mathbf{f}\|^2 = (\mathbf{f},\mathbf{f}) \geq 0$ for any $\mathbf{f} \in \mathcal{L}$. By the Cauchy-Schwarz inequality

$$|\pi_i \mathbf{f}(z)|^2 = |(\mathbf{f}, K(z,\,\cdot\,)\, e_i)|^2 \leq \|\mathbf{f}\|^2 \, \|K(z,\,\cdot\,)\, e_i\|^2,$$

where e_i, $i = 1,\dots,n$, denotes the canonical basis of $\mathbb{C}^{n\times 1}$. This shows that the only function $\mathbf{f} \in \mathcal{L}$ with $\|\mathbf{f}\| = 0$ is the zero function. Thus $(\mathcal{L},(\,\cdot\,,\,\cdot\,))$ is a pre-Hilbert space. Denote its completion by $\overline{\mathcal{L}}$. For any $\mathbf{f} \in \overline{\mathcal{L}}$ define the function

$$\tilde{\mathbf{f}}(z) := \begin{pmatrix} (f, K(z,\,\cdot\,)e_1) \\ \vdots \\ (f, K(z,\,\cdot\,)e_n) \end{pmatrix}, \quad z \in O.$$

For $\mathbf{f} \in \mathcal{L}$, $\tilde{\mathbf{f}}(z) = \mathbf{f}(z)$. Since the closed linear span of all $K(z,\,\cdot\,)$ is equal to $\overline{\mathcal{L}}$, the mapping $\mathbf{f} \mapsto \tilde{\mathbf{f}}$ is one to one, hence we may identify $\mathbf{f}$ with the function $\tilde{\mathbf{f}}(z)$. We show that for any $\mathbf{f} \in \overline{\mathcal{L}}$, $\mathbf{f}(z) = \tilde{\mathbf{f}}(z)$ is continous on O. There exists a sequence $\mathbf{f}_n \in \mathcal{L}$ that converges to $\mathbf{f}$. Therefore

$$|\pi_i(\mathbf{f}_n(z) - \mathbf{f}(z))|^2 = |(\mathbf{f}_n - \mathbf{f}, K(z,\,\cdot\,)e_i)|^2 \leq \|\mathbf{f}_n - \mathbf{f}\|^2 \, |K(z,z))|^2,$$

and since $K(w,z)$ is continous on O^2, $\mathbf{f}_n(z) \to \mathbf{f}(z)$ uniformly on compact subsets of O. Thus $\mathfrak{K}(K) = \overline{\mathcal{L}}$ is a Hilbert space of vector functions continous on O. By construction $K(z,w)$ is its reproducing kernel.

We show that $\mathfrak{K}(K)$ is unique. In any reproducing kernel space H with $K(w,z)$ as reproducing kernel, the scalar product on $\mathcal{L}$ is determined by $K(w,z)$ as in (1.1.1.8), and the closure of $\mathcal{L}$ is the entire space H. Therefore, the identity mapping from $\mathcal{L} \subseteq H$ onto $\mathcal{L} \subseteq \mathfrak{K}(K)$ is an isometric isomorphism, which can be extended to an isometric isomorphism T between H and $\mathfrak{K}(K)$. We then have

$$\pi_i(T\mathbf{f})(z) = (T\mathbf{f}, K(z,\,\cdot\,)e_i) = (\mathbf{f}, K(z,\,\cdot\,)e_i) = \pi_i \mathbf{f}(z), \quad i = 1,\dots,n.$$

hence $H = \mathfrak{K}(K)$. ☐

1.1.7. Remark. If $K(z,w)$ in Theorem 1.1.6 is such that for any $w \in O$, the function $K(w,\,\cdot\,)$ is analytic on O, then it follows from what we said above that $\mathfrak{K}(K)$ actually consists of vector functions $\mathbf{f}(z)$ analytic on O.

Suppose $(\mathfrak{H},(.,.))$ is a reproducing kernel space of vector functions analytic on O, and $K(w,z)$, $w,z \in O$, is its reproducing kernel. It follows from the above considerations that $\mathfrak{H} = \mathfrak{K}(K)$. For any open subset $O' \subseteq O$, denote by $\mathfrak{H}|_{O'}$ the space of all restricted functions $\mathbf{f}|_{O'}$, $\mathbf{f} \in \mathfrak{H}$, endowed with the inner product

$$(\mathbf{f}|_{O'}, \mathbf{g}|_{O'}) = (\mathbf{f}, \mathbf{g}).$$

Since the restriction mapping $\mathbf{f} \mapsto \mathbf{f}|_{O'}$ is one to one, $\mathfrak{H}|_{O'}$ is a Hilbert space. Moreover, by its definition, $\mathfrak{H}|_{O'}$ is a reproducing kernel space and $K(w,z)|_{O'^2}$ is its reproducing kernel. In particular, this shows that

$$\mathfrak{K}(K|_{O'^2}) = \mathfrak{K}(K)|_{O'}. \tag{1.1.1.9}$$

If $\mathfrak{H} = \mathfrak{K}(K)$ is a reproducing kernel space of $\mathbb{C}^{n\times 1}$-valued functions continuous on O, and if $V(z)$ is a continous $n \times n$-matrix valued function, such that $V(z)$ is invertible for each $z \in O$, denote $V(z)\mathfrak{H}$ the Hilbert space of all functions $V(z)\mathbf{f}(z)$, $\mathbf{f}(z) \in \mathfrak{H}$, with the inner product$(V(z)\mathbf{f}(z), V(z)\mathbf{g}(z)) := (\mathbf{f}, \mathbf{g})$. Since for any $\zeta \in \mathbb{C}^{n\times 1}$,

$$(V(z)\mathbf{f}(z), V(z)K(w,z)V(w)^*\zeta) = (\mathbf{f}(z), K(w,z)V(w)^*\zeta) = \zeta^* V(w)\mathbf{f}(w),$$

it follows that $V(z)\mathfrak{H}$ is a reproducing kernel space with $(V(z)K(w,z)V(w)^*)$ as its reproducing kernel.

1.1.8. Proposition. *If $\mathfrak{H}$ is a reproducing kernel space of $\mathbb{C}^n$-valued functions analytic on O, then for any $\zeta \in \mathbb{C}^{n\times 1}$, the function $K_w\zeta = K(w,\cdot)\zeta$ depends antianalytically on $w \in O$ in the norm of $\mathfrak{H}$. Therefore*

$$K_w^{(n)}\zeta = \frac{\partial^n}{(\partial \overline{w})^n} K(w,\cdot)\zeta \in \mathfrak{H},$$

and

$$\zeta^* \mathbf{f}^{(n)}(w) = (\mathbf{f}, K_w^{(n)}\zeta), \quad \mathbf{f} \in \mathfrak{H}, w \in O. \tag{1.1.1.10}$$

Hence point evaluation of any derivative is continuous in $\mathfrak{H}$.

Proof. For any $\mathbf{f} \in \mathfrak{H}$, the function $(K_w\zeta, \mathbf{f}) = \overline{\zeta^* f(w)}$ is antianalytic in w. Thus $w \mapsto K_{\overline{w}}\zeta$ is weakly holomorphic, which implies that $w \mapsto K_{\overline{w}}\zeta$ is holomorphic in the norm of $\mathfrak{H}$ (cf. [Ru]). Thus, the limit

$$\frac{\partial}{\partial \overline{w}} K(w,\cdot) = \lim_{z \to w} \frac{K_z\zeta - K_w\zeta}{\overline{z} - \overline{w}}$$

exists in the norm of $\mathfrak{H}$, and we have for any $\zeta \in C^{n\times 1}$

$$\zeta^*\mathbf{f}^{(1)}(w) = \lim_{z\to w} \frac{(\mathbf{f}, K_z\zeta) - (\mathbf{f}, K_w\zeta)}{z-w} = \lim_{z\to w}(\mathbf{f}, \frac{K_z - K_w}{\overline{z}-\overline{w}}\zeta^*) = (\mathbf{f}, \frac{\partial}{\partial \overline{w}}K_w\zeta).$$

The general assertion follows by induction. □

For any analytic function f we denote $\mathrm{Ord}_w\,\mathbf{f} := \min\{n\ : \mathbf{f}^{(n)}(w) \neq 0\}$. In any reproducing kernel space $\mathfrak{H}$ of functions $\mathbf{f}(z)$ holomorphic in O, define for any $w \in O$

$$\mathrm{Ord}_w\,\mathfrak{H} = \min_{\mathbf{f}\in\mathfrak{H}} \mathrm{Ord}_w\,\mathbf{f}. \tag{1.1.1.11}$$

Note that

$$\mathrm{Ord}_w\,\mathfrak{H} = \mathrm{Ord}_w\,K_w = \min\{n\ : K_w^{(n)} \neq 0\}. \tag{1.1.1.12}$$

If $U(z)$ is an analytic function with $\mathrm{Ord}_z\,f \leq \mathrm{Ord}_z\,\mathfrak{H}$ for all $z \in O$ then $\frac{1}{U(z)}\mathfrak{H}$ consists of functions holomorphic on O, and Proposition 1.1.8 implies that also point evaluation at points with $U(z) = 0$ is continuous, hence $\frac{1}{U(z)}\mathfrak{H}$ is a reproducing kernel space of functions $f(z)$ holomorphic on O.

1.1.9. Proposition. *Suppose O is an connected open subset of $\mathbb{C}$, and M is a subset of O which has a cluster point in O. Assume $\mathfrak{H}$ is a Hilbert space of C^n-valued functions $\mathbf{f}(z)$ analytic on O, such that point evaluation is continous for $m \in M$. Then point evaluation is continous for any $z \in O$, hence $\mathfrak{H}$ is a reproducing kernel space.*

Proof. Choose a sequence K_n of compact subsets of O such that $K_n \subseteq K_{n+1}^\circ$ and $\bigcup_{n\in\mathbb{N}} K_n = O$. For $\mathbf{f}, \mathbf{g} \in \mathcal{H}$ define the metrics $d_n(\mathbf{f},\mathbf{g}) = \sup_{z\in K_n} |\mathbf{f} - \mathbf{g}|(z)$,

$$d(\mathbf{f},\mathbf{g}) = \sum_{n=1}^{\infty} \frac{1}{2^n}\,\frac{d_n(\mathbf{f},\mathbf{g})}{1 + d_n(\mathbf{f},\mathbf{g})},$$

and

$$d_{\mathfrak{H}}(\mathbf{f},\mathbf{g}) := \|\mathbf{f} - \mathbf{g}\| + d(\mathbf{f},\mathbf{g}),$$

where $\|\cdot\|$ is the norm on $\mathfrak{H}$. Then $(\mathfrak{H}, d_{\mathfrak{H}}(\cdot,\cdot))$ is a Frechet space, i.e. a metrizable topological vector space $\mathfrak{H}$ such that $\mathfrak{H}$ is complete with respect to this metric. We shall only show the completeness, the proof of the continuity of the vector space operations is left to the reader. If $\mathbf{f}_n$ is a Cauchy sequence in the metric $d_{\mathfrak{H}}(\cdot,\cdot)$, then it is also a Cauchy sequence in the metric $d(\cdot,\cdot)$, thus it converges uniformly on compact subsets of O to an analytic function $\mathbf{g}(z)$. Since $\mathbf{f}_n$ is also a Cauchy sequence in $(\mathfrak{H}, (\cdot,\cdot))$, it has a limit $\mathbf{f} \in \mathfrak{H}$. Since point evaluation is continuous for points of M, we have

$$g(z) = \lim_{n\to\infty} \mathbf{f}_n(z) = f(z), \quad z \in M.$$

This implies that $\mathbf{f}(z) = \mathbf{g}(z)$ for all $z \in O$, and $\mathbf{f}_n \to \mathbf{f}$ in the metric $d_{\mathfrak{H}}(\cdot,\cdot)$. Since both spaces $(\mathfrak{H}, d_{\mathfrak{H}}(\cdot,\cdot))$ and $(\mathfrak{H}, (\cdot,\cdot))$ are Frechet spaces and since the identity mapping

$$\mathrm{id} : (\mathfrak{H}, d_{\mathfrak{H}}(\cdot,\cdot)) \to (\mathfrak{H}, (\cdot,\cdot)), \quad \mathbf{f} \mapsto \mathbf{f}$$

is continuous, the open mapping theorem shows that its inverse is also continuous, hence the topologies of the spaces coincide. Convergence of $\mathbf{f}_n$ in the norm $\|\cdot\|$ thus implies uniform convergence on compact subsets of O. □

We close this section with a result from complex analyis, the so called Stieltjes inversion formula (see e.g. [GG]).

1.1.10. Theorem. *Let σ be a complex Borel measure which satisfies the condition*

$$\int_{\mathbb{R}} \frac{1}{1+|t|}\, d|\sigma|(t) < \infty.$$

Define a function $f(z)$ holomorphic on $\mathbb{C}\backslash\mathbb{R}$ by

$$f(z) = \int_{\mathbb{R}} \frac{1}{t-z}\, d\sigma(t), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Suppose $g(z)$ is holomorphic on an open set containing the interval $[a,b]$ of the real line. For $\varepsilon > 0$ let Δ_ε denote the path $[a - i\varepsilon, b - i\varepsilon] + [b + i\varepsilon, a + i\varepsilon]$. Then

$$-\lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\Delta_\varepsilon} g(z) f(z)\, dz = \int_{[a,b]} g(t)\, d\sigma(t) + \frac{\sigma(\{a\}) + \sigma(\{b\})}{2}. \tag{1.1.1.13}$$

1.2 Space Triplets

In this section we introduce the spaces H_+ and H_- associated to a closed symmetric relation S with not necessarily dense domain. The results of the present section will be needed for our later work on de Branges spaces and can be easily adapted from the classical case, when the domain of S is dense in H, although the notation becomes a bit more complicated. Most of the results can be found in [KW1] or [St]. For a rigorous treatment of the classical case, see e.g. [Be].

Consider a Hilbert space $(H, (\cdot,\cdot))$ and a closed symmetric relation $S \subseteq H^2$. We define H_+ as the subspace $S^* \subseteq H^2$ equipped with the inner product

$$(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix})_+ := (f_1, g_1) + (f_2, g_2).$$

Since S^* is closed in H^2, $(H_+, (\cdot,\cdot)_+)$ is a Hilbert space. Its dual space is denoted by $(H_-, (\cdot,\cdot)_-)$. We write the action of $u \in H_-$ on $(f;g) \in H_+$ as

$$\left[\begin{pmatrix} f \\ g \end{pmatrix}, u\right]_\pm .$$

Since H_+ is an Hilbert space, any $u \in H_-$ can be represented uniquely by an element $(f;g) \in H_+$ such that $[\,\cdot\,,u]_\pm = (\,\cdot\,,(f;g))_+$. Note that $\|u\|_- = \|(f;g)\|_+$.

Define the continous mappings

$$\pi : H_+ \to H, \qquad \begin{pmatrix} f \\ g \end{pmatrix} \mapsto f,$$
$$\iota : H \to H_-, \qquad f \mapsto (\,\cdot\,, \begin{pmatrix} f \\ 0 \end{pmatrix})_{H^2}.$$

The chain $(H_+,(\cdot,\cdot)_+) \xrightarrow{\pi} (H,(\cdot,\cdot)) \xrightarrow{\iota} (H_-,(\cdot,\cdot)_-)$ is called the space triplet associated to the symmetric relation S.

The proof of the following lemma is obvious.

1.2.1. Lemma. *Regarding $(H,(\cdot,\cdot))$ as its own dual we have $\iota = \pi^*$. Since H_+ is reflexive,*

$$\ker \iota = \operatorname{ran} \pi^\perp = (\operatorname{dom} S^*)^\perp, \tag{1.2.1.1}$$

$$^\perp \operatorname{ran} \iota = \ker \pi = S^*_\infty, \tag{1.2.1.2}$$

with $^\perp \operatorname{ran} \iota = \bigcap_{u \in \operatorname{ran} \iota} \ker u$. Moreover, the orthogonal complement of $\overline{\operatorname{ran} \iota}$ in H_- is given by all functionals u of the form

$$u = \big(\,\cdot, (0;f)\big)_+, \quad f \in S^*(0). \tag{1.2.1.3}$$

1.2.2. Remark. If S is densely defined then we have the classical situation. According to (1.2.1.2) and (1.2.1.1), $\pi : H_+ \to H$ and $\iota : H \to H_-$ are one to one. Moreover $\operatorname{ran} \pi$ and $\operatorname{ran} \iota$ is dense in H and H_-, respectively. Thus π (ι)is a continous embedding of $H_+ (H_-)$ onto a dense subset of H (H_-). We can therefore assume that

$$H_+ \subseteq H \subseteq H_-.$$

1.2.3. Remark. If S is minimal, i.e.

$$\operatorname{cls} \bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(S - z)^\perp = H,$$

then, by the Neumann formula, $\operatorname{dom} S^*$ is dense in H, thus ι is one to one. Thus we can regard H as a subspace of H_-, and we obtain the situation

$$H_+ \xrightarrow{\pi} H \subseteq H_-, \quad \operatorname{codim}_{H_-} H = \dim S^*(0) = \dim \overline{\operatorname{dom} S}^\perp.$$

1.2.4. Lemma. *Suppose $u \in H_-$ and write $u = (\cdot,(h_1;h_2))_+$, where $(h_1;h_2) \in H_+$. Then $u \in \operatorname{ran} \iota$ if and only if $h_2 \in \operatorname{dom} S$, and $u \in \overline{\operatorname{ran} \iota}$ if and only if $h_2 \in \overline{\operatorname{dom} S}$.*

Proof. First, observe that

$$\left(\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) = \left(\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ 0 \end{pmatrix}\right) \quad \text{for all } \begin{pmatrix} f \\ g \end{pmatrix} \in S^*,$$

if and only if $(h_2; h - h_1) \in (S^*)^* = S$. Thus $u \in H$ if and only if there exists a vector $h \in H$ such that $(h_2; h - h_1) \in S$. This is the case if and only if $h_2 \in \operatorname{dom} S$, which proves the first assertion. To prove the second assertion, observe that by (1.2.1.3), we have $u = (\,\cdot, (h_1; h_2))_+ \in \overline{\operatorname{ran} \iota} = (\overline{\operatorname{ran} \iota})^{\perp\perp}$ if and only if

$$(h_1, h_2) \in (S^*_\infty)^\perp.$$

Since $S^*_\infty = \{0\} \times (\operatorname{dom} S)^\perp$, we conclude that $u \in \overline{\operatorname{ran} \iota}$ if and only if $h_2 \in ((\operatorname{dom} S)^\perp)^\perp = \operatorname{dom} S$, which completes the proof. ☐

Consider the space triplet

$$H_+ \xrightarrow{\pi} H \xrightarrow{\iota} H_-$$

associated to the symmetric relation $S \subseteq H^2$. Suppose $A \subseteq H^2$ is a selfadjoint extension of S. We define for $z \in \rho(A)$

$$R_z^+ : H \to H_+, \quad f \mapsto \begin{pmatrix} (A - z)^{-1} f \\ (I + z(A - z)^{-1}) f \end{pmatrix}$$

That R_z^+ is indeed a mapping into H_+ follows from the fact that $((A - z)^{-1} f; f) \in A - z$, and hence $R_z^+ f \in A \subseteq S^*$. Define

$$R_z^- : H_- \to H \quad R_z^- := (R_{\overline{z}}^+)^*.$$

We will give also an explicit description of R_z^-. If we write $u = (\cdot, (f_1, f_2))_+ \in H_-$, then for any $h \in H$

$$(R_z^- u, h) = [u, R_{\overline{z}}^+ h]_\pm = \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} R_{\overline{z}} h \\ (I + \overline{z} R_{\overline{z}}) h \end{pmatrix}\right) = (R_z f_1 + (I + z R_z) f_2, h),$$

and hence

$$R_z u = R_z f_1 + (I + z R_z) f_2, \quad u = (\cdot, (f_1; f_2))_+. \tag{1.2.1.4}$$

If $(E(\,\cdot\,), E(\{\infty\}))$ is the resolution of the identity connected with A, and if $B \subseteq \mathbb{R}$ is a bounded Borel set, then we define

$$E(B)^+ : H \to H_+, \quad f \mapsto \begin{pmatrix} E(B) f \\ A_s E(B) f \end{pmatrix} \in A_s,$$

and

$$E(B)^- : H_- \to H, \quad E(B)^- := (E(B)^+)^*.$$

1.2.5. Lemma. *With the above notation the following relations hold.*

$$R_z^+ - R_w^+ = (z-w)R_z^+ R_w, \qquad z,w \in \rho(A), \tag{1.2.1.5}$$

$$R_z^- - R_w^- = (z-w)R_w R_z^-, \qquad z,w \in \rho(A). \tag{1.2.1.6}$$

If $B \subseteq \mathbb{R}$ is a bounded Borel set, then

$$R_z^+ E(B) = E(B)^+ R_z, \quad z \in \rho(A), \tag{1.2.1.7}$$

$$R_z E(B)^- = E(B) R_z^- \quad z \in \rho(A). \tag{1.2.1.8}$$

The mappings R_z^+ and $E(B)^-$ are extensions of R_z and $E(B)$, respectively, in the following sense:

$$R_z = \pi R_z^+ = R_z^- \iota \tag{1.2.1.9}$$

$$E(B) = \pi E(B)^+ = E(B)^- \iota. \tag{1.2.1.10}$$

Proof. A direct computation gives

$$(z-w)R_z^+ R_w f = \begin{pmatrix} (z-w)R_z R_w f \\ (z-w)(I+zR_z)R_w f \end{pmatrix} =$$

$$= \begin{pmatrix} R_z f - R_w f \\ (z-w)R_w f + z(R_z f - R_w f) \end{pmatrix} = \begin{pmatrix} R_z f - R_w f \\ zR_z f - wR_w f \end{pmatrix} = R_z^+ - R_w^+.$$

Taking the adjoint on both sides in (1.2.1.5), we get (1.2.1.6). If B is a bounded Borel set, then for any $f \in H$ we have

$$R_z^+ E(B) f = \begin{pmatrix} R_z E(B) f \\ E(B)f + zR_z E(B) f \end{pmatrix} \in A.$$

Since $E(B)f \perp \operatorname{ran} E(\{\infty\}) = A(0)$, we have $(E(B)f + zR_z E(B)f, g) = 0$ for any $g \in A(0)$, which shows that $R_z^+ E(B) f \in A_s$ and hence

$$R_z^+ E(B) f = \begin{pmatrix} R_z E(B) f \\ A_s R_z E(B) f \end{pmatrix} = \begin{pmatrix} E(B) R_z f \\ A_s E(B) R_z f \end{pmatrix} = E(B)^+ R_z f.$$

Taking the adjoint on both sides of (1.2.1.7)leads to (1.2.1.8). By definition, $R_z = \pi R_z^+$ and $E(B) = \pi E(B)^+$. Again, taking the adjoints yields (1.2.1.9) and (1.2.1.10). □

Fix a bounded Borel set $B \subseteq \mathbb{R}$. Since the restriction A_s to $E(B)H$ is a bounded operator, we conclude that for any $f \in H$, the set function

$$E^+(\,\cdot\,)f : \quad B' \mapsto E(B')^+ f,$$

which acts on all Borel subsets of B, is countably additive, hence it is a H_+-valued measure. We shall denote this measure by $dE_\lambda^+ f$. By the spectral representation of A_s, we have

$$dE_\lambda^+ f = \begin{pmatrix} dE_\lambda f \\ \lambda dE_\lambda f \end{pmatrix}. \tag{1.2.1.11}$$

For any function g continuous on the closure of B, the integral

$$h = \int_B g(\lambda)\, dE_\lambda^+ f$$

converges in the metric of H_+, hence for any $u \in H_-$, we can write

$$[h,u]_\pm = \int_B g(\lambda)\, d[E_\lambda^+ f, u]_\pm.$$

The following prosition shows how $E(B)^-$ can be represented by $E(B)$.

1.2.6. Proposition. *Suppose $B \subseteq \mathbb{R}$ is a bounded Borel set. Then for any $f \in H$, $z \in \rho(A)$,*

$$dE_\lambda^+ f = (\lambda - z)\, dE_\lambda^+ R_z f \quad \text{on } B, \tag{1.2.1.12}$$

and for any $u \in H_-$

$$E(B)^- u = \int_B (\lambda - z)\, dE_\lambda R_z^- u, \quad z \in \rho(A). \tag{1.2.1.13}$$

Proof. By 1.2.1.11, and $dE_\lambda R_z = \frac{1}{\lambda - z} dE_\lambda$, we obtain

$$dE_\lambda^+ R_z = \frac{1}{\lambda - z} \begin{pmatrix} dE_\lambda \\ \lambda dE\lambda \end{pmatrix} = \frac{1}{\lambda - z} dE_\lambda^+,$$

which proves 1.2.1.12. Next, using 1.2.1.7, it follows that for $f \in H$

$$(E(B)^- u, f) = [u, E(B)^+ f]_\pm = \int_B (\lambda - z)\, d[u, E_\lambda^+ R_{\overline{z}} f]_\pm =$$
$$\int_B (\lambda - z)\, d[u, R_{\overline{z}} E_\lambda f]_\pm = \int_B (\lambda - z) d(E_\lambda R_z^- u, f).$$

Thus (1.2.1.13) holds. □

The representation 1.2.1.13 shows that for any $u \in H_-$, the set function $E^-(\cdot)u$, restricted to any bounded Borel set B is an H_--valued measure. We shall denote this measure by $dE_\lambda^- u$. Now, 1.2.1.13 can be rewritten as

$$dE_\lambda^- u = (\lambda - z)\, dE_\lambda R_z^- u. \tag{1.2.1.14}$$

We shall define $d\|E_\lambda^- u\|^2$ as the measure

$$d\|E_\lambda^- u\|^2 = |\lambda - z|^2 d(E_\lambda R_z^- u, R_z^- u).$$

This notation is justified by the observation that

$$\|E(B)^- u\|^2 = \int_B |\lambda - z|^2\, d(E_\lambda R_z^- u, R_z^- u) = \int_B d\|E_\lambda^- u\|^2. \tag{1.2.1.15}$$

Suppose that $S \subseteq H^2$ is a symmetric relation and $\tilde{A}$ is a selfadjoint extension acting in a (possibly) larger Hilbert space $\tilde{H}$. Consider the space triplet

$$\tilde{H}_+ \overset{\tilde{\pi}}{\to} \tilde{H} \overset{\tilde{\iota}}{\to} \tilde{H}_-$$

associated to S as a subspace of $\tilde{H}^2$. This means that $\tilde{H}_+$ is the adjoint $\tilde{S}^*$ of S computed in $\tilde{H}^2$. If $\tilde{P}$ is the orthoprojector of $\tilde{H}$ onto H and if $(f;g) \in \tilde{S}^*$, then for all $(h_1;h_2) \in S$

$$0 = (f, h_2) - (g, h_1) = (\tilde{P}f, h_2) - (\tilde{P}g, h_1).$$

Thus $(\tilde{P}f;\tilde{P}g) \in S^* = H_+$, and

$$\tilde{P}^+ : \tilde{H}_+ \to H_+, \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} \tilde{P}f \\ \tilde{P}g \end{pmatrix} \tag{1.2.1.16}$$

is the orthogonal projection of $\tilde{H}_+$ onto H_+. For $z \in \rho(\tilde{A})$, we define the compressed (or generalized) resolvent R_z of S generated by $\tilde{A}$, as ($\tilde{R}_z = (\tilde{A} - z)^{-1}$)

$$R_z : H \to H, \quad R_z = \tilde{P}\tilde{R}_z|_H.$$

Moreover, let

$$R_z^+ : H \to H_+, \quad R_z^+ f = \tilde{P}^+ \tilde{R}_z^+ f = \begin{pmatrix} R_z f \\ (I + zR_z)f \end{pmatrix},$$

$$R_z^- : H_- \to H, \quad R_z^- = (R_{\overline{z}}^+)^*.$$

Note that R_z^+ and R_z^- are in fact uniquely determined by the generalized resolvent R_z.

For $n \in N_z = \operatorname{ran}(S - z)^\perp$ we define the vector

$$n_+ := \begin{pmatrix} n \\ z \cdot n \end{pmatrix} \in H_+. \tag{1.2.1.17}$$

If A is a canonical selfadjoint extension of S, R_z^- its extended resolvent and $\varphi(z)$ a parametrization connected with S and A, then

$$\varphi_+(z) = \varphi_+(z_0) + (z - z_0)R_z^+ \varphi(z_0), \tag{1.2.1.18}$$

as one can easily check. For the compressed resolvents R_z^+ and R_z^- the Krein formula reads as follows:

1.2.7. Theorem. *Fix a canonical selfadjoint extension A° of S, a parametrization $\varphi(z)$ as defined in* (1.1.1.6)*, and a q-function $Q(z)$ of S and A°. Then the formulas*

$$R_z^+ = R_z^{\circ +} + \frac{(\,.\,, \varphi(\overline{z}))}{\tau(z) + Q(z)} \varphi_+(z), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{1.2.1.19}$$

and

$$R_z^- = R_z^{\circ -} + \frac{[\,\cdot\,, \varphi_+(\overline{z})]_\pm}{\tau(z) + Q(z)} \varphi(z), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{1.2.1.20}$$

establish a one to one correspondence between all functions $\tau \in \mathcal{N}_0 \cup \{\infty\}$ and all generalized resolvents R_z^+, R_z^- of S. They arise from a canonical selfadjoint extension if and only if τ is a real constant or $\tau \equiv \infty$.

Proof. Formula (1.2.1.19) follows immediately from Theorem 1.1.3. Taking the adjoint on both sides of (1.2.1.19) we obtain formula (1.2.1.20). □

We shall call $u \in H_-$ a boundary value if $[\cdot, u]_\pm$ annihilates S. If we write $u = (\cdot, a)_+$,$a \in H_+$, then u is a boundary value if and only if $a \in \mathcal{H}_+ \ominus S$.

1.2.8. Lemma. *The mapping* $J : H^2 \to H^2$,

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \tag{1.2.1.21}$$

leaves $S^* \ominus S$ *invariant.*

Proof. First recall that $(f; g) \in S^*$ if and only if $J(f; g)$ is orthogonal to S. Therefore, if $(f; g) \in S^* \ominus S$, then $J(f; g)$ and also $J^2(f; g) = -(f; g)$ is orthogonal to S, which proves that $J(f, g)$ is in $S^* \ominus S$. □

The vectors a and ib are skewly linked with respect to the inner product $\langle \cdot, \cdot \rangle$ (as defined in Section 1.1), if and only if

$$\begin{pmatrix} (Ja, a) & (Ja, b) \\ (Jb, a) & (Jb, b) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -J. \tag{1.2.1.22}$$

If $u, v \in H_-$, $u = (\cdot, a)_+$ and $v = (\cdot, b)_+$, then we shall say that $u, v \in H_-$ are skewly linked (with respect to $\langle \cdot, \cdot \rangle$), if the vectors $a, b \in H_+$ are skewly linked (with respect to $\langle \cdot, \cdot \rangle$). Boundary values $(\cdot, a)_+$, $(\cdot, b)_+$ such that a and ib are skewly linked will play a role in 2.3.

1.2.9. Proposition. *Suppose S has deficiency index $(1,1)$, and assume that $a, ib \in H_+ \ominus S$ are skewly linked. The vectors $a_1, ib_1 \in H_+ \ominus S$ are skewly linked (with respect to $\langle \cdot, \cdot \rangle$), if and only if there exists an iJ-unitary matrix U such that*

$$(a_1, b_1) = (a, b)\, U. \tag{1.2.1.23}$$

Moreover, we can always find a orthonormal basis a, ib of $H_+ \ominus S$ such that a and ib are skewly linked.

Proof. Suppose that the vectors $a, ib \in H_+ \ominus S$ are skewly linked. Then that a and b are linear independent, hence they form a basis for $H_+ \ominus S$. If we write $(a_1, b_1) = (a, b)U$, then

$$\begin{pmatrix} (Ja_1, a_1) & (Ja_1, b_1) \\ (Jb_1, a_1) & (Jb_1, b_1) \end{pmatrix} = U(-J)U^*,$$

which proves the first assertion. Since the deficiency index of S is $(1,1)$, we can find a vector $a \in H_+ \ominus S$, $\|a\| = 1$, such that

$$S \oplus \operatorname{span}\{a\}$$

is a selfadjoint relation. According Lemma 1.2.8, $b = Ja \in H_+ \ominus S$. Since $(Ja, a) = 0$, we find

$$(Jb, b) = -(a, Ja) = 0, \quad (Ja, b) = (a, a) = 1, \quad (Jb, a) = -(a, a) = -1.$$

Consequently a, ib form a orthonormal base of skewly linked vectors. □

If the deficiency index of S is $(1, 1)$, then any selfadjoint extension A can be written as

$$A = \ker[\cdot, s]_\pm,$$

where s is a certain boundary value. A is called the extension determined by the boundary condition $[\cdot, s]_\pm = 0$.

1.2.10. Theorem. *Suppose that S has deficiency index $(1, 1)$, and assume that that the vectors $a, ib \in H_+ \ominus S$ are skewly linked (with respect to $\langle \cdot, \cdot \rangle$). The extension of S determined by the boundary condition $(\cdot, s)_+ = 0$, $s \in H_+$, is selfadjoint if and only if*

$$s = \alpha a + \beta b, \quad \alpha, \beta \in \mathbb{C}, \alpha\overline{\beta} \in \mathbb{R}, \tag{1.2.1.24}$$

or equivalently

$$s = c\,(\sin\varphi\, a - \cos\varphi\, b), \quad \varphi \in [0, \pi), c \in \mathbb{C}\backslash\{0\}. \tag{1.2.1.25}$$

Proof. Recall that a, b form a basis for $H_+ \ominus S$. Consider $s \in H_+ \ominus S$, $s = \alpha a + \beta b$, and choose $s' \in H_+ \ominus S$ such that $(s, s')_+ = 0$. Then the extension determined by the boundary condition $(\cdot, s)_+ = 0$ is given by

$$T = S \oplus \operatorname{span}\{s'\}.$$

Observe that, if $(Js', s') = 0$, then Lemma 1.2.8 implies that $s = cJs'$ for some $c \in \mathbb{C}$, hence also $(Js, s) = 0$. Similarly, if $(Js, s) = 0$, then $(Js', s') = 0$. This shows that $(Js', s') = 0$ if and only if $(Js, s) = 0$. Therefore T is selfadjoint if and only if

$$(Js, s) = |\alpha|^2 (Ja, a) + \alpha\overline{\beta}(Ja, b) + \overline{\alpha}(a, Jb) + |\beta|^2 (Jb, b) = \operatorname{Im} \alpha\overline{\beta} = 0.$$

The equivalence of (1.2.1.24) and (1.2.1.25) is proved by an elementary calculation. □

Given skewly linked boundary values $u, iv \in H_-$, $u = (\cdot, a)_+$ and $v = (\cdot, b)_+$, Proposition 1.2.9 characterizes all other pairs of skewly linked boundary values u_1, iv_1, and Theorem 1.2.10 shows that all canonical selfadjoint extensions of S are given by the boundary conditions $(\cdot, s_\alpha)_+ = 0$, where

$$s_\alpha = \sin\alpha\, a - \cos\alpha b, \quad \alpha \in [0, \pi).$$

The extension determined by the boundary condition $(\cdot, s_\alpha)_+ = 0$ is explicitly given by

$$A_\alpha = S \oplus \operatorname{span}\{Js_\alpha\}. \tag{1.2.1.26}$$

1.3 Representation by Spaces of Entire Functions

In this section we introduce the method of directing functionals (originally developed by M.G.Krein). We restrict our considerations to a very special case which is suitable for the treatise in Chapter 2. For a treatise of directing functionals in the usual setting, see [AG2] or [GG].

1.3.1. Definition. Suppose S is a closed symmetric operator acting in a Hilbert space $(H, (.,.))$. A family $\phi(z)$, $z \in \mathbb{C}$, of (not necessarily bounded) linear functionals, defined on the entire space H, will be called an universal directing functional for S if the following conditions are satisfied.

1. $\Phi f(z) = (f, \phi(\overline{z}))$ is an entire function.
2. There is a least one $z \in \mathbb{C}$ such that $\phi(\overline{z}) \neq 0$.
3. $\Phi f(z) = 0$ if and only if $f \in \operatorname{ran}(S - z)$.

We shall denote the vector space of all functions Φf, where $f \in H$, by $\mathcal{H}$.

Condition 2 implies that $\mathcal{H}$ is nontrivial, i.e. there exists at least one nonzero function $\Phi f \in \mathcal{H}$. Observe that if $f \in \operatorname{dom} S$, then $\Phi(Sf)(z) - z\Phi f(z) = \Phi(Sf - zf)(z) = 0$. Thus

$$\Phi(S - w)f(z) = (z - w)\Phi f(z). \tag{1.3.1.1}$$

Conversely, if $\Phi f(w) = 0$, then there exists $g \in \operatorname{dom} S$ such that $(S - w)g = f$, hence

$$(z - w)\ \Phi g(z) = \Phi f(z),$$

and therefore

$$\frac{\Phi f(z)}{z - w} = \Phi g(z) \in \mathcal{H}. \tag{1.3.1.2}$$

By (1.3.1.2) there exists for any compact subset K of $\mathbb{C}$ a function $\Phi g \in \mathcal{H}$ such that

$$\Phi g(z) \neq 0, \quad z \in K.$$

Thus $\phi(\overline{z}) \neq 0$ for any $z \in \mathbb{C}$. Since

$$\ker(\cdot, \phi(\overline{z})) = \operatorname{ran}(S - z),$$

we may conclude that, for nonreal z, the kernel of the linear functional $(\cdot, \phi(\overline{z}))$ is closed, hence it is bounded. Lateron we prove that this is also true for real z. Moreover, by condition 3, the deficiency index of S is $(1, 1)$.

The mapping $\Phi : H \to \mathcal{H},\ f \mapsto \Phi f$, is a kind of Fourier transformation for S. Its inversion theorem reads a follows:

1.3.2. Theorem. *Suppose the family $\phi(z)$ of linear functionals satisfies the conditions 1-3. Let $\Delta = (a,b] \subseteq \mathbb{R}$ and choose $u \in H$ with $\Phi u(z) \neq 0$, $z \in \Delta$. Then for any selfadjoint extension $\tilde{A}$ of S, acting in a possibly larger Hilbert space $\tilde{H} \supseteq H$, we have*

$$\tilde{E}(\Delta)f = \int_\Delta \frac{\Phi f}{\Phi u}(\lambda)\, d\tilde{E}_\lambda u, \tag{1.3.1.3}$$

where $(\tilde{E}(\cdot), \tilde{E}(\{\infty\}))$ is the orthogonal resolution of the identity connected with $\tilde{A}$.

Proof. Suppose $\Phi u \neq 0$ on Δ. For any $f \in H$, define the function

$$F(\lambda) = \Phi f(\lambda)/\Phi u(\lambda), \quad \lambda \in \Delta.$$

If we set $f_\lambda = f - F(\lambda)u$, then $\Phi f_\lambda(\lambda) = 0$, hence there is a $g_\lambda \in \operatorname{dom} S$ such that

$$(S - \lambda)g_\lambda = f_\lambda.$$

Let $\tilde{A}$ be a selfadjoint extension of S acting in a possibly larger Hilbert space $\tilde{H}$ with $(\tilde{E}(\cdot), \tilde{E}(\{\infty\}))$ as its orthogonal resolution of the identity. For $\lambda \in \Delta$ put

$$\Delta_\lambda^h = \begin{cases} (\lambda, \lambda + h], & h > 0 \\ (\lambda - h, \lambda], & h < 0 \end{cases}.$$

Then

$$\|\tilde{E}(\Delta_\lambda^h)f - F(\lambda)\tilde{E}(\Delta_\lambda^h)u\| = \|\tilde{E}(\Delta_\lambda^h)f_\lambda\| =$$
$$= \|\tilde{E}(\Delta_\lambda^h)(S-\lambda)g_\lambda\| = \left(\int_{\Delta_\lambda^h} |t-\lambda|^2\, d(\tilde{E}_t g_\lambda, g_\lambda)\right)^{1/2} = o(h).$$

Also, since F is differentiable at λ,

$$\|F(\lambda)\tilde{E}(\Delta_\lambda^h)u - \int_{\Delta_\lambda^h} F(t)d\tilde{E}_t u\| = \left(\int_{\Delta_\lambda^h} |F(t) - F(\lambda)|^2\, d(\tilde{E}_t u, u)\right)^{1/2} = o(h).$$

From the above formulas it follows that the H-valued function

$$\varphi : x \mapsto \int_{(a,x]} d\tilde{E}_t f - \int_{(a,x]} F(t) d\tilde{E}_t u,$$

defined for $x \in \Delta$, is differentiable and its derivative is equal to 0. Since $\lim_{x \to a+} \varphi(x) = 0$, φ is identically 0, and we arrive at the equality

$$d\tilde{E}_\lambda f = F(\lambda) d\tilde{E}_\lambda u \quad \text{on } \Delta, \tag{1.3.1.4}$$

which yields (1.3.1.3). □

Let $\tilde{A}$ be as in Theorem 1.3.2. For any bounded Borel set B choose an element $u \in H$ such that $\Phi u(z) \neq 0$, $z \in B$, and define

$$d\sigma = \frac{d(\tilde{E}_\lambda u, u)}{|\Phi u(\lambda)|^2} \quad \text{on } B. \tag{1.3.1.5}$$

By (1.3.1.3), the definition does not depend on u. By writing $\mathbb{R}$ as a disjoint union of bounded Borel sets B_n, (1.3.1.5) determines uniquely a nonnegative Borel measure on the real line which we shall also denote by $d\sigma$.

1.3.3. Corollary. *There exists nonnegative measures $d\sigma$ such that*

$$(f,g) = \int_{\mathbb{R}} \Phi f(\lambda)\overline{\Phi g}(\lambda)\, d\sigma(\lambda), \quad f,g \in H. \tag{1.3.1.6}$$

In fact, if $\tilde{A}$ is any selfadjoint extension with $\operatorname{ran}\tilde{E}(\mathbb{R}) \supseteq H$, then we can choose $d\sigma$ as in (1.3.1.5).

Proof. Write $\mathbb{R}$ as a union of disjoint intervals $\Delta_n = (a_n, b_n]$. To each Δ_n there is a $u_n \in H$ such that $\Phi u_n \neq 0$ on Δ_n. Define

$$d\sigma = \frac{d(\tilde{E}_\lambda u_n, u_n)}{|\Phi u_n(\lambda)|^2} \quad \text{on } \Delta_n.$$

Since $\tilde{A}$ is an selfadjoint operator, $\tilde{E}(\mathbb{R}) = \mathrm{id}_{\tilde{H}}$, and (1.3.1.3) gives

$$(f,g) = \int_{\mathbb{R}} \Phi f(\lambda)\overline{\Phi g}(\lambda)\, d\sigma(\lambda).$$

Observe that for any $u \in H$, according to (1.3.1.4), $d\tilde{E}_\lambda u = \frac{\Phi u(\lambda)}{\Phi u_n(\lambda)}\, d\tilde{E}_\lambda u_n$, and thus

$$d(\tilde{E}_\lambda u, u) = |\Phi u(\lambda)|^2\, d\sigma \quad \text{on } \Delta_n,$$

which proves the second assertion. □

1.3.4. Remark. For a more complete version of Corollary 1.3.3, see Theorem 1.3.8.

The preceeding corollary also implies that the mapping $\Phi : H \to \mathcal{H}$, $f \mapsto \Phi f$ is one to one, thus it is an isomorphism of the vector spaces H and $\mathcal{H}$. If $\mathcal{H}$ is endowed with the inner product

$$(F,G) = (\Phi^{-1}F, \Phi^{-1}G), \quad F,G \in \mathcal{H},$$

it therefore is a Hilbert space of entire functions. For nonreal $z \in \mathbb{C}$, the point evaluation functional is bounded, which follows from

$$F(w) = (\Phi^{-1}F, \phi(\overline{w})), \quad w \in \mathbb{C}\backslash\mathbb{R}.$$

and the continuity of $(\cdot, \phi(\overline{w}))$, $w \in \mathbb{C}\backslash\mathbb{R}$. Therefore, by Proposition 1.1.9, point evaluation is continuous for all $w \in \mathbb{C}$. In particular, we may regard $\phi(\overline{z})$ as an element of H. Moreover, since

$$\ker(\cdot, \phi(\overline{z})) = \operatorname{ran}(S - z),$$

the range $\operatorname{ran}(S-z)$ is closed, hence S is regular, and we have $\phi(\overline{z}) \in N_{\overline{z}}$. Via the isomorphism Φ, S is transformed into the multiplication operator $M : F(z) \mapsto z\,F(z)$ in $\mathcal{H}$, defined on its maximal domain

$$\operatorname{dom} M = \{F \in \mathcal{H} \,:\, zF(z) \in \mathcal{H}\}.$$

Alltogether, the space $\mathcal{H}$ satisfies the following conditions:

(H1) $\mathcal{H}$ is a reproducing kernel space of entire functions.

(H2) The multiplication operator $F(z) \mapsto z\,F(z)$ is a closed symmetric operator with deficiency index $(1,1)$.

In Section 2.3 we shall see that any space which satisfies the above conditions is in fact of type $\mathcal{H}(A,B)$.

Since $\Phi : H \to \mathcal{H}$ is one to one, and since $\operatorname{span}\{\phi(z)\} = N_z$, S satisfies the minimality condition

$$\operatorname{cls} \bigcup_{z \in \mathbb{C}\backslash\mathbb{R}} N_z = H. \qquad (1.3.1.7)$$

According to Remark 1.2.3, we may regard H as a subspace of H_- identifying $f \in H$ with the functional $\iota f = (\,\cdot\,, (f;0)) \in H_-$. We are able to extend Φ to H_-. Since $\phi(z) \in N_z$ we have $\phi_+(z) \in H_+$. For $u \in H_-$ we define

$$\Phi u(z) := [u, \phi_+(\overline{z})]_\pm = \left[\, u\,, \begin{pmatrix} \phi(\overline{z}) \\ \overline{z}\phi(\overline{z}) \end{pmatrix} \right]_\pm . \qquad (1.3.1.8)$$

The vector space of all functions Φu, $u \in H_-$ is denoted by $\mathcal{H}_-$.

The following lemma particularly imlplies that $\Phi : H_- \to \mathcal{H}_-$ is one to one.

1.3.5. Lemma. *Suppose that S has deficiency index $(1,1)$ and satisfies the minimality condition (1.3.1.7). Choose any nonzero $\phi(z) \in N_z$, $z \in \mathbb{C}\backslash\mathbb{R}$. Then*

$$\operatorname{cls}\{\phi_+(z) : z \in \mathbb{C}\backslash\mathbb{R}\} = H_+. \qquad (1.3.1.9)$$

Proof. Choose a canonical selfadjoint extension A of S and fix any $w \in \mathbb{C}\backslash\mathbb{R}$. Recall that $\phi(z) + (w-z)(A-w)^{-1}\phi(z) \in N_w$, thus there is a nonzero constant $c_z \in \mathbb{C}$ such that

$$(A-w)^{-1}\phi(z) = \frac{\phi(z) - c_z\phi(w)}{z-w}.$$

Since $(\phi(w);0) \in S^* - w$, $(\phi(w),0) \notin A - w$, and since $\operatorname{codim}_{S^*} A = 1$, we have

$$(S^* - w)^{-1} = (A - w)^{-1} \dot{+} \operatorname{span}(0; \phi(w)).$$

Because S is minimal, $\operatorname{cls}\{\phi(z) : z \in \mathbb{C}\backslash\mathbb{R}, z \neq w\} = H$, and

$$(A-w)^{-1} = \operatorname{cls}\left\{\begin{pmatrix} \phi(z) \\ (A-w)^{-1}\phi(z) \end{pmatrix} : z \notin \mathbb{R} \cup \{w\}\right\} = \operatorname{cls}\{\begin{pmatrix} \phi(z) \\ \frac{\varphi(z) - c_z\varphi(w)}{z-w} \end{pmatrix} : z \notin \mathbb{R} \cup \{w\}\}.$$

In this place we used that $(A-w)^{-1}$ is a bounded operator. Thus, by adding multiples of $(0;\phi(w)) \in (S^* - w)^{-1}$ we find

$$(S^* - w)^{-1} = \operatorname{cls}\{\begin{pmatrix} (z-w)\phi(z) \\ \phi(z) \end{pmatrix} : z \notin \mathbb{R}\},$$

which implies (1.3.1.9). □

We can identify $\mathcal{H}_-$ with $\mathcal{H} + z\mathcal{H}$:

1.3.6. Lemma. *Let $v = (\,\cdot\,, (f;g))_+ \in H_-$. Then, with the above notation,*

$$\Phi v(z) = \Phi f(z) + z\Phi g(z). \tag{1.3.1.10}$$

Conversely, if $h(z) \in \mathcal{H} + z\mathcal{H}$, there exists a unique element $v \in H_-$ with $h(z) = \Phi v(z)$.

Proof. The first assertion follows immediately from

$$[v, \phi_+(\overline{z})]_\pm = \left[\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \phi(\overline{z}) \\ \overline{z}\phi(\overline{z}) \end{pmatrix}\right]_\pm = (f, \phi(\overline{z})) + z(g, \phi(\overline{z})).$$

Suppose $h(z) = F(z) + zG(z)$, $F, G \in \mathcal{H}$. Let P denote the orthogonal projection of H^2 onto $S^* = H_+$. Then, setting $v = (\,\cdot\,, P(\Phi^{-1}F; \Phi^{-1}g))_+$, we have

$$[v, \phi_+(\overline{z})]_\pm = (\begin{pmatrix} \Phi^{-1}F \\ \Phi^{-1}G \end{pmatrix}, \begin{pmatrix} \phi(\overline{z}) \\ \overline{z}\phi(\overline{z}) \end{pmatrix}) = (\Phi^{-1}F, \phi(\overline{z})) + z(\Phi^{-1}G, \phi(\overline{z})),$$

which proves the lemma. □

Suppose $F \in \mathcal{H}_-$, and $F(w) = 0$ for some $w \in \mathbb{C}$. By Lemma 1.3.6, we can write $F(z) = H_1(z) + (z-w)H_2(z)$, $H_1, H_2 \in \mathcal{H}$. As $F(w) = 0$, we also have $H_1(w) = 0$, and therefore $H_1(z) = (z-w)H_3(z)$, $H_3 \in \mathcal{H}$. Hence, with $H = H_3 + H_2$,

$$\frac{F(z)}{z-w} = H(z) \in \mathcal{H}. \tag{1.3.1.11}$$

Next, suppose $\tilde{A}$ is a selfadjoint extension of S, acting in a Hilbert space $\tilde{H} \supseteq H$. In the sequel the mappings $\tilde{R}_z^-$, $E(B)^+$, $\tilde{E}(B)^-$ as defined in Section 1.2 will refer to the space triplet

$$\tilde{H}_+ \xrightarrow{\tilde{\pi}} \tilde{H} \xrightarrow{\tilde{\iota}} \tilde{H}_-$$

associated to S as a symmetric subspace of $\tilde{H}^2$. If a functional $u \in H_-$ has the representation

$$[\,\cdot\,, u]_\pm = (\,\cdot\,, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}), \quad \text{ on } H_+,$$

for certain $h_1, h_2 \in H$, then it can be extended in a canonical way fashion to a continuous functional on $\tilde{H}_+$ by

$$[\,\cdot\,, \tilde{u}]_\pm = (\,\cdot\,, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}).$$

For the sake of simplifying notation, we will not distinguish between u and $\tilde{u}$ whenever this does not cause confusion. Therefore we will write $\tilde{R}_z^- u$, $\tilde{E}(B)^- u$ instead of $\tilde{R}_z^- \tilde{u}$ and $\tilde{E}(B)^- \tilde{u}$, respectively.

We can rewrite Theorem 1.3.2 in terms of H_-.

1.3.7. Theorem. *Suppose that the family $\phi(z)$, $z \in \mathbb{C}$, satisfies the conditions 1-3. Let $\Delta = (a, b] \subseteq \mathbb{R}$ and choose $u \in H$ with $\Phi u(z) \neq 0$, $z \in \Delta$. Then for any selfadjoint extension $\tilde{A}$ of S, acting in a possibly larger Hilbert space $\tilde{H} \supseteq H$, we have for arbitrary $f \in H_-$*

$$\tilde{E}(\Delta)^- f = \int_\Delta \frac{\Phi f}{\Phi u}(\lambda)\, d\tilde{E}_\lambda^- u = \int_\Delta \frac{\Phi f}{\Phi u}(\lambda)(\lambda - i)\, d\tilde{E}_\lambda \tilde{R}_i^- u, \tag{1.3.1.12}$$

where $(\tilde{E}(\,\cdot\,), \tilde{E}(\{\infty\}))$ is the orthogonal resolution of the identity connected with $\tilde{A}$.

Proof. For any $f \in H_-$, define

$$F(\lambda) = \frac{\Phi f}{\Phi u}(\lambda), \quad \lambda \in \Delta.$$

If we set $f_\lambda = f - F(\lambda) u \in H_-$, then $F_\lambda(\lambda) = \Phi f_\lambda(\lambda) = 0$. Thus $\frac{F_\lambda(z)}{z-w} = G_\lambda(z) \in \mathcal{H}$. It is easy to see that with $g_\lambda = \Phi^{-1} G_\lambda$, the functional f_λ can be represented as

$$f_\lambda = (\,\cdot\,, \begin{pmatrix} -\lambda g_\lambda \\ g_\lambda \end{pmatrix}).$$

Suppose $\tilde{A}$ is a selfadjoint extension of S which acts in a possibly larger Hilbert space $\tilde{H}$, and denote $(\tilde{E}(\,\cdot\,), \tilde{E}(\{\infty\}))$ its resolution of the identity. Let Δ_λ^h be defined as in Theorem 1.3.2. For any $h \in H$ we obtain

$$(\tilde{E}(\Delta_\lambda^h)^- f_\lambda, h) = (f_\lambda, \tilde{E}(\Delta_\lambda^h)^+ h) = (\begin{pmatrix} -\lambda g_\lambda \\ g_\lambda \end{pmatrix}, \begin{pmatrix} \tilde{E}(\Delta_\lambda^h) h \\ \tilde{A}_s \tilde{E}(\Delta_\lambda^h) h \end{pmatrix}) =$$

$$= -\int_{\Delta_\lambda^h} \lambda\, d\overline{(\tilde{E}_t h, g_\lambda)} + \int_{\Delta_\lambda^h} t\, d\overline{\tilde{E}_t h, g_\lambda} = \int_{\Delta_\lambda^h} (t - \lambda) d(\tilde{E}_t g_\lambda, h).$$

Hence

$$\|\tilde{E}(\Delta_\lambda^h)^- f_\lambda\| \leq \Big(\int_{\Delta_\lambda^h} |t-\lambda|^2 d(\tilde{E}_t g_\lambda, g_\lambda) \Big)^{1/2} = o(h).$$

The same argument as in the proof of Theorem 1.3.2 yields

$$\|\tilde{E}_(\Delta_\lambda^h)^- f - F(\lambda)\tilde{E}(\Delta_\lambda^h)^- u\| = \|\tilde{E}(\Delta_\lambda^h)^- f_\lambda\| = o(h),$$

and since F is diffentiable at λ,

$$\|F(\lambda)\tilde{E}(\Delta_\lambda^h)^- u - \int_{\Delta_\lambda^h} F(t) d\tilde{E}_t^- u\| = \Big(\int_{\Delta_\lambda^h} |F(t)-F(\lambda)|^2 \, d\|\tilde{E}_t^- u\|^2 \Big)^{1/2} = o(h).$$

It follows from the above estimates that the H-valued function

$$\varphi : x \mapsto \int_{(a,x]} d\tilde{E}_t^- f - \int_{(a,x]} F(t)\, d\tilde{E}_t^- u,$$

defined for $x \in \Delta$, has derivative equal to 0. Since $\lim_{x\to a+} \varphi(x) = 0$, φ must vanish identically, which gives

$$d\tilde{E}_\lambda^- f = F(\lambda)\, d\tilde{E}_\lambda^- u \quad \text{on } \Delta. \tag{1.3.1.13}$$

Thus we proved the first equality sign in (1.3.1.12). The validity of the second follows from (1.2.1.14). □

Let $\tilde{A}$ be as in Theorem 1.3.7. For any bounded Borel set B choose an element $u \in H$ such that $\Phi u(z) \neq 0$, $z \in B$, and define

$$d\sigma(\lambda) = \frac{1}{|\Phi u|^2(\lambda)}\, d\|\tilde{E}_\lambda^- u\|^2 = \frac{1}{|\Phi u|^2(\lambda)}\, |\lambda - i|^2\, d(\tilde{E}_\lambda \tilde{R}_i^- u, \tilde{R}_i^- u) \quad \text{on } B. \tag{1.3.1.14}$$

By (1.3.1.12), we have for any other $s \in H_-$

$$d\|\tilde{E}_\lambda^- s\|^2 = \frac{|\Phi s(\lambda)|^2}{|\Phi u(\lambda)|^2}\, d\|\tilde{E}_\lambda^- u\|^2 \quad \text{on } B, \tag{1.3.1.15}$$

hence the definition of σ does not depend on u. By writing $\mathbb{R}$ as a disjoint union of bounded Borel sets B_n, (1.3.1.14) determines uniquely a nonnegative Borel measure on the real line which we shall also denote by $d\sigma$.

In the next theorem we give a description of all nonegative measures σ such that $\mathcal{H}$ is contained isometrically in $L^2(\sigma)$.

1.3.8. Theorem. *The space $\mathcal{H}$ is contained isometrically in $L^2(\sigma)$ for a nonnegative Borel measure σ if and only if $d\sigma$ is of the form* (1.3.1.14) *with some selfadjoint extension $\tilde{A}$ of S satisfying* $\operatorname{ran}\tilde{E}(\mathbb{R}) \supseteq H$.

Proof. Suppose $\tilde{A}$ is a selfadjoint extension of S, defined in a Hilbert space $\tilde{H} \supseteq H$. As in Corollary 1.3.3, it is easily seen that there exists a nonnegative measure σ such that

$$(\tilde{E}(\mathbb{R})f, \tilde{E}(\mathbb{R})g) = \int_{\mathbb{R}} \Phi f(\lambda)\overline{\Phi g}(\lambda)\, d\sigma(\lambda).$$

Moreover, for any Borel set $B \subseteq \mathbb{R}$ and any $u \in H_-$ with $\Phi u \neq 0$ on B, we have

$$d\sigma(\lambda) = \frac{d\|\tilde{E}_\lambda^- u\|^2}{|\Phi u|^2(\lambda)} \quad \text{on } B.$$

Therefore, if $\tilde{E}(\mathbb{R}) \supseteq H$, $\mathcal{H} \subseteq L^2(\sigma)$ isometrically. Conversely, assume that $L^2(\sigma)$ contains $\mathcal{H}$ isometrically as a subspace. Via the isomorphism Φ, $\mathcal{H}$ is isometrically isomorphic to H, and S is transformed into the multiplication operator M in $\mathcal{H}$. Henceforth, we will not distinguish between H and $\mathcal{H}$ or between S and M. Therefore the multiplication operator $\tilde{A} : f(t) \mapsto tf(t)$ in $\tilde{H} = L^2(\sigma)$ is a selfadjoint extension of S. For any Borel set $B \subseteq \mathbb{R}$,

$$\tilde{E}(B) : f(t) \mapsto \chi_B(t)\, f(t),$$

with

$$\chi_B(t) = \begin{cases} 1 & t \in B \\ 0 & t \notin B \end{cases},$$

defines the corresponding orthogonal resolution of the identity. Choose an arbitrary $u \in H_-$,

$$[\,\cdot\,, u]_\pm = \left(\,\cdot\,, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right), \quad h_1, h_2 \in H.$$

Recall that, according to Lemma 1.3.6, $\Phi u(t) = \Phi h_1(t) + t\Phi h_2(t)$. Let $B \subseteq \mathbb{R}$ be a bounded Borel set. Then for any $f \in L^2(\sigma)$

$$(\tilde{E}(B)^- u, f) = (u, \tilde{E}(B)^+ f) = \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} \chi_B(t) f(t) \\ t\chi_B(t) f(t) \end{pmatrix} \right) =$$
$$= \int_B (\Phi h_1(t) + t\Phi h_2(t))\, \overline{f}(t)\, d\sigma(t) = \int_B \Phi u(t)\, \overline{f}(t)\, d\sigma(t).$$

Hence we have

$$\tilde{E}(B)^- u = \chi_B(t)\, \Phi u(t),$$

and thus also

$$d\|E_\lambda^- u\|^2 = |\Phi u|^2(\lambda)\, d\sigma(\lambda),$$

which completes the proof of the theorem. □

1.3.9. Remark. If S is densely defined in H, then $\operatorname{ran} \tilde{E}(\mathbb{R}) \supseteq H$ for any selfadjoint extension $\tilde{A}$. This follows from $\tilde{E}(\mathbb{R})\tilde{H} = \overline{\operatorname{dom} \tilde{A}} \supseteq \overline{\operatorname{dom} S} = H$.

1.4 Representation by $\mathcal{P}(z)$

Throughout this section we shall assume that S is a closed symmetric operator defined in some Hilbert space H with deficiency index $(1,1)$. In the following we introduce the representation with respect to a generalized gauge. The present results are slight generalizations of what can be found in [GG] or [KL], where only the case of densely defined S is treated. See also [KW1].

From now on we shall assume that S satisfies the minimality condition ($N_z = \operatorname{ran}(S-\overline{z})^\perp$)

$$\operatorname{cls} \bigcup_{z\in\mathbb{C}\setminus\mathbb{R}} N_z = H. \tag{1.4.1.1}$$

Recall that, according to Remark 1.2.3, we can view H as a subspace of H_- by identifying the element $f \in H$ with the functional $(\,\cdot\,,(f;0))$ on $H_+ = S^*$. Recall that for any canonical selfadjoint extension A of S and chosen $\varphi(z_0) \in N_{z_0}$, $z_0 \in \mathbb{C}\setminus\mathbb{R}$, the family

$$\varphi(z) = \left(1 + (z - z_0)(A - z)^{-1}\right) \varphi(z_0) \in N_z \quad , z \in \rho(A), \tag{1.4.1.2}$$

is an analytic parametrization of the deficiency spaces N_z (cf. Section 1.1).

For $u \in H_-$, $u \neq 0$, and define

$$r_u(S) := \{z \in r(S) : [n_+, u]_\pm \neq 0 \text{ for some } n \in N_{\overline{z}}\}.$$

Clearly $r_u(S)$ is an open subset of the complex plane $\mathbb{C}$. By Lemma 1.3.5, for each u the set $r_u(S)$ is nonempty. Suppose $r_u(S)$ has nonempty intersections with each of the halfplanes $\mathbb{C}^+$ and $\mathbb{C}^-$. Then, with $\varphi(z)$ as defined above, the function

$$[u, \varphi(\overline{z})]_\pm$$

is holomorphic on each halfplane and does not vanish identically. Thus the zeros of this function form a countable discrete subset of $\mathbb{C}\setminus\mathbb{R}$, and, as $z \mapsto \varphi(z)$ is continuous,

$$\operatorname{cls}\{\varphi(z) \, : z \in r_u(S)\} = \operatorname{cls}\{\varphi(z) \, : z \in \mathbb{C}\setminus\mathbb{R}\} = H. \tag{1.4.1.3}$$

For the same reason we have (compare Lemma 1.3.5)

$$\operatorname{cls}\{\varphi_+(z) \, : z \in r_u(S)\} = \operatorname{cls}\{\varphi_+(z) \, : z \in \mathbb{C}\setminus\mathbb{R}\} = H_+. \tag{1.4.1.4}$$

In the following theorem we give a representation of H by a space functions holomorphic on $r_u(S)$, such that the operator S is transformed into multiplication by the independent variable.

1.4.1. Theorem. *Suppose $r_u(S)$ has a nonempty intersection with each of the halfplanes $\mathbb{C}^+$ and $\mathbb{C}^-$. Moreover, let $f \in H$. Choose any canonical selfadjoint extension A of S. Then the function*

$$\mathcal{P}(z)f := \frac{(f, \varphi(\overline{z}))}{[u, \varphi_+(\overline{z})]_\pm}, \quad z \in r_u(S) \cap \rho(A), \tag{1.4.1.5}$$

has an analytic continuation f_u to $r_u(S)$ which does not depend on the particular choice of A. The mapping

$$\Phi : H \to \mathfrak{H}_u, \quad f \mapsto f_u$$

is a linear isomorphism of H onto $\mathfrak{H}_u = \{f_u : f \in H\}$. Via this isomorphism S is transformed into the multiplication operator $M : F(z) \mapsto z \cdot F(z)$, $\operatorname{dom} M = \{F \in \mathfrak{H}_u : z \cdot F(z) \in \mathfrak{H}_u\}$.

Proof. Obviously the function $f_u(z)$ is analytic on $r_u(S) \cup \rho(A)$. If A' is another selfadjoint extension of S and $\varphi'(z) \in N_z$ a corresponding parametrization of the defect spaces, clearly $\varphi'(z)$ is a multiple of $\varphi(z)$, i.e.

$$\varphi(z) = c(z)\, \varphi'(z), \quad z \in \rho(A) \cap \rho(A').$$

Thus for $z \in r_u(S) \cap \rho(A) \cap \rho(A_1)$

$$\frac{(f, \varphi(\overline{z}))}{[\, u\, , \varphi_+(\overline{z})]_\pm} = \frac{(f, \varphi'(\overline{z}))}{[\, u\, , \varphi'_+(\overline{z})]_\pm}.$$

By Lemma 1.1.4, for any $z \in r(S)$ there exists a selfadjoint extension A' with $z \in \rho(A')$. Hence $f_u(z)$ has an analytic continuation to $r_u(S)$. Because of the relation (1.4.1.3), the mapping $\Phi : f \mapsto f_u$ is one to one, and hence an isomorphism of the spaces H and $\mathfrak{H}_u$. Consider $f \in \operatorname{dom} S$. For $z \in r_u(S) \cap \rho(A)$ we have

$$0 = (Sf - zf, \varphi(\overline{z})) = (Sf, \varphi(\overline{z})) - z(f, \varphi(\overline{z})),$$

which shows

$$\Phi Sf(z) = z\Phi(z). \tag{1.4.1.6}$$

On the other hand, if $\Phi f \in \operatorname{dom} M$, then $(z-w)\Phi f(z) = \Phi g(z)$ for some (uniquely determined) $g \in H$. We keep $w \in r_u(S)$ fixed. Since $\Phi g(w) = 0$, i.e. $(g, \varphi(\overline{w})) = 0$, we have $g \in \operatorname{ran}(S - w)$ and $g = (S - w)h$ for some $h \in \operatorname{dom} S$. Using (1.4.1.6),

$$(z - w)\Phi f(z) = (z - w)\Phi h(z),$$

whence $\Phi f = \Phi h$ and thus $f = h \in \operatorname{dom} S$. ◻

1.4.2. Remark. We can extend the mapping $f \mapsto f_u$ to H_-: For $v \in H_-$ the function

$$\mathcal{P}(z)v := \frac{[\, v\, , \varphi_+(\overline{z})]_\pm}{[\, u\, , \varphi_+(\overline{z})]_\pm}, \quad z \in r_u(S) \cap \rho(A), \tag{1.4.1.7}$$

has an analytic continuation v_u to $r_u(S)$. Moreover, due to Lemma 1.3.5, the mapping

$$\Phi^- : H_- \to \mathfrak{H}_u^-,\ v \mapsto \Phi v = v_u$$

is again a linear isomorphism of H_- onto $\mathfrak{H}_u^- = \{v_u : v \in H_-\}$. Note that $\mathfrak{H}_u \subseteq \mathfrak{H}_u^-$ and that $\Phi u(z) \equiv 1 \in \mathfrak{H}_u^-$.

We will call $u \in H_-$ the gauge of the representation of H given by Theorem 1.4.1. Note that, if the gauge vector u belongs to H, formula (1.4.1.5) simplifies to

$$f_u(z) = \mathcal{P}(z)f = \frac{(f, \varphi(\overline{z}))}{(u, \varphi(\overline{z}))}, \quad z \in r_u(S) \cap \rho(A).$$

In this case the functional $\mathcal{P}(z)$ has also a geometrical interpretation: Since $(f - f_u(z)u, \varphi(\overline{z})) = 0$, the element $f_u(z)u$ is the projection of f onto $U = \operatorname{span}\{u\}$ in the direction of $M_z = \operatorname{ran}(S - z)$. The subspace U is sometimes called the modul of the representation.

Observe that $\mathfrak{H}_u$, equipped with the inner product $(f, g) = (\Phi^{-1}f, \Phi^{-1}g)$, is a Hilbert space of functions analytic on $r_u(S)$. If we put

$$K_w(z) = \frac{1}{[u, n_+]_\pm} \, \Phi n(z),$$

where $n \in N_{\overline{w}}$, then

$$(\Phi f, K_w) = \Phi f(w), \quad w \in r_u(S),$$

thus $\mathfrak{H}_u$ is a reproducing kernel space, i.e. is a Hilbert space of analytic functions such that point evaluation is continuous. It has $K_w(z)$ as its reproducing kernel. Via the tranformation Φ, the space $\mathfrak{H}_u$ is an isomorphic copy of H, and the multiplication operator M is an isomorphic copy of S. Thus the deficiency numbers of M are equal to 1, and

$$\mathfrak{H}_u = \operatorname{ran}(M - w) \oplus \operatorname{span}\{K_w\} = \{F \in \mathfrak{H}_u : F(w) = 0\} \oplus \operatorname{span}\{K_w\}.$$

Note that, if $f \in \mathfrak{H}_u$, $w \in r_u(S)$ and $f(w) = 0$, then also the function $\frac{f(z)}{z-w}$ belongs to $\mathfrak{H}_u$.

Suppose $\tilde{A}$ is a selfadjoint extension of S, acting in a Hilbert space $\tilde{H} \supseteq H$. In the sequel the mappings $\tilde{R}_z^-$, $E(B)^+$, $\tilde{E}(B)^-$ as defined in Section 1.2 will refer to the space triplet

$$\tilde{H}_+ \xrightarrow{\tilde{\pi}} \tilde{H} \xrightarrow{\tilde{\iota}} \tilde{H}_-$$

associated to S as a symmetric subspace of $\tilde{H}^2$. As in section 1.3 we will not distinguish between the functionals $u \in H_-$ and $\tilde{u} = [\tilde{P}^+ \cdot, u]_\pm \in \tilde{H}_-$, where $\tilde{P}^+$ is the orthoprojector of $\tilde{H}_+$ onto H_+.

1.4.3. Lemma. *Suppose $\tilde{A}$ is a selfadjoint extension of S acting in $\tilde{H} \supseteq H$. For some $z_0 \in \mathbb{C}\backslash\mathbb{R}$, choose a nontrivial element $\psi(z_0) \in \tilde{N}_{z_0} = \operatorname{ran}(S - \overline{z})^\perp \subseteq \tilde{H}$ and define*

$$\psi(z) = (I + (z - z_0)(\tilde{A} - z)^{-1})\psi(z_0) \in \tilde{N}_z, \quad z \in \rho(\tilde{A}).$$

Then for any $u \in H_-$

$$[u, \psi_+(\overline{z})]_\pm \, f_u(z) = (f, \psi(\overline{z})), \quad f \in H. \tag{1.4.1.8}$$

Proof. As usual we denot by $\tilde{P}$ the orthogonal projection of $\tilde{H}$ onto H and by $\tilde{P}^+$ the orthogonal projection of $\tilde{H}_+$ onto H_+. Choose a canonical selfadjoint extension A and an analytic parametrization $\varphi(z)$ of the defect spaces N_z associated to S and A. Since $\psi(z) \in \tilde{N}_z$ and therefore $\tilde{P}\psi(z) \in N_z$, we have

$$\tilde{P}\psi(z) = c(z)\varphi(z), \quad c(z) \in \mathbb{C}.$$

Thus also

$$\tilde{P}^+\psi_+(\overline{z}) = \begin{pmatrix} \tilde{P}\psi(\overline{z}) \\ \overline{z}\tilde{P}\psi(\overline{z}) \end{pmatrix} = c(z) \begin{pmatrix} \varphi(\overline{z}) \\ \overline{z}\varphi(\overline{z}) \end{pmatrix} = c(z)\, \varphi_+(\overline{z}), \quad z \in \rho(A) \cap \rho(\tilde{A}),$$

and we obtain

$$[\, u\,, \tilde{P}^+\psi_+(\overline{z})]_\pm\, f_u(z) = \overline{c}(\overline{z})[\, u\,, \varphi_+(\overline{z})]_\pm\, f_u(z) = \overline{c}(\overline{z})(f, \varphi(\overline{z})) = (f, \psi(\overline{z})), \quad z \in \rho(A) \cap \rho(\tilde{A}),$$

which proves the lemma. □

The following theorem is a kind of inversion theorem for the tranformation Φ. One could prove this result similar as Theorem 1.3.7. However, we shall use a different method.

1.4.4. Theorem. *Assume that f_u is analytic on an open set $\Omega \subseteq C$. Choose a selfadjoint extension $\tilde{A}$ acting in $\tilde{H} \supseteq H$, and denote by $(\tilde{E}(\cdot), \tilde{E}(\{\infty\}))$ its orthogonal resolution of the identity. Then for all Borel subsets $B \subseteq \Omega \cap \mathbb{R}$ the following identity holds:*

$$\tilde{E}(B)f = \int_B f_u(\lambda)(\lambda - i)\, d\tilde{E}_\lambda \tilde{R}_i^- u, \quad f \in H. \tag{1.4.1.9}$$

Proof. Define $\psi(z) \in \tilde{N}_z$ as in Lemma 1.4.3. Using Lemma 1.2.5, a simple calculation gives

$$\begin{aligned} \begin{pmatrix} \psi(z) \\ z\psi(z) \end{pmatrix} &= \begin{pmatrix} \psi(z_0) \\ z_0\psi(z_0) \end{pmatrix} + (z - z_0)\tilde{R}_z^+\psi(z_0) = \\ &= \begin{pmatrix} \psi(z_0) \\ z_0\psi(z_0) \end{pmatrix} + (z - z_0)\tilde{R}_{-i}^+\psi(z_0) + (z - z_0)(z + i)\tilde{R}_z^+\tilde{R}_{-i}\psi(z_0). \end{aligned}$$

Thus, with $\tilde{R}_z^+\tilde{R}_{-i} = \tilde{R}_{-i}^+\tilde{R}_z$,

$$\left[u, \begin{pmatrix} \psi(\overline{z}) \\ \overline{z}\psi(\overline{z}) \end{pmatrix}\right]_\pm = \left[u, \begin{pmatrix} \psi(z_0) \\ z_0\psi(z_0) \end{pmatrix}\right]_\pm + (z - \overline{z}_0)(\tilde{R}_i^- u, \psi(z_0)) + (z - \overline{z}_0)(z - i)(\tilde{R}_z\tilde{R}_i^- u, \psi(z_0)),$$

Substituting

$$\tilde{R}_z\tilde{R}_i^- u = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\tilde{E}_\lambda \tilde{R}_i^- u,$$

in the formula above, we get

$$\begin{aligned} \left[u, \begin{pmatrix} \psi(z) \\ z\psi(z) \end{pmatrix}\right]_\pm = \left[u, \begin{pmatrix} \psi(z_0) \\ z_0\psi(z_0) \end{pmatrix}\right]_\pm &+ (z - \overline{z}_0)(\tilde{E}(\{\infty\})\tilde{R}_i^- u, \psi(z_0)) + \\ &+ \int_{\mathbb{R}} \frac{z - \overline{z}_0}{\lambda - z}(\lambda - i)\, d(\tilde{E}_\lambda \tilde{R}_i^- u, \psi(z_0)). \end{aligned} \tag{1.4.1.10}$$

Similar,

$$\begin{aligned}(f,\psi(\overline{z}))) &= (f,\psi(z_0)) + (z-\overline{z}_0)(\tilde{R}_z f,\psi(z_0)) \\ &= (\tilde{E}_\lambda(\{\infty\})f,\psi(z_0)) + \int_{\mathbb{R}} \frac{\lambda-\overline{z}_0}{\lambda-z} d(\tilde{E}_\lambda f,\psi(z_0)).\end{aligned} \tag{1.4.1.11}$$

Since $f_u(z)$ is holomorphic on an open subset Ω, we may choose an interval $[a,b] \subseteq \Omega$ whose endpoints are points of continuity of both measures

$$\begin{aligned} d\sigma_1 &= (\lambda - i) d(\tilde{E}_\lambda \tilde{R}_i^- u, \psi(z_0)), \\ d\sigma_2 &= (\lambda - \overline{z}_0) d(\tilde{E}_\lambda f, \psi(z_0)).\end{aligned}$$

Recall that by Lemma 1.4.3,

$$(f,\psi(\overline{z})) = f_u(z)\left[u, \begin{pmatrix}\psi(\overline{z}) \\ \overline{z}\psi(\overline{z})\end{pmatrix}\right]_\pm .$$

Using the Stieltjes inversion formula (cf. Theorem 1.1.10) and (1.4.1.10), (1.4.1.11), we conclude that

$$\begin{aligned} -\int_{[a,b]} (\lambda-\overline{z}_0) d(\tilde{E}_\lambda \tilde{R}_i^- u,\psi(z_0)) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Delta_\varepsilon}(f,\psi(\overline{z}))dz = \\ = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Delta_\varepsilon} f_u(z)\left[u, \begin{pmatrix}\psi(\overline{z}) \\ \overline{z}\psi(\overline{z})\end{pmatrix}\right]_\pm dz &= -\int_{[a,b]} f(\lambda)(\lambda-\overline{z}_0)(\lambda-i) d(\tilde{E}_\lambda\tilde{R}_i^- u,\psi(z_0))).\end{aligned}$$

Notice that for $\varepsilon \to 0$ the terms in front of the integral of (1.4.1.10) and (1.4.1.11) are neglectible. Since the set of all points of continuity of both measures σ_1, σ_2 is dense in $\mathbb{R}$, it follows that

$$d(\tilde{E}_\lambda f,\psi(z_0)) = f_u(\lambda)(\lambda-i)\, d(\tilde{E}_\lambda \tilde{R}_i^- u,\psi(z_0)) \quad \text{ on } \Omega\cap\mathbb{R}.$$

Remember that in the definition of $\psi(z)$ the choice $z_0 \in \mathbb{C}\backslash R$ and $\psi(z_0) \in \tilde{N}_{z_0}$ was arbitrary. Since $\tilde{N}_z = N_z \oplus (\tilde{H} \ominus H)$ and since S is simple, it follows easily that

$$\text{cls} \bigcup_{z\in\mathbb{C}\backslash\mathbb{R}} \tilde{N}_z = (\text{cls} \bigcup_{z\in\mathbb{C}\backslash\mathbb{R}} N_z) \oplus (\tilde{H}\ominus H) = H \oplus (\tilde{H}\ominus H) = \tilde{H}.$$

Thus we can write

$$d\tilde{E}_\lambda f = f_u(\lambda)(\lambda-i)\, d\tilde{E}_\lambda \tilde{R}_i^- u \quad \text{ on } \Omega\cap\mathbb{R}$$

and we proved (1.4.1.9). □

As an immediate consequence of Theorem 1.4.4 we have:

1.4.5. Corollary. *If f_u and g_u are holomorphic on an open set containing the interval $\Delta \subseteq \mathbb{R}$, and if $\tilde{A}$ is as in Theorem 1.4.4, then*

$$(\tilde{E}(\Delta)f,g) = \int_\Delta f_u(\lambda)\overline{g}_u(\lambda)\,|\lambda-i|^2\, d(\tilde{E}_\lambda\tilde{R}_i^-\tilde{u},\tilde{R}_i^-\tilde{u}) \tag{1.4.1.12}$$

1.4.6. Remark. Assume that f_u is holomorphic on a neighborhood of Δ. If the element $u \in H$ in Theorem 1.4.4 belongs to H, then $\tilde{u} = u \in H$, $\tilde{R}_i^- \tilde{u} = \tilde{R}_i u$ and, since $(\lambda - i) d\tilde{E}_\lambda \tilde{R}_i u = d\tilde{E}_\lambda u$, formula (1.4.1.9) writes as

$$\tilde{E}(\Delta) f = \int_\Delta f_u(\lambda)\, d\tilde{E}_\lambda u.$$

If also $g_u(z)$ is holomorphic on a neighborhood of Δ, then

$$(\tilde{E}(\Delta) f, g) = \int_\Delta f_u(\lambda) \overline{g}_u(\lambda)\, d(\tilde{E}_\lambda u, u).$$

1.4.7. Definition. Let S be a simple symmetric operator with deficiency index $(1,1)$ and let $u \in H_-$. Then S is said to be entire with respect to the generalized gauge u, if $r_u(S) = \mathbb{C}$. The generalized element u is then called an entire gauge for S.

If S is entire and $u \in H_-$ is an entire gauge for S, then the family of functionals $\phi(z)$,

$$(\,\cdot\,, \phi(\overline{z})) : \; f \mapsto f_u(z), \quad z \in \mathbb{C}$$

is a universal directing functional for S as defined in Section 1.3. The spaces $\mathfrak{H}_u$ and $\mathfrak{H}_u^-$ coincide with $\mathcal{H}$ and $\mathcal{H}_-$, respectively. In particular we have $\Phi u \equiv 1 \in \mathcal{H}_-$. Conversely, assume that $\phi(z)$ is a familiy of functionals which satisfies the conditions 1-3 in Section 1.3 and consider the transformation

$$\Phi : H \to \mathcal{H}, \quad f \mapsto \Phi f(z) = (f, \phi(\overline{z}))$$

connected with ϕ. Choose a nonzero $u \in H_-$. Then

$$r_u(S) = \{ z \in \mathbb{C} \;: \Phi u(z) = (u, \phi_+(\overline{z})) \neq 0 \}$$

and, by the definition of f_u,

$$f_u(z) = \frac{\Phi f(z)}{\Phi u(z)}.$$

Thus S is entire if and only if there exists a $u(z) \in \mathcal{H}_-$ such that $u(z) \neq 0$ on $\mathbb{C}$.

1.5 Generalized u-Resolvent Matrices

In the present section we introduce the notion of resolvent matrices of generalized elements. In the case of densely defined operators these result are wellknown (cf. [AG2], [GG] or [KL]. For not densely defined symmetric relations, see [KW1].

Throughout this section S is assumed to be a symmetric relation with deficiency index $(1,1)$ and $\tilde{A}$ denotes a selfadjoint extension of S acting in a possibly larger Hilbert space $\tilde{H} \supseteq H$.

As in Section 1.2, let $\tilde{P}$ be the orthogonal projection of $\tilde{H}$ onto H and let $\tilde{P}^+ = (P;P)$ project $\tilde{H}_+$ onto H_+. Moreover, denote by $(\tilde{E}(.), \tilde{E}(\{\infty\}))$ the orthogonal resolution of the identity associated to $\tilde{A}$ and by $R_z = \tilde{P}\tilde{R}_z|_H$, $z \in \rho(\tilde{A})$, its generalized resolvent. In the sequel the mappings $\tilde{R}_z^-$, $\tilde{E}(B)^+$, $\tilde{E}(B)^-$ as defined in Section 1.2 will refer to the space triplet

$$\tilde{H}_+ \stackrel{\tilde{\pi}}{\to} \tilde{H} \stackrel{\tilde{\iota}}{\to} \tilde{H}_-$$

associated to S as a symmetric subspace of $\tilde{H}^2$. As in Section 1.3 we will not distinguish between the functionals $u \in H_-$ and $\tilde{u} = [\tilde{P}^+ \cdot , u]_\pm \in \tilde{H}_-$.

If $u \in H$, then the function

$$r_u(z) := (R_z u, u), \quad z \in \rho(\tilde{A},) \tag{1.5.1.1}$$

is called the u-resolvent of S induced by $\tilde{A}$. Note that

$$(R_z u, u) = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d(\tilde{E}_\lambda u, u) = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\|\tilde{E}_\lambda^- \iota u\|^2.$$

Thus we can describe all the measures $(\tilde{E}_\lambda u, u)$ once we know all compressed resolvents of S. The latter is effectively done by the Krein formula. For $u \in H_-$ wether $(R_z^- u, u)$ has a meaning, since $R_z^- u \notin H_+$, nor the integral

$$\int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\|\tilde{E}_\lambda^- u\|^2 = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, |\lambda - i|^2 d(\tilde{E}_\lambda \tilde{R}_i^- u, \tilde{R}_i^- u)$$

does converge. Note that, for $u \in H$, we find

$$\int_{\mathbb{R}} \frac{1+\lambda z}{\lambda - z}\, \frac{d(\tilde{E}_\lambda u, u)}{\lambda^2+1} = \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{1}{2}\left(\frac{1}{\lambda - i} + \frac{1}{\lambda + i}\right) d(\tilde{E}_\lambda u, u) = ((R_z - \frac{R_i + R_{-i}}{2})u, u).$$

1.5.1. Definition. Suppose $u \in H$ and R_z is the compressed resolvent generated by some extension $\tilde{A}$, we define

$$\hat{R}_z u := (R_z - \frac{R_i + R_{-i}}{2})u.$$

Any function

$$\hat{r}_u(z) := (\hat{R}_z u, u) + \alpha, \quad \alpha \in \mathbb{R},$$

is called the regularized u-resolvent of S generated by the selfadjoint extension $\tilde{A}$.

We are going to extend this notion to $u \in H_-$. Define, for any generalized resolvent $R_z = \tilde{P}\tilde{R}_z|_H$ of S generated by an extension $\tilde{A}$, a mapping $H_- \to H^2$ by

$$\hat{R}_z^- u := \begin{pmatrix} R_z^- - \frac{R_i^- + R_{-i}^-}{2} \\ zR_z^- - \frac{iR_i^- - iR_{-i}^-}{2} \end{pmatrix} u, \quad z \in \rho(\tilde{A}). \tag{1.5.1.2}$$

1.5.2. Lemma. *Suppose $u \in H_-$. Then, with the above notation,*

$$\hat{R}_z^- u = z\tilde{P}^+ \tilde{R}_{-i}^+ \tilde{R}_i^- u + (z+i)(z-i)\tilde{P}^+ \tilde{R}_{-i}^+ \tilde{R}_z \tilde{R}_i^- u,\ z \in \rho(\tilde{A}), \tag{1.5.1.3}$$

hence $\hat{R}_z^-$ maps H_- into H_+.

Proof. For any $z, w_0, w_1 \in \rho(\tilde{A})$ we find, using Lemma 1.2.5 , that

$$\tilde{R}_z^- - \tilde{R}_{w_0}^- = (z - w_0)\tilde{R}_{w_1}\tilde{R}_{w_0}^- + (z - w_0)(z - w_1)\tilde{R}_{w_1}\tilde{R}_z\tilde{R}_{w_0}^-,$$

and a straightforward computation gives

$$\begin{pmatrix} \tilde{R}_z^- - \tilde{R}_{w_0}^- \\ z\tilde{R}_z^- - w_0\tilde{R}_{w_0}^- \end{pmatrix} = (z - w_0)\tilde{R}_{w_1}^+\tilde{R}_{w_0}^- + (z - w_0)(z - w_1)\tilde{R}_{w_1}^+\tilde{R}_z\tilde{R}_{w_0}^-.$$

Substituting $w_0 = i$, $w_1 = -i$ and vice versa, and summing up gives

$$\begin{pmatrix} \tilde{R}_z^- - \frac{\tilde{R}_i^- + \tilde{R}_{-i}^-}{2} \\ z\tilde{R}_z^- - \frac{i\tilde{R}_i^- - i\tilde{R}_{-i}^-}{2} \end{pmatrix} = z\tilde{R}_{-i}^+\tilde{R}_i^- + (z+i)(z-i)\tilde{R}_{-i}^+\tilde{R}_z\tilde{R}_i^-.$$

Note that $\tilde{P}\tilde{R}_z^- u = R_z^- u$ as one can easily check. Thus, applying $\tilde{P}^+ = (\tilde{P}; \tilde{P})$ to both sides of the formula above, we get (1.5.1.3). ☐

1.5.3. Definition. For $u \in H_-$ and any selfadjoint extension $\tilde{A}$ of S, each function

$$\hat{r}_u(z) := [\hat{R}_z^- u, u]_\pm + \alpha, \quad z \in \rho(\tilde{A}), \alpha \in \mathbb{R},$$

is called a generalized u-resolvent of S induced by the extension $\tilde{A}$.

If $u \in H$ then Definition 1.5.3 is equivalent as Definition 1.5.1 in the sense that $\hat{r}_u(z) = \hat{r}_{\iota u}(z) + \alpha$ with appropriate $\alpha \in \mathbb{R}$.

1.5.4. Theorem. *Suppose $u \in H_-$, $\hat{r}_u(z)$ is the generalized u-resolvent induced by the selfadjoint extension $\tilde{A}$, having $(\tilde{E}(.), \tilde{E}(\{\infty\})$ as its orthogonal resolution of the identity. Then*

$$[\hat{R}_z^- u, u]_\pm = z\,(\tilde{E}(\{\infty\})\tilde{R}_i^- u, \tilde{R}_i^- u) + \int_{\mathbb{R}} \frac{1+\lambda z}{\lambda - z}\frac{d\|\tilde{E}_\lambda^- u\|^2}{\lambda^2+1},\ z \in \rho(\tilde{A}). \tag{1.5.1.4}$$

If $u \in \overline{H} \subseteq H_-$, the term in front of the integral vanishes.

Proof. Using (1.5.1.3), we get

$$\begin{aligned} [\hat{R}_z^- u, u]_\pm &= z(\tilde{R}_i^- \tilde{u}, \tilde{R}_i^- u) + (z+i)(z-i)(\tilde{R}_z\tilde{R}_i^- u, \tilde{R}_i^- u) = \\ &= z(\tilde{E}(\{\infty\})\tilde{R}_i^- u, \tilde{R}_i^- u) + \int_{\mathbb{R}} \left[z + \frac{(z+i)(z-i)}{\lambda - z} \right] d(\tilde{E}_\lambda \tilde{R}_i^- u, \tilde{R}_i^- u) = \\ &= z(\tilde{E}(\{\infty\})\tilde{R}_i^- u, \tilde{R}_i^- u) + \int_{\mathbb{R}} \frac{1+\lambda z}{\lambda - z}\, d(\tilde{E}_\lambda \tilde{R}_i^- u, \tilde{R}_i^- u), \end{aligned}$$

which implies (1.5.1.4). Suppose $u \in H$, then $\tilde{R}_i u \in \operatorname{dom} A$ and since $\operatorname{ran} \tilde{E}(\{\infty\}) = A(0)$ is orthogonal to $\operatorname{dom} A$, the term in front of the integral vanishes. Since $(\tilde{E}(\infty)\tilde{R}_i^- u, \tilde{R}_i^- u)$ depends continuously on $u \in H_-$, the assertion follows. □

1.5.5. Lemma. *Let $\tilde{A}$ be a selfadjoint extension of S acting in a possibly larger Hilbert space $\tilde{H} \supseteq H$. Denote by $(\tilde{E}(\,\cdot\,), \tilde{E}(\{\infty\}))$ its orthogonal resolution of the identity. Then $H \subseteq \tilde{E}(\mathbb{R})$ if and only if for all $u \in H_-$*

$$\tilde{E}(\{\infty\})\tilde{R}_i^- u = 0.$$

Proof. As in Theorem 1.5.4 we have $\tilde{E}(\{\infty\})R_i^- u = 0$ at least for all $u \in \overline{H}$. Recall that by Lemma 1.2.4 we have $u \in H_- \ominus \overline{H}$ if and only if u has the representation $u = (\,\cdot, (0; h))_+$ with $h \in (\operatorname{dom} S)^\perp$. Using (1.2.1.4), we obtain

$$\tilde{E}(\{\infty\})\tilde{R}_i^- u = \tilde{E}(\{\infty\})(h + i\tilde{R}_i h) = \tilde{E}(\{\infty\})h.$$

Since $\tilde{E}(\mathbb{R})$ always contains $\overline{\operatorname{dom} S}$, $H \subseteq \tilde{E}(\mathbb{R})$ if and only if $\tilde{E}(\{\infty\})\tilde{R}_i^- u = 0$ for all $u \in H_- \ominus \overline{H}$, and therefore for all $u \in \overline{H}$. □

1.5.6. Remark. If $\operatorname{dom} S$ is dense in H, then we have $H \subseteq \tilde{E}(\mathbb{R})$ for any selfadjoint extension $\tilde{A}$. In this case we also have $(\tilde{E}(\mathbb{R})\tilde{R}_i^- u, \tilde{R}_i^- u) = 0$ for all $u \in H_- = \overline{H}$. If $\operatorname{dom} S$ is not dense in H, then we can take any $u \in H_-$, $u \notin \overline{H}$, to decide whether or not $\tilde{E}(\mathbb{R})$ contains H: Since S has deficiency index $(1,1)$, we have $\dim H_- \ominus \overline{H} = \dim S^*(0) = 1$, and therefore

$$(\tilde{E}(\{\infty\})\tilde{R}_i^- u, \tilde{R}_i^- u) = 0$$

if and only if $\tilde{E}(\{\infty\})\tilde{R}_i^- v = 0$ for all $v \in H_-$. This implies that $H \subseteq \tilde{E}(\mathbb{R})$ if and only if $(\tilde{E}(\{\infty\})\tilde{R}_i^- u, \tilde{R}_i^- u) = 0$.

Using theorem Theorem 1.2.7, one immediately gets

1.5.7. Theorem. *Fix a canonical selfadjoint extension A° of S, and let $\varphi(z)$ and $Q(z)$ have the same meaning as in Theorem 1.2.7. Then the formula*

$$\begin{aligned}\hat{R}_z^- = \hat{R}_z^{\circ -} &- \frac{[\,\cdot\,, \varphi_+(\overline{z})]_\pm}{Q(z) + \tau(z)}\,\varphi_+(z) + \\ &+ \frac{1}{2}\frac{[\,\cdot\,, \varphi_+(-i)]_\pm}{Q(i) + \tau(i)}\,\varphi_+(i) + \frac{1}{2}\frac{[\,\cdot\,, \varphi_+(i)]_\pm}{Q(-i) + \tau(-i)}\,\varphi_+(-i)\end{aligned}$$

establishes a one to one correspondence between all regularized resolvents $\hat{R}_z^-$ and all functions $\tau \in \mathcal{N}_0 \cup \{\infty\}$.

For any 2×2-matrix valued function $M(z)$ and any complex valued function $\tau(z)$, we define

$$M \circ \tau(z) := \frac{m_{11}(z)\tau(z) + m_{12}(z)}{m_{21}(z)\tau(z) + m_{22}(z)}$$

wherever this expression is meaningful. Note that for 2×2-matrix valued functions $M_1(z)$, $M_2(z)$ and any $\tau(z)$,

$$M_1\circ(M_2\circ\tau)(z)=(M_1M_2)\circ\tau(z).$$

A 2×2 matrix U is called iJ-unitary if

$$UJU^*=J, \tag{1.5.1.5}$$

where

$$J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{1.5.1.6}$$

The following lemma characterizes all iJ-unitary matrices.

1.5.8. Lemma. *A 2×2-matrix $U=(u_{ij})_{i,j=1}^2$ is iJ-unitary if and only if* $\det U=1$ *and the fractional linear transformation*

$$U\circ\tau=\frac{u_{11}\,\tau+u_{12}}{u_{21}\,\tau+u_{22}},\quad \tau\in\mathbb{C}$$

maps the closed upper halfplane onto the closed upper halfplane.

1.5.9. Definition. Let $u\in H_-$. Choose a canonical selfadjoint extension A of S, a parametrization $\varphi(z)$ associated to S and A, a q-function $Q(z)$ of S and A and a generalized u-resolvent $r(z)$ induced by A. For $z\in r_u(S)\cap\rho(A)$, we define a 2×2-matrix function $W(z)=(w_{ij})_{i,j=1}^2$ by

$$w_{11}(z):=\frac{r(z)}{[u,\varphi_+(\overline{z})]_\pm} \tag{1.5.1.7}$$

$$w_{12}(z):=\frac{r(z)Q(z)-[u,\varphi_+(\overline{z})]_\pm[\varphi_+(z),u]_\pm}{[u,\varphi_+(\overline{z})]_\pm} \tag{1.5.1.8}$$

$$w_{21}(z):=\frac{1}{[u,\varphi_+(\overline{z})]_\pm} \tag{1.5.1.9}$$

$$w_{22}(z):=\frac{Q(z)}{[u,\varphi_+(\overline{z})]_\pm} \tag{1.5.1.10}$$

Note that $W(z)$ is holomorphic on $r_u(S)\cap\rho(A)$.

$W(z)$ clearly depends on the choice of A, $\varphi(z)$, $Q(z)$ and $r(z)$. However, this dependence is not essential. If we choose another u-resolvent $r'(z)=r(z)+\alpha$, $\alpha\in\mathbb{R}$, then the modified matrix $W'(z)$ writes

$$W'(z)=\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}W(z). \tag{1.5.1.11}$$

If we choose another Q-function $Q'(z) = Q(z) + \beta$, $\beta \in \mathbb{R}$, then $W(z)$ changes into

$$W'(z) = W(z) \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{1.5.1.12}$$

Observe that the additional factor is iJ-unitary. Next, consider two different selfadjoint extensions A and A' of S, both canonical. By adding appropriate real constants, we can assume that the functions $\varphi(z)$, $Q(z)$, $r(z)$ associated to A and $\varphi'(z)$, $Q'(z)$, $r'(z)$ associated to A' are chosen such that the following conditions are satisfied:

$$\begin{aligned}
(A - z)^{-1} &= (A' - z)^{-1} - \frac{(\,\cdot\,, \varphi'(\overline{z}))}{Q'(z)}\varphi'(z), \quad z \in \rho(A) \cap \rho(A'),\\
(A' - z)^{-1} &= (A - z)^{-1} - \frac{(\,\cdot\,, \varphi(\overline{z}))}{Q(z)}\varphi(z), \quad z \in \rho(A) \cap \rho(A'),\\
r'(z) &= r(z) - \frac{[u, \varphi_+(\overline{z})]_\pm [\varphi_+(z), u]_\pm}{Q(z)},\\
\varphi(i) &= \varphi'(i).
\end{aligned}$$

An elementary calculation leads to the relations

$$\begin{aligned}
\varphi'(z) &= \frac{Q(i)}{Q(z)}\,\varphi(z), \quad z \in \rho(A') \cap \rho(A),\\
\varphi(z) &= \frac{Q'(i)}{Q'(z)}\,\varphi'(z), \quad z \in \rho(A') \cap \rho(A),
\end{aligned}$$

and thus

$$Q'(z) \cdot Q(z) = Q'(i) \cdot Q(i), \quad z \in \rho(A') \cap \rho(A).$$

Substituting the first condition in the second one, using $\varphi(i) = \varphi'(i)$ and setting $z = i$, we obtain

$$\varphi'(-i) = -\frac{Q'(-i)}{Q(-i)}\,\varphi(-i),$$

and hence $Q(i) = -Q'(-i)$. According to the definition of $W'(z)$ and $W(z)$, a straightforward computation using the above relations shows

$$W'(z) = W(z) \begin{pmatrix} 0 & -Q(i) \\ \overline{Q(i)}^{-1} & 0 \end{pmatrix}, \quad z \in \rho(A') \cap \rho(A). \tag{1.5.1.13}$$

Note that the matrix on the right hand side, normed by its determinant, is iJ-unitary. We thus arrive at the following conclusion:

1.5.10. Lemma. *The 2×2-matrix $W(z)$ as defined above can be extended analytically to $r_u(S)$.*

Proof. According to Lemma 1.1.4 there exists to any $z \in r_u(S)$ a canonical selfadjoint extension A' such that $z \in \rho(A')$. Choose $\varphi'(z)$, $Q'(z)$ and $r'(z)$ associated to A' and define $W'(z)$

correspondingly. Then, by the preceeding results, there exists an iJ-unitary 2×2-matrix U and numbers $\alpha \in \mathbb{R}$, $c \in \mathbb{C}\backslash\{0\}$ such that

$$W(z) = c \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} W'(z)\, U, \quad z \in \rho(A) \cap \rho(A'). \tag{1.5.1.14}$$

which proves the assertion. □

We will not distinguish between $W(z)$ and its continuation to $r_u(S)$.

1.5.11. Theorem. *Let $u \in H_-$. Choose any canonical selfadjoint extension A of S and let $W(z)$ be defined correspondingly. Then, for any $\tau \in \mathcal{N}_0 \cup \{\infty\}$,*

$$W \circ \tau(z) = [\hat{R}_z^- u, u]_\pm + \alpha, \quad z \in r_u(S), \alpha \in \mathbb{R},$$

for some selfadjoint extension $\tilde{A}$ of S, where α depends on τ. Conversely, any generalized u-resolvent can be represented in this way.

Proof. Choose a parametrization $\varphi(z)$ and a q-function $Q(z)$ of A and S, and set $r(z) = [\hat{R}_z^{\circ -} u, u]_\pm$, where $\hat{R}_z^{\circ -}$ is induced by the canonical resolvent $R_z^\circ = (A - z)^{-1}$. Then, with $W(z)$ as in Definition 1.5.9, Theorem 1.5.7 gives

$$[\hat{R}_z^- u, u]_\pm = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_2 1(z)\tau(z) + w_{22}(z)} + \mathrm{Re}\, \frac{[u, \varphi_+(-i)]_\pm}{\tau(i)Q(i)} [\varphi_+(i), u]_\pm, \quad z \in \rho(A) \cap \rho(A') \cap r_u(S),$$

where $\hat{R}_z^-$ is generated by the extension $\tilde{A}$ which is associated to the parameter function $\tau(z) \in \mathcal{N}_0 \cup \{\infty\}$ via the Krein formula. □

Observe that for the 2×2-matrix function $W(z)$ as in Definition 1.5.9

$$\det W(z) = \frac{[\varphi_+(z), u]_\pm}{[u, \varphi_+(\overline{z})]_\pm} \neq 0, \quad z \in r_u(S) \cap \rho(A),$$

and hence, due to Lemma 1.5.10, $\det W(z) \neq 0$ for all $z \in r_u(S)$. Since for any generalized u-resolvent $r(z)$ we have $\mathrm{Im}\, r(z) \geq 0$ for $z \in \mathbb{C}^+$, Theorem 1.5.11 yields that for $z \in r_u(S)$ the mapping

$$\zeta \mapsto \frac{w_{11}(z)\zeta + w_{12}(z)}{w_{21}(z)\zeta + w_{22}(z)}$$

is a fractional linear transformation that maps the closed upper halfplane $\overline{\mathbb{C}^+}$ onto a circle lying in $\overline{\mathbb{C}^+}$.

1.5.12. Definition. Let $u \in H_-$. Any 2×2-matrix valued function $W(z)$, holomorphic on $r_u(S)$ possessing the properties of Theorem 1.5.11 is called a generalized u-resolvent matrix.

In the following theorem we characterize all u-resolvent matrices. For a proof see [KW1].

1.5.13. Theorem. *Let $W(z)$ be defined as in Definition 1.5.9. If $W_1(z)$ is a generalized u-resolvent matrix, then there is a function $\gamma(z)$ holomorphic on $r_u(S)$, $\gamma(z) \neq 0$, a constant $\alpha \in \mathbb{R}$, and an iJ-unitary matrix U such that*

$$W_1(z) = \gamma(z) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} W(z)\, U, \quad z \in r_u(S). \tag{1.5.1.15}$$

Conversely, any 2×2-matrix function defined by (1.5.1.15) *is a generalized u-resolvent matrix.*

Next let us consider the situation where a universal directing functional $\phi(z)$, $z \in \mathbb{C}$, for S as defined in Section 1.3 is given. Recall that for $u \in H_-$

$$r_u(S) = \{z \in \mathbb{C} \, : \Phi u(z) = [u, \phi_+(\overline{z})]_\pm \neq 0\}.$$

Thus $W(z)$ as in Definition 1.5.9 is meromorphic in $\mathbb{C}$, with possible poles only at the zeros of $[u, \phi_+(\overline{z})]_\pm$.

1.5.14. Theorem. *Let $W(z)$ be as in Definition 1.5.9. Then the u-resolvent matrix $W_d(z) = [u, \phi_+(\overline{z})]_\pm\, W(z)$ is entire.*

Proof. By Lemma 1.5.10, the matrix function $W_d(z)$ is analytic on $r_u(S) = \{z \in \mathbb{C}[u, \phi_+(\overline{z})]_\pm \neq 0\}$. Suppose $a \in \mathbb{C}$ and $[u, \phi_+(\overline{z})]_\pm = 0$. Choose a canonical selfadjoint extension A' of S such that $a \in \rho(A')$. This is always possible by Lemma 1.1.4, since $r(S) = \mathbb{C}$. As in Lemma 1.5.10 we have

$$W(z) = U\, W'(z)\, V$$

for certain iJ-unitary matrices U and V. The functions $\varphi'(z)$, $Q'(z)$, $r'(z)$ connected with A' are analytic on $\rho(A')$. We can write $\varphi'(z) = c(z)\phi(z)$ for an analytic function $c(z) \neq 0$ on $\rho(A')$. Since

$$\frac{[u, \phi_+(\overline{z})]_\pm}{[u, \varphi'_+(\overline{z})]_\pm} = \frac{1}{\overline{c(\overline{z})}}$$

is analytic on $\rho(A')$, the matrix function $W'_d(z) = [u, \phi_+(\overline{z})]_\pm\, W'(z)$ is analytic on $\rho(A')$. In particular $W_d(z)$ is analytic in a neighborhood of a. □

1.6 Representation by $(-\mathcal{Q}(z); \mathcal{P}(z))$

The results presented here can also be found in [KL], [St] or [KW1]. As in Section 1.4, we assume througout the following that S is a closed symmetric operator with deficiency index $(1,1)$.

Suppose $u \in H_-$. In Section 1.4 we showed that, chosen a canonical selfadjoint extension A of S and a parametrization $\varphi(z) \in N_z$ connected with S and A, the function

$$\mathcal{P}(z) = \frac{(f, \varphi(\overline{z}))}{[u, \varphi_+(\overline{z})]_\pm}, \quad z \in r_u(S) \cap \rho(A),$$

has an analytic continuation to $r_u(S)$ and that this continuation does not depend on the choice of A and $\varphi(z)$. Choose a u-resolvent $r(z)$ generated by the extension A and define the vector function

$$\psi(z) = R_z^- u - \frac{r(z)}{[u, \varphi(z)]_\pm} \varphi(z), \quad z \in \rho(A) \cap r_u(S).$$

Moreover, let

$$\mathcal{Q}(z)f = (f, \psi(\overline{z})) = [R_z^+ f, u]_\pm - (P(z)f)r(z), \quad z \in \rho(A) \cap r_u(S).$$

1.6.1. Lemma. *The function $\mathcal{Q}(z)$ defined above has an analytic continuation to $r_u(S)$. This continuation does not depend on the choice of the canonical selfadjoint extension A and the parametrization $\varphi(z)$.*

Proof. If A' is another canonical extension of S, then by the Krein formula

$$R'_z = R_z - \frac{(\,\cdot\,, \varphi(\overline{z}))}{\tau + Q(z)} \varphi(z), \quad z \in r_u(S) \cap \rho(A) \cap \rho(A'),$$

for a certain real parameter τ. Suppose $r'(z)$ is a u-resolvent generated by A' such that

$$r'(z) = r(z) - \frac{[u, \varphi_+(\overline{z})]_\pm}{\tau + Q(z)} [\varphi_+(z), u]_\pm, \quad z \in r_u(S) \cap \rho(A) \cap \rho(A').$$

We then have

$$\begin{aligned} \mathcal{Q}'(z)f = [R'^{\,+}_z f, u]_\pm - (\mathcal{P}(z)f)\; r'(z) = [R_z^+ f - \frac{(f, \varphi(\overline{z}))}{\tau + Q(z)} \varphi_+(z), u]_\pm - \\ - \mathcal{P}(z)f \left(r(z) - \frac{(f, \varphi(\overline{z}))}{\tau + Q(z)} [\varphi_+(z), u]_\pm \right) = \mathcal{Q}(z)f. \end{aligned}$$

□

In the following lemma we show that we can view $\{\varphi(z), \psi(z)\}$ as a parametrization of the defect spaces of a certain restriction S_u of S.

1.6.2. Lemma. *Let S be a symmetric operator with deficiency index $(1, 1)$. If $u \in H_-$ is chosen such that $S \not\subseteq \ker u$, then the symmetric operator*

$$S_u = S \cap \ker u = \{ \begin{pmatrix} f \\ g \end{pmatrix} \in S\ : [\begin{pmatrix} f \\ g \end{pmatrix}, u]_\pm = 0 \}$$

has deficiency index $(2,2)$*. Moreover, if* A *is any canonical selfadjoint extension of* S *and* $\varphi(z)$ *and* $\psi(z)$ *are defined correspondingly, then*

$$\operatorname{ran}(S_u - z)^{\perp} = \operatorname{span}\{\varphi(\overline{z}), R^-_{\overline{z}} u\} = \operatorname{span}\{\varphi(\overline{z}), \psi(\overline{z})\}, \quad z \in r_u(S) \cap \rho(A).$$

If $S \subseteq \ker u$ *(and therefore* $S_u = S$*), then there exists a holomorphic function* $c(z)$ *on* $r_u(S)$ *such that* $\psi(z) = \overline{c(\overline{z})}\varphi(z)$*. We then have in particular*

$$\mathcal{Q}(z) = c(z)\, \mathcal{P}(z), \quad z \in r_u(S).$$

Proof. Let A be a canonical selfadjoint extension of S. Observe that for $(f; g) \in S$ we have

$$(g - zf, R^-_{\overline{z}} u) = [R^+_z (g - zf), u]_{\pm} = [\begin{pmatrix} f \\ g \end{pmatrix}, u]_{\pm}. \tag{1.6.1.1}$$

This implies that $R^-_{\overline{z}} u$ belongs to the defect space $\operatorname{ran}(S_u - z)^{\perp}$. If $S \not\subseteq \ker u$, then $\operatorname{codim}_S S_u = 1$ and hence S_u has deficiency index $(2,2)$. Moreover, according to (1.6.1.1), the vector $R^-_{\overline{z}} u$ does not belong to $(S - z)^{\perp}$ (otherwise $S \subseteq \ker u$). Thus $\varphi(\overline{z})$ and $R^-_{\overline{z}} u$ (and therefore $\varphi(\overline{z})$ and $\psi(\overline{z})$) are linearly independent and

$$\operatorname{ran}(S_u - z)^{\perp} = \operatorname{span}\{\varphi(\overline{z}), R^-_{\overline{z}} u\} = \operatorname{span}\{\varphi(\overline{z}), \psi(\overline{z})\}.$$

If $S \subseteq \ker u$, then $S_u = S$ has deficiency index $(1,1)$. Thus $R^-_z u$ and therefore $\psi(z)$ must be a multiple of $\varphi(z)$, i.e. $\psi(z) = \overline{c(\overline{z})}\varphi(z)$, $z \in \rho(A)$, for some holomorphic function $c(z)$. Hence

$$\mathcal{Q}(z) = c(z)\mathcal{P}(z), \quad z \in \rho(A).$$

Since for each $f \in H$ both functions $\mathcal{P}(z)f$ and $\mathcal{Q}(z)f$ can be continued analytically to $r_u(S)$, also $c(z)$ has a holomorphic continuation to $r_u(S)$. ☐

From now on, we shall assume that S_u is minimal, i.e. the following relation holds:

$$\operatorname{cls} \bigcup_{z \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(S_u - z)^{\perp} = H. \tag{1.6.1.2}$$

Note that this is always the case if S itself is minimal. Denote by Φ the mapping that assigns to each $f \in H$ the analytic vector function

$$\Phi f = \mathbf{f}(z) = \begin{pmatrix} -Q(z)f \\ \mathcal{P}(z)f \end{pmatrix}, \quad z \in r_u(S).$$

It follows from the minimality of S_u that, if $u \in H_-$ is chosen such that $r_u(S)$ has nonempty intersection with both halfplanes $\mathbb{C}^+$ and $\mathbb{C}^-$, Φ is one to one. Define $\mathfrak{H}_u$ as the space of all vector functions $\mathbf{f} = \Phi f$, $f \in H$, equipped with the inner product

$$(\mathbf{f}, \mathbf{g}) = (f, g).$$

The space $\mathfrak{H}_u$ is a Hilbert space of functions analytic on $r_u(S)$, and $\Phi : H \to \mathfrak{H}_u$ is an isomorphism of the Hilbert spaces H and $\mathfrak{H}_u$. It is easy to verify that, via the mapping Φ, the operator S_u is transformed into the operator of multiplication by the independent variable. According to Lemma 1.6.2, the multiplication operator in $\mathcal{H}_u$ may have deficiency index $(1, 1)$ or $(2, 2)$, depending on whether or not $S \subseteq \ker u$.

For $\mathbf{f} \in \mathfrak{H}_u$ and $a \in r_u(S)$ define the difference quotient by

$$\mathcal{R}_1(a)\mathbf{f} = \frac{\mathbf{f}(z) - \mathbf{f}(a)}{z - a}. \tag{1.6.1.3}$$

Via the isomorphism $\Phi : H \to \mathfrak{H}_u$, the operator S is transformed as follows:

1.6.3. Theorem. *Let $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ be defined as above. Then for any $f \in \operatorname{dom} S$*

$$\mathcal{Q}(z)Sf = z\,\mathcal{Q}(z)f + \left[\begin{pmatrix} f \\ Sf \end{pmatrix}, u \right]_{\pm}. \tag{1.6.1.4}$$

Hence the operator $\Phi(S)$ is determined by

$$(\Phi(S) - a)^{-1} = \mathcal{R}_1(a)|_{\{\mathbf{f}:\, \pi_2\mathbf{f}(a)=0\}}, \quad a \in r_u(S). \tag{1.6.1.5}$$

Proof. Suppose $f \in \operatorname{dom} S$. Note that for any selfadjoint extension A of S,

$$\begin{aligned}[R_z^+ Sf, u]_{\pm} &= z\,[R_z^+ f, u]_{\pm} + [R_z^+(S - z)f, u]_{\pm} = \\ &= z\,[R_z^+ f, u]_{\pm} + \left[\begin{pmatrix} f \\ Sf \end{pmatrix}, u \right]_{\pm}.\end{aligned}$$

Since $\mathcal{P}(z)Sf = z\,\mathcal{P}(z)f$, the above formula implies (1.6.1.4). Fix $a \in r_u(S)$ and choose a canonical selfadjoint extension A of S with $a \in \rho(A)$. Since $\mathbf{f} \in \operatorname{ran}(\Phi(S) - a)$ if and only if $(f, \varphi(\overline{a})) = \pi_2\mathbf{f}(a) = 0$ we get

$$\operatorname{ran}(\Phi(S) - a) = \{\mathbf{f} \,:\, \pi_2\mathbf{f}(a) = 0\}, \quad a \in r_u(S).$$

Together with (1.6.1.4) this immediately implies (1.6.1.5). □

1.6.4. Remark. If S is minimal, then $\Phi f(z)$ is uniquely determined by its second component $\pi_2\Phi f(z) = \mathcal{P}(z)f$. According Theorem 1.4.1, S corresponds via Φ to the operator

$$\Phi(S) = \{(\mathbf{f}; \mathbf{g}) \,:\, z\pi_2\mathbf{f}(z) = \pi_2\mathbf{g}(z)\}.$$

Observe that for any $w \in r_u(S)$, $x, y \in \mathbb{C}$, the linear functional

$$f \mapsto -x\,\mathcal{Q}(w)f + y\,\mathcal{P}(z)f$$

is bounded. This implies that in $\mathfrak{H}_u$ the point evaluation functional $\mathbf{f} \mapsto x\,\pi_1\mathbf{f}(w) + y\,\pi_2\mathbf{f}(w)$ is bounded, i.e. $\mathfrak{H}_u$ is reproducing kernel space. Its reproducing kernel is given by a 2×2-matrix function $K(w, z)$ such that for any $(x; y) \in \mathbb{C}^2$ and $w \in r_u(S)$ the function

$$K(w, \,\cdot\,) \begin{pmatrix} x \\ y \end{pmatrix}$$

belongs to $\mathfrak{H}_u$ and satisfies

$$(\mathbf{f}, K(w,\,\cdot\,)\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \end{pmatrix}^* \mathbf{f}(w), \quad \mathbf{f} \in \mathfrak{H}_u.$$

The kernel of $\mathfrak{H}_u$ can be computed explicitly:

1.6.5. Theorem. *For $w, z \in r_u(S)$ and $W(z)$ as in Definition 1.5.9, the relation*

$$H_W(w,z) = \frac{W(z)JW(w)^* - J}{z - \overline{w}} = \begin{pmatrix} -\mathcal{Q}(z) \\ \mathcal{P}(z) \end{pmatrix} (-\mathcal{Q}(w)^*, \mathcal{P}(w)^*), \quad z, w \in r_u(S), \qquad (1.6.1.6)$$

holds. Hence $H_W(w,z)$ is the reproducing kernel of $\mathfrak{H}_u$.

Proof. For $w \in r_u(S)$ the adjoints of the linear functionals $f \mapsto \mathcal{P}(w)f$ and $f \mapsto \mathcal{Q}(w)f$ are given by

$$\mathcal{P}(w)^* y = y\, \frac{1}{[\varphi_+(\overline{w}), u]_\pm}\, \varphi(\overline{w}), \quad y \in \mathbb{C}, \qquad (1.6.1.7)$$

and

$$\mathcal{Q}(w)^* x = x\, (R^-_{\overline{w}} u - \frac{r(\overline{w})}{[\varphi_+(\overline{w}), u]_\pm}\, \varphi(\overline{w})), \quad x \in \mathbb{C}, \qquad (1.6.1.8)$$

respectively. By Definition 1.5.9 we have

$$(z - \overline{w})\mathcal{P}(z)\mathcal{P}(w)^* = (z - \overline{w})\, \frac{(\varphi(\overline{w}), \varphi(\overline{z}))}{[\varphi_+(\overline{w}), u]_\pm [u, \varphi_+(\overline{z})]_\pm} =$$

$$= \frac{Q(z) - \overline{Q(w)}}{[u, \varphi_+(\overline{z})]_\pm \overline{[u, \varphi_+(\overline{w})]_\pm}} = w_{22}(z)\overline{w_{21}(w)} - w_{21}(z)\overline{w_{21}(w)}.$$

Since $(w - \overline{z})R^+_w \varphi(\overline{z}) = \varphi_+(w) - \varphi_+(\overline{z})$,

$$-(z - \overline{w})\mathcal{P}(z)\mathcal{Q}(w)^* = -(z - \overline{w})\frac{(R^-_{\overline{w}} u, \varphi(\overline{z}))}{[u, \varphi_+(\overline{z})]_\pm} + \frac{r(\overline{w})\,(z - \overline{w})(\varphi(\overline{w}), \varphi(\overline{z}))}{[\varphi_+(\overline{w}), u]_\pm [u, \varphi_+(\overline{z})]_\pm} =$$

$$= \frac{[u, \varphi_+(w) - \varphi_+(\overline{z})]_\pm}{[u, \varphi_+(\overline{z})]_\pm} + \frac{\overline{r(w)}\,(Q(z) - \overline{Q(w)})}{[\varphi_+(\overline{w}), u]_\pm [u, \varphi_+(\overline{z})]_\pm} = -1 + \frac{Q(z)}{[u, \varphi_+(\overline{z})]_\pm} \frac{\overline{r(w)}}{\overline{[u, \varphi_+(\overline{w})]_\pm}} -$$

$$- \frac{\overline{r(w)Q(w)} - \overline{[\varphi_+(w), u]_\pm} \overline{[u, \varphi_+(\overline{w})]_\pm}}{[u, \varphi_+(\overline{z})]_\pm \overline{[u, \varphi_+(\overline{w})]_\pm}} = w_{22}(z)\overline{w_{11}(w)} - w_{21}(z)\overline{w_{12}(w)} - 1,$$

and

$$-(z - \overline{w})\mathcal{Q}(z)\mathcal{P}(w)^* = -(z - \overline{w})\frac{[R^+_z \varphi(\overline{w}), u]_\pm}{[\varphi_+(\overline{w}), u]_\pm} + \frac{r(z)\,(z - \overline{w})(\varphi(\overline{w}), \varphi(\overline{z}))}{[\varphi_+(\overline{w}), u]_\pm [u, \varphi_+(\overline{z})]_\pm} =$$

$$= -\frac{[\varphi_+(z) - \varphi_+(w), u]_\pm}{\overline{[u, \varphi_+(\overline{w})]_\pm}} + \frac{r(z)\,(Q(z) - Q(w))}{[u, \varphi_+(\overline{z})]_\pm \overline{[u, \varphi_+(\overline{w})]_\pm}} = \frac{r(z)Q(z) - [\varphi_+(z), u]_\pm [u, \varphi_+(\overline{z})]_\pm}{[u, \varphi_+(\overline{z})]_\pm \overline{[u, \varphi_+(\overline{w})]_\pm}} -$$

$$- \frac{r(z)}{[u, \varphi_+(\overline{z})]_\pm} \frac{\overline{Q(w)}}{\overline{[u, \varphi_+(\overline{w})]_\pm}} + 1 = w_{12}(z)\overline{w_{21}(w)} - w_{11}(z)\overline{w_{22}(w)} + 1.$$

Moreover, since $(z-\overline{w})\,[R_z^+R_{\overline{w}}^-u,u]_\pm = r(z)-\overline{r(w)}$,

$$(z-\overline{w})\mathcal{Q}(z)\mathcal{Q}(w)^* = \frac{\overline{r(w)}r(z)\,(z-\overline{w})(\varphi(\overline{w}),\varphi(\overline{z}))}{[u,\varphi_+(\overline{z})]_\pm\overline{[u,\varphi_+(\overline{w})]}_\pm} + (z-\overline{w})[R_z^+R_{\overline{w}}^-u,u]_\pm -$$

$$-\frac{\overline{r(w)}(z-\overline{w})[R_z^+\varphi(\overline{w}),u]_\pm}{[\varphi_+(\overline{w}),u]_\pm} - \frac{r(z)(z-\overline{w})[u,R_w^+\varphi(\overline{z})]_\pm}{[u,\varphi_+(\overline{z})]_\pm} = \frac{\overline{r(w)}r(z)(Q(z)-\overline{Q(w)})}{[u,\varphi_+(\overline{z})]_\pm\overline{[u,\varphi_+(\overline{w})]}_\pm} +$$

$$+r(z)-\overline{r(w)} - \frac{\overline{r(w)}[\varphi_+(z)-\varphi_+(\overline{w}),u]_\pm}{[\varphi_+(\overline{w}),u]_\pm} - \frac{r(z)[u,\varphi_+(w)-\varphi_+(\overline{z})]_\pm}{[u,\varphi_+(\overline{z})]_\pm} =$$

$$= \frac{\big(r(z)Q(z)-[\varphi_+(z),u]_\pm[u,\varphi_+(\overline{z})]_\pm\big)\overline{r(w)}}{[u,\varphi_+(\overline{z})]_\pm\overline{[u,\varphi_+(\overline{w})]}_\pm} - \frac{r(z)\big(\overline{r(w)Q(w)}-\overline{[\varphi_+(w),u]}_\pm\overline{[u,\varphi_+(\overline{w})]}_\pm\big)}{[u,\varphi_+(\overline{z})]_\pm\overline{[u,\varphi_+(\overline{w})]}_\pm} =$$

$$= w_{12}(z)\overline{w_{11}(w)} - w_{11}(z)\overline{w_{12}(w)}.$$

This proves (1.6.1.6). A short computation shows that $H_W(w,z)$ is the reproducing kernel for $\mathfrak{H}_u$. □

We give some properties of the difference quotient $\mathcal{R}_1(a)$:

1.6.6. Theorem. *The space $\mathfrak{H}_u$ is invariant under forming difference quotients. Moreover, the operator $\mathcal{R}_1(a):\mathfrak{H}_u\to\mathfrak{H}_u$ is the resolvent operator of a certain linear relation extending S. In particular, $\mathcal{R}_1(a)$ is bounded and $\mathcal{R}_1(a)\mathbf{f}$ depends analytically on $a\in r_u(S)$. For $\mathbf{f},\mathbf{g}\in\mathfrak{H}_u$ and $a,b\in r_u(S)$, the following identity holds.*

$$\mathbf{g}(b)^*J\mathbf{f}(a) = (\mathbf{f},\mathcal{R}_1(b)\mathbf{g}) - (R_1(a)\mathbf{f},\mathbf{g}) + (a-\overline{b})\,(\mathcal{R}_1(a)\mathbf{f},\mathcal{R}_1(b)\mathbf{g}) \tag{1.6.1.9}$$

Proof. Let $a\in r_u(S)$ and choose a canonical selfadjoint extension A of S such that $a\in\rho(A)$. Consider the element $R_a^-(\iota f-\mathcal{P}(a)f\,u)$. We shall show that

$$\mathcal{R}_1(a)\mathbf{f} = \frac{\mathbf{f}(z)-\mathbf{f}(a)}{z-a} = \Phi(R_a^-(\iota f-\mathcal{P}(a)f\,u)). \tag{1.6.1.10}$$

Since, by (1.2.1.18), we have $R_a^+\varphi(z)=\frac{\varphi_+(z)-\varphi_+(a)}{z-a}$, it follows that

$$\mathcal{P}(z)(R_a^-(\iota f-\mathcal{P}(a)f\,u)) = \frac{(R_a^-(\iota f-\mathcal{P}(a)f\,u),\varphi(\overline{z})}{[u,\varphi_+(\overline{z})]_\pm} =$$

$$\frac{[\iota f-\mathcal{P}(a)f\,u,\frac{\varphi_+(\overline{z})-\varphi_+(\overline{a})}{\overline{z}-\overline{a}}]_\pm}{[u,\varphi_+(\overline{z})]_\pm} = \frac{\mathcal{P}(z)f-\mathcal{P}(a)f}{z-a}.$$

As one can easily verify, the resolvent identity implies $\hat{R}_z^- - \hat{R}_a^- = (z-a)\,R_z^+R_a^-$. Also, since $R_a^-\iota f = R_af$,

$$Q(z)(R_a^-(\iota f-\mathcal{P}(a)f\,u)) = [R_z^+R_a^-(\iota f-\mathcal{P}(a)f\,u),u]_\pm -$$

$$-r(z)\,\mathcal{P}(z)(R_a^-(\iota f-\mathcal{P}(a)f\,u)) = \frac{[R_z^+f,u]_\pm-[R_a^+f,u]_\pm-\mathcal{P}(a)f\,r(z)+\mathcal{P}(a)f\,r(a)}{z-a} -$$

$$-\frac{\mathcal{P}(z)f-\mathcal{P}(a)f}{z-a} = \frac{\mathcal{Q}(z)f-\mathcal{Q}(a)f}{z-a}.$$

It follows from (1.6.1.10) that $\mathcal{R}_1(a) : \mathfrak{H}_u \to \mathfrak{H}_u$ is bounded and depends analytically on $a \in r_u(S)$. Since the difference quotient obviously satisfies the resolvent identity $\mathcal{R}_1(a) - \mathcal{R}_1(b) = (a-b)\,\mathcal{R}_1(a)\mathcal{R}_1(b)$, it is the resolvent operator of a certain relation extending S (cf. Theorem 1.1.2). Finally, a computation will prove the identity (1.6.1.9)

$$\begin{gathered}\big(f, R_b^-(\iota g - \mathcal{P}(b)g\,u)\big) - \big(R_a^-(\iota f - \mathcal{P}(a)f\,u), g\big) + \\ +(a-\overline{b})\,\big(R_a^-(\iota f - \mathcal{P}(a)f\,u), R_b^-(\iota g - \mathcal{P}(b)g\,u)\big) = (R_{\overline{b}}f, g) - (R_a f, g) + (a-\overline{b})\,(R_{\overline{b}}R_a f, g) + \\ +\mathcal{P}(a)f\,\big([u, R_{\overline{a}}^+ g]_\pm + (\overline{b}-a)\,[u, R_{\overline{a}}^+ R_b g]_\pm\big) - P(b)g\,\big([R_{\overline{b}}^+ f, u]_\pm + (a-\overline{b})\,[R_{\overline{b}}^+ R_a f, u]_\pm\big) + \\ +(a-\overline{b})\mathcal{P}(a)f\overline{\mathcal{P}(b)g}\,[R_{\overline{b}}^+ R_a^- u, u]_\pm = \mathcal{P}(a)f\,\big([u, R_b^+ g]_\pm - \overline{\mathcal{P}(b)g}\,\overline{r(b)}\big) - \\ -\overline{\mathcal{P}(b)g}\,\big([R_a^+ f, u]_\pm + \mathcal{P}(a)f\,r(a)\big) = \mathbf{g}(b)^* J f(a).\end{gathered}$$

□

For the remainder of this section we investigate u-resolvent matrices of symmetric extensions of S. Most of the proofs will not be carried out in all details, for a more rigorous treatment see e.g. [KW1].

Suppose S_1 is a symmetric extension of S defined in a larger Hilbert space H_1. Both S and S_1 are assumed to have deficiency index $(1,1)$. As before, the space triplet

$$H_+ \stackrel{\pi}{\to} H \stackrel{\iota}{\to} H_-$$

refers to the symmetric operator S, and

$$H_{1,+} \stackrel{\pi_1}{\to} H_1 \stackrel{\iota_1}{\to} H_{1,-}$$

denotes the space triplet associated to $S_1 \subseteq H_1^2$. For our purposes, we need to embed H_- into $H_{1,-}$: Let P be the orthogonal projection of H_1 onto H. As in Section 1.2, it is easy to see that the projection

$$P \oplus P : H_1^2 \to H^2, \quad \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} Pf \\ Pg \end{pmatrix},$$

maps $H_{1,+}$ into H_+. Denote by P_+ the restriction of $P \oplus P$ to $H_{1,+}$. The adjoint P_+^* of $P_+ : H_{1,+} \to H_+$ is a mapping of H_- into $H_{1,-}$.

1.6.7. Lemma. *Denote by $r(S,S_1)$ the set $r(S,S_1) = \{z \in r(S_1) \ : H \not\subseteq \operatorname{ran}(S_1 - z)\}$. We then have*

$$P(\operatorname{ran}(S_1 - z)^\perp) = \operatorname{ran}(S - z)^\perp, \quad z \in r(S,S_1).$$

Moreover, if $u \in H_-$ then

$$r_u(S) \cap r(S,S_1) = r_{P_+^* u}(S_1) \cap r(S,S_1).$$

Let A and A_1 be canonical selfadjoint extensions of S and S_1, respectively. Denote by $\hat{R}_z : H_- \to H_+$ the regularized generalized resolvent generated by the extension A_1, and by $\hat{R}_1 : H_{1,-} \to H_{1,+}$ the regularized resolvent that belongs to A_1. It is not difficult to verify that

$$P_+ \hat{R}_{1,z} P_+^* = \hat{R}_z. \tag{1.6.1.11}$$

Fix $u \in H_-$. Let $r(z)$ be the generalized u-resolvent generated by A and $r_1(z)$ the generalized $P_+^* u$-resolvent generated by the extension A_1. By Theorem 1.5.7 and (1.6.1.11), $r(z)$ and $r_1(z)$ are connected as follows:

$$r_1(z) = r(z) - \frac{[u, \varphi_+(\overline{z})]_\pm [\varphi_+(z), u]_\pm}{Q(z) + \tau(z)} + \beta, \tag{1.6.1.12}$$

where $\tau(z)$ is the parameter function that corresponds to A_1 and β is a real constant. Denote by $W(z)$ ($W_1(z)$) the generalized u- ($P_+^* u$-) resolvent matrix as introduced in Definition 1.5.9 using $r(z)$ and $r_1(z)$. The following statement shows the connection between $W(z)$ and $W_1(z)$. For the definition of $\mathcal{M}_0$ and $\mathfrak{H}(M)$, see Section 2.2.

1.6.8. Theorem. *Let $u \in H_-$. With the above notation, the following relations hold:*

$$\mathcal{P}_1(z) f = \mathcal{P}(z) f, \quad \mathcal{Q}_1(z) f = \mathcal{Q}(z) f + \beta, \qquad f \in H, z \in r_u(S) \cap r(S, S_1).$$

Assume that S_u and $S_{1,P_+^* u}$ satisfy the minimality condition (1.6.1.2) *and that $r_u(S)$ has nonempty intersection with both halfplanes $\mathbb{C}^+$ and C^-. If $\beta = 0$, then there exists a matrix function $M(z) \in \mathcal{M}_0$ analytic on $r(S, S_1)$ such that*

$$W_1(z) = W(z) M(z), \quad z \in r_u(S) \cap r(S, S_1).$$

The matrix function $M(z)$ does not depend on the choice of u.

Proof. Denote by O the set $O = r_u(S) \cap r(S, S_1)$. Recall that, according to Lemma 1.6.7, we have

$$O = r_u(S) \cap r(S, S_1) = r_{P_+^* u}(S_1) \cap r(S, S_1).$$

Lemma 1.6.7 also implies that for $z \in O$ we have (with chosen nonzero $n_{1,\overline{z}} \in \operatorname{ran}(S_1 - z)^\perp$)

$$\mathcal{P}_1(z) f = \frac{(f, n_{1,\overline{z}})}{[P_+^* u, n_{1,\overline{z}}^+]_\pm} = \frac{(f, P n_{1,\overline{z}})}{[u, P_+ n_{1,\overline{z}}^+]_\pm} = \mathcal{P}(z) f$$

and also

$$\mathcal{Q}_1(z) f = \mathcal{Q}(z) f + \beta,$$

where β is as in (1.6.1.12). Assume that both S_u and $S_{1,P_+^* u}$ satisfy the minimality condition (1.6.1.2), and also that $r_u(S) \cap \mathbb{C}^+ \neq \emptyset$ and $r_u(S) \cap \mathbb{C}^- \neq \emptyset$. Consider the reproducing kernel spaces $\mathfrak{H}_u$ and $\mathfrak{H}_{1,P_+^* u}$. As follows from Theorem 1.6.5, the reproducing kernel of $\mathfrak{H}_u$ and $\mathfrak{H}_{1,P_+^* u}$ is given by $H_W(w,z)$ and $H_{W_1}(w,z)$, respectively. If $\beta = 0$, then obviously $\mathfrak{H} = \mathfrak{H}_u|_O$ is contained isometrically in $\mathfrak{H}_1 = \mathfrak{H}_{1,P_+^* u}|_O$. Define a holomorphic matrix function $M(z)$ on the set $r_u(S) \cap r(S, S_1)$ by

$$M(z) = W(z)^{-1} W_1(z).$$

The orthogonal projection of $\mathfrak{H}_1$ onto $\mathfrak{H}$ maps the reproducing kernel of $\mathfrak{H}_1$ to the reproducing kernel of $\mathfrak{H}$. Thus

$$H_{W_1}(w,z) - H_W(w,z) = \frac{W_1(z)JW_1(w)^* - J}{z - \overline{w}} - \frac{W_1(z)JW_1(w)^* - J}{z - \overline{w}} = \\ = W(z)\frac{M(z)JM(w)^* - J}{z - \overline{w}} W(w)^*, \quad w, z \in O,$$

is the reproducing kernel of $\mathfrak{H}_1 \ominus \mathfrak{H}$. In particular, this implies $M \in \mathcal{M}_0$. Once it is proved that $M(z)$ is independent on the choice of u, it follows immediately that $M(z)$ has an analytic continuation to $r(S, S_1)$, since to any $z \in r(S, S_1)$ we can find an element $u \in H_-$ such that $z \in r_u(S)$. For a proof of the independence see [KW1]. □

In the proof of Theorem 1.6.8 we also showed:

1.6.9. Corollary. *Let $u \in H_-$, S and S_1 be as in Theorem 1.6.8, and denote $O = r_u(S) \cap r(S, S_1)$. If $\beta = 0$ in Theorem 1.6.8, then $\mathfrak{H} = \mathfrak{H}_u|_O$ is contained isometrically in $\mathfrak{H}_1 = \mathfrak{H}_{1,P_+^* u}|_O$. Moreover, with $W(z)$ and $M(z)$ as in Theorem 1.6.8,*

$$W(z)\frac{M(z)JM(w)^* - J}{z - \overline{w}} W(w)^*, \quad w, z \in O,$$

is the reproducing kernel of $\mathfrak{H}_1 \ominus \mathfrak{H}$. In particular, the mapping $\mathbf{f}(z) \mapsto W(z)\mathbf{f}(z)$ is an isomorphism of the spaces $\mathfrak{H}(M)$ and $\mathfrak{H}_1 \ominus \mathfrak{H}$.

Chapter 2

Reproducing Kernel Spaces of Entire Functions

2.1 Functions of class $\mathcal{N}_0$

2.1.1. Definition. Suppose O is an open subset of $\mathbb{C}$. A function $f(z)$ analytic on O is said to belong to the class $\mathcal{N}_0$ if

$$Q(z) = \overline{Q(\overline{z})}, \quad z, \overline{z} \in O, \tag{2.1.2.1}$$

and if the kernel

$$K(w,z) = \frac{Q(z) - \overline{Q(w)}}{z - \overline{w}}, \quad w, z \in O, \tag{2.1.2.2}$$

is positive definite.

Since $K(w,z)$ is positive definite on O, there exists a unique reproducing kernel space $\mathfrak{K}(Q)$ of functions analytic on O, such that $K(w,z)$ is its reproducing kernel. Since $\operatorname{Im} Q(z) = \operatorname{Im} z\, K(z,z)$, we have $\operatorname{Im} Q(z) \leq 0$ for $z \in O \cap \mathbb{C}^-$, and $\operatorname{Im} Q(z) \geq 0$ for $z \in O \cap \mathbb{C}^+$, both inequalities are strict if $Q(z)$ is nonconstant. Thus $\mathfrak{K}(Q)$ is nontrivial if and only if Q is nonconstant.

Any function $f(z)$ analytic in $\mathbb{C}\backslash\mathbb{R}$, with $\operatorname{Im} f(z) \geq 0$ for $z \in \mathbb{C}^+$, belongs to the class $\mathcal{N}_0$. This is easily seen from the Poisson representation

$$f(z) = \alpha + \beta z + \frac{1}{\pi}\int_{\mathbb{R}} \frac{1 + \lambda z}{\lambda - z}\,\frac{d\sigma(\lambda)}{1 + \lambda^2},$$

where $\alpha \in R$ and $\beta > 0$. In fact, it follows that for any choice of $z_i \in \mathbb{C}\backslash\mathbb{R}$, $\zeta_i \in \mathbb{C}$, $i = 1, \dots, n$,

$$\sum_{i,j=1}^{n} \zeta_i \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z_j}} \overline{\zeta_j} = \beta \sum_{i,j=1}^{n} \zeta_i \overline{\zeta_j} + \frac{1}{\pi} \int_{\mathbb{R}} \sum_{i,j=1}^{n} \frac{\zeta_i \overline{\zeta_j}}{(\lambda - z_i)(\lambda - \overline{z_j})} \, d\sigma(\lambda) =$$
$$= \beta \Big| \sum_{i=1}^{n} \zeta_i \Big|^2 + \frac{1}{\pi} \int_{\mathbb{R}} \Big| \sum_{i=1}^{n} \frac{\zeta_i}{\lambda - z_i} \Big|^2 \, d\sigma(\lambda).$$

For any function $Q(z)$ analytic on $\mathbb{C}\backslash\mathbb{R}$, denote by $\rho(Q)$ the largest open set $O \supseteq \mathbb{C}\backslash\mathbb{R}$ such that Q can be continued analytically to O. The kernel (2.1.2.2) is positive definite on the entire set $\rho(Q)$ as is seen by an easy continuity argument. Moreover, denote by $r(Q)$ the largest open set O such that Q has a meromorphic continuation to O.

Let us state some important properties of Q-functions.

2.1.2. Proposition. *Suppose $S \subseteq H^2$ is a symmetric relation with deficiency index $(1, 1)$ and let A be a canonical selfadjoint extension of S. Any Q-function of S and A as defined in (1.1.1.7) is of class $\mathcal{N}_0$ on $\rho(A)$. Assume that S is H-minimal. Then $\rho(A) = \rho(Q)$ and $r(S) = r(Q)$. Moreover, Q has only real poles in $r(Q)$, and*

$$Q'(t) > 0, \quad t \in \rho(Q) \cap \mathbb{R}.$$

If $S - \lambda$ is one to one for any $\lambda \in r(S) \cap \mathbb{R}$, then the poles of Q in $r(Q)$ are simple, and the spectrum in $r(S)$ of any other canonical selfadjoint extension A_τ is given by

$$\sigma(A_\tau) \cap r(S) = \{t \in r(S) \cap \mathbb{R} \, : Q(t) = -\tau\}, \tag{2.1.2.3}$$

where τ is the parameter associated with A_τ by the Krein formula (Theorem 1.1.3).

Proof. If $Q(z)$ is a Q-function of S and A, then by definition

$$\frac{Q(z) - \overline{Q(w)}}{z - \overline{w}} = (\varphi(z), \varphi(w)), \quad z, w \in \rho(A),$$

where $\varphi(z)$ is connected with A. Thus $Q \in \mathcal{N}_0$. Obviously, $\rho(A) \subseteq \rho(Q)$. We prove that $\rho(Q) \subseteq \rho(A)$. Let $(E(\,\cdot\,), E(\{\infty\}))$ be the spectral resolution of H connected with A. Choose $z_0 \in \mathbb{C}\backslash\mathbb{R}$ and $\varphi(z_0) \in N_{z_0}$. Then

$$\varphi(z) = \varphi(z_0) + (z - z_0)(A - z)^{-1}\varphi(z_0) = E(\{\infty\})\varphi(z_0) + \int_{\mathbb{R}} \frac{\lambda - z_0}{\lambda - z} \, dE_\lambda \varphi(z_0), \quad z \in \rho(A),$$

and we find for $x \in \mathbb{R}$, $y > 0$,

$$\operatorname{Im} Q(x + iy) = y \|E(\{\infty\})\varphi(z_0)\|^2 + \int_{\mathbb{R}} \frac{y}{(\lambda - x)^2 + y^2} \, |\lambda - z_0|^2 d(E_\lambda \varphi(z_0), \varphi(z_0)).$$

We have $Q(z) = \overline{Q(\overline{z})}$ for $z \in \rho(A)$ and hence, by continuation, also for $z \in \rho(Q)$. If $x \in \rho(Q) \cap \mathbb{R}$, then $\operatorname{Im} Q(x) = 0$. Thus the left side of the last equation tends to zero as $y \to 0$. Since the functions

$$f_y(\lambda) = \frac{1}{\pi} \frac{y}{(\lambda - x)^2 + y^2}$$

form an approximate identity, we may conclude that

$$|\lambda - z_0|^2 d(E_\lambda \varphi(z_0), \varphi(z_0)) \equiv 0 \quad \text{ on } \rho(Q) \cap \mathbb{R}.$$

Since the linear span of the $\varphi(z_0)$, $z_0 \in \mathbb{C}\backslash\mathbb{R}$, is dense in H, we can conclude that $E(\cdot)$ vanishes identically on $\rho(Q) \cap \mathbb{R}$ and therefore $\rho(Q) \subseteq \rho(A)$. In particular, we have

$$Q'(t) = \lim_{s \to t} \frac{Q(s) - Q(t)}{s - t} = (\varphi(t), \varphi(t)) > 0, \quad t \in \rho(Q) \cap \mathbb{R}. \tag{2.1.2.4}$$

The spectrum of A in discrete in $r(Q)$. Let A_τ be the canonical selfadjoint extension of S determined by

$$(A_\tau - z)^{-1} = (A - z)^{-1} - \frac{(\cdot, \varphi(\overline{z}))}{Q(z) + \tau} \varphi(z), \quad z \in \rho(A) \cap \rho(A_\tau).$$

Obviously,

$$\sigma(A_\tau) \subseteq \sigma(A) \cup \{t \in \rho(Q) \, : Q(t) = -\tau\}.$$

Observe that the set $\{t \in \rho(Q) : Q(t) = -\tau\}$ and therefore $\sigma(A_\tau)$ is discrete in $r(Q)$, which implies that $r(Q) \subseteq r(S)$ (cf. [AG2]). On the other hand, since S has finite deficiency numbers, any canonical selfadjoint extension has discrete spectrum in $r(S)$ (cf. [AG2]), thus $r(S) \subseteq r(Q)$. If $S - \lambda$ is one to one for any $\lambda \in r(Q) \cap \mathbb{R}$, we conclude that, since S has deficiency index 1, the spectrum in $r(S)$ of any canonical selfadjoint extension consists only of simple eigenvalues, and any two different selfadjoint extensions have no eigenvalues $t \in r(S)$ in common. Thus $\varphi(z)$ and therefore $Q(z)$ have a simple poles at $t \in \sigma(A) \cap r(S)$, and

$$\sigma(A_\tau) \cap \sigma(A) \cap r(Q) = \emptyset.$$

From this (2.1.2.3) follows, and the proof is complete. □

The following construction of $\mathfrak{K}(Q)$ is standard and can also be found in [ABDS]. For a similar method see [LT].

2.1.3. Theorem. *Suppose $Q \in \mathcal{N}_0$ on O and Q is nonconstant. Then $\mathfrak{H} = \mathfrak{K}(Q)$ is closed under forming difference quotients*

$$\mathcal{R}_1(a) f(z) = \frac{f(z) - f(a)}{z - a}, \quad f \in \mathfrak{H}, a \in O.$$

For any $f, g \in \mathfrak{H}$, $a, b \in O$, the following identity holds.

$$(\mathcal{R}_1(a) f, g) - (f, \mathcal{R}_1(b) g) = (a - \overline{b}) \, (\mathcal{R}_1(a) f, \mathcal{R}_1(b) g). \tag{2.1.2.5}$$

The operator $\mathcal{S} : f(z) \mapsto z\, f(z)$ with $\operatorname{dom}\mathcal{S} = \{f \in \mathfrak{H} : zf(z) \in \mathfrak{H}\}$ is a closed symmetric operator with deficieny index $(1,1)$, and $\mathcal{R}_1(a)$ is the resolvent operator of a canonical selfadjoint extension A of $\mathcal{S}$. Hence the operator $\mathcal{R}_1(a) : \mathfrak{H} \to \mathfrak{H}$ is bounded and $\mathcal{R}_1(a)f$ depends analytically on a, in the norm of $\mathfrak{H}$. $Q(z)$ is a Q-function of A and $\mathcal{S}$.

Proof. Consider $\mathcal{L} = \operatorname{span}\{K(w,\,\cdot\,) : w \in \rho(Q)\}$, endowed with the inner product of $\mathfrak{K}(Q)$. $\mathcal{L}$ is a pre-Hilbert space, and $\overline{\mathcal{L}} = \mathfrak{K}(Q)$. A straightforward computation shows that

$$K_a(z) - K_b(z) = (\overline{a} - \overline{b})\frac{K_b(z) - K_b(\overline{a})}{z - \overline{a}} = (\overline{a} - \overline{b})\,\mathcal{R}_1(\overline{a})K_b(z), \quad a, b \in O. \tag{2.1.2.6}$$

Hence $\mathcal{L}$ is closed under $R_1(a)$. First we show that (2.1.2.5) is valid for $f, g \in \mathcal{L}$. Using (2.1.2.6), we obtain that for any $w, z \in O$

$$(K_w, \mathcal{R}_1(b)K_z) - (\mathcal{R}_1(a)K_w, K_z) + (a - \overline{b})\,(\mathcal{R}_1(a)K_w, \mathcal{R}_1(b)K_z) = \frac{(K_w, K_{\overline{b}}) - (K_w, K_z)}{\overline{b} - z} -$$
$$-\frac{(K_{\overline{a}}, K_z) - (K_w, K_z)}{a - \overline{w}} + (a - \overline{b})\frac{(K_{\overline{a}}, K_{\overline{b}}) - (K_w, K_{\overline{b}}) - (K_{\overline{a}}, K_z) + (K_w, K_z)}{(a - \overline{w})(\overline{b} - z)} =$$
$$= \frac{(\overline{b} - \overline{w})K(w, \overline{b}) + (z - a)K(\overline{a}, z) - (z - \overline{w})K(w, z) - (\overline{b} - a)K(\overline{a}, \overline{b})}{(a - \overline{w})(\overline{b} - z)} = 0.$$

This implies that (2.1.2.5) holds for $f, g \in \mathcal{L}$, $a, b \in O$. The operator $R(a) = \mathcal{R}_1(a) : \mathcal{L} \to \mathcal{L}$ satisfies

$$(R(a)f, g) = (f, R(\overline{a})g), \quad f, g \in \mathcal{L},$$

and therefore $R(a)^* = R(\overline{a})$ as linear manifolds in $\mathcal{L}^2$. Moreover, the resolvent identity holds:

$$R(a)f - R(b)f = (a - b)\,R(b)R(a)f, \quad f \in \mathcal{L}.$$

According to Theorem 1.1.2, $R(a) = (A_{\mathcal{L}} - a)^{-1}$ for a certain selfadjoint relation $A_{\mathcal{L}}$ in $\mathcal{L}^2$. Since $\mathcal{L}$ is dense in $\mathfrak{H}$, the closure A of $A_{\mathcal{L}}$ is selfadjoint in $\mathfrak{H}^2$. Thus its resolvent $R(z) = (A - z)^{-1}$ satisfies the identity

$$(R(a)f, g) - (f, R(b)g) = (a - \overline{b})(R(a)f, R(b)g), \quad f, g \in \mathfrak{H},\ a, b \in \rho(A)$$

For any $a \in \rho(A)$, the mapping $f \mapsto (A_{\mathcal{L}} - a)^{-1}f = \mathcal{R}_1(a)f$, $f \in \mathcal{L}$, is continuous. Let $f \in \mathfrak{H}$ and choose a sequence $f_n \in \mathcal{L}$ that converges to $f \in \mathfrak{H}$. Then $R_1(a)f_n$ is convergent, say to $g \in \mathfrak{H}$. It follows that

$$\frac{f(z) - f(a)}{z - a} = \lim_{n\to\infty} \mathcal{R}_1(a)f_n(z) = g(z),$$

which shows that for any $a \in \rho(A)$, $\mathfrak{H}$ is closed under forming difference quotients $\mathcal{R}_1(a)f$, $f \in \mathfrak{H}$. Moreover, $\mathcal{R}_1(a)f = (A - a)^{-1}f$ possesses all the asserted properties. For any $z \in \mathbb{C}\backslash\mathbb{R}$,

$$(\mathcal{S} - z)^{-1} = (A - z)^{-1}|_{\{f : f(z) = 0\}},$$

hence $\mathcal{S}$ is a closed symmetric operator. Unless Q is constant, $K(z,z) > 0$ for all nonreal $z \in O$, whence

$$\mathfrak{H} = \operatorname{span}\{K_z\} \oplus \{f \in \mathfrak{H} : f(z) = 0\} = \operatorname{span}\{K_z\} \oplus \operatorname{ran}(\mathcal{S} - z), \quad z \in O \backslash \mathbb{R}. \tag{2.1.2.7}$$

Thus $\mathcal{S}$ has deficiency index $(1,1)$. If we put $K_{\overline{z}} = \varphi(z)$, $z \in O$, then by (2.1.2.6) the identity $\varphi(z) = (1 + (z - z_0)(A - z)^{-1})\varphi(z_0)$ holds. Since

$$\frac{Q(z) - \overline{Q(w)}}{z - \overline{w}} = \frac{Q(\overline{w}) - \overline{Q(\overline{z})}}{\overline{w} - z} = K(\overline{w}, \overline{z}) = (\varphi(z), \varphi(w)), \quad z, w \in O,$$

$Q(z)$ is a Q-function of A and S. Since $\mathcal{S}$ is $\mathcal{H}$-minimal, Proposition 2.1.2 shows $\rho(A) = \rho(Q) \supseteq O$ and the proof is complete. ☐

2.1.4. Remark. Since $\mathfrak{H}$ is closed under forming difference quotients $\mathcal{R}_1(a)f$, $a \in O$, for each $a \in O$ there exists $f \in \mathfrak{H}$ with $f(a) \neq 0$. Thus $K(z,z) > 0$ for all $z \in O$ and hence

$$\mathfrak{H} = \operatorname{span}\{K_z\} \oplus \{f \in \mathfrak{H} : f(z) = 0\} = \operatorname{span}\{K_z\} \oplus \operatorname{ran}(\mathcal{S} - z), \quad z \in O. \tag{2.1.2.8}$$

The following corollary is obvious.

2.1.5. Corollary. *If $Q(z) : O \to \mathbb{C}$ belongs to the class $\mathcal{N}_0$, then $Q(z)$ has an analytic continuation to $\mathbb{C} \backslash \mathbb{R}$ which also belongs to the class $\mathcal{N}_0$.*

2.2 The Space $\mathcal{H}_S(M)$

In the present section we introduce the classes $\mathcal{M}_0$ and $\mathcal{M}_0^S$ of matrix functions, and the reproducing kernel spaces $\mathfrak{K}(M)$ and $\mathcal{H}_S(M)$ which occur in this context. For our later work we will just need few results on these spaces. See also [KL], [KW1] or [KW2].

2.2.1. Definition. An analytic 2×2-matrix function $M(z)$, defined in some open set $O \subseteq \mathbb{C}$, is said to belong to the class $\mathcal{M}_0$, if

$$M(z)\, J\, M(\overline{z})^* = J, \quad z, \overline{z} \in O, \tag{2.2.2.1}$$

and if the matrix kernel

$$H_M(w, z) = \frac{M(z) J M(w)^* - J}{z - \overline{w}}, \quad z, w \in O, \tag{2.2.2.2}$$

is positive definite.

The condition (2.2.2.1) implies that $H_M(w,z)$ is continuous on O^2 and that for each $w \in O$, the function $H_M(w, \cdot\,)$ in analytic in O. Note that, if U and V are iJ-unitary matrices, then $U\, M(z)\, V$ belongs to $\mathcal{M}_0$ if and only if $M(z) \in \mathcal{M}_0$.

To any 2×2-matrix function $M(z)\in\mathcal{M}_0$ there exists a unique Hilbert space $(\mathfrak{K}(M),(\cdot,\cdot))$ of vector functions analytic in O, such that $H_M(z,w)$ is its reproducing kernel (cf. Section 1.1). Theorem 1.6.5 shows that the generalized u-resolvent matrix $W(z)$ belongs to the class $\mathcal{M}_0$. The following theorem, which is proved in [KW1] characterizes all $\mathcal{M}_0$-matrix functions.

2.2.2. Theorem. *Suppose that the 2×2-matrix function $M(z)$, $z\in O$, belongs to $\mathcal{M}_0$. Then there exists a Hilbert space H and a symmetric relation $S\subseteq H^2$ with deficiency index $(1,1)$, a canonical selfadjoint extension A of S, an element $u\in H$ with $r_u(S)\cap O\neq\emptyset$, and iJ-unitary matrices U and V, such that*

$$M(z)=V\,W(z)\,U,\quad z\in r_u(S)\cap O,$$

where $W(z)$ is a generalized u-resolvent matrix associated with S and A.

In paticular this implies

2.2.3. Theorem. *For any $M(z)\in\mathcal{M}_0$, the space $\mathfrak{K}(M)$ is closed under forming difference quotients*

$$\mathcal{R}_1(a)\mathbf{f}=\frac{\mathbf{f}(z)-\mathbf{f}(a)}{z-a},\quad \mathbf{f}\in\mathfrak{K}(M).$$

The operator $\mathcal{R}_1(a):\mathfrak{K}(M)\to\mathfrak{K}(M)$ is bounded, $\mathcal{R}_1(a)\mathbf{f}$ depends analytically on a, in the norm of $\mathfrak{K}(M)$, and satisfies the identity

$$\mathbf{g}(b)^*J\mathbf{f}(a)=(\mathbf{f},\mathcal{R}_1(b)\mathbf{g})-(R_1(a)\mathbf{f},\mathbf{g})+(a-\overline{b})\,(\mathcal{R}_1(a)\mathbf{f},\mathcal{R}_1(b)\mathbf{g}),\quad \mathbf{f},\mathbf{g}\in\mathfrak{K}(M).\qquad(2.2.2.3)$$

Proof. Let be $W(z)$ a conveniently chosen u-resolvent matrix such that $M(z)=V\,W(z)U$ with iJ-unitary matrices U and V. Since $\mathfrak{K}(W)=\mathfrak{H}_u$, the space $\mathfrak{K}(W)$ is closed under application of $\mathcal{R}_1(a)$, $a\in O$, and the operator $\mathcal{R}_1(a)$ possesses all the properties listed above. We have

$$H_M(w,z)=V\,H_W(w,z)\,V^*,$$

hence, as discussed in Section 1.1, the mapping $\mathbf{f}(z)\mapsto V\mathbf{f}(z)$ is an isomorphism between the Hilbert spaces $\mathfrak{K}(W)$ and $\mathfrak{K}(M)$. Since

$$\mathcal{R}_1(a)\mathbf{f}=V^{-1}\mathcal{R}_1(a)V\mathbf{f},\quad \mathbf{f}\in\mathfrak{K}(M),$$

also $\mathfrak{K}(M)$ is closed with respect to $\mathcal{R}_1(a)$, and the operator $\mathcal{R}_1(a)$ has all asserted properties. □

2.2.4. Definition. Suppose $S(z)$ is a scalar valued entire function. A 2×2-matrix valued function $M(z)$ whose entries are entire functions belongs to the class $\mathcal{M}_0^S$, if

$$M(z)JM(\overline{z})^*=S(z)J\overline{S(\overline{z})},\quad z\in\mathbb{C},\qquad(2.2.2.4)$$

and if the matrix kernel

$$H_M(w,z)=\frac{M(z)JM(w)^*-S(z)J\overline{S(w)}}{z-\overline{w}},\quad z,w\in\mathbb{C},\qquad(2.2.2.5)$$

is positive definite.

Note that for iJ-unitary matrices U and V, we have $UM(z)V \in \mathcal{M}_0^S$ if and only if $M(z) \in M_0^S$. For any $M(z) \in \mathcal{M}_0^S$, there is a unique reproducing kernel space $\mathcal{H}_S(M)$ such that H_M is its reproducing kernel. If the matrix $M(z)$ is given by

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

the kernel $H_M(w,z)$ can be written as

$$H_M(w,z) = \begin{pmatrix} \frac{B(z)\overline{A(w)}-A(z)\overline{B(w)}}{z-\overline{w}} & \frac{B(z)\overline{C(w)}-A(z)\overline{D(w)}+S(z)\overline{S(w)}}{z-\overline{w}} \\ \frac{D(z)\overline{A(w)}-C(z)\overline{B(w)}+S(z)\overline{S(w)}}{z-\overline{w}} & \frac{D(z)\overline{C(w)}-C(z)\overline{D(w)}}{z-\overline{w}} \end{pmatrix}. \tag{2.2.2.6}$$

2.2.5. Corollary. *Let $M(z) \in \mathcal{M}_0^S$, and assume that $a \in \mathbb{C}$ is such that $S(a) \neq 0$. Then the space $\mathfrak{H} = \mathcal{H}_S(M)$ is closed with respect to forming difference quotients*

$$\mathcal{R}_S(a)\mathbf{f}(z) = \frac{\mathbf{f}(z) - \frac{S(z)}{S(a)}\mathbf{f}(a)}{z-a}, \quad \mathbf{f} \in \mathfrak{H}. \tag{2.2.2.7}$$

The operator $\mathcal{R}_1(a) : \mathfrak{H} \to \mathfrak{H}$ is bounded and $\mathcal{R}_1(a)\mathbf{f}$ depends analytically on a, in the norm of $\mathfrak{H}$. Moreover, it satisfies the identity

$$\frac{1}{S(a)\overline{S(b)}}\mathbf{g}(b)^*J\mathbf{f}(a) = (\mathbf{f}, \mathcal{R}_S(b)\mathbf{g}) - (R_S(a)\mathbf{f}, \mathbf{g}) + (a - \overline{b})\,(\mathcal{R}_S(a)\mathbf{f}, \mathcal{R}_S(b)\mathbf{g}), \tag{2.2.2.8}$$

for all $a, b \in \mathbb{C}$, $\mathbf{f}, \mathbf{g} \in \mathfrak{H}$.

Proof. Since $M'(z) := \frac{1}{S(z)}M(z) \in \mathcal{M}_0$ on the set $O = \{z \in \mathbb{C} : S(z) \neq 0\}$, and since the mapping $\mathbf{f}(z) \to S(z)\mathbf{f}(z)$ is an isomorphism of the Hilbert spaces $\mathfrak{K}(M'(z))$ and $\mathfrak{K}(M(z))|_O$, the present result is an immediate consequence of Theorem 2.2.3. ☐

2.3 The Space $\mathcal{H}(A, B)$

2.3.1. Definition. Suppose $A(z)$ is an entire scalarly valued function. An entire function $B(z)$ belongs to the class $\mathcal{N}_0^A$, if

$$B(z)\overline{A(\overline{z})} - A(z)\overline{B(\overline{z})} = 0, \quad z \in \mathbb{C}, \tag{2.3.2.1}$$

and if the scalar kernel

$$K(w,z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{z - \overline{w}}, \quad w, z \in \mathbb{C}, \tag{2.3.2.2}$$

is positive definite. The reproducing kernel space generated by $K(w,z)$ is denoted by $\mathcal{H}(A, B)$.

2.3.2. Remark. If both A and B are real functions, i.e. $\overline{A(\overline{z})} = A(z)$ and $\overline{B(\overline{z})} = B(z)$, and if A and B have no common zeros in the upper halfplane, then $K(w,z)$ as defined above is the reproducing kernel of a de Branges space $\mathcal{H}(E)$, with $E(z) = A(z) - iB(z)$.

Obviously, $B \in \mathcal{N}_0^A$ if and only if $A \in \mathcal{N}_0^B$. Note that

$$\frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{z - \overline{w}} = A(z)\frac{\frac{B(z)}{A(z)} - \overline{\frac{B(w)}{A(w)}}}{z - \overline{w}}\overline{A(w)}.$$

Hence $B(z) \in \mathcal{N}_0^A$ if and only if $B(z)/A(z) \in \mathcal{N}_0$. The transformation $f(z) \to A(z)f(z)$ is an isomorphism between the Hilbert spaces $\mathfrak{K}(B/A)$ and $\mathcal{H}(A,B)|_{\rho(B/A)}$. Therefore $\mathcal{H}(A,B)$ is nontrivial if and only if A and B are linearly independent. Via this isomorphism the mulitplication operator by the independent variable in $\mathfrak{K}(B/A)$ is transformed into the mulitplication operator $\mathcal{S} : F(z) \mapsto zF(z)$ in $\mathcal{H}(A,B)$. According to Theorem 2.1.3, $\mathcal{S}$ is a closed symmetric operator with deficiency index $(1,1)$. Moreover, $B(z)/A(z)$ is a Q-function of a certain selfadjoint extension of $\mathcal{S}$, Proposition 2.1.2 shows that $r(\mathcal{S}) = rB/A = \mathbb{C}$. We summarize these facts in the following theorem:

2.3.3. Theorem. *Suppose $B \in \mathcal{N}_0^A$. Then $\mathcal{H}(A,B)$ is nontrivial if and only if A and B are linear independent. In this case, $\mathcal{H} = \mathcal{H}(A,B)$ has the following properties:*

$(H1)$ *$\mathcal{H}$ is a reproducing kernel space of entire functions.*

$(H2)$ *$\mathcal{S} : F(z) \mapsto z\,F(z)$ is a closed symmetric operator with deficiency index $(1,1)$.*

Moreover, any $z \in \mathbb{C}$ is a point of regular type for $\mathcal{S}$.

2.3.4. Remark. Note that, since any $w \in \mathbb{C}$ is a point of regular type for $\mathcal{S}$,

$$\mathcal{H} = \operatorname{ran}(S - w) \oplus \operatorname{span}\{K_w^{(n)}\}, \quad n = \operatorname{Ord}_w \mathcal{H}.$$

Suppose $\mathcal{H}$ is any Hilbert space of entire functions which possesses the properties $(H1)$ and $(H2)$. Consider the space triplet

$$\mathcal{H}_+ \xrightarrow{\pi} \mathcal{H} \xrightarrow{\iota} \mathcal{H}_-$$

associated to $\mathcal{S}$. Since $\mathcal{S}$ satisfies the minimality condition (1.3.1.7), we can regard $\mathcal{H}$ as a subspace of $\mathcal{H}_-$ by identifying $F \in \mathcal{H}$ with the linear functional $(\cdot, (F;0)) \in \mathcal{H}_-$. If convinient, we will not distinguish between $(F;G) \in \mathcal{H}_+$ and the functional $(\cdot, (F;G))_+ \in \mathcal{H}_-$. With K_z^+ we denote the element $(K_z; \overline{z}\,K_z) \in \mathcal{H}_+$. As in Lemma 1.3.5, we have

$$\operatorname{cls}\{K_z^+ : z \in \mathbb{C}\} = \mathcal{H}_+. \tag{2.3.2.3}$$

The following lemma enables us to identify $\operatorname{Assoc}(\mathcal{H}) = \mathcal{H} + z\mathcal{H}$ with $\mathcal{H}_-$. Its proof is exactly the same as the one of Lemma 1.3.6.

2.3.5. Lemma. *Let $v = (\,\cdot\,, (F;G))_+ \in \mathcal{H}_-$. Then, with the above notation,*

$$[v, K_z^+]_\pm = F(z) + zG(z) \in \operatorname{Assoc}(\mathcal{H}). \tag{2.3.2.4}$$

Conversely, to any $H(z) \in \operatorname{Assoc}(\mathcal{H})$ there exists a unique element $v \in \mathcal{H}_-$, such that $H(z) = [v, K_z^+]_\pm$.

Note that $L_w(z) = (z - \overline{w})K_w(z) \in \operatorname{Assoc}(\mathcal{H})$. Let P denote the orthoprojector of $\mathcal{H}^2$ onto $\mathcal{H}_+$. Define

$$l_w = PJK_w^+ \in \mathcal{H}_+. \tag{2.3.2.5}$$

Since

$$(l_w, K_z^+) = \big(P\begin{pmatrix} -\overline{w}K_w \\ K_w \end{pmatrix}, K_z^+\big)_+ = \big(\begin{pmatrix} -\overline{w}K_w \\ K_w \end{pmatrix}, K_z^+\big) = (z - \overline{w})K_w(z), \tag{2.3.2.6}$$

the functional $(\cdot, l_w)_+$ is the element of $\mathcal{H}_-$ that corresponds to $L_w(z)$. Since $K_w^+ \in \mathcal{H}_+$, JK_w^+ and therefore PJK_w^+ is orthogonal to S, hence $(\cdot, l_w)_+$ is a boundary value. We now give an explicit description of the kernel $K_w(z)$.

2.3.6. Theorem. *Assume that a Hilbert space $\mathcal{H}$ satisfies $(H1)$ and $(H2)$. Then there exist entire functions A and B such that*

$$(z - \overline{w})\, K_w(z) = \overline{A(w)}B(z) - \overline{B(w)}A(z), \quad w, z \in \mathbb{C}. \tag{2.3.2.7}$$

Explicitly, for entire functions A and B, relation (2.3.2.7) holds if and only if $A, B \in \operatorname{Assoc}(\mathcal{H})$, and the corresponding functionals $(\cdot, a)_+$ and $(\cdot, b)_+$ are boundary values, such that the vectors $a, ib \in H_+$ are skewly linked (with respect to $\langle\,\cdot,\cdot\,\rangle$ as defined in Section 1.1).

Proof. First observe that (2.3.2.7) can be written in the form

$$(z - \overline{w})\, K_w(z) = (A(z), B(z))J(A(w), B(w))^*.$$

Thus, if (2.3.2.7) holds for entire functions A, B, then it also holds for the functions

$$(A_1, B_1) = (A, B)U,$$

where U is an arbitrary iJ-unitary matrix. Assume that $(\cdot, a)_+$ and $(\cdot, ib)_+$ boundary values, and that a and ib are skewly linked with respect to the inner product $\langle\,\cdot,\cdot\,\rangle$. According to Proposition 1.2.9, such boundary values always exist. Moreover, it follows from above that we may assume that a and ib form an orthonormal basis for $\mathcal{H}_+ \ominus \mathcal{S}$. Recall that for such $a, b \in \mathcal{H}_+ \ominus S$ we have $b = Ja$ and $Jb = -a$. Since l_w is a boundary value, we can write

$$l_w = c_1(w)b + c_2(w)a.$$

Now a simple computation gives

$$c_1(w) = (l_w, b)_+ = (JK_w^+, b) = -(K_w^+, Jb) = (K_w^+, a) = \overline{A(w)},$$

and

$$c_2(w) = (l_w, a)_+ = (JK_w^+, a) = -(K_w^+, Ja) = -(K_w^+, b) = \overline{B(w)},$$

which proves (2.3.2.7). Conversely, suppose A and B are entire functions such that (2.3.2.7) holds. Since $\mathcal{H}$ is nontrivial, the function $\frac{B}{A}$ is nonconstant, or equivalently A and B are linear independent. Hence we can find $w_1, w_2 \in \mathbb{C}$ such that

$$\overline{B(w_1)A(w_2)} - \overline{A(w_1)B(w_2)} \neq 0,$$

which implies that both A and B are a linear combination of L_{w_1} and L_{w_2}. Therefore $A, B \in \mathrm{Assoc}(\mathcal{H})$. Let $(\cdot, a)_+$ and $(\cdot, b)_+$ be the corresponding elements of $\mathcal{H}_-$. Since both a and b are a linear combination of l_{w_1} and l_{w_2}, they actually are boundary values. From

$$l_w = \overline{A(w)}b - \overline{B(w)}a$$

we obtain

$$(Jl_w, b) = \overline{A(w)}(Jb, b) - \overline{B(w)}(Ja, b).$$

Since Jb is also in $H_+ \ominus S$,

$$(Jl_w, b) = -(PJK_w^+, Jb) = -(JK_w^+, Jb) = -(K_w^+, b) = -\overline{B(w)}.$$

Together with the above formula, this implies

$$\overline{B(w)}\,((Ja, b) - 1) = \overline{A(w)}\,(Jb, b) \quad \text{for all } w \in \mathbb{C}.$$

Since A and B are linearly independent, we arrive at the conclusion that $(Jb, b) = 0$ and $(Ja, b) = 1$. An analoguous calculation of (Jl_w, a) gives $(Ja, a) = 0$ and $(Jb, a) = -1$. This shows that a and ib are skewly linked with respect to $\langle \cdot, \cdot \rangle$. □

Together with Theorem 2.3.3, Theorem 2.3.6 immediately yields the following characterization of the spaces $\mathcal{H}(A, B)$:

2.3.7. Theorem. *Suppose $B \in \mathcal{N}_0^A$ and that A and B are linearly independent. Then $\mathcal{H} = \mathcal{H}(A, B)$ posesses the properties $(H1)$ and $(H2)$. Conversely, if $\mathcal{H}$ is a (notrivial) Hilbert space of entire functions satisfying $(H1)$ and $(H2)$, then there exist (linearly independent) entire functions A and B, $B \in \mathcal{N}_0^A$, such that $\mathcal{H} = \mathcal{H}(A, B)$.*

Let $\mathcal{H} = \mathcal{H}(A, B)$ be nontrivial. By Theorem 2.3.6 A and B are associated functions which correspond to boundary values $(\cdot, a)_+$ and $(\cdot, b)_+$, respectively, where a and ib are skewly linked. This observation enables us to characterize all entire functions A and B, $B \in \mathcal{N}_0^A$, which generate the same space $\mathcal{H}(A, B)$:

2.3.8. Proposition. *Suppose $\mathcal{H}(A, B)$ is nontrivial. Then $\mathcal{H}(A, B) = \mathcal{H}(A_1, B_1)$ for entire functions A_1 and B_1, $B_1 \in \mathcal{N}_0^{A_1}$, if and only if there exists an iJ-unitary matrix U such that*

$$(A_1, B_1) = (A, B)\, U. \tag{2.3.2.8}$$

Proof. Assume that (2.3.2.8) holds. Then

$$(A(z), B(z))J(A(w), B(w))^* = (A(z), B(z))UJU^*(A(w), B(w))^* = \\ = (A_1(z), B_1(z))J(A_1(w), B_1(w))^*,$$

which shows that $\mathcal{H}(A, B)$ and $\mathcal{H}(A_1, B_1)$ have the same reproducing kernel, and hence they are equal. Conversely, assume that $\mathcal{H}(A, B) = \mathcal{H}(A_1, B_1)$. Then

$$K(w, z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{z - \overline{w}} = \frac{B_1(z)\overline{A_1(w)} - A_1(z)\overline{B_1(w)}}{z - \overline{w}},$$

and Theorem 2.3.6 together with Proposition 1.2.9 immediately yields (2.3.2.8). □

2.3.9. Remark. In the situation of de Branges spaces the generating functions A and B (and also A_1 and B_1) are real functions. Thus, by the linear independence of A and B, the matrix U in (2.3.2.8) has to be real. Note that a real 2×2-matrix is iJ-unitary if and only if $\det U = 1$.

Suppose $F \in \mathcal{H}$ and $\operatorname{Ord}_w F > \operatorname{Ord}_w \mathcal{H}$. By Remark 2.3.4, we also have

$$\frac{F(z)}{z - w} \in \mathcal{H}.$$

Next, assume $S \in \operatorname{Assoc}(\mathcal{H})$ and $\operatorname{Ord}_w S > \operatorname{Ord}_w \mathcal{H}$. We can write $S(z) = (z - w)F(z) + G(z)$, with $F, G \in \mathcal{H}$. Since $\operatorname{Ord}_w F \geq \operatorname{Ord}_w \mathcal{H}$, a simple argument shows that also $\operatorname{Ord}_w G > \operatorname{Ord}_w \mathcal{H}$. Thus

$$\frac{S(z)}{z - w} = F(z) + \frac{G(z)}{z - w} \in \mathcal{H}.$$

2.3.10. Theorem. *Suppose $\mathcal{H} = \mathcal{H}(A, B)$ is nontrivial. For arbitrary $S \in \operatorname{Assoc}(\mathcal{H})$, $\mathcal{H}$ is closed under forming difference quotients*

$$\mathcal{R}_S(a)F(z) = \frac{F(z) - \frac{S(z)}{S(a)}F(a)}{z - a}, \quad F \in \mathcal{H}, a \in \{w : \operatorname{Ord}_w S = \operatorname{Ord}_w \mathcal{H}\}.$$

The operator $R_S(a)$ is the resolvent operator of a certain one dimensional extension T of $\mathcal{S}$. Hence $\mathcal{R}_S(a) : \mathcal{H} \to \mathcal{H}$ is bounded, and $\mathcal{R}_S(a)f$ depends analytically on a,in the norm of $\mathcal{H}$. If $(\cdot, s)_+$, $s \in \mathcal{H}_+$, is the element of $\mathcal{H}_-$ that corresponds to S, then the extension T is explicitly given by

$$T = \mathcal{S} \oplus \operatorname{span}\{Js\},$$

and has spectrum $\sigma(T) = \{w : \operatorname{Ord}_w S > \operatorname{Ord}_w \mathcal{H}\}$.

Proof. Choose any $(H_1; H_2) \in \mathcal{H}_+$. First observe that, since $\mathcal{S}$ is symmetric, the element $(-H_2; H_1)$ is orthogonal to $\mathcal{S}$. Consider the linear relation $T = \mathcal{S} \oplus \operatorname{span}\{(-H_2; H_1)\}$. We have

$$(T - a)^{-1} = (\mathcal{S} - a)^{-1} \dot{+} \operatorname{span}\{\begin{pmatrix} H_1 + aH_2 \\ -H_2 \end{pmatrix}\}.$$

Observe that the relation $(T-a)^{-1}$ is an operator if and only if $H_1 + aH_2 \notin \operatorname{ran}(\mathcal{S}-a)$. By Remark 2.3.4, the latter condition is equivalent to

$$(H_1 + aH_2, K_a^{(n)}) = H_1^{(n)}(a) + aH_2^{(n)}(a) = S^{(n)}(a) \neq 0,$$

where $n = \operatorname{Ord}_a \mathcal{H}$. In this case $(\mathcal{S}-a)$ and therefore $(T-a)^{-1}$ is bounded. This shows

$$\rho(T) = \{w : \operatorname{Ord}_w S = \operatorname{Ord}_w \mathcal{H}\}.$$

We now compute the resolvent. For any $F \in \mathcal{H}$, $a \in \rho(T)$,

$$F - \frac{F(a)}{H_1(a) + aH_2(a)}\,(H_1 + aH_2) \in \operatorname{ran}(\mathcal{S}-a).$$

Since $(T-a)^{-1}$ maps $H_1 + aH_2$ to $-H_2$,

$$\begin{aligned}(T-a)^{-1}F(z) &= \frac{F(z) - \frac{F(a)}{H_1(a)+aH_2(a)}(H_1(z)+aH_2(z))}{z-a} - \frac{F(a)}{H_1(a)+aH_2(a)}\,H_2(z) = \\ &= \frac{F(z) - \frac{F(a)}{S(a)}S(z)}{z-a}, \quad a \in \rho(T),\end{aligned}$$

which completes the proof. □

We finish this section with some elementary considerations. Recall that $K_w^{(n)}$ denotes the kernel that reproduces the n-th derivative of $f \in \mathcal{H}$ at the point $w \in \mathbb{C}$, i.e.

$$(f, K_w^{(n)}) = f^{(n)}(w), \quad f \in \mathcal{H}.$$

As follows from Proposition 1.1.8, the kernel $K_w^{(n)}(z)$ is given by

$$K_w^{(n)}(z) = \frac{\partial^n}{\partial \overline{w}^n} K(w,z).$$

2.3.11. Proposition. *Suppose $\mathcal{H}(A,B)$ is nontrivial. For any $w \in \mathbb{C}$ the relation* $\operatorname{Ord}_w \mathcal{H} = \min(\operatorname{Ord}_w A, \operatorname{Ord}_w B)$ *holds. If $n = \operatorname{Ord}_w \mathcal{H}$, we have*

$$K_w^{(n)}(z) = \frac{\overline{A^{(n)}(w)}B(z) - \overline{B^{(n)}(w)}A(z)}{z - \overline{w}}, \tag{2.3.2.9}$$

and

$$\|K_w^{(n)}\|^2 = \begin{cases} \dfrac{\overline{A^{(n)}(w)}B^{(n)}(w) - \overline{B^{(n)}(w)}A^{(n)}(w)}{w - \overline{w}}, & w \notin \mathbb{R} \\ \frac{1}{n+1}\left(\overline{A^{(n)}(w)}B^{(n+1)}(w) - \overline{B^{(n)}(w)}A^{(n+1)}(w)\right), & w \in \mathbb{R}. \end{cases} \tag{2.3.2.10}$$

Proof. Recall that $\mathrm{Ord}_w\,\mathcal{H} = \min\{n : K_w^{(n)} \neq 0\}$. A straight forwardcomputation gives

$$\frac{\partial^n}{\partial \overline{w}^n} K(w,z) = \overline{\frac{\partial^n}{\partial w^n} K(z,w)} = \sum_{k=0}^{n} \binom{n}{k} (\overline{B^{(k)}(w)}A(z) - \overline{A^{(k)}(w)}B(z)) \frac{(-1)^{n-k}}{(\overline{w}-z)^{n-k+1}} =$$
$$= \sum_{k=0}^{n} \binom{n}{k} \frac{\overline{A^{(k)}(w)}B(z) - \overline{B^{(k)}(w)}A(z)}{(z-\overline{w})^{n-k+1}}, \quad z \neq \overline{w}.$$

Now, if $n = \min(\mathrm{Ord}_w\,A, \mathrm{Ord}_w\,B)$, then $K_w^{(k)}(z) = \frac{\partial^k}{\partial \overline{w}^k} K(w,z) = 0$ for $k < n$ and

$$K_w^{(n)}(z) = \frac{\partial^n}{\partial \overline{w}^n} K(w,z) = \frac{\overline{A^{(n)}(w)}B(z) - \overline{B^{(n)}(w)}A(z)}{z-\overline{w}},$$

where not both $A^{(n)}(w)$ and $B^{(n)}(w)$ are zero. Since A and B are linearly independent, $K_w^{(n)} \neq 0$ and we obtain $\mathrm{Ord}_w\,\mathcal{H} = \min(\mathrm{Ord}_w\,A, \mathrm{Ord}_w\,B)$. Formula (2.3.2.10) follows immediately from $\|K_w^{(n)}\|^2 = (K_w^{(n)}, K_w^{(n)}) = \frac{d^n}{dz^n} K_w^{(n)}(z)|_{z=w}$. □

2.3.12. Lemma. *Suppose $\mathcal{H}(A,B)$ is nontrivial. The functions A and B have the properties*

$$|\,\mathrm{Ord}_t\,A - \mathrm{Ord}_t\,B| \leq 1, \quad t \in \mathbb{R},$$
$$\mathrm{Ord}_w\,A = \mathrm{Ord}_w\,B = \mathrm{Ord}_w\,\mathcal{H}, \quad w \in \mathbb{C}\backslash\mathbb{R}.$$

The function $E(z) = A(z) - iB(z) \in \mathrm{Assoc}(\mathcal{H})$ satisfies $\mathrm{Ord}_t\,E = \mathrm{Ord}_t\,\mathcal{H}$ for all real t.

Proof. Recall that the function$\frac{B}{A} \in \mathcal{N}_0$. Thus it has only real poles of order one, which immediately proves the assertions on A and B. Recall that, according to Proposition 2.3.11, $\mathrm{Ord}_t\,\mathcal{H} = \min\{\mathrm{Ord}_t\,A, \mathrm{Ord}_t\,B\}$. If we set $n = \mathrm{Ord}_t\,H$, then

$$E^{(k)}(t) = A^{(k)}(t) - iB^k(t) = 0, \quad k < n,$$

and

$$E^{(n)}(t) = A^{(n)}(t) - iB^n(t) \neq 0,$$

since $B(t)/A(t)$ and $A(t)/B(t)$ are real for $t \in \rho(B/A)\cap\mathbb{R}$ and $t \in \rho(A/B)\cap\mathbb{R}$, respectively. □

2.4 Selfadjoint Extensions of $\mathcal{S}$

In this section we will investigate all canonical selfadjoint extensions of $\mathcal{S}$. Throughout the following we assume that $\mathcal{H}(A,B)$ is nontrivial.

Given $\mathcal{H}(A,B)$, we define the functions

$$S_\alpha(z) = \sin\alpha A(z) - \cos\alpha B(z), \quad \alpha \in \mathbb{R}. \qquad (2.4.2.1)$$

Note that S_α is associated to $\mathcal{H}(A,B)$ and that

$$(S_\alpha, S_{\alpha+\frac{\pi}{2}}) = (A,B)\begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}.$$

Since the matrix on the right hand side is iJ-unitary, S_α and $S_{\alpha+\frac{\pi}{2}}$ correspond to boundary values $(\cdot, s_\alpha)_+$ and $(\cdot, s_{\alpha+\frac{\pi}{2}})$, respectively, where s_α and $is_{\alpha+\frac{\pi}{2}}$ are skewly linked. In particular, we can write

$$K(w,z) = \frac{\overline{S_\alpha(w)}S_{\alpha+\frac{\pi}{2}}(z) - \overline{S_{\alpha+\frac{\pi}{2}}(w)}S_\alpha(z)}{z-\overline{w}}, \quad w,z\in\mathbb{C}. \tag{2.4.2.2}$$

Recall that, by Theorem 1.2.10, A_α determined by the boundary condition $(\cdot, s_\alpha)_+ = 0$ is a canonical selfadjoint extension of $\mathcal{S}$ which can be written explicitly as

$$A_\alpha = \mathcal{S} \oplus \operatorname{span}\{Js_\alpha\}. \tag{2.4.2.3}$$

Moreover, any canonical selfadjoint extension of $\mathcal{S}$ is obtained in this way by choosing $\alpha\in[0,\pi)$. By Theorem 2.3.10,

$$\rho(A_\alpha) = \{w : \operatorname{Ord}_w S_\alpha = \operatorname{Ord}_w \mathcal{H}\}$$

and its resolvent is given by

$$\mathcal{R}_{S_\alpha}(a)F(z) = \frac{F(z) - \frac{S_\alpha(z)}{S_\alpha(w)}F(w)}{z-a}, \quad F\in\mathcal{H}, a\in\rho(A_\alpha). \tag{2.4.2.4}$$

Note that the selfadjoint extension A_α is an operator extension if and only if $S_\alpha\notin\mathcal{H}$. If $\mathcal{S}$ is densely defined, then all canonical selfadjoint extensions are operator extensions, hence $S_\alpha\notin\mathcal{H}$ for all $\alpha\in[0,\pi)$. If $\overline{\operatorname{dom} S}\neq\mathcal{H}$, then there is one (and only one) selfadjoint extension A_{α_0} wich is not an operator extension. The number α_0 is the only number α of $[0,\pi)$ with $S_\alpha\in\mathcal{H}$. An accurate description of the spectral properties will be given in Theorem 2.4.3.

We show that the function $Q(z) = S_{\alpha+\frac{\pi}{2}}(z)/S_\alpha(z)$ is a Q-function of A_α and $\mathcal{S}$. Choose any $w_0\in\mathbb{C}\backslash\mathbb{R}$ such that $\operatorname{Ord}_{\overline{w_0}}\mathcal{H}=0$, and set

$$\varphi(w_0) = \frac{1}{\overline{S_\alpha(\overline{w}_0)}}K_{\overline{w}_0}\in N_{w_0}.$$

A straightforward computation using (2.4.2.4) and (2.4.2.2) shows that for $w\in\rho(A_\alpha)$

$$\varphi(w) = \varphi(w_0) + (w-w_0)(A_\alpha - z)^{-1}\varphi(w_0) = \frac{1}{\overline{S_\alpha^{(n)}(\overline{w})}}K_{\overline{w}}^{(n)}, \tag{2.4.2.5}$$

where $n=\operatorname{Ord}_{\overline{w}}\mathcal{H}$. It is also easy to verify that

$$\frac{Q(z)-\overline{Q(w)}}{z-\overline{w}} = (\varphi(z),\varphi(w)), \quad w,z\in\rho(A_\alpha).$$

Note that implies in particular

$$Q'(t) = (\varphi(t), \varphi(t)) = \frac{1}{|S_\alpha^{(n)}(t)|^2} \, \|K_t^{(n)}\|^2, \quad t \in \rho(Q) \cap \mathbb{R}, \tag{2.4.2.6}$$

again with $n = \operatorname{Ord}_t \mathcal{H}$.

2.4.1. Lemma. *The (real-valued) function $\varphi(t) = \arctan B(t)/A(t)$, with an appropriate choice of the branches of* arctan, *is a C^∞-function on the entire real line. Its derivative equals*

$$\varphi'(t) = \frac{1}{|A^{(n)}(t)|^2 + |B^{(n)}(t)|^2} \, \|K_t^{(n)}\|^2 = \frac{1}{|E^{(n)}(t)|^2} \, \|K_t^{(n)}\|^2,$$

where $n = \operatorname{Ord}_t \mathcal{H}$. Hence $\varphi(t)$ is strictly increasing.

Proof. Write $A_1 = S_\alpha$ and $B_1 = S_{\alpha+\frac{\pi}{2}}$ and set $Q(t) = B_1(t)/A_1(t)$. Since $\operatorname{Ord}_t \mathcal{H}$ is the minimum of the orders $\operatorname{Ord}_t A_1$ and $\operatorname{Ord}_t B_1$, we have $Q(t) = B_1^{(n)}(t)/A_1^{(n)}(t)$, where $n = \operatorname{Ord}_t \mathcal{H}$. Now (2.4.2.6) implies

$$\frac{d}{dt} \arctan Q(t) = \frac{1}{1 + Q^2(t)} \, Q'(t) = \frac{1}{|A_1^{(n)}(t)|^2 + |B_1^{(n)}(t)|^2} \, \|K_t^{(n)}\|^2, \quad t \in \rho(Q).$$

An elementary computation shows that $|A_1^{(n)}(t)|^2 + |B_1^{(n)}(t)|^2 = |A^{(n)}(t)|^2 + |B^{(n)}(t)|^2 = |E^{(n)}(t)|^2$, hence there exists a real number β such that

$$\arctan \frac{B(t)}{A(t)} = \arctan \frac{B_1(t)}{A_1(t)} + \beta, \quad t \in \rho\Big(\frac{B}{A}\Big) \cap \rho\Big(\frac{B_1}{A_1}\Big).$$

This already proves the theorem, since to any real t there exists always a value of α such that $t \in \rho(S_{\alpha+\frac{\pi}{2}}/S_\alpha)$. ☐

We shall call $\varphi(t)$ as defined in Lemma 2.4.1 the phase function of $\mathcal{H}(A, B)$. Note that the phase function $\varphi(t)$ depends on A and B.

2.4.2. Lemma. *Suppose that $\mathcal{H}(A, B) = \mathcal{H}(A_1, B_1)$. Then there exist a real constant α such that $\varphi_1(t) = \varphi(t) + \alpha$, $t \in \mathbb{R}$, if and only if*

$$(A_1, B_1) = (A, B)U$$

for an iJ-unitary matrix U which satisfies $UU^* = 1$.

Proof. By Proposition 2.3.8, there exists an iJ-unitary matrix U such that

$$(A_1, B_1) = (A, B)U.$$

By Lemma 2.4.1, $\varphi_1'(t) = \varphi'(t)$ holds for all real t if and only if

$$|A_1^{(n)}(t)|^2 + |B_1^{(n)}(t)|^2 = |A^{(n)}(t)|^2 + |B^{(n)}(t)|^2, \quad t \in \mathbb{R},$$

where $n = \mathrm{Ord}_t \mathcal{H}$. This condition can be rewritten as

$$(A^{(n)}(t), B^{(n)}(t))UU^*(A^{(n)}(t), B^{(n)}(t))^* = (A^{(n)}(t), B^{(n)}(t))(A^{(n)}(t), B^{(n)}(t))^*, \quad t \in \mathbb{R}.$$

If $UU^* = 1$ then the latter equation obviously holds. On the other hand, since $B(t)/A(t)$ assumes every real value, every vector $(c_1, c_2) \in \mathbb{C}^2$ with $c_1 \neq 0$ can be represented as $(c_1, c_2) = c\,(A(t), B(t))$ for appropriate $t \in \rho(B/A)$ and $c > 0$. This implies that $UU^* = 1$. □

2.4.3. Theorem. *Choose any $\alpha \in [0, \pi)$. If $S_\alpha \notin \mathcal{H}$, then A_α is an operator extension and its spectrum is given by*

$$\sigma(A_\alpha) = \{t \in \mathbb{R} : \varphi(t) \equiv \alpha \mod \pi\}.$$

If $S_\alpha \in \mathcal{H}$ then A_α is not an operator and its spectrum is

$$\sigma(A_\alpha) = \{t \in \mathbb{R} : \varphi(t) \equiv \alpha \mod \pi\} \cup \{\infty\}.$$

Any $t \in \sigma(A_\alpha)$ (including $t = \infty$) is a simple eigenvalue of A_α. For finite $t \in \sigma(A_\alpha)$ the corresponding eigenvector is

$$K_t^{(n)}(z) = -\overline{S^{(n)}_{\alpha+\frac{\pi}{2}}(t)}\,\frac{S_\alpha(z)}{z-t}, \quad \|K_t^{(n)}\|^2 = \frac{|E^{(n)}(t)|^2}{\varphi'(t)},$$

where $n = \mathrm{Ord}_t \mathcal{H}$. If A_α is not an operator, then the eigenvector that corresponds to the eigenvalue $t = \infty$ is the function $S_\alpha(z)$ itself.

Proof. Since the resolvent of A_α is given by (2.4.2.4), A_α is an operator if and only if $S_\alpha \notin \mathcal{H}$. By Theorem 2.3.10, the (real) spectrum of A_α consists exactly of all points $t \in \mathbb{R}$ with $\mathrm{Ord}_t S_\alpha > \mathrm{Ord}_t \mathcal{H}$. Note that $\mathrm{Ord}_t S_\alpha > \mathrm{Ord}_t \mathcal{H}$ if and only if one of the expressions

$$\frac{S_\alpha(t)}{A(t)}, \quad \frac{S_\alpha(t)}{B(t)},$$

is equal to zero. Since $S_\alpha(t) = \sin\alpha A(t) - \cos\alpha B(t)$, this is the case if and only if

$$\frac{B(t)}{A(t)} = \tan\alpha,$$

which proves the assertion on the spectrum. Since $\mathcal{S}$ is one to one and has deficiency index $(1, 1)$, all eigenvalues $t \in \sigma(A_\alpha)$ (including $t = \infty$ in case of a proper relational extension) are simple. In fact the eigenspace that corresponds to $t \in \sigma(A_\alpha) \cap \mathbb{R}$ is

$$N_t = \mathrm{ran}(\mathcal{S} - t)^\perp = \mathrm{span}\{K_t^{(n)}\},$$

where $n = \mathrm{Ord}_t \mathcal{H}$. Lemma 2.4.1 gives $\|K_t^{(n)}\|^2$ expressed as above. If A_α is not an operator, then S_α is (the only) vector of $\mathcal{H}$ that is mapped to zero by the resolvent of A_α. This shows that S_α is the eigenvector that corresponds to $t = \infty$. □

The following corollary is an immediate consequence of Theorem 2.4.3. For a more complete version see Theorem 2.5.4.

2.4.4. Corollary. *Let $\varphi(t)$ be the phase function of $\mathcal{H}(A,B)$. Choose any $\alpha \in [0,\pi)$. If $S_\alpha \notin \mathcal{H}$, then*

$$\|F\|^2 = \sum_{\varphi(t)\equiv\alpha \mod \pi} \left|\frac{F(t)}{E(t)}\right|^2 \frac{1}{\varphi'(t)}, \quad F \in \mathcal{H}.$$

If $S_\alpha \in \mathcal{H}$, which only can happen if $\mathcal{S}$ is not densely defined, then

$$\|F\|^2 \geq \sum_{\varphi(t)\equiv\alpha \mod \pi} \left|\frac{F(t)}{E(t)}\right|^2 \frac{1}{\varphi'(t)}, \quad F \in \mathcal{H}.$$

Equality holds if and only if $F \in \overline{\operatorname{dom}\mathcal{S}}$.

After all, we compute the extended resolvent $(A_\alpha - w)_-^{-1} : \mathcal{H}_- \to \mathcal{H}$. In this place it seems to be more convinient to identify $\mathcal{H}_-$ with $\operatorname{Assoc}(\mathcal{H})$ and write $(A_\alpha - w)_-^{-1}$ as a mapping from $\operatorname{Assoc}(\mathcal{H})$ into $\mathcal{H}$.

2.4.5. Proposition. *Identifying* $\operatorname{Assoc}(\mathcal{H})$ *with $\mathcal{H}_-$, the (extended) resolvent $(A_\alpha - w)_-^{-1}$ is given by*

$$(A_\alpha - w)_-^{-1}F(z) = \frac{F(z) - \frac{S_\alpha(z)}{S_\alpha(w)}F(w)}{z-w}, \quad w \in \rho(A_\alpha), F \in \operatorname{Assoc}(\mathcal{H}). \tag{2.4.2.7}$$

Proof. Let $(\cdot, f)_+$, $f = (F_1; F_2) \in \mathcal{H}_+$, denote the element of $\mathcal{H}_-$ that corresponds to F, and write $R_w = (A_\alpha - w)^{-1}$. By (1.2.1.4), we have $R_w^- f = R_w F_1 + (I + wR_w)F_2$, and therefore, identifying F and $f \in \mathcal{H}_-$,

$$R_w^- F(z) = \frac{F_1(z) - \frac{S_\alpha(z)}{S_\alpha(w)}F_1(w)}{z-w} + F_2(z) + w\frac{F_2(z) - \frac{S_\alpha(z)}{S_\alpha(w)}F_2(w)}{z-w} =$$
$$= \frac{F_1(z) + zF_2(z) - \frac{S_\alpha(z)}{S_\alpha(w)}(F_1(w) + wF_2(w))}{z-w}.$$

Since $F(z) = F_1(z) + zF_2(z)$, this proves (2.4.2.7). □

2.5 Measures Associated to $\mathcal{H}(A,B)$

2.5.1. Definition. Consider the space $\mathcal{H} = \mathcal{H}(A,B)$, and put $E(z) = A(z) - iB(z)$. We shall call a nonnegative Borel measure σ associated to $\mathcal{H}$, if

$$\|F\|^2 = \int_{\mathbb{R}} \left|\frac{F(t)}{E(t)}\right|^2 d\sigma(t), \quad F \in \mathcal{H}.$$

Recall that, by Lemma 2.3.12, $\operatorname{Ord}_t E = \operatorname{Ord}_t \mathcal{H}$, $t \in \mathbb{R}$, thus F/E has no real poles.

Write $\mathcal{H}(A,B) = \mathcal{H}$. Choose an entire function U such that $\operatorname{Ord}_w U = \operatorname{Ord}_w \mathcal{H}$ for all $w \in \mathbb{C}$. Such functions do always exist, they can be constructed as a Weierstrass product. Note that the function F/U is entire for every $F \in \mathcal{H}$, and that $(F/U)(w) = 0$ if and only if $F \in \operatorname{ran}(\mathcal{S} - w)$. Thus the family of point evaluation functionals $F \mapsto (F/U)(w)$, $w \in \mathbb{C}$, is a universal directing functional for $\mathcal{S}$, in the sense of Section 1.3. The transformation Φ is given by

$$\Phi F(z) = \frac{F}{U}(z), \quad F \in \mathcal{H}.$$

and its extension to $\mathcal{H}_-$ is

$$\Phi s(z) = \frac{S(z)}{U(z)}, \quad s \in \mathcal{H}_-,$$

where $S(z) = [s, K_z^+]_\pm$ is the associated function that corresponds to s. The function $E = A - iB \in \operatorname{Assoc}(\mathcal{H})$ corresponds to an element $e \in \mathcal{H}_-$, which is a boundary value for $\mathcal{S}$. By Lemma 2.3.12, $\operatorname{Ord}_t E = \operatorname{Ord}_t U$ for all $t \in \mathbb{R}$, hence

$$\frac{E}{U}(t) \neq 0, \quad t \in \mathbb{R}.$$

By Theorem 1.3.8, all measures associated to $\mathcal{H}$ are given by

$$d\sigma(t) = d\|\tilde{E}_t^- e\|^2 = |t-i|^2 \, d(\tilde{E}_t \tilde{R}_i^- e, \tilde{R}_i^- e), \tag{2.5.2.1}$$

where $\tilde{E}_i^-$ and $\tilde{R}_i^-$ correspond to a selfadjoint extension $\tilde{A}$ acting in a possibly larger Hilbert space $\tilde{H}$, with $\tilde{E}(\mathbb{R}) \supseteq \mathcal{H}$. Note that, for any $s \in \mathcal{H}_-$, we have

$$d\|\tilde{E}_t^- s\|^2 = \frac{|S(t)|^2}{|E(t)|^2} \, d\|\tilde{E}_t^- e\|^2. \tag{2.5.2.2}$$

To describe all those measures, we compute the generalized resolvent matrix of an element $s \in \mathcal{H}_-$.

Fix $S \in \operatorname{Assoc}(\mathcal{H})$, and denote by $s \in \mathcal{H}_-$ its corresponding element of $\mathcal{H}_-$. We are going to compute the matrix $W(z)$ as defined in Definition 1.5.9. There it is required to choose a canonical selfadjoint extension A of $\mathcal{S}$. We use the one which has the difference quotient

$$R_w F(z) = \frac{F(z) - \frac{A(z)}{A(w)} F(w)}{z - w}$$

as its resolvent. Taking $\varphi(z) = \frac{1}{A^\#(z)} K_{\overline{z}} \in N_z$ as appropriate paramatrization of the defect spaces and $Q(z) = B(z)/A(z)$ as a Q-function , we have

$$W(z) = \frac{1}{S(z)} \begin{pmatrix} A(z) r(z) & r(z) B(z) - \frac{S(z) S^\#(z)}{A^\#(z)} \\ A(z) & B(z) \end{pmatrix}, \tag{2.5.2.3}$$

where $r(z)$ is a s-resolvent generated by the extension A. Consider the matrix

$$S(z)W(z) = \begin{pmatrix} A(z)r(z) & r(z)B(z) - \frac{S(z)S^{\#}(z)}{A^{\#}(z)} \\ A(z) & B(z) \end{pmatrix}. \tag{2.5.2.4}$$

Note that all entries of $S(z)W(z)$ are entire function, since $S(z)W(z) = U(z)W_d(z)$, with $W_d(z)$ as defined in Theorem 1.5.14. Since $W(z) \in \mathcal{M}_0$, we obviously have $S(z)W(z) \in \mathcal{M}_0^S$. By its construction, the entire matrix function $S(z)W(z)$ is an s-resolvent matrix. For any other s-resolvent matrix $M_S(z)$ of the form

$$M_S(z) = \begin{pmatrix} C(z) & D(z) \\ A(z) & B(z) \end{pmatrix},$$

where $C(z)$ and $D(z)$ are entire functions, there exists a real constant α such that

$$M_S(z) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} S(z)W(z) \tag{2.5.2.5}$$

This is an immediate consequence of Theorem 1.5.13, and the fact that A and B are linear independent. Thus $C(z)$ and $D(z)$ are uniquely determined up to adding real multiples of $A(z)$ and $B(z)$:

$$C(z) = S(z)w_{11}(z) + \alpha A(z), \quad D(z) = S(z)w_{12}(z) + \alpha B(z).$$

Note that by (2.5), the entire matrix function $M_S(z)$ belongs to the class $\mathcal{M}_0^S$.

2.5.2. Lemma. *Given $\mathcal{H} = \mathcal{H}(A, B)$, let $E = A - iB$, and denote by $e \in \mathcal{H}_-$ the element of $\mathcal{H}_-$ that corresponds to E. The function $B(z)/A(z)$ is an e-resolvent generated by the extension A as defined above. With $S(z) = E(z)$ and $r(z) = B(z)/A(z)$ in (2.5.2.4) we obtain*

$$S(z)W(z) = \begin{pmatrix} B(z) & -A(z) \\ A(z) & B(z) \end{pmatrix}. \tag{2.5.2.6}$$

Proof. by Proposition 2.4.5, we have in the above notation

$$(R_w^- e)(z) = \frac{E(z) - \frac{A(z)}{A(w)}E(w)}{z - w} = -i\,\frac{B(z) - \frac{B(w)}{A(w)}A(z)}{z - w} =$$
$$= -i\,\frac{B(z) - \frac{B^{\#}(w)}{A^{\#}(w)}A(z)}{z - w} = \frac{1}{iA^{\#}(w)}K_{\overline{w}}(z).$$

Hence, by the definition of $\hat{R}_z^-$, it follows that

$$(\hat{R}_w^- e)(z) = \frac{1}{iA^{\#}(w)}K_{\overline{w}}^+ - \frac{1}{2i}\Big(\frac{1}{A^{\#}(i)}K_{-i}^+(z) + \frac{1}{A^{\#}(-i)}K_i^+(z)\Big),$$

and therefore

$$[\hat{R}_w^- e, e]_\pm = \frac{E^{\#}(w)}{iA^{\#}(w)} - \frac{1}{2i}\Big(\frac{E^{\#}(i)}{A^{\#}(i)} + \frac{E^{\#}(-i)}{A^{\#}(-i)}\Big) = \frac{B^{\#}(w)}{A^{\#}(w)} - \frac{1}{2}\Big(\frac{B^{\#}(i)}{A^{\#}(i)} + \frac{B^{\#}(-i)}{A^{\#}(-i)}\Big).$$

Since $B^\#(z)/A^\#(z) = B(z)/A(z)$, the constant $B^\#(i)/A^\#(i) + B^\#(-i)/A^\#(-i)$ is real, hence we proved the first assertion. Next, if we substitute $S(z) = E(z)$ and $r(z) = B(z)/A(z)$ in (2.5.2.4), we get

$$S(z)W(z) = \begin{pmatrix} B(z) & \frac{B(z)B^\#(z)-E(z)E^\#(z)}{A^\#(z)} \\ A(z) & B(z) \end{pmatrix} = \begin{pmatrix} B(z) & -\frac{A(z)A^\#(z)}{A^\#(z)} \\ A(z) & B(z) \end{pmatrix} = \begin{pmatrix} B(z) & -A(z) \\ A(z) & B(z) \end{pmatrix}.$$

□

Before we are able to describe all measures associated to $\mathcal{H}(A, B)$, we need one more lemma which states a crucial property of E.

2.5.3. Lemma. *Let e denote the element of $\mathcal{H}_-$ which corresponds to $E(z) = A(z) + iB(z)$. If $\mathcal{S}$ is not densely defined in $\mathcal{H}$, then*

$$\mathcal{H}_- = \overline{\mathcal{H}} \dot{+} \operatorname{span}\{e\}.$$

In particular, e does not belong to the closure of $\mathcal{H}$ in $\mathcal{H}_-$.

Proof. It follows from Remark 1.2.3 that $\operatorname{codim}_{\mathcal{H}_-} \overline{\mathcal{H}} = 1$. Recall that there is (one and only one) $\alpha \in [0, \pi)$ such that $S_\alpha \in \mathcal{H}$, and therefore its corresponding element $s_\alpha \in \mathcal{H}_-$ also belongs to H. Let a and b denote the boundary values that correspond to $A(z)$ and $B(z)$, respectively. We have

$$e = a - ib, \quad s_\alpha = \sin\alpha\, a - \cos\alpha\, b,$$

hence $\operatorname{span}\{e, s_\alpha\} = \operatorname{span}\{a, b\}$, which is the space of all boundary values. Therefore e cannot belong to $\overline{\mathcal{H}}$, otherwise $\overline{\mathcal{H}}$ is the entire space $\mathcal{H}_-$. □

2.5.4. Theorem. *Suppose $\mathcal{H} = \mathcal{H}(A, B)$ is nontrivial and set $E(z) = A(z) - iB(z)$. Then for any function $\tau \in \mathcal{N}_0$ we can write*

$$\frac{B(z)\tau(z) - A(z)}{A(z)\tau(z) + B(z)} = \alpha + \beta z + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{d\sigma(t)}{t^2 + 1} \tag{2.5.2.7}$$

for certain constants $\alpha \in \mathbb{R}$, $\beta \geq 0$ and some nonnegative Borel measure σ. The measure σ is an associated measure if and only if $\beta = 0$, which is always the case when $\mathcal{S}$ is densely defined. Conversely, to any associated measure σ there exists a function $\tau \in \mathcal{N}_0$ such that (2.5.2.7) *holds (with $\beta = 0$). If $\beta \neq 0$ (which can only happen when $\overline{\operatorname{dom}\mathcal{S}} \neq \mathcal{H}$) then we have at least*

$$\int_{\mathbb{R}} \left|\frac{F(t)}{E(t)}\right|^2 d\sigma(t) \leq \|F\|^2, \quad F \in \mathcal{H}, \tag{2.5.2.8}$$

with equality if and only if $F \in \overline{\operatorname{dom}\mathcal{S}}$.

Proof. Denote by e the element of $\mathcal{H}_-$ that corresponds to the (associated) function $E(z)$. Since the matrix (2.5.2.6) is an e-resolvent matrix, the mapping

$$\tau \mapsto M_E \circ \tau(z) = \frac{B(z)\tau(z) - A(z)}{A(z)\tau(z) + B(z)}$$

establish a one to one correspondence of all functions $\tau \in \mathcal{N}_0 \cup \{\infty\}$ and all generalized e-resolvents. Recall that by Theorem 1.5.4, any generalized u-resolvent generated by some selfadjoint extension $\tilde{A}$ of $\mathcal{S}$ has the integral representation

$$\alpha + (\tilde{E}(\{\infty\})\tilde{R}_i^- e, \tilde{R}_i^- e)z + \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{d\|\tilde{E}_t^- e\|^2}{t^2+1},$$

where $(\tilde{E}_t^-, \tilde{E}(\{\infty\}))$ and $\tilde{R}_i^-$ correspond to the extension $\tilde{A}$ and $\alpha \in \mathbb{R}$. This shows (2.5.2.7). It follows from the discussion in the beginning of this section that, if $\tilde{A}$ is any selfadjoint extension of $\mathcal{S}$, acting in a possibly larger Hilbert space $\tilde{H}$, the relation

$$\|\tilde{E}(\mathbb{R})F\|^2 = \int_{\mathbb{R}} \left|\frac{F(t)}{E(t)}\right|^2 d\|\tilde{E}_t^- e\|^2, \quad F \in \mathcal{H}, \tag{2.5.2.9}$$

holds, where $\tilde{E}_t^-$ and $\tilde{R}_i^-$ belong to the selfadjoint extension $\tilde{A}$. Thus the measure $d\sigma(t) = d\|\tilde{E}_t^- e\|^2$ is associated to $\mathcal{H}$ if and only if $\mathcal{H} \subseteq \operatorname{ran} \tilde{E}(\mathbb{R})$. Conversely, by Theorem 1.3.8, all associated measures are obtained in this way. We now consider two cases. If $\mathcal{S}$ is densely defined, we always have $\mathcal{H} \subseteq \operatorname{ran} \tilde{E}(\mathbb{R})$. We also have $\overline{\mathcal{H}} = \mathcal{H}_-$, and therefore $\beta = (\tilde{E}(\{\infty\})\tilde{R}_i^- e, \tilde{R}_i^- e) = 0$. This proves the theorem in the case $\overline{\operatorname{dom} \mathcal{S}} = \mathcal{H}$. Now assume that $\overline{\operatorname{dom} \mathcal{S}} \neq \mathcal{H}$. By Lemma 2.5.3, e does not belong to the closure of $\mathcal{H}$ in $\mathcal{H}_-$. According to Lemma 1.5.5, we have $\mathcal{H} \subseteq \tilde{E}(\mathbb{R})$ if and only if

$$\beta = (\tilde{E}(\{\infty\})\tilde{R}_i^- e, \tilde{R}_i^- e) = 0.$$

If $\beta \neq 0$, then (2.5.2.9) immedately gives the inequality (2.5.2.8). Since $\overline{\operatorname{dom} \mathcal{S}}$ has codimension 1 and since $\operatorname{ran} \tilde{E}(\mathbb{R}) \neq \mathcal{H}$, equality holds in (2.5.2.8) if and only if $F \in \overline{\operatorname{dom} \mathcal{S}}$. ☐

2.5.5. Remark. Fix $\alpha \in [0, \pi)$. If we choose for $\tau(z)$ the constant fucntion $\tau(z) = -\tan \alpha$, then we get

$$\frac{B(z)\tau(z) - A(z)}{A(z)\tau(z) + B(z)} = \frac{\sin \alpha B(z) + \cos \alpha A(z)}{\sin \alpha A(z) - \cos \alpha B(z)} = \frac{S_{\alpha+\frac{\pi}{2}}(z)}{S_\alpha(z)}.$$

Hence (2.5.2.7) can be rewritten as

$$\frac{S_{\alpha+\frac{\pi}{2}}(z)}{S_\alpha(z)} = \alpha + \beta z + \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{d\sigma(t)}{t^2+1}$$

The measure σ is exactly the discrete measure which occurs in Corollary 2.4.4.

2.6 Transfer Matrices of Subspaces

Suppose $\mathcal{H} = \mathcal{H}(A, B)$ is nontrivial. In this section we investigate how the generating functions A_1 and B_1 of a subspace $\mathcal{H}_1 = \mathcal{H}(A_1, B_1)$ of $\mathcal{H}$ are related to A and B. The present results follow easily from the results of Section 1.6.

Suppose $S \in \operatorname{Assoc}(\mathcal{H})$ and denote by s the element of $\mathcal{H}_-$ which corresponds to S. As in Section 2.5, there exist entire functions $C(z)$ and $D(z)$ such that

$$M_S(z) = \begin{pmatrix} C(z) & D(z) \\ A(z) & B(z) \end{pmatrix}$$

is an s-resolvent matrix. Moreover, if $W(z)$ denotes the resolvent matrix associated to the canonical selfadjoint extension which has the difference quotient $\mathcal{R}_A(a)$ as its resolvent as in Definition 1.5.9, we have

$$M_S(z) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} S(z)\, W(z)$$

for an appropriate real constant α. Thus $M_S \in \mathcal{M}_0^S$. Using s as the gauge of the representation $(-\mathcal{Q}(z); \mathcal{P}(z))$, the mapping

$$\Phi : F \mapsto \begin{pmatrix} F_- \\ F_+ \end{pmatrix} = S(z) \begin{pmatrix} -\mathcal{Q}(z)F \\ \mathcal{P}(z)F \end{pmatrix}$$

is an isomorphism between the spaces $\mathcal{H}$ and $\mathcal{H}_S = \mathcal{H}_S(M_S)$ as defined in Section 2.2. Note that $r_s(\mathcal{S}) = \{z \in \mathbb{C} : \operatorname{Ord}_z S = \operatorname{Ord}_z \mathcal{H}\}$, and, with $n = \operatorname{Ord}_z \mathcal{H}$,

$$\mathcal{P}(z)F = \frac{(F, K_z^{(n)})}{[s, K_{z,+}^{(n)}]_\pm} = \frac{F^{(n)}(z)}{S^{(n)}(z)} = \left(\frac{F}{S}\right)(z), \quad z \in r_s(\mathcal{S}).$$

Hence the second component F_+ equals F, and the mapping $\pi_+ : (F_-; F_+) \mapsto F_+$ is an isomorphism between the spaces $\mathcal{H}_S$ and $\mathcal{H}$. In particular, for $(F_-; F_+) \in \mathcal{H}_S$, the first component F_- is uniquely determined by F_+.

The following theorem is the central result of the present section.

2.6.1. Theorem. *Suppose that $\mathcal{H}_1 = \mathcal{H}(A_1, B_1)$ is a subspace of $\mathcal{H} = \mathcal{H}(A, B)$, and that $\operatorname{Ord}_z \mathcal{H}_1 = \operatorname{Ord}_z \mathcal{H}$ for all $z \in \mathbb{C}$. Then there exist an entire 2×2-matrix function $M(z) \in \mathcal{M}_0$ such that*

$$(A(z), B(z)) = (A_1(z), B_1(z))\ M(z). \tag{2.6.2.1}$$

Denote by $\mathfrak{K}(M)$ be the space that is generated by the matrix function $M(z) \in \mathcal{M}_0$. Then the mapping

$$\begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix} \mapsto F_1(z)A_1(z) + F_2(z)B_1(z)$$

is an isomorphism between the spaces $\mathfrak{K}(M)$ and $\mathcal{H} \ominus \mathcal{H}_1$.

Proof. Choose $S \in \operatorname{Assoc}(\mathcal{H}_1)$ and denote by s the element of $\mathcal{H}_{1,-}$ that corresponds to $S(z)$. Note that the multiplication operator $\mathcal{S}$ in $\mathcal{H}$ is a symmetric extension of the multiplication operator $\mathcal{S}_1$ in $\mathcal{H}_1$, and, since $\operatorname{Ord}_z \mathcal{H}_1 = \operatorname{Ord}_z \mathcal{H}$, we obviously have $r(\mathcal{S}_1, \mathcal{S}) = \mathbb{C}$. If P denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$, then $P_+ = P \oplus P$ is a mapping of $\mathcal{H}_+$ into $\mathcal{H}_{1,+}$. The element $P_+^* s \in \mathcal{H}_-$ belongs to the associated function S:

$$[P^* s, K_z^+]_\pm = [s, P K_z^+]_\pm = [s, K_{1,z}^+]_\pm = S(z),$$

since P maps the reproducing kernel K_z of $\mathcal{H}$ to the reproducing kernel $K_{1,z}$ of $\mathcal{H}_1$. Let $W(z)$ and $W_1(z)$ denote the corresponding $P_+^* s$- and s-resolvent matrix, respectively, as introduced above. Set $M_S(z) = S(z)W(z)$ and $M_{1,S}(z) = S(z)W_1(z)$. By an appropriate choice of the the generalized s-resolvent in the definition of $W_1(z)$, we can assume that $\mathfrak{K}(W_1)$ is contained isometrically in $\mathfrak{K}(W)$, and therefore that $\mathcal{H}_{S,1} = \mathcal{H}_S(M_{1,S})$ is contained isometrically in $\mathcal{H}_S = \mathcal{H}_S(M_S)$. By Theorem 1.6.8, there exists a matrix function $M(z) \in M_0$ which is holomorphic on $r(\mathcal{S}, \mathcal{S}_1) = \mathbb{C}$ such that

$$W(z) = W_1(z)\, M(z),$$

and therefore

$$M_S(z) = S(z)W(z) = S(z)W_1(z)\, M(z) = M_{1,S}(z)\, M(z)$$

This shows (2.6.2.1). Moreover, by Corollary 1.6.9, the mapping

$$\mathbf{f}(z) \mapsto W_1(z)\, \mathbf{f}(z)$$

is an isomorphism between $\mathfrak{K}(M)$ and $\mathfrak{K}(W) \ominus \mathfrak{K}(W_1)$. This implies that

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \mapsto M_{1,S}(z) \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

is an isomorphism between the spaces $\mathfrak{K}(M)$ and $\mathcal{H}_S \ominus \mathcal{H}_{1,S}$. Since the projection π_+ is an isomorphism between $\mathcal{H}_S$ and $\mathcal{H}$, this completes the proof of the theorem. ☐

Bibliography

[AG1] N. I. Akhiezer, M. I. Glazman: *Theory of linear operators in Hilbert space Vol. 1,* Monographs and Studies in Math., Pitman Press, Bath 1981.

[AG2] N. I. Akhiezer, M. I. Glazman: *Theory of linear operators in Hilbert space Vol. 2,* Monographs and Studies in Math., Pitman Press, Bath 1981.

[ABDS] D. Alpay, P. Bruinsma, A. Dijksma, H. de Snoo: *A Hilbert space associated with a Nevanlinna function,* Signal processing, scattering and operator theory, and numerical methods (Amsterdam 1989), 115–122, Progr.Systems Control Theory, **5**, Birkhäuser, Boston, MA, 1990.

[Ar] N. Aronszaijn: *Theory of reproducing kernels,* Trans. Amer. Math. Soc. **68**, (1950), 337–404.

[Be] Ju. M. Berezanski: *Expansions in eigenfunctions of selfadjoint operators ,* (Russian), Naukova Dymka, Kiev 1965, Engl. Transl., Trans. Math. Monographs **17**, Amer. Math. Soc. 1968.

[dB] L. de Branges: *Hilbert spaces of entire functions,* Prentice-Hall, London 1968.

[DS1] A. Dijksma, H. de Snoo: *Symmetric and selfadjoint relations in Krein spaces I,* Operator Theory: Advances and Applications Vol. **24**(1978), Birkäuser, Basel.

[DS2] A. Dijksma, H. de Snoo: *Selfadjoint extensions of symmetric subspaces,* Pacific Journal of Mathematics **54** (1974) 71–99.

[GG] M. L. Gorbachuk, V. I. Gorbachuk: *M.G.Krein's lectures on entire operators,* Oper. Theory Adv. Appl. 97, Birkhäuser Verlag, Basel 1997.

[KW1] M. Kaltenbäck, H. Woracek: *Generalized resolvent matrices and spaces of analytic functions,* Integral Equations Operator Theory **32** (1998), 282–318.

[KW2] M. Kaltenbäck, H. Woracek: *Pontryagin spaces of entire functions I,* Integral Equations Operator Theory **33** (1999), 34–97.

[Kr] M. G. Krein: *Fundamental propositions of the representation theory of hermitian operators with deficiency index (m, m),* (Russian) Ukrain.Mat.Žurnal **1** (1949), no.2, 3–66; (Tranlation) Amer.Math.Soc.Transl., (2) **97** (1970), 75–143.

[KL] M. G. Krein, H. Langer: *On some continuation problems which are closely related to the theory of operators in spaces Π_κ. II. Generalized resolvents, u-resolvents and entire operators,* Journal of Functional Analysis **30** (1978), 390–447.

[LT] H. Langer, B. Textorius : *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space,*
Pacific Journal of Mathematics **72** (1977).

[St] A. V. Štraus: *Extensions and generalized resolvents of a non-densely defined symmetric operator,*
(Russian) Izv. Akad. Nauk SSSR. Ser. Mat. **34** (1970), 175–202.

[Ru] W. Rudin: *Functional analyis, 2nd ed.,*
Int. Series in Pure and Appl. Math., McGraw-Hill, New York 1991.