DIPLOMARBEIT

Slice hyperholomorphic functions and the quaternionic functional calculus

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Preface

It was during the work on my bachelor thesis with the topic "*Der Holomorphiebegriff für Clifford-Algebra-wertige Funktionen*" that I first got into contact with non-commutative analysis. I discovered that the notion of Cauchy-Fueter-regularity allowed to generalize most of the classical results on holomorphic functions to the higher-dimensional case of quaternion- or Clifford-algebra-valued functions. The associated function theory had been well developed for a long time and played a fundamental role in the field of Clifford-analysis.

After the course "*Functional analysis 2*" I was fascinated by the idea of a functional calculus: the fact that it was possible to extend functions from a complex to an operator argument. I wondered whether it was possible to define an analogue theory for operators on "Banach spaces" over quaternions or Clifford-numbers and I chose this question to be the topic of my master thesis.

Defining a functional calculus in the quaternionic setting had been an open problem for a long time. Several mathematicians had considered it over the years without achieving satisfactory results. Fortunately, the discovery of the notion of slice hyperholomorphicity gave new impact to this field and hence, during the last decade, mathematicians have made great progress in answering the related questions. When I started to work on my master thesis, the foundations of the theory of slice hyperholomorphic functions and the associated S-functional calculus were well established. Many related results had been developed by Fabrizio Colombo and his collaborators at the *Politecnico di Milano*. Encouraged by Michael Kaltenbäck, my supervisor at Vienna, I contacted Fabrizio Colombo and we started a very interesting and fruitful cooperation.

The aim of my master thesis is to give an overview on the fundamentals of the theory of quaternion-valued slice hyperholomorphic functions and the S-functional calculus for quaternionic linear operators. The analogue of the classical resolvent equation, Theorem 4.16, was found during my cooperation with Fabrizio Colombo. The presented proofs of the product rule and the existence of Riesz-projectors, which are based on this equation, were also developed in this period.

To keep my master thesis within reasonable bounds, I expect the reader to be familiar with the fundamentals of complex analysis and functional analysis, as they are taught in introductory courses at university.

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Chapter 1

Introduction

In the 1930s, Birkhoff and von Neumann showed that the Schrödinger equation can be written using either complex- or quaternion-valued functions [5]. Since then, many attempts have been made to develop a quaternionic version of quantum theory. However, in contrast to the complex case, the mathematical foundations of quaternionic quantum mechanics have been an open question for a long time. In particular, the identification of the correct notion of spectrum of a quaternionic linear operator and the definition of a functional calculus that is useful in applications caused difficulties for mathematicians [1]. As a consequence, it was also not possible to formulate the spectral theorem for quaternionic linear operators precisely. The theory of slice hyperholomorphic functions and the related S-functional calculus, which are presented in this master thesis, answer these questions.

In order to explain the difficulties in the quaternionic case, we give a short overview on the situation in the complex case. This overview is not a complete discussion. It is rather meant to be a motivation of the approach in the quaternionic case. The proofs of the presented results can be found for instance in [16, Chapter VII].

1.1 The Riesz-Dunford functional calculus

The most basic functional calculus is the polynomial functional calculus for linear operators on a finite dimensional Banach space. For a linear operator A on $\mathbb{C}^k$ and any $p(z) = \sum_{n=0}^{N} a_n z^n$ in the set of complex polynomials $\mathbb{C}[z]$, we define

$$p(A) = \sum_{n=0}^{N} a_n A^n,$$

where $A^0 = \mathcal{I}$ and $\mathcal{I}$ denotes the identity operator as usual.

This polynomial functional calculus is consistent with algebraic operations such as addition and multiplication and gives a lot of useful information about the operator A. For instance, we may consider the minimal polynomial m_A of A, that is, the polynomial with leading coefficient 1 of lowest degree such that $m_A(A) = 0$. Then $a \in \mathbb{C}$ is an eigenvalue of A if and only if a is a root of m_A as it is well known from linear algebra. Moreover, for any $p \in \mathbb{C}[z]$ there exist polynomials $q, r \in \mathbb{C}[z]$ with $0 \leq \deg(r) < \deg(m_A)$ such that $p(z) = q(z)m_A(z) + r(z)$. Hence, we obtain $p(A) = q(A)m_A(A) + r(A) = r(A)$. It is even possible to specify this observation.

Lemma 1.1. *Let A be a linear operator on $\mathbb{C}^k$, let $\sigma(A) = \{\lambda_1, \dots, \lambda_N\}$ be the set of eigenvalues of A and let $\nu_1, \dots, \nu_N \in \mathbb{N}$ be such that $m_A(z) = \prod_{n=1}^{N}(z - \lambda_n)^{\nu_n}$. A polynomial $p \in \mathbb{C}[z]$ satisfies $p(A) = 0$ if and only if p has a zero of order at least ν_n at λ_n for any $n = 1, \dots, N$.*

In particular, $p(A) = q(A)$ for two polynomials $p, q \in \mathbb{C}[z]$ if and only if $p - q$ has a zero of order at least ν_n at λ_n for any $n = 1, \dots, N$, that is, if and only if

$$p(\lambda_n) = q(\lambda_n),\ p'(\lambda_n) = q'(\lambda_n),\ \dots,\ p^{(\nu_n)}(\lambda_n) = q^{(\nu_n)}(\lambda_n) \quad \textit{for } n = 1, \dots, N.$$

It is possible to extend this polynomial calculus to functions that are analytic on an open set that contains $\sigma(A)$. For such a function f, we can choose a polynomial $p_f \in \mathbb{C}[z]$ such that

$$p_f(\lambda_n) = f(\lambda_n),\ p_f'(\lambda_n) = f'(\lambda_n),\ \dots,\ p_f^{(\nu_n)}(\lambda_n) = f^{(\nu_n)}(\lambda_n) \quad \text{for } n = 1, \dots, N$$

and set $f(A) = p_f(A)$. Lemma 1.1 implies that this calculus is well defined and independent of the choice of the polynomial p_f. Note that the set, on which f is holomorphic, does not have to be connected because $f(\lambda_n)$, $f'(\lambda_n)$, ..., $f^{(\nu_n)}(\lambda_n)$ only depend on the values of f on a neighborhood of λ_n. Nevertheless, the essential information is given by the polynomials of degree lower or equal to $\deg(m_A)$.

If we consider a bounded linear operator T on an infinite dimensional complex Banach space V, then T does not necessarily satisfy a polynomial equation $p(T) = 0$. Therefore, the set of polynomials is too small to provide a complete picture of the operator T. A natural approach to enlarge the class of admissible functions would be to consider power series of the form $P(z) = \sum_{n=0}^{\infty} a_n z^n$ and to define $P(T) = \sum_{n=0}^{\infty} a_n T^n$. However, this method is not satisfactory because it requires additional assumptions on the radius of convergence of P, which are often too restrictive, in order to ensure the convergence of the series $\sum_{n=0}^{\infty} a_n T^n$. Moreover, this series does not converge at all if the operator T is unbounded. Therefore this approach can not be generalized to the case of unbounded operators.

To find a different approach, we first recall that, in the finite dimensional case, $f(A)$ depends only on the values of the function f on a neighborhood of the set of eigenvalues of A. In the infinite dimensional case, the set of eigenvalues is not sufficient to characterize an operator, as it is well known. It must be replaced by its spectrum, which coincides with the set of eigenvalues in the finite dimensional case.

Definition 1.2. *Let T be a bounded operator on a complex Banach space V. The set $\rho(T)$ of all $\lambda \in \mathbb{C}$ such that $(\lambda\mathcal{I} - T)^{-1}$ exists as a bounded operator on V is called the* resolvent set *of T. The set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the* spectrum *of T.*

Lemma 1.3. *Let T be a bounded operator on a complex Banach space V. The spectrum $\sigma(T)$ is a nonempty, compact set that is contained in the ball $B_{\|T\|}(0)$.*

Let $U \subset \mathbb{C}$ be an open set such that its boundary ∂U consists of a finite number of rectifiable Jordan curves. If f is a function that is holomorphic on an open set that contains $\overline{U}$, then Cauchy's integral formula states that

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\xi)}{\xi - z}\, d\xi$$

for any $z \in U$. The idea of the Riesz-Dunford-functional calculus is to replace the variable z in this formula by the operator T and to define

$$f(T) = \frac{1}{2\pi i} \int_{\partial U} (\xi\mathcal{I} - T)^{-1} f(\xi)\, d\xi. \tag{1.1}$$

The question is whether this procedure makes any sense.

Lemma 1.4. *Let T be a bounded operator on a complex Banach space V. The function $\mu \mapsto R_\mu(T) = (\mu\mathcal{I} - T)^{-1}$ is holomorphic on $\rho(T)$. It is called the* resolvent *of T.*

This lemma and Cauchy's integral theorem imply that the integral in (1.1) does not depend on the set U.

Let us consider the Cauchy kernel $\frac{1}{\xi - z}$. For $|z| < |\xi|$ it allows the expansion

$$\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1 - \xi^{-1}z} = \frac{1}{\xi}\sum_{n=0}^{\infty}(\xi^{-1}z)^n = \sum_{n=0}^{\infty} z^n \xi^{-n-1} \tag{1.2}$$

because the geometric series $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ converges for $|q| < 1$. Now recall that the Neumann series $\sum_{k=0}^{\infty} T^k = (\mathcal{I} - T)^{-1}$ converges for $\|T\| < 1$. Thus, for $\|T\| < |\lambda|$, we obtain the analogous series expansion of the resolvent operator, namely

$$(\lambda\mathcal{I} - T)^{-1} = \frac{1}{\lambda}(\mathcal{I} - \lambda^{-1}T) = \frac{1}{\lambda}\sum_{n=0}^{\infty}(\lambda^{-1}T)^n = \sum_{n=0}^{\infty} T^n \lambda^{-n-1}. \tag{1.3}$$

Let $f(z) = z^m$ with $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let U be a ball $B_r(0)$ with radius $\|T\| < r$. Then the series (1.3) converges uniformly on ∂U. Hence,

$$\frac{1}{2\pi i}\int_{\partial U} (\xi\mathcal{I} - T)^{-1}\xi^m\, d\xi = \frac{1}{2\pi i}\int_{\partial U} \sum_{n=0}^{\infty} T^n \xi^{-n-1}\xi^m\, d\xi = \sum_{n=0}^{\infty} T^n \frac{1}{2\pi i}\int_{\partial U} \xi^{-n+m-1}\, d\xi = T^m \tag{1.4}$$

because

$$\frac{1}{2\pi i}\int_{\partial B_r(0)} \xi^{-n+m-1}\,d\xi = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.}\end{cases}$$

Therefore, replacing the variable z by the operator T in Cauchy's integral formula is consistent with the polynomial functional calculus, which justifies the following definition.

Definition 1.5 (Riesz-Dunford functional calculus)**.** *Let T be a bounded operator on a complex Banach space and let f be holomorphic on an open set O with $\sigma(T) \subset O$. Then we define*

$$f(T) = \frac{1}{2\pi i}\int_{\partial U} R_\xi(T) f(\xi)\,d\xi,$$

where U is an arbitrary open set such that $\sigma(T) \subset U$ and $\overline{U} \subset O$ and such that ∂U consists of a finite number of rectifiable Jordan curves.

Lemma 1.6. *Let T be a bounded operator on a complex Banach space. Let f and g be holomorphic functions on an open set O with $\sigma(T) \subset O$ and let α and β be complex numbers. Then $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$ and $(fg)(T) = f(T)g(T)$.*

Moreover, if f_n is a sequence of holomorphic functions on O that converges uniformly to f, then $f_n(T)$ converges to $f(T)$ in the uniform operator topology.

An important application of the Riesz-Dunford-functional calculus is that it allows to identify invariant subspaces. Let us assume that $\sigma(T) = \sigma_1(T) \cup \sigma_2(T)$ with $\operatorname{dist}(\sigma_1(T), \sigma_2(T)) > 0$. Then we can choose open sets U_1 and U_2 such that $\sigma_i(T) \subset U_i, i = 1, 2$ and $\overline{U_1} \cap \overline{U_2} = \emptyset$. Since the indicator functions $\mathbb{1}_{U_i}$ are holomorphic on $U_1 \cup U_2$, we can apply the functional calculus and define $P_i = \mathbb{1}_{U_i}(T)$ for $i = 1, 2$. We obtain

$$P_i^2 = \mathbb{1}_{U_i}(T)\mathbb{1}_{U_i}(T) = (\mathbb{1}_{U_i}\cdot \mathbb{1}_{U_i})(T) = \mathbb{1}_{U_i}(T) = P_i \tag{1.5}$$

and

$$P_i T = \mathbb{1}_{U_i}(T) z(T) = (\mathbb{1}_{U_i} z)(T) = (z\mathbb{1}_{U_i})(T) = z(T)\mathbb{1}_{U_i}(T) = TP_i, \tag{1.6}$$

where z denotes the identity function $z \mapsto z$. Thus, the operators P_i are projections and they commute with T. Therefore, the subspaces $V_i = P(V), i = 1, 2$ of the Banach space V are invariant under T. Indeed, for any $\boldsymbol{v} \in V_i$, we have

$$T(\boldsymbol{v}) = TP_i(\boldsymbol{v}) = P_i T(\boldsymbol{v}) \in V_i.$$

The operators P_1 and P_2 are called the *Riesz-projections* associated with $\sigma_1(T)$ and $\sigma_2(T)$.

We conclude our discussion with two important properties of the spectrum of an operator.

Theorem 1.7 (Spectral Radius Theorem)**.** *Let T be a bounded operator on a complex Banach space. The* spectral radius *of T is defined as $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. It satisfies*

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T\|^n}.$$

Note that $r(T) \le \|T\|$ because of Lemma 1.3. Moreover, the resolvent of T is holomorphic on $\{\lambda \in \mathbb{C} : r(T) < |\lambda|\} \subset \rho(T)$ by Lemma 1.4. Since $\lim_{\lambda\to\infty} R_\lambda(T) = 0$, the Taylor series expansion of the resolvent at infinity, that is, the series $R_\lambda(T) = \sum_{n=0}^\infty T^n \lambda^{-n-1}$, converges in the uniform operator topology not only for λ with $\|T\| < |\lambda|$ but even for λ with $r(T) < |\lambda|$.

Theorem 1.8 (Spectral mapping theorem)**.** *Let T be a bounded operator on a complex Banach space and let f be holomorphic on an open set O with $\sigma(T) \subset O$. Then*

$$f(\sigma(T)) = \sigma(f(T)).$$

Theorem 1.9. *Let T be a bounded operator on a complex Banach space, let f be holomorphic on an open set O with $\sigma(T) \subset O$ and let g be holomorphic on an open set that contains $f(\sigma(T))$. Then*

$$g(f(T)) = (g \circ f)(T).$$

1.2 Difficulties in the quaternionic setting

We introduce the quaternions, quaternionic vector spaces etc. in the next chapter. For the moment it is enough to know that the quaternions $\mathbb{H}$ are the 4-dimensional real vector space with basis $\{1, e_1, e_2, e_3\}$ that is endowed with a non-commutative product such that $e_i^2 = -1$ for $i = 1, 2, 3$ and $e_i e_j = -e_j e_i$ for $i \neq j$ and $1 \leq i, j \leq 3$. We shall not be too much concerned about the details of the definitions of quaternionic vector spaces, quaternionic linear operators etc. since these details are not essential to understand the following discussion.

When we want to generalize the Riesz-Dunford-functional calculus to the quaternionic setting, we meet several problems. As in the complex case, we can consider the finite-dimensional case and try to generalize it to the infinite dimensional one. However, since the quaternionic multiplication is not commutative, we have to distinguish whether we multiply a vector $\boldsymbol{v}$ with a scalar $a \in \mathbb{H}$ from the left or from the right. This leads to two different notions of eigenvalues.

Definition 1.10. *Let T be a right linear operator on a quaternionic vector space V, that is, an operator that is linear with respect to the multiplication with scalars from the right. A quaternion $\lambda \in \mathbb{H}$ is called a* left eigenvalue *of T if there exists a vector $\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}$ such that*

$$T(\boldsymbol{v}) = \lambda \boldsymbol{v}.$$

It is called a right eigenvalue *if there exists a vector $\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}$ such that*

$$T(\boldsymbol{v}) = \boldsymbol{v}\lambda.$$

We denote the set of left and right eigenvalues of T by $\sigma_L(T)$ and $\sigma_R(T)$, respectively.

When one tries to generalize these notions of eigenvalues to a notion of spectrum, one is faced with a paradoxical situation.

It is the notion of right eigenvalues that is relevant in applications such as quaternionic quantum theory [1]. Moreover, it allows to prove the spectral theorem for quaternionic matrices [17]. However, the mapping $\mathcal{R}_\lambda(T) : \boldsymbol{v} \to T(\boldsymbol{v}) - \boldsymbol{v}\lambda$ is not right linear because

$$\mathcal{R}_\lambda(T)[\boldsymbol{v}a] = T(\boldsymbol{v}a) - \boldsymbol{v}a\lambda \neq T(\boldsymbol{v})a - \boldsymbol{v}\lambda a = \mathcal{R}_\lambda(T)[\boldsymbol{v}]a$$

if λ and a do not commute. Therefore, it is not possible to associate a right linear resolvent operator to the set of right eigenvalues and to define a generalized notion of right spectrum as Colombo and Sabadini point out in [9].

On the contrary, the operator $\mathcal{L}_\lambda(T) : \boldsymbol{v} \to T(\boldsymbol{v}) - \lambda\boldsymbol{v}$ is right linear. Therefore, one can consider the left resolvent operator $\mathcal{L}_\lambda^{-1}(T) = (T - \lambda\mathcal{I})^{-1}$ and define the *left spectrum* $\sigma_L(T)$ as the set of quaternions $\lambda \in \mathbb{H}$ such that $\mathcal{L}_\lambda(T) = T - \lambda\mathcal{I}$ is not invertible. Unfortunately, the left spectrum does not seem to be of any relevance in applications.

It is therefore not at all clear how to generalize the notion of eigenvalues of a quaternionic linear operator to a meaningful notion of spectrum if one starts from the finite dimensional case.

Another approach to define a quaternionic functional calculus is to consider a notion of generalized holomorphicity and to directly replace the quaternionic variable by an operator in the respective Cauchy formula. The most successful notion of holomorphicity in the quaternionic setting was the notion of *Cauchy-Fueter-regularity*, which is discussed for instance in [20].

Cauchy-Fueter-regularity is based on the observations of the *Wirtinger calculus*. Let us identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ and let $f(z) = u(z_0, z_1) + iv(z_0, z_1)$ be a real differentiable function from an open set $U \subset \mathbb{C}$ to $\mathbb{C}$, where $z = z_0 + iz_1$ for any $z \in \mathbb{C}$. For small $h = h_0 + ih_1$, we have

$$\begin{aligned} f(z+h) - f(z) =& df(z)h + o(\|h\|) = \frac{\partial f}{\partial z_0}(z)h_0 + \frac{\partial f}{\partial z_1}(z)h_1 + o(\|h\|) = \\ =& \frac{1}{2}\left(\frac{\partial f}{\partial z_0}(z)h_0 - i\frac{\partial f}{\partial z_1}(z)ih_1\right) + \frac{1}{2}\left(\frac{\partial f}{\partial z_0}(z)h_0 + i\frac{\partial f}{\partial z_1}(z)(-ih_1)\right) + o(\|h\|) = \\ =& \frac{1}{2}\left(\frac{\partial f}{\partial z_0}(z)h_0 - i\frac{\partial f}{\partial z_1}(z)h_0 + \frac{\partial f}{\partial z_0}(z)ih_1 - i\frac{\partial f}{\partial z_1}(z)ih_1\right) + \\ &+ \frac{1}{2}\left(\frac{\partial f}{\partial z_0}(z)h_0 + i\frac{\partial f}{\partial z_1}(z)h_0 + \frac{\partial f}{\partial z_0}(z)(-ih_1) + i\frac{\partial f}{\partial z_1}(z)(-ih_1)\right) + o(\|h\|). \end{aligned}$$

Hence,

$$f(z+h)-f(z)=\frac{1}{2}\left(\frac{\partial f}{\partial z_0}(z)-i\frac{\partial f}{\partial z_1}(z)\right)h+\frac{1}{2}\left(\frac{\partial f}{\partial z_0}(z)+i\frac{\partial f}{\partial z_1}(z)\right)\overline{h}+o(\|h\|). \tag{1.7}$$

This observation justifies the following definition.

Definition 1.11 (Wirtinger derivatives)**.** *The differential operators*

$$\partial_z=\frac{1}{2}\left(\frac{\partial}{\partial z_0}-i\frac{\partial}{\partial z_1}\right)\qquad \textit{and}\qquad \partial_{\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial z_0}+i\frac{\partial}{\partial z_1}\right)$$

are called the Wirtinger derivatives *with respect to the complex and the complex conjugate variable, respectively.*

With this definition, the equation (1.7) turns into

$$f(z+h)-f(z)=\partial_z f(z)h+\partial_{\bar z}f(z)\overline{h}+o(\|h\|).$$

Since a function f is complex differentiable at z if and only if there exists a complex number $f'(z)$, the derivative of f at z, such that

$$f(z+h)-f(z)=f'(z)h+o(\|h\|),$$

we obtain the following lemma.

Lemma 1.12. *A real differentiable function $f: U\subset\mathbb{C}\to\mathbb{C}$ is holomorphic if and only if $\partial_{\bar z}f\equiv 0$ on U. Moreover, in this case, we have $f'(z)=\partial_z f(z)$ for any $z\in U$.*

The idea of Cauchy-Fueter-regularity is to generalize the Wirtinger derivatives and to consider the operator $\overline{\partial}=\frac{\partial}{\partial x_0}+\sum_{k=1}^{3}\frac{\partial}{\partial x_i}e_i$ instead.

Definition 1.13. *For a real differentiable function $f: U\subset\mathbb{H}\to\mathbb{H}$, we define*

$$\overline{\partial}f(x)=\frac{\partial f}{\partial x_0}(x)+\sum_{k=1}^{3}e_i\frac{\partial f}{\partial x_i}(x).$$

A function $f: U\subset\mathbb{H}\to\mathbb{H}$ is called is called Cauchy-Fueter-(left)-regular *on U, if $\overline{\partial}f\equiv 0$ on U.*

Cauchy-Fueter-regularity allows to generalize a huge part of the classical theory of holomorphic functions. In particular, Cauchy-Fueter-regular functions allow a series expansion based on Fueter-polynomials and they satisfy a version of Cauchy's integral formula. Thus, it is actually possible to define a functional calculus based on this notion of generalized holomorphicity, if one follows the ideas in [21]. Nevertheless, this functional calculus has several disadvantages. We just want to point out the most obvious one: the class of Cauchy-Fueter-regular functions is very specific and does not contain many of the most important functions in mathematics. In particular, it does not contain polynomials and power series of the form $\sum_{n=0}^{\infty}x^n a_n$ with $a_n\in\mathbb{H}$. Indeed, not even the identity function $x\mapsto x$ is Cauchy-Fueter-regular since

$$\overline{\partial}\,x=\frac{\partial}{\partial x_0}\left(x_0+\sum_{i=1}^{3}x_ie_i\right)+\sum_{j=1}^{3}e_j\frac{\partial}{\partial x_j}\left(x_0+\sum_{i=1}^{3}x_ie_i\right)=1+\sum_{i=1}^{3}e_i^2=-2\neq 0$$

because $e_i^2=-1$ for $i=1,\dots,3$. For this reason, the theory of Cauchy-Fueter-regular functions was not the appropriate starting point for the development of a quaternionic functional calculus that is useful in applications either.

The development of a useful functional calculus for quaternionic linear operators required a new notion of generalized holomorphicity, so-called *slice hyperholomorphicity.* Actually, special cases of slice hyperholomorphic functions were already considered in the 1930s by Fueter in [18] and [19], who used them to generate Cauchy-Fueter-regular functions, and later for instance by Cullen in [15]. Nevertheless, it took more than 70 years until their potential to define an associated functional calculus was understood; see [10].

Slice hyperholomorphic functions satisfy an integral formula of Cauchy-type with a modified kernel. This kernel naturally leads to a new notion of spectrum, the *S-spectrum*, which coincides with the set of right eigenvalues in the finite-dimensional case. The associated *S-functional calculus* for slice hyperholomorphic functions can be considered as the most natural generalization of the Riesz-Dunford-calculus to the quaternionic setting since it shares almost all its properties. We point out that, although only bounded operators are considered in this master thesis, the S-functional calculus can also be defined for unbounded operators [12].

Moreover, the class of slice hyperholomorphic functions contains polynomials and power series in the quaternion variable. In particular, it contains the exponential function $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Thus, the S-functional calculus allows to define the quaternionic evolution operator and to generalize the classical Hille-Yoshida-theory of strongly continuous semi-groups [4, 11]. Moreover, recently, a proof of the spectral theorem for unitary operators on quaternionic Hilbert spaces based on the S-spectrum has been provided in [3].

We point out that slice hyperholomorphicity can even be defined in a more general setting for functions defined on the real space $\mathbb{R}^{n+1}$ with values in the Clifford-algebra that is generated by n imaginary units. The theory of these functions is then analogue to the quaternionic case and it allows to define a functional calculus for n-tuples of not necessarily commuting operators. In the literature, slice hyperholomorphic functions defined on the space $\mathbb{R}^{n+1}$ are also called *slice monogenic*, whereas slice hyperholomorphic functions of a quaternion variable are referred to as *slice regular*. We follow this convention in this master thesis.

Chapter 2

Fundamentals of quaternions

In this chapter, we introduce the algebra of quaternions and discuss their main algebraic properties. Then we consider vector spaces and linear mappings in a quaternionic setting and extend certain results of classical linear algebra. The proof of these results follow the lines of the classical case, but since usually only vectors spaces over a field are considered in introductory linear algebra courses at university, we give the proofs for the sake of completeness. Finally, we introduce several well-known objects studied in classical functional analysis in the quaternionic setting. In particular, we proof the quaternionic version of the Hahn–Banach theorem.

Most of the results presented in this section can be found in [24]. They can also be found for a more general setting in [7] and [6]. For basic considerations on quaternionic two-sided vector spaces and on quaternionic Banach spaces see [23]. Note that therein, quaternionic two-sided vector spaces are referred to simply as "quaternionic vector spaces".

2.1 The algebra of quaternions

Definition 2.1. *The* algebra of quaternions $\mathbb{H}$ *is defined as the 4-dimensional real vector space with basis* 1, e_1, e_2 *and* e_3, *that is,*

$$\mathbb{H} = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 : x_i \in \mathbb{R}\},$$

endowed with the associative $\mathbb{R}$*-bilinear product with unity* 1 *that satisfies*

$$e_1^2 = e_2^2 = e_3^2 = -1, \tag{2.1}$$

$$e_1e_2 = e_3 = -e_2e_1, \quad e_2e_3 = e_1 = -e_3e_2 \quad \text{and} \quad e_3e_1 = e_2 = -e_1e_3. \tag{2.2}$$

Sometimes, when it is more convenient, we will write e_0 instead of 1. Moreover, note that (2.2) is equivalent to

$$e_1e_2e_3 = -1. \tag{2.3}$$

As in the complex case, we will identify the subalgebra $\operatorname{span}\{1\}$ with the field of real numbers $\mathbb{R}$. Moreover, we will identify $\mathbb{R}^3$ with $\operatorname{span}\{e_1, e_2, e_3\}$. The following definitions are formulated in analogy to the case of complex numbers.

Definition 2.2. *Let* $x = x_0 + \sum_{i=1}^3 x_ie_i \in \mathbb{H}$.

(i) We call $\operatorname{Re} x = x_0$ *the* real part *of the quaternion* x *and* $\operatorname{Im} x = \underline{x} = \sum_{i=1}^3 x_ie_i$ *the* imaginary *or* vector part *of the quaternion. We call a quaternion* x real, *if* $\operatorname{Im} x = 0$ *and we call it* (purely) imaginary *if* $\operatorname{Re} x = 0$.

(ii) We call $\overline{x} = x_0 - \sum_{i=1}^3 x_ie_i$ *the* conjugate *of* x.

(iii) The norm *or* absolute value *of* x *is defined as* $|x| = \sqrt{\sum_{i=0}^3 x_i^2}$.

Proposition 2.3. *(i) The quaternionic conjugation is an $\mathbb{R}$-linear involutive antiautomorphism, that is, for all $x, y \in \mathbb{H}$ and all $\lambda \in \mathbb{R}$, we have*

$$\overline{x+y} = \overline{x} + \overline{y}, \qquad \overline{\lambda x} = \lambda \overline{x}, \qquad \overline{\overline{x}} = x, \qquad \text{and} \qquad \overline{xy} = \overline{y}\,\overline{x}.$$

Moreover, $\overline{x} = x$ if and only if $x \in \mathbb{R}$ and $\overline{x} = -x$ if and only if x is purely imaginary.

(ii) Let $x, y \in \mathbb{H}$. Similar to the complex case, the following identities hold true:

- $\operatorname{Re} x = \frac{1}{2}(x + \overline{x})$ *and* $\operatorname{Im} x = \frac{1}{2}(x - \overline{x})$,
- $\overline{x}x = x\overline{x} = |x|^2$,
- $|xy| = |x|\,|y|$.

Proof. From the definition, it is clear that the quaternionic conjugation is $\mathbb{R}$-linear and an involution. Moreover, we have

$$\overline{1e_i} = \overline{e_i} = \overline{e_i}\,1 = \overline{e_i}\,\overline{1}, \qquad i = 1, \ldots, 3,$$

and

$$\overline{e_1 e_2} = \overline{e_3} = -e_3 = e_2 e_1 = (-e_2)(-e_1) = \overline{e_2}\,\overline{e_1}.$$

Similarly, we get $\overline{e_2 e_3} = \overline{e_3}\,\overline{e_2}$ and $\overline{e_1 e_3} = \overline{e_3}\,\overline{e_1}$. Thus, $\overline{xy} = \overline{y}\,\overline{x}$ holds if x and y are elements of the basis of $\mathbb{H}$. Hence, it holds for any $x, y \in \mathbb{H}$ because of the $\mathbb{R}$-bilinearity of the quaternionic product.

It is also clear that $\overline{x} = x$ if and only if $\operatorname{Im} x = 0$, that is, if and only if x is real, and that $\overline{x} = -x$ if and only if $\operatorname{Re} x = 0$, that is, if and only if x is purely imaginary. Therefore, *(i)* holds true.

The identities in *(ii)* are also easy to show. We have

$$x + \overline{x} = \operatorname{Re} x + \operatorname{Im} x + \operatorname{Re} x - \operatorname{Im} x = 2\operatorname{Re} x$$

and

$$x - \overline{x} = \operatorname{Re} x + \operatorname{Im} x - (\operatorname{Re} x - \operatorname{Im} x) = 2\operatorname{Im} x.$$

Since $e_i e_j = -e_j e_i$ for $i \neq j \in \{1, 2, 3\}$, we get

$$\overline{x}x = \left(x_0 - \sum_{i=1}^{3} x_i e_i\right)\left(x_0 + \sum_{j=1}^{3} x_j e_j\right) = x_0^2 - \sum_{i=1}^{3} x_i x_0 e_i + \sum_{j=1}^{3} x_0 x_j e_j - \sum_{i,j=1}^{3} x_i x_j e_i e_j =$$

$$= x_0^2 - \sum_{i=1}^{3} \sum_{j>i} (x_i x_j - x_j x_i) e_i e_j - \sum_{i=1}^{3} x_i^2 e_i^2 = x_0^2 + \sum_{i=1}^{3} x_i^2 = |x|^2.$$

Similarly, we obtain $x\overline{x} = |x|^2$. Finally,

$$|xy|^2 = xy\overline{xy} = xy\overline{y}\,\overline{x} = |y|^2 x\,\overline{x} = |y|^2 |x|^2.$$

Hence, $|xy| = |x|\,|y|$.

□

Corollary 2.4. *Every quaternion $x \in \mathbb{H} \setminus \{0\}$ has an multiplicative inverse, namely*

$$x^{-1} = \frac{1}{|x|^2}\,\overline{x}.$$

In particular, the quaternions form a skew field.

Although the quaternionic multiplication is not commutative, if x or y is real, then $xy = yx$. As the next Lemma shows, reals are the only quaternions that commute with any other quaternion. We will specify this result later.

Lemma 2.5. *A quaternion commutes with every other quaternion if and only if it is real. That is, the center of $\mathbb{H}$ is the real line $\mathbb{R}$.*

Proof. Since 1 is the multiplicative neutral element of $\mathbb{H}$, it commutes with every quaternion. Moreover, as the multiplication is $\mathbb{R}$-bilinear, any $x \in \mathbb{R} = \operatorname{span}\{1\}$ commutes with every other quaternion, too.

Now, let $x = x_0 + \sum_{i=1}^3 x_i e_i \in \mathbb{H}$ be such that $xy = yx$ for all $y \in \mathbb{H}$. In particular, $xe_1 = e_1 x$. But since

$$e_1 x = x_0 e_1 + x_1 e_1^2 + x_2 e_1 e_2 + x_3 e_1 e_3 = x_0 e_1 - x_1 + x_2 e_3 - x_3 e_2$$

and

$$x e_1 = x_0 e_1 + x_1 e_1^2 + x_2 e_2 e_1 + x_3 e_3 e_1 = x_0 e_1 - x_1 - x_2 e_3 + x_3 e_2,$$

this implies $x_2 = 0$ and $x_3 = 0$. Similarly $xe_2 = e_2 x$ together with

$$x e_2 = x_0 e_2 + x_1 e_1 e_2 = x_0 e_2 + x_1 e_3 \qquad \text{and} \qquad e_2 x = x_0 e_2 + x_1 e_2 e_1 = x_0 e_2 - x_1 e_3$$

yields $x_1 = 0$. Thus, x is real.

□

Lemma 2.6. *Let $x, y \in \mathbb{H} \setminus \{0\}$. Then x and y satisfy $xy = -yx$ if and only if* $\operatorname{Re} x = \operatorname{Re} y = 0$ *and x and y are orthogonal as vectors in $\mathbb{R}^3$.*

Proof. Suppose that $x = x_0 + \sum_{i=1}^3 x_i e_i$ and $y = y_0 + \sum_{j=1}^3 y_j e_j$ belong to $\mathbb{H} \setminus \{0\}$ and satisfy $xy = -yx$. Then

$$xy = x_0 y_0 + \sum_{i=1}^3 x_i y_0 e_i + \sum_{j=1}^3 x_0 y_j e_j + \sum_{i,j=1}^3 x_i y_j e_i e_j \tag{2.4}$$

and

$$yx = y_0 x_0 + \sum_{j=1}^3 y_j x_0 e_j + \sum_{i=1}^3 y_0 x_i e_i + \sum_{i,j=1}^3 y_j x_i e_j e_i \tag{2.5}$$

give

$$0 = xy + yx = 2x_0 y_0 + 2\sum_{i=1}^3 x_i y_0 e_i + 2\sum_{j=1}^3 x_0 y_j e_j + \sum_{\substack{i,j=1 \\ i \neq j}}^3 x_i y_j (e_i e_j + e_j e_i) + 2\sum_{i=1}^3 x_i y_i e_i^2.$$

From $e_i^2 = -1$ for $i \in \{1,2,3\}$ and $e_i e_j = -e_j e_i$ for $1 \leq i, j \leq 3$ with $i \neq j$, we conclude

$$0 = 2\left(x_0 y_0 - \sum_{i=1}^3 x_i y_i\right) + 2\sum_{i=1}^3 (x_i y_0 + x_0 y_i) e_i, \tag{2.6}$$

which implies $x_0 y_0 - \sum_{i=1}^3 x_i y_i = 0$ and $x_i y_0 + x_0 y_i = 0$ for $i = 1, 2, 3$. If $x_0 \neq 0$ and $y_0 \neq 0$, then we have $y_i = -y_0 \frac{x_i}{x_0}$ for $i = 1, 2, 3$. Hence,

$$0 = x_0 y_0 - \sum_{i=1}^3 x_i y_i = x_0 y_0 + \sum_{i=1}^3 y_0 \frac{x_i^2}{x_0} = x_0 y_0 \left(1 + \sum_{i=1}^3 \frac{x_i^2}{x_0^2}\right),$$

which is a contradiction because $1 + \sum_{i=1}^3 \frac{x_i^2}{x_0^2} > 0$. Therefore, x_0 and y_0 cannot both be unequal to zero.

If on the other hand $y_0 = 0$, then (2.6) simplifies to

$$0 = -2\sum_{i=1}^3 x_i y_i + \sum_{i=1}^3 2x_0 y_i e_i. \tag{2.7}$$

Since $y \neq 0$, there exists an index $1 \leq i_0 \leq 3$ such that $y_{i_0} \neq 0$. But (2.7) implies $2x_0 y_{i_0} = 0$. Hence, $x_0 = 0$. Similarly, $x_0 = 0$ implies $y_0 = 0$. Therefore, we get $x_0 = y_0 = 0$ and (2.6) turns into $0 = -2\sum_{i=1}^3 x_i y_i$. Hence, x and y are orthogonal vectors in $\mathbb{R}^3$.

If on the other hand x and y are orthogonal vectors in $\mathbb{R}^3$, then $\sum_{i=1}^{3} x_i y_i = 0 = -\sum_{i=1}^{3} x_i y_i$. Since $e_i^2 = -1$ for $i \in \{1,2,3\}$ and $e_i e_j = -e_j e_i$ for $1 \leq i, j \leq 3$ with $i \neq j$, we obtain

$$
\begin{aligned}
xy = \sum_{i,j=1}^{3} x_i y_j e_i e_j = -\sum_{i=1}^{3} x_i y_i + \sum_{\substack{i,j=1\\ i\neq j}}^{3} x_i y_j e_i e_j = \\
= \sum_{i=1}^{3} x_i y_i - \sum_{\substack{i,j=1\\ i\neq j}}^{3} y_j x_i e_j e_i = -\sum_{i,j=1}^{3} y_j x_i e_j e_i = -yx.
\end{aligned}
$$

□

Finally, we show that the quaternions do not only form a real vector space. They can also be considered as a 2-dimensional complex vector space. Indeed, this fact is fundamental for the theory that we establish.

Definition 2.7. *A purely imaginary quaternion with absolute value* 1 *is called an* imaginary unit*. We denote the set of all imaginary units by* $\mathbb{S}$*, that is,*

$$
\mathbb{S} = \left\{ \sum_{i=1}^{3} x_i e_i \in \mathbb{H} : \sum_{i=1}^{3} x_i^2 = 1 \right\}.
$$

The name imaginary unit is justified by the fact that, for any $I \in \mathbb{S}$, we have

$$
I^2 = -\overline{I}\, I = -|I|^2 = -1.
$$

Corollary 2.8. *For* $I \in \mathbb{S}$*, the plane* $\mathbb{C}_I = \{x_0 + x_1 I : x_0, x_1 \in \mathbb{R}\}$ *is isomorphic to the field of complex numbers* $\mathbb{C}$*.*

Lemma 2.9. *Let* $I, J \in \mathbb{S}$ *with* $I \perp J$ *and set* $K = IJ$*. Then* K *is an imaginary unit and* $\{1, I, J, K\}$ *is a basis of* $\mathbb{H}$ *that satisfies the defining relations of the quaternionic product, that is*

$$
I^2 = J^2 = K^2 = IJK = -1.
$$

Proof. As I and J are orthogonal, they satisfy $IJ = -JI$ by Lemma 2.6. Hence, $\overline{K} = \overline{J}\,\overline{I} = JI = -IJ = -K$. Therefore, K is purely imaginary by Proposition 2.3. Because of $|K| = |I|\,|J| = 1$, it is even an imaginary unit.

Since $IK = IIJ = -IJI = -KI$ and $JK = JIJ = -IJJ = -KJ$, the quaternions I, J and K form an orthogonal basis of $\mathbb{R}^3$ by Lemma 2.6. Hence, $\{1, I, J, K\}$ is a basis of $\mathbb{H}$.

Finally, as I, J and K are imaginary units, we obtain $I^2 = J^2 = K^2 = -1$ and $IJK = IJIJ = -IIJJ = -1$.

□

Note that the previous lemma states that the basis 1, e_1, e_2 and e_3 is not canonical. In fact, each triple I, J and K forms, together with 1, a generating basis of $\mathbb{H}$.

Now let $x \in \mathbb{H}$ and let us write x in terms of the basis defined in the previous lemma. Then we have

$$
x = x_0 + x_1 I + x_2 J + x_3 K = x_0 + x_1 I + (x_2 + x_3 I)J = z_1 + z_2 J, \tag{2.8}
$$

where $z_1 = x_0 + x_1 I$ and $z_2 = x_2 + x_3 I$ are in $\mathbb{C}_I$. Moreover, since $-K = JI$, we also have

$$
x = x_0 + x_1 I + x_2 J - x_3(-K) = x_0 + x_1 I + J(x_2 - x_3 I) = \widetilde{z}_1 + J\widetilde{z}_2, \tag{2.9}
$$

where $\widetilde{z}_1 = x_0 + x_1 I$ and $\widetilde{z}_2 = x_2 - x_3 I$ are in $\mathbb{C}_I$.

Corollary 2.10. *Let* $I \in \mathbb{S}$*. The operations*

$$
\langle \cdot, \cdot \rangle_L : \begin{cases} \mathbb{C}_I \times \mathbb{H} & \to \mathbb{H} \\ (a, x) & \mapsto ax \end{cases} \qquad \text{and} \qquad \langle \cdot, \cdot \rangle_R : \begin{cases} \mathbb{C}_I \times \mathbb{H} & \to \mathbb{H} \\ (a, x) & \mapsto xa \end{cases}
$$

define complex scalar multiplications on $\mathbb{H}$, i.e., $\mathbb{H}$ is a complex vector space over $\mathbb{C}_I$ if it is endowed either with $\langle\cdot,\cdot\rangle_L$ or with $\langle\cdot,\cdot\rangle_R$. In these cases, we call $\mathbb{H}$ a left and right vector space over $\mathbb{C}_I$, respectively.

Moreover, $\mathbb{H}$ is isomorphic to the two-dimensional complex vector space $\mathbb{C}_I^2$. For any $J \in \mathbb{S}$ with $I \perp J$, the mappings

$$\iota_L : \begin{cases} \mathbb{C}_I \times \mathbb{C}_I & \to \mathbb{H} \\ (z_1, z_2) & \mapsto z_1 + z_2 J \end{cases} \qquad \text{and} \qquad \iota_R : \begin{cases} \mathbb{C}_I \times \mathbb{C}_I & \to \mathbb{H} \\ (z_1, z_2) & \mapsto z_1 + J z_2 \end{cases}$$

are isomorphism from $\mathbb{C}_I^2$ to $(\mathbb{H}, \langle\cdot,\cdot\rangle_L)$ and $(\mathbb{H}, \langle\cdot,\cdot\rangle_R)$, respectively.

Proof. It is straight forward to check that $\langle\cdot,\cdot\rangle_L$ and $\langle\cdot,\cdot\rangle_R$ are actually vector space scalar multiplications. Moreover, we saw in (2.8) and (2.9) that the mappings ι_L and ι_R are bijective.

Let $\langle\cdot,\cdot\rangle_{\mathbb{C}_I}$ denote the usual scalar multiplication on $\mathbb{C}_I^2$, that is, $\langle a, \boldsymbol{z}\rangle_{\mathbb{C}_I} = (az_1, az_2)^T$ for any $\boldsymbol{z} = (z_1, z_2)^T \in \mathbb{C}_I^2$ and any $a \in \mathbb{C}_I$. For two vectors $\boldsymbol{z} = (z_1, z_2)$ and $\tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2)$ in $\mathbb{C}_I^2$ and $a \in \mathbb{C}_I$, we have

$$\iota_L(\langle a, \boldsymbol{z}\rangle_{\mathbb{C}_I} + \tilde{\boldsymbol{z}}) = az_1 + \tilde{z}_1 + (az_2 + \tilde{z}_2)J = a(z_1 + z_2 J) + \tilde{z}_1 + \tilde{z}_2 J = \langle a, \iota_L(\boldsymbol{z}), \rangle_L + \iota_L(\tilde{\boldsymbol{z}}).$$

Therefore, ι_L is an isomorphism from $\mathbb{C}_I^2$ to $(\mathbb{H}, \langle\cdot,\cdot\rangle_L)$. Similarly,

$$\iota_R(\langle a, \boldsymbol{z}\rangle_{\mathbb{C}_I} + \tilde{\boldsymbol{z}}) = az_1 + \tilde{z}_1 + J(az_2 + \tilde{z}_2) = (z_1 + Jz_2)a + \tilde{z}_1 + J\tilde{z}_2 = \langle a, \iota_R(\boldsymbol{z})\rangle_R + \iota_R(\tilde{\boldsymbol{z}}).$$

Therefore, ι_R is an isomorphism from $\mathbb{C}_I^2$ to $(\mathbb{H}, \langle\cdot,\cdot\rangle_R)$.

□

We will omit the notation $\langle\cdot,\cdot\rangle_L$ and $\langle\cdot,\cdot\rangle_R$ and simply write ax and xa instead of $\langle a, x\rangle_L$ and $\langle a, x\rangle_R$ whenever we consider $\mathbb{H}$ as a vector space over $\mathbb{C}_I$. However, we have to keep in mind that the vector space structures do not coincide. The following corollary clarifies their relation.

Corollary 2.11. *Let $I, J \in \mathbb{S}$ with $I \perp J$ and let $z \in \mathbb{C}_I$. Then $zJ = J\overline{z}$. Moreover, the identity*

$$\iota_L((z_1, z_2)^T) = \iota_R((z_1, \overline{z_2})^T)$$

holds true for any $z_1, z_2 \in \mathbb{C}_I$.

Proof. Let $z = x_0 + Ix_1 \in \mathbb{C}_I$. Then

$$zJ = x_0 J + IJx_1 = Jx_0 - JIx_1 = J(x_0 - Ix_1) = J\overline{z}$$

because $IJ = -JI$ by Corollary 2.6. Hence,

$$\iota_L((z_1, z_2)^T) = z_1 + z_2 J = z_1 + J\overline{z_2} = \iota_R((z_1, \overline{z_2})^T).$$

□

We can now specify Lemma 2.5.

Corollary 2.12. *Let $x, y \in \mathbb{H}$. Then x an y commute if and only if they belong to the same complex plane $\mathbb{C}_I$.*

Proof. If x or y is real, then x and y commute by Lemma 2.5 and obviously they lie in the same complex plane. Thus, we assume $x, y \notin \mathbb{R}$. Let $I \in \mathbb{S}$ such that $x = x_0 + Ix_1$ and let $J \in \mathbb{S}$ with $I \perp J$. Then, there exist $y_1, y_2 \in \mathbb{C}_I$ such that $y = y_1 + y_2 J$ because of Corollary 2.10. Hence,

$$xy = x(y_1 + y_2 J) = y_1 x + y_2 xJ = y_1 x + y_2 J\,\overline{x} \qquad \text{and} \qquad yx = y_1 x + y_2 Jx,$$

where we used that $xJ = J\overline{x}$; see Corollary 2.11. Thus, x and y commute if and only if $y_2 = 0$, that is, if and only if $y \in \mathbb{C}_I$.

□

2.2 Quaternionic vector spaces

We establish now some aspects of the theory of quaternionic vectors spaces. The fact that the quaternionic multiplication is not commutative requires some modifications of the classical theory. In particular, we have to distinguish between left and right vector spaces over $\mathbb{H}$. Nevertheless, much of the classical theory of vector spaces extends to the quaternionic setting.

Definition 2.13. *A* quaternionic right vector space *is an additive group* $(V,+)$ *together with a right scalar multiplication* $V \times \mathbb{H} \to V, (\boldsymbol{v}, a) \mapsto \boldsymbol{v}a$ *such that, for any* $\boldsymbol{u}, \boldsymbol{v} \in V$ *and for any* $a, b \in \mathbb{H}$*, the identities*

$$(\boldsymbol{u}+\boldsymbol{v})a = \boldsymbol{u}a + \boldsymbol{v}a \qquad \boldsymbol{u}1 = \boldsymbol{u}$$
$$\boldsymbol{u}(a+b) = \boldsymbol{u}a + \boldsymbol{u}b \qquad (\boldsymbol{u}a)b = \boldsymbol{u}(ab)$$

hold true. A quaternionic left vector space *is an additive group* $(V,+)$ *together with a left scalar multiplication* $\mathbb{H} \times V \to V, (a, \boldsymbol{v}) \mapsto a\boldsymbol{v}$ *such that for any* $\boldsymbol{u}, \boldsymbol{v} \in V$ *and any* $a, b \in \mathbb{H}$ *the identities*

$$a(\boldsymbol{u}+\boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v} \qquad 1\boldsymbol{u} = \boldsymbol{u}$$
$$(a+b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u} \qquad a(b\boldsymbol{u}) = (ab)\boldsymbol{u}$$

hold. A quaternionic two-sided vector space *is an additive group* $(V,+)$ *together with a left and a right scalar multiplication such that* $(V,+)$ *together with the left scalar multiplication is a quaternionic left vector space and* $(V,+)$ *together with the right scalar multiplication is a quaternionic right vector space and such that* $a\boldsymbol{v} = \boldsymbol{v}a$ *for all* $a \in \mathbb{R}$.

Remark 2.14. Note that any quaternionic right, left or two-sided vector space is a real vector space if we restrict the scalar multiplication to $\mathbb{R}$. Moreover, any quaternionic right or left vector space is also a complex vector space if we restrict the scalar multiplication to the complex plane $\mathbb{C}_I$ for some $I \in \mathbb{S}$. In general, the restrictions to different complex planes $\mathbb{C}_I$ do not lead to the same complex vector spaces. Moreover, note that a quaternionic two-sided vector space V is not a complex vector space if we restrict the left and the right scalar multiplication to a complex plane $\mathbb{C}_I$ because in general $a\boldsymbol{v} \neq \boldsymbol{v}a$ for $\boldsymbol{v} \in V$ and $a \in \mathbb{C}_I$. Therefore, the complex scalar multiplication is not well defined. We have to consider either the left or the right vector space structure to obtain a complex vector space.

Starting from a real vector space $V_{\mathbb{R}}$, it is easy to construct a quaternionic two-sided vector space. Let us consider the space $V_{\mathbb{R}}^4$ formally written as

$$V_{\mathbb{R}}^4 = \left\{ \boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i \otimes e_i : \boldsymbol{v}_i \in V_{\mathbb{R}} \right\}.$$

This space is an additive group if we define the addition componentwise by $(\boldsymbol{u}+\boldsymbol{v}) = \sum_{i=0}^{3} (\boldsymbol{u}_i + \boldsymbol{v}_i) \otimes e_i$. Moreover, we can define a right and a left scalar multiplication on $V_{\mathbb{R}}^4$ by

$$\boldsymbol{v}a = \sum_{i,j=0}^{3} (a_j \boldsymbol{v}_i) \otimes (e_i e_j) \qquad \text{and} \qquad a\boldsymbol{v} = \sum_{i,j=0}^{3} (a_j \boldsymbol{v}_i) \otimes (e_j e_i)$$

for $a = \sum_{j=0}^{3} a_j e_j$. These expressions can be written more compactly as

$$\boldsymbol{v}a = \sum_{i=0}^{3} \boldsymbol{v}_i \otimes (e_i a) \qquad \text{and} \qquad a\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v} \otimes (a e_i), \tag{2.10}$$

where the terms in the brackets have to be understood as multiplications within $\mathbb{H}$ and where $\boldsymbol{v} \otimes (e_i \alpha)$ is identified with $(\boldsymbol{v}\alpha) \otimes e_i$ for $\alpha \in \mathbb{R}$.

It is easy to check that $V_{\mathbb{R}}^4$ is actually a quaternionic left resp. right vector space with these scalar multiplications. Moreover, for $a \in \mathbb{R}$, we have $a\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v} \otimes (a e_i) = \sum_{i=0}^{3} \boldsymbol{v}_i \otimes (e_i a) = \boldsymbol{v}a$. Altogether, we even obtain a quaternionic two-sided vector space.

Definition 2.15. *Let $V_{\mathbb{R}}$ be a real vector space. We denote the quaternionic two-sided vector space $V_{\mathbb{R}}^4$ endowed with the left and right scalar multiplications* (2.10) *by $V_{\mathbb{R}} \otimes \mathbb{H}$.*

Example 2.16. Let us consider the set of n-tuples of quaternions $\mathbb{H}^n$. It is easy to show that $\mathbb{H}^n$ is a quaternionic two-sided vector space if we define the addition, the left and the right scalar multiplication componentwise, that is,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \qquad a \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} ax_1 \\ \vdots \\ ax_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} a = \begin{pmatrix} x_1 a \\ \vdots \\ x_n a \end{pmatrix}$$

for $(x_1, \dots, x_n)^T, (y_1, \dots, y_n)^T \in \mathbb{H}^n$ and $a \in \mathbb{H}$.

As a two-sided vector space, $\mathbb{H}^n$ is isomorphic to $\mathbb{R}^n \otimes \mathbb{H}$. If $\boldsymbol{v} = (v_1, \dots, v_n)^T \in \mathbb{H}^n$ with $v_i = \sum_{j=0}^3 v_{i,j} e_j \in \mathbb{H}$ for $i = 1, \dots, n$, then

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^3 v_{1,j} e_j \\ \vdots \\ \sum_{j=0}^3 v_{n,j} e_j \end{pmatrix} = \sum_{j=0}^3 \begin{pmatrix} v_{1,j} \\ \vdots \\ v_{n,j} \end{pmatrix} e_j = \sum_{j=0}^3 \boldsymbol{v}_j e_j,$$

where $\boldsymbol{v}_j = (v_{1,j}, \dots, v_{n,j})^T \in \mathbb{R}^n$. For $a \in \mathbb{H}$, we have

$$\boldsymbol{v} a = \begin{pmatrix} v_1 a \\ \vdots \\ v_n a \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^3 v_{1,j} e_j a \\ \vdots \\ \sum_{j=0}^3 v_{n,j} e_j a \end{pmatrix} = \sum_{j=0}^3 \begin{pmatrix} v_{1,j} \\ \vdots \\ v_{n,j} \end{pmatrix} e_j a = \sum_{j=0}^3 \boldsymbol{v}_j (e_j a)$$

and

$$a\boldsymbol{v} = \begin{pmatrix} av_1 \\ \vdots \\ av_n \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^3 v_{1,j} a e_j \\ \vdots \\ \sum_{j=0}^3 v_{n,j} a e_j \end{pmatrix} = \sum_{j=0}^3 \begin{pmatrix} v_{1,j} \\ \vdots \\ v_{n,j} \end{pmatrix} a e_j = \sum_{j=0}^3 \boldsymbol{v}_j (a e_j),$$

because the real components $v_{i,j}$ commute with a. The mapping

$$\psi : \begin{cases} \mathbb{H}^n & \to \mathbb{R}^n \otimes \mathbb{H} \\ \sum_{j=0}^3 \boldsymbol{v}_i e_i & \mapsto \sum_{j=0}^3 \boldsymbol{v}_i \otimes e_i \end{cases}$$

is a two-sided vector space isomorphism. It is obviously bijective and satisfies $\psi(\boldsymbol{u} + \boldsymbol{v}) = \psi(\boldsymbol{u}) + \psi(\boldsymbol{v})$ and $\psi(\alpha \boldsymbol{v}) = \alpha \psi(\boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}^n$ and $\alpha \in \mathbb{R}$. For $a = \sum_{k=0}^3 a_k e_k \in \mathbb{H}$, we have

$$\begin{aligned} \psi(\boldsymbol{v} a) = \psi\left(\sum_{j=0}^3 \boldsymbol{v}_j (e_j a) \right) &= \psi\left(\sum_{j=0,k=0}^3 a_k \boldsymbol{v}_j e_j e_k \right) = \sum_{j=0,k=0}^3 a_k \psi\left(\boldsymbol{v}_j e_j e_k \right) = \\ &= \sum_{j=0,k=0}^3 a_k \boldsymbol{v}_j \otimes (e_j e_k) = \left(\sum_{j=0}^3 \boldsymbol{v}_j \otimes e_j \right) a = \psi(\boldsymbol{v}) a \end{aligned}$$

and

$$\begin{aligned} \psi(a\boldsymbol{v}) = \psi\left(\sum_{j=0}^3 \boldsymbol{v}_j (a e_j) \right) &= \psi\left(\sum_{j=0,k=0}^3 a_k \boldsymbol{v}_j e_k e_j \right) = \sum_{j=0,k=0}^3 a_k \psi\left(\boldsymbol{v}_j e_k e_j \right) = \\ &= \sum_{j=0,k=0}^3 a_k \boldsymbol{v}_j \otimes (e_k e_j) = a \left(\sum_{j=0}^3 \boldsymbol{v}_j \otimes e_j \right) = a\psi(\boldsymbol{v}). \end{aligned}$$

The following result, Lemma 1.5 in [23], implies that any quaternionic two-sided vector space is isomorphic to a quaternionic two-sided vector space of the form $V_{\mathbb{R}} \otimes \mathbb{H}$ for some real vector space $V_{\mathbb{R}}$.

Lemma 2.17. *Let V be a quaternionic two-sided vector space and let $V_{\mathbb{R}} = \{\boldsymbol{v} \in V : a\boldsymbol{v} = \boldsymbol{v}a \ \forall a \in \mathbb{H}\}$. Then $V_{\mathbb{R}}$ is a real vector space, called the* real part of V, *and*

$$\mathsf{Re}(\boldsymbol{v}) = \frac{1}{4}\sum_{i=0}^{3} \overline{e_i}\boldsymbol{v}e_i$$

defines an $\mathbb{R}$-linear mapping from V onto $V_{\mathbb{R}}$ that satisfies $\mathsf{Re} \circ \mathsf{Re} = \mathsf{Re}$. Moreover, any $\boldsymbol{v} \in V$ satisfies the polarization identity

$$\boldsymbol{v} = \sum_{i=0}^{3} \mathsf{Re}(\overline{e_i}\boldsymbol{v})e_i = \sum_{i=0}^{3} e_i\mathsf{Re}(\overline{e_i}\boldsymbol{v}). \tag{2.11}$$

Proof. It is clear that $V_{\mathbb{R}}$ is a real vector space and that Re is $\mathbb{R}$-linear. Obviously, for $\boldsymbol{v} \in V$, we have $e_0\mathsf{Re}(\boldsymbol{v}) = \mathsf{Re}(\boldsymbol{v}) = \mathsf{Re}(\boldsymbol{v})e_0$. Moreover,

$$e_1\mathsf{Re}(\boldsymbol{v}) = e_1\frac{1}{4}(\boldsymbol{v} - e_1\boldsymbol{v}e_1 - e_2\boldsymbol{v}e_2 - e_3\boldsymbol{v}e_3) = \frac{1}{4}(e_1\boldsymbol{v} + \boldsymbol{v}e_1 - e_3\boldsymbol{v}e_2 + e_2\boldsymbol{v}e_3)$$

and

$$\mathsf{Re}(\boldsymbol{v})e_1 = \frac{1}{4}(\boldsymbol{v} - e_1\boldsymbol{v}e_1 - e_2\boldsymbol{v}e_2 - e_3\boldsymbol{v}e_3)e_1 = \frac{1}{4}(\boldsymbol{v}e_1 + e_1\boldsymbol{v} + e_2\boldsymbol{v}e_3 - e_3\boldsymbol{v}e_2).$$

Hence, $e_1\mathsf{Re}(\boldsymbol{v}) = \mathsf{Re}(\boldsymbol{v})e_1$. Analogous calculations show that $e_2\mathsf{Re}(\boldsymbol{v}) = \mathsf{Re}(\boldsymbol{v})e_2$ and $e_3\mathsf{Re}(\boldsymbol{v}) = \mathsf{Re}(\boldsymbol{v})e_3$. Thus, for $a = \sum_{i=0}^{3} a_ie_i \in \mathbb{H}$, we have

$$a\mathsf{Re}(\boldsymbol{v}) = \sum_{i=0}^{3} a_ie_i\mathsf{Re}(\boldsymbol{v}) = \sum_{i=0}^{3} \mathsf{Re}(\boldsymbol{v})a_ie_i = \mathsf{Re}(\boldsymbol{v})a,$$

that is, $\mathsf{Re}(\boldsymbol{v}) \in V_{\mathbb{R}}$. Moreover, if $\boldsymbol{v} \in V_{\mathbb{R}}$, then

$$\mathsf{Re}(\boldsymbol{v}) = \frac{1}{4}\sum_{i=0}^{3} \overline{e_i}\boldsymbol{v}e_i = \frac{1}{4}\sum_{i=0}^{3} \boldsymbol{v}\overline{e_i}e_i = \frac{1}{4}\sum_{i=0}^{3} \boldsymbol{v} = \boldsymbol{v}.$$

Therefore, $\mathsf{Re} : V \to V_{\mathbb{R}}$ is onto and $\mathsf{Re} \circ \mathsf{Re} = \mathsf{Re}$.

Finally, we have

$$\sum_{i=0}^{3} \mathsf{Re}(\overline{e_i}\boldsymbol{v})e_i = \sum_{i=0}^{3} \frac{1}{4}\sum_{j=0}^{3} \overline{e_j}\,\overline{e_i}\boldsymbol{v}e_je_i = \frac{1}{4}\sum_{i=0}^{3} \overline{e_i}^2\boldsymbol{v}e_i^2 + \frac{1}{4}\sum_{\substack{i,j=0\\ i\neq j}}^{3} \overline{e_j}\,\overline{e_i}\boldsymbol{v}e_je_i.$$

For the second sum $S_2 = \sum_{i\neq j} \overline{e_j}\,\overline{e_i}\boldsymbol{v}e_je_i$, we obtain

$$\begin{aligned} S_2 = &- e_1\boldsymbol{v}e_1 - e_2\boldsymbol{v}e_2 - e_3\boldsymbol{v}e_3 - e_1\boldsymbol{v}e_1 + e_1e_2\boldsymbol{v}e_1e_2 + e_1e_3\boldsymbol{v}e_1e_3 - \\ &- e_2\boldsymbol{v}e_2 + e_2e_1\boldsymbol{v}e_2e_1 + e_2e_3\boldsymbol{v}e_2e_3 - e_3\boldsymbol{v}e_3 + e_3e_1\boldsymbol{v}e_3e_1 + e_3e_2\boldsymbol{v}e_3e_2 = \\ = &- e_1\boldsymbol{v}e_1 - e_2\boldsymbol{v}e_2 - e_3\boldsymbol{v}e_3 - e_1\boldsymbol{v}e_1 + e_3\boldsymbol{v}e_3 + e_2\boldsymbol{v}e_2 - \\ &- e_2\boldsymbol{v}e_2 + e_3\boldsymbol{v}e_3 + e_1\boldsymbol{v}e_1 - e_3\boldsymbol{v}e_3 + e_2\boldsymbol{v}e_2 + e_1\boldsymbol{v}e_1 = \mathbf{0}. \end{aligned}$$

Thus,

$$\sum_{i=0}^{3} \mathsf{Re}(\overline{e_i}\boldsymbol{v})e_i = \frac{1}{4}\sum_{i=0}^{3} \overline{e_i}^2\boldsymbol{v}e_i^2 = \frac{1}{4}\sum_{i=0}^{3} \boldsymbol{v} = \boldsymbol{v}.$$

Since $\mathsf{Re}(\overline{e_i}\boldsymbol{v}) \in V_{\mathbb{R}}$, it commutes with any scalar. Hence, we also obtain

$$\boldsymbol{v} = \sum_{i=0}^{3} \mathsf{Re}(\overline{e_i}\boldsymbol{v})e_i = \sum_{i=0}^{3} e_i\mathsf{Re}(\overline{e_i}\boldsymbol{v})$$

□

Corollary 2.18. *Let V be a quaternionic two-sided vector space and let $V_{\mathbb{R}}$ be the real part of V. Then V is isomorphic to $V_{\mathbb{R}} \otimes \mathbb{H}$.*

Proof. Let $\boldsymbol{v} \in V$. The polarization identity (2.11) implies $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i$, with $\boldsymbol{v}_i = \mathsf{Re}(\overline{e_i}\boldsymbol{v}) \in V_{\mathbb{R}}$. The mapping

$$\psi : \begin{cases} V & \to V_{\mathbb{R}} \otimes \mathbb{H} \\ \boldsymbol{v} & \mapsto \sum_{j=0}^{3} \boldsymbol{v}_i \otimes e_i \end{cases}$$

is obviously $\mathbb{R}$-linear. For $a = \sum_{k=0}^{3} a_k e_k \in \mathbb{H}$, we have

$$\psi(\boldsymbol{v}a) = \psi\left(\sum_{j=0}^{3} \boldsymbol{v}_j(e_j a)\right) = \psi\left(\sum_{j=0,k=0}^{3} a_k \boldsymbol{v}_j e_j e_k\right) = \sum_{j=0,k=0}^{3} a_k \psi\left(\boldsymbol{v}_j e_j e_k\right) =$$

$$= \sum_{j=0,k=0}^{3} a_k \boldsymbol{v}_j \otimes (e_j e_k) = \left(\sum_{j=0}^{3} \boldsymbol{v}_j \otimes e_j\right) a = \psi(\boldsymbol{v})a$$

and

$$\psi(a\boldsymbol{v}) = \psi\left(\sum_{j=0}^{3} \boldsymbol{v}_j(a e_j)\right) = \psi\left(\sum_{j=0,k=0}^{3} a_k \boldsymbol{v}_j e_k e_j\right) = \sum_{j=0,k=0}^{3} a_k \psi\left(\boldsymbol{v}_j e_k e_j\right) =$$

$$= \sum_{j=0,k=0}^{3} a_k \boldsymbol{v}_j \otimes (e_k e_j) = a\left(\sum_{j=0}^{3} \boldsymbol{v}_j \otimes e_j\right) = a\psi(\boldsymbol{v}).$$

Hence, ψ is a two-sided vector space homomorphism. It is surjective because for any vector $\boldsymbol{v}^{\otimes} = \sum_{i=0}^{3} \boldsymbol{v}_i \otimes e_i$ in $V_{\mathbb{R}} \otimes \mathbb{H}$, the vector $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i$ belongs to V and satisfies $\boldsymbol{v}^{\otimes} = \psi(\boldsymbol{v})$. It is even bijective, and therefore an isomorphism, since $\psi(\boldsymbol{v}) = \boldsymbol{0}$ implies $\boldsymbol{v}_i = \boldsymbol{0}, i = 0, \ldots, 3$, and in turn, $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i = \boldsymbol{0}$.

□

In the following, we identify any tow-sided vector space V with $V_{\mathbb{R}} \otimes \mathbb{H}$. We also omit the symbol $\otimes$ and write $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_j e_j$ instead of $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i \otimes e_i$.

We show now that certain well known results from linear algebra also hold true for quaternionic vector spaces. We will mainly restrict our discussion to the case of quaternionic right vector spaces, but it will be clear that all results are also true for quaternionic left vector spaces if we replace the word "right" by the word "left" and vice versa.

The following definitions are similar to those for vector spaces over a field known from linear algebra.

Definition 2.19. *Let V be a quaternionic right vector space.*

- *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be a finite number of vectors in V with $\boldsymbol{v}_i \neq \boldsymbol{v}_j$ for $i \neq j$. A vector of the form $\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{v}_i a_i$ with $a_i \in \mathbb{H}$ is called a* right linear combination *of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.*

- *Let $A \subset V$. A vector $\boldsymbol{v} \in V$ is called* right linearly dependent *on A, if it can be represented as a right linear combination of vectors in A. Otherwise, $\boldsymbol{v}$ is called* right linearly independent *of A.*

- *A set $A \subset V$ is called* right linearly independent*, if every $\boldsymbol{v} \in A$ is not right linearly dependent on $A \setminus \{\boldsymbol{v}\}$.*

As in classical linear algebra, we have the following characterization of right linear independence.

Corollary 2.20. *Let V be a quaternionic right vector space. A set $A \subset V$ is right linearly independent if and only if every right linear combination of the zero vector is trivial, that is, $\sum_{i=1}^{n} \boldsymbol{v}_i a_i = \boldsymbol{0}$ with $\boldsymbol{v}_i \in A$ and $a_i \in \mathbb{H}$ implies $a_i = 0$ for all $i = 1, \ldots, n$.*

Proof. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in A$ and let $\sum_{i=1}^{n} \boldsymbol{v}_i a_i = \mathbf{0}$ with $a_j \neq 0$ for some $j \in \{1, \ldots, n\}$. Then we have $\boldsymbol{v}_j = -\sum_{i \in \{1,\ldots,n\} \setminus \{j\}} \boldsymbol{v}_i a_i a_j^{-1}$. Thus, the set A is not right linearly independent. On the other hand, if the set A is linearly dependent, then there exist $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n \in A$ with $\boldsymbol{v}_i \neq \boldsymbol{v}_j$ for $i \neq j$, such that $\boldsymbol{v}_0 = \sum_{i=1}^{n} \boldsymbol{v}_i a_i$. But then $\mathbf{0} = \sum_{i=1}^{n} \boldsymbol{v}_i a_i - \boldsymbol{v}_0$. Hence, the zero vector can be represented as a non-trivial right linear combination of vectors in A.

□

Let $\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{v}_i a_i$ be a right linear combination of vectors $\boldsymbol{v}_i \in A$, where the set A is right linearly independent. Then the previous corollary implies that the coefficients a_i and the vectors $\boldsymbol{v}_i$ are unique.

Definition 2.21. *Let V be a quaternionic right vector space.*

- *A subset $U \subset V$ is called a* quaternionic right vector subspace *if it is closed under addition and right scalar multiplication, that is, if for any $\boldsymbol{u}, \boldsymbol{v} \in U$ and any $a \in \mathbb{H}$ the vectors $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{v}a$ are in U, too.*
- *Let $A \subset V$. The* right linear span *of A is the set of all right linear combinations of vectors in A, that is,* $\operatorname{span}_R(A) = \{\sum_{i=1}^{n} \boldsymbol{v}_i a_i : \boldsymbol{v}_i \in A, a_i \in \mathbb{H}, n \in \mathbb{N}\}$.
- *A right linearly independent set A with* $\operatorname{span}_R(A) = V$ *is called a* right basis *of V.*

Similar to the classical case, every right vector space has a right basis.

Corollary 2.22. *Let V be a quaternionic right vector space and let A be a right linearly independent subset of V. Then there exists a right basis B of V such that $A \subset B$. In particular, every quaternionic right vector space has a basis.*

Proof. Let $\mathcal{B}$ be the set of all right linearly independent supersets of A. The set $\mathcal{B}$ is nonempty as it contains A itself and it is partially ordered by the set inclusion $\subset$. Moreover, for every chain $B_i, i \in I$, in $\mathcal{B}$, we can consider $B^* = \bigcup_{i \in I} B_i$. This set must be right linearly independent. Otherwise there would exist right linearly dependent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in B^*$. But since $B_i, i \in I$, is totally ordered with respect to $\subset$, there would exist an index i_0 such that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in B_{i_0}$, which is a contradiction because $B_{i_0} \in \mathcal{B}$. Thus, B^* is a right linearly independent superset of A and of every set $B_i, i \in I$, that is, B^* is an upper bound of the chain $B_i, i \in I$, in $\mathcal{B}$.

Therefore, we can apply Zorn's lemma to obtain a maximal element $B \in \mathcal{B}$. The set B is right linearly independent because it belongs to $\mathcal{B}$.

Let us assume that there exists a vector $v_0 \in V \setminus \operatorname{span}_R(V)$. Then the set $\{v_0\} \cup B$ is a right linearly independent superset of B. But this is a contradiction to the maximality of B. Hence, $\operatorname{span}_R(B)$ must already be the entire space V, that is, B is a right basis of V.

□

Remark 2.23. As pointed out before, all these definitions and results can be obtained for quaternionic left vector spaces by analogous arguments. Nevertheless, when we are working with a quaternionic two-sided vector space, it is important to keep in mind that the left linear and the right linear structure of this space do not coincide. Thus, a left linear subspace is not necessarily a right linear subspace, a right basis of the space is not necessarily a left basis etc. For imaginary units I, J and K as in Corollary 2.9, let us consider the vectors $\boldsymbol{u} = (1, I)^T$ in $\mathbb{H}^2$ and $\boldsymbol{v} = (J, K)^T \in \mathbb{H}^2$. Then $\boldsymbol{u} = \boldsymbol{v}J$. On the other hand $a\boldsymbol{u} + b\boldsymbol{v} = \mathbf{0}$ implies $a + bJ = \mathbf{0}$ and $aI + bK = (a - bJ)I = \mathbf{0}$. Consequently, $a = b = 0$ and $\boldsymbol{u}$ and $\boldsymbol{v}$ are left linearly independent but right linearly dependent. Therefore, not even the left and the right linearly independent subsets of a two-sided quaternionic vector space coincide.

The next lemma specifies the relation between the left and the right vector space structure of a quaternionic two-sided vector space.

Lemma 2.24. *Let $V = V_{\mathbb{R}} \otimes \mathbb{H}$ be a two-sided quaternionic vector space, where $V_{\mathbb{R}}$ is the real part of V as in Corollary 2.18.*

(i) A quaternionic right subspace of U is a left subspace of V if and only if there exists a real subspace $U_{\mathbb{R}}$ of $V_{\mathbb{R}}$ such that $U = U_{\mathbb{R}} \otimes \mathbb{H}$.

(ii) Let B be a left and a right basis, that is, any vector $\boldsymbol{v} \in V$ can be represented uniquely as a right linear combination $\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{b}_i a_i$ and as a left linear combination $\boldsymbol{v} = \sum_{i=1}^{n} \widetilde{a}_i \boldsymbol{b}_i$ of vectors $\boldsymbol{b}_i \in B$. The coefficitents a_i and $\widetilde{a}_i$ coincide for any vector $\boldsymbol{v} \in V$ if and only if B is a basis of the real vector space $V_{\mathbb{R}}$. Moreover, any basis of the real vector space $V_{\mathbb{R}}$ is a left and a right basis of V.

Proof. If $U \subset V$ is a left and a right subspace of V, then it is a quaternionic two-sided vector space. Therefore, by Lemma 2.17 and Corollary 2.18, the set $U_{\mathbb{R}} = \{\boldsymbol{u} \in U : a\boldsymbol{u} = \boldsymbol{u}a \ \forall a \in \mathbb{H}\}$ is a real vector space and $U = U_{\mathbb{R}} \otimes \mathbb{H}$. It is obvious that $U_{\mathbb{R}}$ is a subspace of $V_{\mathbb{R}} = \{\boldsymbol{v} \in V : a\boldsymbol{v} = \boldsymbol{v}a \ \forall a \in \mathbb{H}\}$.

Conversely, if $U_{\mathbb{R}}$ is a subspace of $V_{\mathbb{R}}$, then $U = U_{\mathbb{R}} \otimes \mathbb{H}$ is a left and a right subspace of $V = V_{\mathbb{R}} \otimes \mathbb{H}$ by definition. Hence, *(i)* holds true.

To prove *(ii)*, we assume that the coefficients a_i and $\widetilde{a}_i$ coincide for any $v \in V$. Then $a\boldsymbol{b} = \boldsymbol{b}a$ for any $a \in \mathbb{H}$ and any $\boldsymbol{b} \in B$. Hence, $B \subset V_{\mathbb{R}}$. Let $\boldsymbol{v}$ be a vector in $V_{\mathbb{R}}$ and let $\boldsymbol{v} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i = \sum_{i=1}^{n} \boldsymbol{b}_i a_i$ be its representation as a left linear combination of vectors in B. Then $\beta\boldsymbol{v} = \boldsymbol{v}\beta$ for any $\beta \in \mathbb{H}$. Hence,

$$\sum_{i=1}^{n} \beta a_i \boldsymbol{b}_i = \beta\boldsymbol{v} = \boldsymbol{v}\beta = \sum_{i=1}^{n} \boldsymbol{b}_i a_i.\beta$$

Since the coefficients of any left or right linear combination of vectors in B are unique, we obtain $\beta a_i = a_i \beta, i = 1, \ldots, n$, for any $\beta \in \mathbb{H}$. Therefore, the coefficients a_i are real by Lemma 2.5, and in turn, B is a basis of the real vector space $V_{\mathbb{R}}$.

If on the other hand B is a basis of $V_{\mathbb{R}}$, then B is right linearly independent in V. Indeed, if $\sum_{i=0}^{n} \boldsymbol{b}_i a_i = \boldsymbol{0}$ with $a_i = \sum_{i=0}^{3} a_{i,j} e_j \in \mathbb{H}$ and $\boldsymbol{b}_i \in B, i = 1, \ldots, n$, then $\boldsymbol{0} = \sum_{i=1}^{n} \boldsymbol{b}_i \left(\sum_{j=0}^{3} a_{i,j}\right) e_j = \sum_{j=0}^{3} \left(\sum_{i=1}^{n} \boldsymbol{b}_i a_{i,j}\right) e_j$. Since $\sum_{i=0}^{n} \boldsymbol{b} a_{i,j} \in V_{\mathbb{R}}$, this implies $\sum_{i=0}^{n} \boldsymbol{b} a_{i,j} = \boldsymbol{0}$ for $j = 0, \ldots, 3$, and we obtain $a_{i,j} = 0, i = 1, \ldots, n, j = 0, \ldots, 3$ because the vectors $\boldsymbol{b}_i, i = 1, \ldots, n$ are linearly independent in $V_{\mathbb{R}}$. Hence, $a_i = 0, i = 1, \ldots, n$ and Corollary 2.20 implies that B is right linearly independent in V. Similarly, we obtain that B is left linearly independent in V.

Let $\boldsymbol{v} \in V$. Then we can apply the polarization identity (2.11) to obtain $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_3 \in V_{\mathbb{R}}$ such that $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i$. Since B is a basis of $V_{\mathbb{R}}$, there exist $\boldsymbol{b}_j \in B, j = 1, \ldots, n$, and real coefficients $a_{i,j}, i = 0, \ldots, 3$, for any $j = 1, \ldots, n$ such that $\boldsymbol{v}_i = \sum_{j=1}^{n} \boldsymbol{b}_j a_{i,j}$. Hence,

$$\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i = \sum_{i=0}^{3} \sum_{j=1}^{n} \boldsymbol{b}_j a_{i,j} e_i = \sum_{j=1}^{n} \boldsymbol{b}_j \sum_{i=0}^{3} a_{i,j} e_i = \sum_{j=1}^{n} \boldsymbol{b}_j a_j$$

with $a_j = \sum_{i=0}^{3} a_{i,j} e_i \in \mathbb{H}$. Therefore, $\boldsymbol{v} \in \operatorname{span}_R(B)$ and B is a right basis of V. Moreover, since $B \subset V_{\mathbb{R}}$, the vectors $\boldsymbol{b}_j, j = 1, \ldots, n$ commute with the coefficients a_j and we obtain $\boldsymbol{v} = \sum_{j=1}^{n} a_j \boldsymbol{b}_j$. Thus, B is also a left basis of V and the coefficients of the representation as a left and as a right linear combination of vectors in B coincide for any $\boldsymbol{v} \in V$.

□

Remark 2.25. Note that not any left and right basis of a quaternionic two-sided vector space V is of the type considered in *(ii)* of Lemma 2.24. Let for instance I, J and K be imaginary units as in Lemma 2.9 and consider the vectors $(1, I)^T$ and $(I, 1)^T$, which form a left and a right basis of $\mathbb{H}^2$. Then the coefficients of the representation of $(J, K)^T$ as a left and as right linear combination of this basis do not coincide because

$$\begin{pmatrix} J \\ K \end{pmatrix} = \begin{pmatrix} 1 \\ I \end{pmatrix} J + \begin{pmatrix} I \\ 1 \end{pmatrix} 0 \qquad \text{but} \qquad \begin{pmatrix} J \\ K \end{pmatrix} = 0 \begin{pmatrix} 1 \\ I \end{pmatrix} + K \begin{pmatrix} I \\ 1 \end{pmatrix}.$$

Lemma 2.26. *Let V be a quaternionic right vector space, let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a finite subset of V such that $\operatorname{span}_R(B) = V$ and let A be a right linearly independent subset of V. Then A is finite and its cardinality satisfies $\#A \leq \#B$.*

Proof. By induction, we replace k elements $\boldsymbol{b}_{i_1}, \ldots \boldsymbol{b}_{i_k}$ of B by k elements $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ of A such that the resulting set B_k still satisfies $\operatorname{span}_R(B_k) = V$. For $k = 0$, we can choose $B_0 = B$. Assume that $B_k = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\} \cup (B \setminus \{\boldsymbol{b}_{i_1}, \ldots \boldsymbol{b}_{i_k}\})$ is defined for some $k \leq \#B$. If $A \setminus B_k = \emptyset$, then $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$, which implies $\#A = k \leq \#B_k$. Otherwise there exists $\boldsymbol{a}_{k+1} \in A \setminus B_k$. Since $\operatorname{span}_R(B_k) = V$, we have

$$\boldsymbol{a}_{k+1} = \sum_{i=1}^{k} \boldsymbol{a}_i \alpha_i + \sum_{i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}} \boldsymbol{b}_i \beta_i$$

with $\alpha_i, \beta_i \in \mathbb{H}$. Moreover, there exists an index $i_{k+1} \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ such that $\beta_{k+1} \neq 0$ because the set A is right linearly independent. In particular, the set $\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ is nonempty if $A \setminus B_k \neq \emptyset$.

We set $B_{k+1} = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{k+1}\} \cup (B \setminus \{\boldsymbol{b}_{i_1}, \dots \boldsymbol{b}_{i_{k+1}})$. Since

$$\boldsymbol{b}_{i_{k+1}} = \boldsymbol{a}_{k+1}\beta_{i_{k+1}}^{-1} - \sum_{i=1}^{k} \boldsymbol{a}_i \alpha_i \beta_{i_{k+1}}^{-1} - \sum_{i \in \{1,\dots,n\} \setminus \{i_1,\dots,i_{k+1}\}} \boldsymbol{b}_i \beta_i \beta_{i_{k+1}}^{-1},$$

we can transform any right linear combination of vectors in B_k into a right linear combination of vectors in B_{k+1}. Hence, $\operatorname{span}_R(B_{k+1}) = V$.

Since $A \setminus B_k \neq \emptyset$ implies that $\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ is nonempty, this algorithm stops after at most n steps and we obtain $\#A \leq n = \#B$.

□

Corollary 2.27. *Let V be a quaternionic right vector space and let B be a finite right basis of V. Then any right basis B' of V satisfies $\#B' = \#B$.*

Proof. Since $\operatorname{span}_R(B) = V$ and since B' is right linearly independent, we can apply Lemma 2.26 and obtain $\#B' \leq \#B$. In particular, B' is finite. Moreover, $\operatorname{span}_R(B') = V$ and since B is right linearly independent. Thus, we can apply Lemma 2.26 again and obtain $\#B \leq \#B'$. Hence, $\#B' = \#B$.

□

Definition 2.28. *Let V be a quaternionic right vector space and let B be a right basis of V. If B is finite, then we call $\#B$ the* dimension *of V over $\mathbb{H}$. We say that V has* infinite dimension *if B is infinite.*

The dimension of a quaternionic left vector space is defined analogously. If V is a quaternionic two-sided vector space, let $V_{\mathbb{R}}$ be the real part of V as in Lemma 2.17 and let B be a basis of $V_{\mathbb{R}}$. Then B is a left and a right basis of V by Lemma 2.24. Hence, the dimension of V as a left and as a right vector space coincide.

Moreover, if B is a right basis of a quaternionic right vector space, then the set $\{\boldsymbol{b}e_i : \boldsymbol{b} \in B, 0 \leq i \leq 3\}$ is a basis of V as real vector space. Thus, the dimension of V as a quaternionic right vector space and the dimension of V as a real vector space satisfy

$$\dim_{\mathbb{R}}(V) = 4 \dim_{\mathbb{H}}(V).$$

Corollary 2.29. *Let V be a quaternionic right vector space of dimension n. Any right linearly independent subset B of V with $\#B = n$ is a right basis of n.*

Proof. Let $B \subset V$ with $\#B = n$ be right linearly independent. By Corollary 2.22, there exists a right basis B' of V with $B \subset B'$. But Corollary 2.27 implies $\#B' = n$, and therefore, $B' = B$.

□

Definition 2.30. *Let V and W be quaternionic right vector spaces. A mapping $T : V \to W$ is called* quaternionic right linear *if $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ and $T(\boldsymbol{v}a) = T(\boldsymbol{v})a$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and all $a \in \mathbb{H}$. We will denote the set of quaternionic right linear mappings from V to W by $L_R(V, W)$.*

As in usual linear algebra, the two conditions for right linearity can be written more compactly as

$$T(\boldsymbol{u}a + \boldsymbol{v}) = T(\boldsymbol{u})a + T(\boldsymbol{v})$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and all $a \in \mathbb{H}$.

Remark 2.31. Note that, in general, for two quaternionic right linear vector spaces V and W, the set $L_R(V, W)$ is not a right linear vector space. Indeed, if $T : V \to W$ is a right linear mapping, we can define the multiplication of T with a scalar $a \in \mathbb{H}$ from the right pointwise by $[Ta](\boldsymbol{v}) = T(\boldsymbol{v})a$ for all $\boldsymbol{v} \in V$. But if $a, b \in \mathbb{H}$ do not commute and $T(\boldsymbol{v}) \neq \boldsymbol{0}$, then

$$[Ta](\boldsymbol{v}b) = T(\boldsymbol{v}b)a = T(\boldsymbol{v})ba \neq T(\boldsymbol{v})ab = [Ta](\boldsymbol{v})b.$$

Thus, the mapping Ta is not right linear.

On the other hand, if W is a quaternionic two-sided vector space—in particular, if $W = \mathbb{H}$—we can define a left scalar multiplication pointwise. Then, we have

$$[aT](\boldsymbol{v}b) = aT(\boldsymbol{v}b) = aT(\boldsymbol{v})b = [aT](\boldsymbol{v})b \tag{2.12}$$

for all $a, b \in \mathbb{H}$ and all $\boldsymbol{v} \in V$. Therefore, aT is still right linear and $L_R(V, W)$ is a quaternionic left vector space.

Finally, if V and W are both two-sided quaternionic vector spaces, then we can define the multiplication of a mapping $T \in L_R(V, W)$ with a scalar $a \in \mathbb{H}$ on the left and on the right by

$$[aT](\boldsymbol{v}) = aT(\boldsymbol{v}) \qquad \text{and} \qquad [Ta](\boldsymbol{v}) = T(a\boldsymbol{v}). \tag{2.13}$$

Then the mapping aT is right linear because of (2.12). Moreover, for any $\boldsymbol{u}, \boldsymbol{v} \in V$ and any $b \in \mathbb{H}$, we have

$$[Ta](\boldsymbol{u}b + \boldsymbol{v}) = T(a\boldsymbol{u}b + a\boldsymbol{v}) = T(a\boldsymbol{u})b + T(a\boldsymbol{v}) = [Ta](\boldsymbol{u})b + [Ta](\boldsymbol{v}).$$

Thus, the mapping Ta is right linear, too. For $a \in \mathbb{R}$, we have

$$[aT](\boldsymbol{v}) = aT(\boldsymbol{v}) = T(\boldsymbol{v})a = T(\boldsymbol{v}a) = T(a\boldsymbol{v}) = [Ta](\boldsymbol{v}).$$

Hence, $L_R(V, W)$ is a two-sided vector space over $\mathbb{H}$.

Note that the identity mapping $\mathcal{I} : \boldsymbol{v} \mapsto \boldsymbol{v}$ on a quaternionic two-sided vector space V commutes with any scalar $a \in \mathbb{H}$. Indeed, due to the definition of the scalar multiplications in (2.13), we have

$$[a\mathcal{I}](\boldsymbol{v}) = a\mathcal{I}(\boldsymbol{v}) = a\boldsymbol{v} = \mathcal{I}(a\boldsymbol{v}) = [\mathcal{I}a](\boldsymbol{v}) \tag{2.14}$$

for any $\boldsymbol{v} \in V$ and any $a \in \mathbb{H}$.

As in the classical case, a right linear mapping is uniquely determined by its values on a basis of the space.

Corollary 2.32. *Let V and W be quaternionic right vector spaces, let B be a basis of V and let f be a function from B to W. Then there exists a unique quaternionic right linear mapping $T : V \to W$ with $T|_B = f$.*

Proof. Let $\boldsymbol{v} \in V$. Then there exist unique $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in B$ and $a_1, \ldots, a_n \in \mathbb{H}$ such that $\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{v}_1 a_i$. Let us define $T(\boldsymbol{v}) = \sum_{i=1}^{n} f(\boldsymbol{v}_i)a_i$. Obviously, T is a quaternionic right linear mapping from V to W that extends f.

On the other hand, if S is an arbitrary quaternionic right linear mapping that extends f, we have

$$T(\boldsymbol{v}) - S(\boldsymbol{v}) = \sum_{i=1}^{n} T(\boldsymbol{v}_i)a_i - \sum_{i=1}^{n} S(\boldsymbol{v}_i)a_i = \sum_{i=1}^{n} f(\boldsymbol{v}_i)a_i - \sum_{i=1}^{n} f(\boldsymbol{v}_i)a_i = \boldsymbol{0}.$$

Thus, the mapping T is unique.

□

Note that, as a consequence of this corollary, any right linear mapping defined on a subspace U of a quaternionic right vector space V over $\mathbb{H}$ can be extended to a right linear mapping on V.

Corollary 2.33. *Let V be a quaternionic right vector space of finite dimension. A right linear mapping $T : V \to V$ is injective if and only if it is surjective.*

Proof. If $T(\boldsymbol{u}) = T(\boldsymbol{v})$ for $\boldsymbol{u} \neq \boldsymbol{v}$, then $T(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{0}$. Therefore, the mapping T is injective if and only if the pre-image of $\{\boldsymbol{0}\}$ is trivial, that is, $T^{-1}(\{\boldsymbol{0}\}) = \{\boldsymbol{0}\}$.

Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ be a right basis of V. If $\sum_{i=1}^{n} T(\boldsymbol{b}_i)a_i = \boldsymbol{0}$, then

$$\boldsymbol{0} = \sum_{i=1}^{n} T(\boldsymbol{b}_i)a_i = T\left(\sum_{i=1}^{n} \boldsymbol{b}_i a_i\right).$$

Therefore, the vectors $T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)$ allow a nontrivial right linear combination of the zero, if and only if the pre-image of the zero is non-trivial, that is, if and only if T is not injective. Hence, by

Corollary 2.20, the mapping T is injective if and only if the vectors $T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)$ are right linearly independent.

If T is injective, then the vectors $T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)$ are right linearly independent. Therefore, by Corollary 2.29, they form a right basis of V. Hence, for any $\boldsymbol{v} \in V$, there exist $a_1, \ldots, a_n \in \mathbb{H}$ such that $\boldsymbol{v} = \sum_{i=1}^{n} T(\boldsymbol{b}_i)a_i$ and we deduce $\boldsymbol{v} = T\left(\sum_{i=1}^{n} \boldsymbol{b}_i a_i\right)$. Thus, T is surjective.

If on the other hand T is surjective, then $\operatorname{span}_R(\{T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)\}) = T(V) = V$. In particular, the set $\{T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)\}$ contains a right basis B of V. Since Corollary 2.27 and Definition 2.28 imply $\#B = n$, we obtain $B = \{T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)\}$. Therefore, the vectors $T(\boldsymbol{b}_1), \ldots, T(\boldsymbol{b}_n)$ are right linearly independent, which implies that T is injective.

□

Finally, we show that any right linear mapping between two quaternionic two-sided vector spaces V and W allows a representation in terms of $\mathbb{R}$-linear mappings from $V_{\mathbb{R}}$ to $W_{\mathbb{R}}$, where $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ are the real parts of V and W, respectively, as defined in Lemma 2.17.

Lemma 2.34. *Let V and W be quaternionic two-sided vector spaces. A mapping $T : V \to W$ is quaternionic right linear if and only if, there exist $\mathbb{R}$-linear mappings $T_i : V_{\mathbb{R}} \to W_{\mathbb{R}},\ i = 0, \ldots, 3$, such that*

$$T(\boldsymbol{v}) = \sum_{i,j=0}^{3} T_i(\boldsymbol{v}_j)e_i e_j, \tag{2.15}$$

where $\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{j=1}^{3} \boldsymbol{v}_j e_j$ with $\boldsymbol{v}_j \in V_{\mathbb{R}}, j = 0, \ldots, 3$ as in (2.11).

Proof. Let $T : V \to W$ be quaternionic right linear and let $\mathsf{Re}_W : W \to W_{\mathbb{R}}$ be the projection onto the real part of W as defined in Lemma 2.17. If we set $T_i(\boldsymbol{v}) = \mathsf{Re}_W(\overline{e_i}\, T(\boldsymbol{v}))$ for any $\boldsymbol{v} \in V_{\mathbb{R}}$ and $i = 0, \ldots, 3$, then the mappings T_i are $\mathbb{R}$-linear and the polarization identity (2.11) implies $T(\boldsymbol{v}) = \sum_{i=0}^{3} T_i(\boldsymbol{v})e_i$ for $\boldsymbol{v} \in V_{\mathbb{R}}$. For $\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{j=1}^{3} \boldsymbol{v}_j e_j \in V$ with $\boldsymbol{v}_j \in V_{\mathbb{R}}$, we obtain

$$T(\boldsymbol{v}) = \sum_{i=0}^{3} T(\boldsymbol{v}_j)e_j = \sum_{i,j=0}^{3} T_i(\boldsymbol{v}_j)e_i e_j.$$

If on the other hand T has the form (2.15), then T is $\mathbb{R}$-linear. For two vectors $\boldsymbol{u} = \boldsymbol{u}_0 + \sum_{j=1}^{3} \boldsymbol{u}_j e_j$ and $\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{j=0}^{3} \boldsymbol{v}_j e_j \in V$ and $a \in \mathbb{R}$, we have $\boldsymbol{u}a + \boldsymbol{v} = \boldsymbol{u}_0 a + \boldsymbol{v}_0 + \sum_{j=1}^{3}(\boldsymbol{u}_j a + \boldsymbol{v}_j)e_j$. Hence,

$$T(\boldsymbol{u}a + \boldsymbol{v}) = \sum_{i,j=0}^{3} T_i(\boldsymbol{u}_j a + \boldsymbol{v}_j)e_i e_j = \sum_{i,j=0}^{3} T_i(\boldsymbol{u}_j)e_i e_j a + \sum_{i,j=0}^{3} T_i(\boldsymbol{u}_j)e_i e_j = T(\boldsymbol{u})a + T(\boldsymbol{v}),$$

because the mappings T_i are $\mathbb{R}$-linear.

Moreover, $\boldsymbol{v}e_1 = \boldsymbol{v}_0 e_1 - \boldsymbol{v}_1 - \boldsymbol{v}_2 e_3 + \boldsymbol{v}_3 e_2$ implies

$$\begin{aligned} T(\boldsymbol{v}e_1) = &- T_0(\boldsymbol{v}_1)e_0e_0 + T_0(\boldsymbol{v}_0)e_0e_1 + T_0(\boldsymbol{v}_3)e_0e_2 - T_0(\boldsymbol{v}_2)e_0e_3 - \\ &- T_1(\boldsymbol{v}_1)e_1e_0 + T_1(\boldsymbol{v}_0)e_1e_1 + T_1(\boldsymbol{v}_3)e_1e_2 - T_1(\boldsymbol{v}_2)e_1e_3 - \\ &- T_2(\boldsymbol{v}_1)e_2e_0 + T_2(\boldsymbol{v}_0)e_2e_1 + T_2(\boldsymbol{v}_3)e_2e_2 - T_2(\boldsymbol{v}_2)e_2e_3 - \\ &- T_3(\boldsymbol{v}_1)e_3e_0 + T_3(\boldsymbol{v}_0)e_3e_1 + T_3(\boldsymbol{v}_3)e_3e_2 - T_3(\boldsymbol{v}_2)e_3e_3. \end{aligned}$$

If we factor out e_1 on the right, we obtain

$$\begin{aligned} T(\boldsymbol{v}e_1) = [\ &T_0(\boldsymbol{v}_1)e_0e_1 + T_0(\boldsymbol{v}_0)e_0e_0 + T_0(\boldsymbol{v}_3)e_0e_3 + T_0(\boldsymbol{v}_2)e_0e_2 + \\ &+ T_1(\boldsymbol{v}_1)e_1e_1 + T_1(\boldsymbol{v}_0)e_1e_0 + T_1(\boldsymbol{v}_3)e_1e_3 + T_1(\boldsymbol{v}_2)e_1e_2 + \\ &+ T_2(\boldsymbol{v}_1)e_2e_1 + T_2(\boldsymbol{v}_0)e_2e_0 + T_2(\boldsymbol{v}_3)e_2e_3 + T_2(\boldsymbol{v}_2)e_2e_2 + \\ &+ T_3(\boldsymbol{v}_1)e_3e_1 + T_3(\boldsymbol{v}_0)e_3e_0 + T_3(\boldsymbol{v}_3)e_3e_3 + T_3(\boldsymbol{v}_2)e_3e_2\ ]\, e_1 = T(\boldsymbol{v})e_1. \end{aligned}$$

Similar computations show $T(\boldsymbol{v}e_2) = T(\boldsymbol{v})e_2$ and $T(\boldsymbol{v}e_3) = T(\boldsymbol{v})e_3$. Thus, for $a \in \mathbb{H}$ with $a =$

$a_0 + \sum_{j=1}^{3} a_j e_j$, we have

$$T(\boldsymbol{v}a) = \sum_{j=0}^{3} T(\boldsymbol{v}a_j e_j) = \sum_{j=0}^{3} T(\boldsymbol{v}) a_j e_j = T(\boldsymbol{v})a.$$

Hence, T is quaternionic right linear.

□

Lemma 2.35. *Let V and W be quaternionic two-sided vector spaces. Then*

$$L_R(V,W) \cong L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H},$$

where $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ are the real parts of V and W defined in Lemma 2.17, respectively, and $L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}})$ denotes the set of $\mathbb{R}$-linear mappings from V to W.

Proof. Because of Lemma 2.34, we have $L_R(V,W) \cong L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}})^4 \cong L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H}$ as real vector spaces. Let $T \in L_R(V,W)$ and let $a = \sum_{j=0}^{3} a_j e_j \in \mathbb{H}$. By the definition of the right scalar multiplication on $L_R(V,W)$ in Remark 2.31 and by (2.15), we have

$$[Ta](\boldsymbol{v}) = T(a\boldsymbol{v}) = T\left(\sum_{j,k=0}^{3} a_j \boldsymbol{v}_k e_j e_k\right) = \sum_{j=0}^{3} a_j T\left(\sum_{k=0}^{3} \boldsymbol{v}_k e_j\right) e_k = \sum_{i,j,k=0}^{3} a_j T_i(\boldsymbol{v}_k) e_i e_j e_k$$

for any $\boldsymbol{v} = \sum_{k=0}^{3} \boldsymbol{v}_k e_k \in V$, where $\boldsymbol{v}_k \in V_{\mathbb{R}}, k = 0, \ldots, 3$.

If on the other hand we identify T with $\sum_{i=0}^{3} T_i e_i \in L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H}$ by Lemma 2.34 and multiply it with $a = \sum_{j=0}^{3} a_j e_j \in \mathbb{H}$ from the right, we obtain

$$\begin{aligned}
Ta = &\sum_{ij=0}^{3} T_i a_j e_i e_j = \\
=&T_0 a_0 e_0 e_0 + T_1 a_0 e_1 e_0 + T_2 a_0 e_2 e_0 + T_3 a_0 e_3 e_0 + \\
&+ T_0 a_1 e_0 e_1 + T_1 a_1 e_1 e_1 + T_2 a_1 e_2 e_1 + T_3 a_1 e_3 e_1 + \\
&+ T_0 a_2 e_0 e_2 + T_1 a_2 e_1 e_2 + T_2 a_2 e_2 e_2 + T_3 a_2 e_3 e_2 + \\
&+ T_0 a_3 e_0 e_3 + T_1 a_3 e_1 e_3 + T_2 a_3 e_2 e_3 + T_3 a_3 e_3 e_3 = \\
=&(a_0 T_0 - a_1 T_1 - a_2 T_2 - a_3 T_3) e_0 + (a_1 T_0 + a_0 T_1 + a_3 T_2 - a_2 T_3) e_1 + \\
&+ (a_2 T_0 - a_3 T_1 + a_0 T_2 + a_1 T_3) e_2 + (a_3 T_0 + a_2 T_1 - a_1 T_2 + a_0 T_3) e_3
\end{aligned}$$

by the definition of the right scalar multiplication on $L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H}$ in (2.10). If we apply this operator to $\boldsymbol{v} = \sum_{k=0}^{3} \boldsymbol{v}_k e_k \in V$, we get

$$\begin{aligned}
[Ta](\boldsymbol{v}) = \sum_{k=0}^{3} \Big(&(a_0 T_0(\boldsymbol{v}_k) - a_1 T_1(\boldsymbol{v}_k) - a_2 T_2(\boldsymbol{v}_k) - a_3 T_3(\boldsymbol{v}_k)) e_0 e_k + \\
&+ (a_1 T_0(\boldsymbol{v}_k) + a_0 T_1(\boldsymbol{v}_k) + a_3 T_2(\boldsymbol{v}_k) - a_2 T_3(\boldsymbol{v}_k)) e_1 e_k + \\
&+ (a_2 T_0(\boldsymbol{v}_k) - a_3 T_1(\boldsymbol{v}_k) + a_0 T_2(\boldsymbol{v}_k) + a_1 T_3(\boldsymbol{v}_k)) e_2 e_k + \\
&+ (a_3 T_0(\boldsymbol{v}_k) + a_2 T_1(\boldsymbol{v}_k) - a_1 T_2(\boldsymbol{v}_k) + a_0 T_3(\boldsymbol{v}_k)) e_3 e_k \Big),
\end{aligned}$$

and hence,

$$\begin{aligned}
[Ta](\boldsymbol{v}) = \sum_{k=0}^{3} \Big(&a_0 T_0(\boldsymbol{v}_k) e_0 e_0 e_k + a_1 T_1(\boldsymbol{v}_k) e_1 e_1 e_k + a_2 T_2(\boldsymbol{v}_k) e_2 e_2 e_k + a_3 T_3(\boldsymbol{v}_k) e_3 e_3 e_k + \\
&+ a_1 T_0(\boldsymbol{v}_k) e_0 e_1 e_k + a_0 T_1(\boldsymbol{v}_k) e_1 e_0 e_k + a_3 T_2(\boldsymbol{v}_k) e_2 e_3 e_k - a_2 T_3(\boldsymbol{v}_k) e_3 e_2 e_k + \\
&+ a_2 T_0(\boldsymbol{v}_k) e_0 e_2 e_k + a_3 T_1(\boldsymbol{v}_k) e_1 e_3 e_k + a_0 T_2(\boldsymbol{v}_k) e_2 e_0 e_k + a_1 T_3(\boldsymbol{v}_k) e_3 e_1 e_k + \\
&+ a_3 T_0(\boldsymbol{v}_k) e_0 e_3 e_k + a_2 T_1(\boldsymbol{v}_k) e_1 e_2 e_k + a_1 T_2(\boldsymbol{v}_k) e_2 e_1 e_k + a_0 T_3(\boldsymbol{v}_k)) e_3 e_0 e_k \Big) = \\
= \sum_{i,j,k=0}^{3} &a_j T_i(\boldsymbol{v}_k) e_i e_j e_k.
\end{aligned}$$

Thus, the two right scalar multiplications coincide. A similar computation shows that the left scalar multiplications coincide, too. Hence, we obtain $L_R(V,W) \cong L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H}$ even as quaternionic two-sided vector spaces.

□

A result analogue to Lemma 2.35 holds for the space $L_L(V,W)$ of quaternionic left linear mappings from V to W. Hence, $L_L(V,W) \cong L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H} \cong L_R(V,W)$. Let $T = \sum_{i=0}^{3} T_i e_i \in L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H}$. If we consider T as a right linear operator, then Lemma 2.34 states that

$$T(\boldsymbol{v}) = \sum_{i,j=0}^{3} T_i(\boldsymbol{v}_j) e_i e_j$$

for any $\boldsymbol{v} = \sum_{j=0}^{3} \boldsymbol{v}_j e_j \in V$. The analogous result for left linear operators states that

$$T(\boldsymbol{v}) = \sum_{i,j=0}^{3} T_i(\boldsymbol{v}_j) e_j e_i$$

for any $\boldsymbol{v} = \sum_{j=0}^{3} \boldsymbol{v}_j e_j \in V$. A comparison of these two formulas with the product of two quaternions $x = \sum_{i=0}^{3} x_i e_i$ and $y = \sum_{j=0}^{3} y_j e_j$, that is,

$$xy = \sum_{i,j=0}^{3} x_i y_j e_i e_j,$$

suggests the following notation.

Definition 2.36. *Let V and W be two-sided quaternionic vector spaces and let $T = T_0 + \sum_{i=1}^{3} T_i e_i$ belong to $L^{\mathbb{R}}(V_{\mathbb{R}}, W_{\mathbb{R}}) \otimes \mathbb{H}$. If we consider T as a quaternionic right linear operator, then we say that T* acts on the right *and we denote $T(\boldsymbol{v})$ by the formal multiplication of T by $\boldsymbol{v}$ on the right, that is, $T(\boldsymbol{v}) = T\boldsymbol{v}$. If we consider T as a left linear operator, we say that* T acts on the left *and we denote $T(\boldsymbol{v})$ by the formal multiplication of T by $\boldsymbol{v}$ on the left, that is, $T(\boldsymbol{v}) = \boldsymbol{v}T$.*

If $T = \sum_{i=0}^{3} T_i e_i$ and $S = \sum_{j=0}^{3} S_i e_i$ belong to $L^{\mathbb{R}}(V_{\mathbb{R}}, V_{\mathbb{R}}) \otimes \mathbb{H}$, then we can consider their formal product $TS = \sum_{i,j=0}^{3} T_i S_j e_i e_j$. As in the classical case, we can interpret this product as the composition of T and S. However, note the convention introduced in Definition 2.36 implies that

$$TS\boldsymbol{v} = \sum_{i,j,k=0}^{3} T_i S_j \boldsymbol{v}_k e_i e_j e_k = \sum_{i,j,k} T_i(S_j(\boldsymbol{v}_k)) e_i e_j e_k$$

and

$$\boldsymbol{v}TS = \sum_{i,j,k=0}^{3} \boldsymbol{v}_k T_i S_j e_k e_i e_j = \sum_{i,j,k=0}^{3} S_j(T_i(\boldsymbol{v}_k)) e_k e_i e_j.$$

Thus, TS denotes $T \circ S$ if we consider T and S as right linear operators, but it denotes $S \circ T$ if we consider them as left linear operators.

2.3 Quaternionic functional analysis

In this section, we extend certain basic results of classical functional analysis to the quaternionic case. Again, results for right linear operators also hold for the left linear case with obvious modifications.

Definition 2.37. *Let V be a quaternionic two-sided vector space. A function $\|\cdot\| : V \to [0,\infty)$ is called a norm on V, if it satisfies*

(i) $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$

(ii) $\|a\boldsymbol{v}\| = \|\boldsymbol{v}a\| = |a|\|\boldsymbol{v}\|$ for all $\boldsymbol{v} \in V$ and all $a \in \mathbb{H}$

(iii) $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ *for all* $\boldsymbol{u}, \boldsymbol{v} \in V$.

If V *is complete with respect to the metric induced by* $\|\cdot\|$, *we call* V *a* quaternionic Banach space.

Corollary 2.38. *Let* V *be a quaternionic Banach space. Then* V *is a real Banach space, if we restrict the left and the right scalar multiplication to* $\mathbb{R}$. *Moreover, if we restrict either the left or the right scalar multiplication to the complex plane* $\mathbb{C}_I$ *for some imaginary unit* $I \in \mathbb{S}$, *then* V *is a complex Banach space over* $\mathbb{C}_I$.

Proof. We know from Remark 2.14 that V is a real vector space if we restrict the left and the right scalar multiplication to $\mathbb{R}$. In this case, the norm $\|\cdot\|$ on V satisfies the axioms of a norm on a real vector space and V is complete with respect to the induced metric. Hence, it is a real Banach space.

Similarly, we know form Remark 2.14 that V is a complex vector space over $\mathbb{C}_I$ if we restrict either the left or the right scalar multiplication to $\mathbb{C}_I$. Then, the norm $\|\cdot\|$ on V satisfies the axioms of a norm on a complex vector space. Since V is complete with respect to the induced metric, it is a complex Banach space.

□

Note that we only consider spaces with a left and a right vector space structure. We are interested in (right or left) linear operators on a quaternionic Banach space. Hence, it is of minor interest to consider only left or right vector spaces over $\mathbb{H}$ and endow them with a norm. As we pointed out in Remark 2.31, in this case, the set of right linear operators does not have the structure of a quaternionic vector space.

Definition 2.39. *Let* V *and* W *be two quaternionic Banach spaces. We denote the space of all continuous right linear mappings from* V *to* W *by* $\mathcal{B}_R^{\mathbb{H}}(V,W)$ *and also by* $\mathcal{B}_R(V,W)$ *if there is no possibility of confusion. We will denote the space of all quaternionic left linear operators by* $\mathcal{B}_L^{\mathbb{H}}(V,W)$ *resp.* $\mathcal{B}_L(V,W)$.

Let V *and* W *be two real Banach spaces. Then we denote the space of all continuous linear mappings from* V *to* W *by* $\mathcal{B}^{\mathbb{R}}(V,W)$ *and we endow it with the usual operator norm.*

Finally, if $V = W$, *we will write* $\mathcal{B}_R^{\mathbb{H}}(V)$, $\mathcal{B}_L^{\mathbb{H}}(V)$ *and* $\mathcal{B}^{\mathbb{R}}(V)$ *instead of* $\mathcal{B}_R^{\mathbb{H}}(V,W)$, $\mathcal{B}_L^{\mathbb{H}}(V,W)$ *and* $\mathcal{B}^{\mathbb{R}}(V,W)$, *respectively.*

Remark 2.40. By Corollar 2.38, a quaternionic Banach space is also a real Banach space if we restrict the scalar multiplication to the real numbers. Moreover, any quaternionic right linear mapping is also $\mathbb{R}$-linear. Thus, as a real vector space, $\mathcal{B}_R^{\mathbb{H}}(V,W)$ is a subspace of $\mathcal{B}^{\mathbb{R}}(V,W)$. Moreover, the limit of a sequence of quaternionic right linear operators in $\mathcal{B}^{\mathbb{R}}(V,W)$ must be quaternionic right linear, too. Hence, $\mathcal{B}_R^{\mathbb{H}}(V,W)$ is a closed subspace of $\mathcal{B}^{\mathbb{R}}(V,W)$, and therefore, a Banach space over $\mathbb{R}$. Finally, the operator norm satisfies the axiom (ii) in Definition 2.37, because

$$||aT|| = \sup_{\boldsymbol{v}\neq\boldsymbol{0}} \frac{\|aT(\boldsymbol{v})\|_W}{\|\boldsymbol{v}\|_V} = |a| \sup_{\boldsymbol{v}\neq\boldsymbol{0}} \frac{\|T(\boldsymbol{v})\|_W}{\|\boldsymbol{v}\|_V} = |a|\,\|T\|$$

and

$$\|Ta\| = \sup_{\boldsymbol{v}\neq\boldsymbol{0}} \frac{\|T(a\boldsymbol{v})\|_W}{\|\boldsymbol{v}\|_V} = |a| \sup_{\boldsymbol{v}\neq\boldsymbol{0}} \frac{\|T(a\boldsymbol{v})\|_W}{\|a\boldsymbol{v}\|_V} = |a|\,\|T\|$$

for any $a \in \mathbb{H}$ and any $T \in \mathcal{B}_R^{\mathbb{H}}(V,W)$. Therefore, $\mathcal{B}_R^{\mathbb{H}}(V,W)$ is a quaternionic Banach space if it is endowed with the usual operator norm.

We conclude this chapter with the proof of the quaternionic Hahn–Banach theorem, Theorem 4.10.1 in [12], which was first proved by Suchomlinov in [27]. For a linear functional ϕ and a vector $\boldsymbol{v}$, we use the notation $\langle \phi, \boldsymbol{v}\rangle := \phi(\boldsymbol{v})$.

Theorem 2.41 (Hahn–Banach)**.** *Let* V *be a quaternionic right vector space and let* V_0 *be a right subspace of* V. *Let* $p : V \to [0,\infty)$ *satisfy* $p(\boldsymbol{u}+\boldsymbol{v}) \leq p(\boldsymbol{u}) + p(\boldsymbol{v})$ *and* $p(\boldsymbol{u}a) = p(\boldsymbol{u})|a|$ *for all* $\boldsymbol{u}, \boldsymbol{v} \in V$ *and all* $a \in \mathbb{H}$. *Moreover, let* $\phi : V_0 \to \mathbb{H}$ *be a quaternionic right linear functional on* V_0 *such that* $|\langle \phi, \boldsymbol{v}\rangle| \leq p(\boldsymbol{v})$ *for all* $\boldsymbol{v} \in V_0$. *Then there exists a right linear functional* $\Phi : V \to \mathbb{H}$ *such that* $\Phi|_{V_0} = \phi$ *and such that*

$$|\langle \Phi, \boldsymbol{v}\rangle| \leq p(\boldsymbol{v}), \qquad \text{for all } \boldsymbol{v} \in V. \tag{2.16}$$

Proof. Let $I, J \in \mathbb{S}$ with $I \perp J$. Then V is a complex vector space over $\mathbb{C}_I$ by Remark 2.14, the space V_0 is a complex linear subspace of V and p is a seminorm on V. Moreover, $\mathbb{H}$ is a complex left vector space over $\mathbb{C}_I$ by Corollary 2.10. Therefore, we can consider the coordinate functions $z_1, z_2 : \mathbb{H} \to \mathbb{C}_I$ with respect to the basis $\{1, J\}$ as they are defined in (2.8), such that $x = z_1(x) + z_2(x)J$ for all $x \in \mathbb{H}$. Since $xJ = -z_2(x) + z_1(x)J$, we obtain $z_2(x) = -z_1(xJ)$.

Let $\phi_1 = z_1 \circ \phi$ and $\phi_2 = z_2 \circ \phi$. Then $\phi = \phi_1 + \phi_2 J$, and hence,

$$\begin{aligned}\langle \phi, \boldsymbol{v}\rangle = \langle \phi_1, \boldsymbol{v}\rangle + \langle \phi_2, \boldsymbol{v}\rangle J = z_1(\langle \phi, \boldsymbol{v}\rangle) + z_2(\langle \phi, \boldsymbol{v}\rangle)J = \\ = z_1(\langle \phi, \boldsymbol{v}\rangle) - z_1(\langle \phi, \boldsymbol{v}J\rangle)J = \langle \phi_1, \boldsymbol{v}\rangle - \langle \phi_1, \boldsymbol{v}J\rangle J.\end{aligned}$$

Moreover, for any $a \in \mathbb{C}_I$ and any $\boldsymbol{u}, \boldsymbol{v} \in V_0$, we have

$$\begin{aligned}&\langle \phi_1, \boldsymbol{u}a + \boldsymbol{v}\rangle + \langle \phi_2, \boldsymbol{u}a + \boldsymbol{v}\rangle J = \langle \phi, \boldsymbol{u}a + \boldsymbol{v}\rangle = \langle \phi, \boldsymbol{u}\rangle a + \langle \phi, \boldsymbol{v}\rangle = \\ &\quad = \langle \phi_1, \boldsymbol{u}\rangle a + \langle \phi_2, \boldsymbol{u}\rangle Ja + \langle \phi_1, \boldsymbol{v}\rangle + \langle \phi_2, \boldsymbol{v}\rangle J = \underbrace{\langle \phi_1, \boldsymbol{u}\rangle a + \langle \phi_1, \boldsymbol{v}\rangle}_{\in \mathbb{C}_I} + \underbrace{\langle \phi_2, \boldsymbol{u}\rangle Ja + \langle \phi_2, \boldsymbol{v}\rangle J}_{\in \mathbb{C}_I J}.\end{aligned}$$

Since 1 and J are linearly independent over $\mathbb{C}_I$, we obtain $\langle \phi_1, \boldsymbol{u}a + \boldsymbol{v}\rangle = \langle \phi_1, \boldsymbol{u}\rangle a + \langle \phi_1, \boldsymbol{v}\rangle$. Thus, ϕ_1 is $\mathbb{C}_I$-linear. By the classical Hahn–Banach theorem, there exists a $\mathbb{C}_I$-linear functional $\Phi_1 : V \to \mathbb{C}_I$ that extends ϕ_1 and satisfies $|\langle \Phi_1, \boldsymbol{v}\rangle| \leq p(\boldsymbol{v})$ for all $\boldsymbol{v} \in V$. If we set

$$\langle \Phi, \boldsymbol{v}\rangle = \langle \Phi_1, \boldsymbol{v}\rangle - \langle \Phi_1, \boldsymbol{v}J\rangle J \qquad \text{for all } \boldsymbol{v} \in V,$$

then Φ extends ϕ to V and satisfies $\langle \Phi, \boldsymbol{u} + \boldsymbol{v}\rangle = \langle \Phi, \boldsymbol{u}\rangle + \langle \Phi, \boldsymbol{v}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$. Moreover, for any $a = a_1 + a_2 J \in \mathbb{H}$, we have $a_i J = J\,\overline{a_i}$ by Corollary 2.11 because $a_i \in \mathbb{C}_I$. Therefore,

$$\begin{aligned}\langle \Phi, \boldsymbol{v}a\rangle &= \langle \Phi_1, \boldsymbol{v}a\rangle - \langle \Phi_1, \boldsymbol{v}aJ\rangle J = \langle \Phi_1, \boldsymbol{v}(a_1 + a_2 J)\rangle - \langle \Phi_1, \boldsymbol{v}(a_1 J - a_2)\rangle J = \\ &= \langle \Phi_1, \boldsymbol{v}\rangle a_1 + \langle \Phi_1, \boldsymbol{v}J\rangle \overline{a_2} - \langle \Phi_1, \boldsymbol{v}J\rangle J a_1 + \langle \Phi_1, \boldsymbol{v}\rangle a_2 J = \\ &= \langle \Phi_1, \boldsymbol{v}\rangle a_1 - \langle \Phi_1, \boldsymbol{v}J\rangle J a_2 J - \langle \Phi_1, \boldsymbol{v}J\rangle J a_1 + \langle \Phi_1, \boldsymbol{v}\rangle a_2 J = \\ &= (\langle \Phi_1, \boldsymbol{v}\rangle - \langle \Phi_1, \boldsymbol{v}J\rangle J)(a_1 + a_2 J) = \langle \Phi, \boldsymbol{v}\rangle a.\end{aligned}$$

Hence, Φ is a quaternionic right linear functional on V.

Finally, if $\boldsymbol{v} \in V$ with $\langle \Phi, \boldsymbol{v}\rangle \neq 0$, then set $x = \langle \Phi, \boldsymbol{v}\rangle$. Let $I_x \in \mathbb{S}$ and $x_0, x_1 \in \mathbb{R}$ be such that $x = x_0 + I_x x_1$. If we set $x_I = x_0 + I x_1$, then $|x^{-1}x_I| = 1$ and $\langle \Phi, \boldsymbol{v}x^{-1}x_I\rangle = \langle \Phi, \boldsymbol{v}\rangle x^{-1}x_I = x_I$ belongs to $\mathbb{C}_I$. Therefore, $\langle \Phi, \boldsymbol{v}x^{-1}x_I\rangle = \langle \Phi_1, \boldsymbol{v}x^{-1}x_I\rangle$ and we obtain

$$|\langle \Phi, \boldsymbol{v}\rangle| = |\langle \Phi, \boldsymbol{v}\rangle x^{-1}x_I| = |\langle \Phi, \boldsymbol{v}x^{-1}x_I\rangle| = |\langle \Phi_1, \boldsymbol{v}x^{-1}x_I\rangle| \leq p(\boldsymbol{v}x^{-1}x_I) = p(\boldsymbol{v})|x^{-1}x_I| = p(\boldsymbol{v}).$$

□

Definition 2.42. *Let V be a quaternionic Banach space. We define $V_R' = \mathcal{B}_R(V, \mathbb{H})$ and call V_R'* the right dual space of V. *Similarly, we define $V_L' = \mathcal{B}_L(V, \mathbb{H})$ and call V_L'* the left dual space of V.

Corollary 2.43. *Let V be a quaternionic Banach space. Then the right dual space and the left dual space of V separate points.*

Proof. For $\boldsymbol{u}, \boldsymbol{v} \in V$ with $\boldsymbol{u} \neq \boldsymbol{v}$, we set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ and $f(\boldsymbol{w}) = \|\boldsymbol{w}\|$. Because of Lemma 2.32, there exists a unique quaternionic right linear functional ϕ on the right subspace $\operatorname{span}_R\{\boldsymbol{w}\} = \{\boldsymbol{w}a : a \in \mathbb{H}\}$ that extends f. Moreover, we have

$$|\langle \phi, \boldsymbol{w}a\rangle| = |\langle \phi, \boldsymbol{w}\rangle|\,|a| = \|\boldsymbol{w}\|\,|a| = \|\boldsymbol{w}a\|.$$

Thus, we can apply the quaternionic Hahn–Banach theorem to obtain a quaternionic right linear functional Φ that extends ϕ to V and satisfies $|\langle \phi, \boldsymbol{t}\rangle| \leq \|\boldsymbol{t}\|$ for all $\boldsymbol{t} \in V$. We have found $\Phi \in V_R'$ such that $\langle \Phi, \boldsymbol{v}\rangle - \langle \Phi, \boldsymbol{u}\rangle = \langle \Phi, \boldsymbol{w}\rangle = \|\boldsymbol{w}\| \neq 0$. Hence, V_R' separates points.

□

Chapter 3

Slice regular functions

We introduce now the notion of slice regularity and develop the theory of slice regular functions as far as it is necessary to define the associated functional calculus. We follow Chapters 2 and 4 of [12], except for the discussion of Runge's Theorem, which can be found in [13].

3.1 The definition of slice regular functions

Let $I \in \mathbb{S}$ be an imaginary unit. By ∂_I and $\bar{\partial}_I$, we denote the Wirtinger derivatives with respect to the complex and the complex conjugate variable on the plane $\mathbb{C}_I$, that is, the operators

$$\partial_I = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - I\frac{\partial}{\partial x_1}\right) \qquad \text{and} \qquad \bar{\partial}_I = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1}\right).$$

In accordance with Definition 2.36, they act as follows: Let $U \subset \mathbb{C}_I$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. If ∂_I and $\bar{\partial}_I$ act on the right, then

$$\partial_I f(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x_0 + Ix_1) - I\frac{\partial}{\partial x_1} f_I(x_0 + Ix_1)\right)$$

and

$$\bar{\partial}_I f(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x_0 + Ix_1) + I\frac{\partial}{\partial x_1} f_I(x_0 + Ix_1)\right)$$

for $x = x_0 + Ix_1 \in U$. If they act on the left, then

$$(f\partial_I)(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x_0 + Ix_1) - \frac{\partial}{\partial x_1} f_I(x_0 + Ix_1)I\right)$$

and

$$(f\bar{\partial}_I)(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x_0 + Ix_1) + \frac{\partial}{\partial x_1} f_I(x_0 + Ix_1)I\right).$$

Recall from Corollary 2.10 that $\mathbb{H}$ is a left and a right vector space over the complex plane $\mathbb{C}_I$ for any $I \in \mathbb{S}$. In accordance with Lemma 1.12, the characterization of holomorphicity in the Wirtinger calculus, we give the following definition.

Definition 3.1. *Let $I \in \mathbb{S}$, let $U \subset \mathbb{C}_I$ be open and let $f : U \to \mathbb{H}$ be real differentiable. The function f is called* left holomorphic*, if $\bar{\partial}_I f = 0$ on U, that is, if f is a holomorphic function with values in $\mathbb{H}$, where $\mathbb{H}$ is considered a left vector space over $\mathbb{C}_I$. It is called* right holomorphic*, if $f\bar{\partial}_I = 0$ on U, that is, if f is a holomorphic function with values in $\mathbb{H}$, where $\mathbb{H}$ is considered a right vector space over $\mathbb{C}_I$.*

Definition 3.2. *Let $U \subset \mathbb{H}$ be open and let $f : U \to \mathbb{H}$ be a real differentiable function. For $I \in \mathbb{S}$, we denote the restriction of f to the set $U \cap \mathbb{C}_I$ by f_I, that is, $f_I = f|_{U\cap\mathbb{C}_I}$. The function f is called* left slice regular *on U if f_I is left holomorphic for any $I \in \mathbb{S}$, that is, if*

$$\bar{\partial}_I f_I = 0$$

on $U \cap \mathbb{C}_I$ for all $I \in \mathbb{S}$. It is called right slice regular *if f_I is right holomorphic for any $I \in \mathbb{S}$, that is,*

$$f_I \bar{\partial}_I = 0$$

on $U \cap \mathbb{C}_I$ for all $I \in \mathbb{S}$. We will denote the set of left slice regular and right slice regular functions on an open set U by $\mathcal{M}^L(U)$ and $\mathcal{M}^R(U)$, respectively.

Moreover, we say that f is left slice regular on a closed set $C \subset \mathbb{H}$, if there exists an open set O with $C \subset O$, such that f is left slice regular on O. We say that f is right slice regular on C, if there exists an open set O with $C \subset O$, such that f is right slice regular on O.

The theory of right slice regular functions is analogous to the one of left slice regular functions. We will state the results for both cases, but we will only give the proofs for the left slice regular one, because the results for right slice regular functions follow with obvious modifications from these proofs.

Corollary 3.3. *Let U be an open subset of $\mathbb{H}$. The set $\mathcal{M}^L(U)$ of left slice regular functions on U endowed with the pointwise addition $(f+g)(z) = f(z) + g(z)$ and the right scalar multiplication $(fa)(z) = f(z)a$ for $a \in \mathbb{H}$ is a quaternionic right vector space.*

The set $\mathcal{M}^R(U)$ of right slice regular functions on U endowed with the pointwise addition and the left scalar multiplication $(af)(z) = af(z)$ for $a \in \mathbb{H}$ is a quaternionic left vector space.

Proof. Let $f, g \in \mathcal{M}^L(U)$ and $a \in \mathbb{H}$. Then, for $x = x_0 + Ix_1 \in U$, we have

$$\bar{\partial}_I(f_I + g_I)(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + \frac{\partial}{\partial x_0} g_I(x) + I\frac{\partial}{\partial x_1} f_I(x) + I\frac{\partial}{\partial x_1} g_I(x)\right) = \bar{\partial}_I f_I(x) + \bar{\partial}_I g_I(x) = 0$$

and

$$\bar{\partial}_I(f_I a)(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x)a + I\frac{\partial}{\partial x_1} f_I(x)a\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + I\frac{\partial}{\partial x_1} f_I(x)\right)a = (\bar{\partial}_I f(x))a = 0.$$

Thus, $\mathcal{M}^L(U)$ is closed under the pointwise addition and the multiplication with scalars from the right. □

Note that $\mathcal{M}^L(U)$ is not closed under multiplication with scalars on the left because, in general, $a \in \mathbb{H}$ and $\bar{\partial}_I$ do not commute as the following example shows.

Example 3.4. One important difference between the notion of slice regularity and the notion of Cauchy-Fueter-regularity considered in Definition 1.13 is that polynomials of a quaternionic variable are slice regular. Indeed, any monomial of the form $x^n a$ with $n \in \mathbb{N}$ and $a \in \mathbb{H}$ is left slice regular as

$$\bar{\partial}_I x^n a = \frac{1}{2}\left(\frac{\partial}{\partial x_0} x^n a + I\frac{\partial}{\partial x_1} x^n a\right) = \frac{1}{2}\left(nx^{n-1}a + I^2 nx^{n-1}a\right) = 0$$

for $x = x_0 + Ix_1$. By Corollary 3.3, also polynomials of the form $\sum_{n=0}^{N} x^n a_n$ with $a_n \in \mathbb{H}$ are left slice regular. On the contrary, monomials of the form ax^n are in general not left slice regular. If $x = x_0 + Ix_1$ and $a \notin \mathbb{C}_I$, then a and I do not commute because of Corollary 2.12. Hence,

$$\bar{\partial}_I ax^n = \frac{1}{2}\left(\frac{\partial}{\partial x_0} ax^n + I\frac{\partial}{\partial x_1} ax^n\right) = \frac{1}{2}\left(anx^{n-1} + IaInx^{n-1}\right) \neq \frac{1}{2}\left(anx^{n-1} + I^2 anx^{n-1}\right) = 0.$$

Similarly, polynomials of the form $\sum_{n=0}^{N} a_n x^n, a_n \in \mathbb{H}$ are right slice regular, but not left slice regular.

Furthermore, as in the case of Cauchy-Fueter-regularity, the product and the composition of two left slice regular functions are in general not left slice regular and the product and the composition of two right slice regular functions are in general not right slice regular. An easy counterexample is the function $f(x) = xaxa$ with $a \in \mathbb{H} \setminus \mathbb{R}$. Then $f(x)$ can either be considered as the left slice regular function $x \mapsto xa$ multiplied by itself or as the composition of the left slice regular functions $x \mapsto xa$ and $x \mapsto x^2$. But for $x = x_0 + Ix_1$ and $a \notin \mathbb{C}_I$, a and I do not commute, and we obtain

$$\bar{\partial}_I f_I(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} xaxa + I\frac{\partial}{\partial x_1} xaxa\right) = \frac{1}{2}\left(axa + xa^2 + I^2 axa + IxaIa\right) =$$

$$= \frac{1}{2}\left(xa^2 + IxaIa\right) \neq \frac{1}{2}\left(xa^2 + I^2 xa^2\right) = 0.$$

Hence, f itself is not left slice regular.

3.2 Representation formulas and extension theorems

By definition, a function is slice regular, if it is holomorphic on every complex plane $\mathbb{C}_I$ in $\mathbb{H}$. The following Lemma specifies this idea.

Lemma 3.5 (Splitting Lemma)**.** *Let $U \subset \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be real differentiable. Then f is left slice regular if and only if, for any $I, J \in \mathbb{S}$ with $I \perp J$, there exist holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that*

$$f_I(x) = f_1(x) + f_2(x)J \qquad \text{for all } x \in U \cap \mathbb{C}_I.$$

Similarly, f is right slice regular if and only if, for any $I, J \in \mathbb{S}$ with $I \perp J$, there exist holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that

$$f_I(x) = f_1(x) + Jf_2(x) \qquad \text{for all } x \in U \cap \mathbb{C}_I.$$

Proof. Let f be left slice regular and let $I, J \in \mathbb{S}$ with $I \perp J$. By Corollary 2.10, there exist functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I(x) = f_1(x) + f_2(x)J$ for all $x \in U \cap \mathbb{C}_I$. Since f is real differentiable, f_1 and f_2 must be real differentiable, too. Moreover, if f is left slice regular, it satisfies $\bar{\partial}_I f = 0$. Thus, for $x = x_0 + Ix_1$, we have

$$0 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + I\frac{\partial}{\partial x_1} f_I(x)\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_1(x) + I\frac{\partial}{\partial x_1} f_1(x)\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_2(x) + I\frac{\partial}{\partial x_1} f_2(x)\right) J,$$

which implies

$$\frac{1}{2}\left(\frac{\partial}{\partial x_0} f_1(x) + I\frac{\partial}{\partial x_1} f_1(x)\right) = 0 \qquad \text{and} \qquad \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_2(x) + I\frac{\partial}{\partial x_1} f_2(x)\right) = 0.$$

Hence, f_1 and f_2 are holomorphic by Lemma 1.12.

On the contrary, if $f_I = f_1 + f_2 J$, where f_1 and f_2 are holomorphic functions from $U \cap \mathbb{C}_I \to \mathbb{C}_I$, then $\bar{\partial}_I f_1 = 0$ and $\bar{\partial}_I f_2 = 0$, and in turn,

$$\bar{\partial}_I f_I = \bar{\partial}_I f_1 + \bar{\partial}_I f_2 J = 0.$$

Hence, if such a decomposition of f_I exists for any $I, J \in \mathbb{S}$ with $I \perp J$, then f is left slice regular . □

Slice regular functions allow the development of a rich function theory, if their domains of definition satisfy certain regularity assumptions.

Definition 3.6. *Let U be an open subset of $\mathbb{H}$. Then U is called slice domain if $U \cap \mathbb{R}$ is nonempty and if $U \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$, that is an open and connected subset of $\mathbb{C}_I$ for all $I \in \mathbb{S}$.*

Corollary 3.7. *Let $U \subset \mathbb{H}$ be a slice domain. Then U is a domain in $\mathbb{H}$.*

Proof. The set $U = \bigcup_{I\in\mathbb{S}}(U \cap \mathbb{C}_I)$ is the union of connected sets, whose intersection is nonempty because $\bigcap_{I\in\mathbb{S}}(U \cap \mathbb{C}_I) = U \cap \mathbb{R} \neq \emptyset$. Hence, it is connected itself. Since it is also open by definition, it is a domain. □

The condition $U \cap \mathbb{R} \neq 0$ in Definition 3.6 is essential as can been seen from the proof of the following theorem.

Theorem 3.8 (Identity Principle)**.** *Let $U \subset \mathbb{H}$ be a slice domain, let $f : U \to \mathbb{H}$ be a left or right slice regular function and let $Z = \{x \in U : f(x) = 0\}$. If there exists an imaginary unit $I \in \mathbb{S}$ such that $Z \cap \mathbb{C}_I$ has an accumulation point in $U \cap \mathbb{C}_I$, then $f \equiv 0$ on U.*

Proof. Let f be left slice regular on U, let $I \in \mathbb{S}$ be an imaginary unit such that $Z \cap \mathbb{C}_I$ has an accumulation point in $U \cap \mathbb{C}_I$ and let $J \in \mathbb{S}$ with $I \perp J$. By applying the Splitting Lemma, Lemma 3.5, we obtain holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I(x) = f_1(x) + f_2(x)J$. Since $Z_1 = \{x \in U \cap \mathbb{C}_I : f_1(x) = 0\}$ and $Z_2 = \{x \in U \cap \mathbb{C}_I : f_2(x) = 0\}$ are supersets of $Z \cap \mathbb{C}_I$, they have an accumulation point in $U \cap \mathbb{C}_I$, too. Thus, from the identity theorem for holomorphic functions, it follows that $f_1 \equiv 0$ and $f_2 \equiv 0$. Hence, $f \equiv 0$ on $U \cap \mathbb{C}_I$. In particular, we obtain that $f \equiv 0$ on $U \cap \mathbb{R}$.

Now let $K \in \mathbb{S}$ be an arbitrary imaginary unit. Then the set of zeros of f_K has an accumulation point in $\mathbb{C}_K$ as $f_K|_{\mathbb{R}} \equiv 0$. Therefore, we can repeat the above arguments to show that $f_K \equiv 0$ on $U \cap \mathbb{C}_K$, and in turn, $f \equiv 0$ on U.

□

By applying the Identity Principle to $f - g$, we can easily deduce the following corollary.

Corollary 3.9. *Let $U \subset \mathbb{H}$ be a slice domain, let $f, g \in \mathcal{M}^L(U)$ or $f, g \in \mathcal{M}^R(U)$ and let $Z = \{z \in U : f(z) = g(z)\}$. If there exists an imaginary unit $I \in \mathbb{S}$ such that $Z \cap \mathbb{C}_I$ has an accumulation point, then $f \equiv g$ on U.*

The second important regularity assumption on the domain of definition of slice regular functions is axial symmetry.

Definition 3.10. *For a quaternion $x \in \mathbb{H}$, we set*

$$I_x = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0 \\ \text{any element of } \mathbb{S} & \text{otherwise,} \end{cases}$$

where $\underline{x}$ denotes the vector part of x as in Definition 2.2.

Corollary 3.11. *Let $x \in \mathbb{H}$. Then $I_x \in \mathbb{S}$ and $x \in \mathbb{C}_{I_x}$. More precisely, $x = x_0 + I_x x_1$ with $x_0 = \mathrm{Re}[x]$ and $x_1 = |\underline{x}|$.*

Proof. Let $x \in \mathbb{R}$. Then $x \in \mathbb{C}_I$ and $x = x_0 + Ix_1$ with $x_0 = \mathrm{Re}[x] = x$ and $x_1 = 0 = |\underline{x}|$ for any $I \in \mathbb{S}$. Otherwise, if $x \notin \mathbb{R}$, then $|\underline{x}| \neq 0$ and we have $\mathrm{Re}[I_x] = \mathrm{Re}[\underline{x}]/|\underline{x}| = 0$ and $|I_x| = |\underline{x}|/|\underline{x}| = 1$. Hence, $I_x \in \mathbb{S}$. Moreover, we have

$$x = \mathrm{Re}[x] + \underline{x} = x_0 + \frac{\underline{x}}{|\underline{x}|}|\underline{x}| = x_0 + I_x x_1.$$

□

Definition 3.12. *For $x = x_0 + I_x x_1 \in \mathbb{H}$, we define*

$$[x] = x_0 + \mathbb{S}x_1 = \{x_0 + Ix_1 : I \in \mathbb{S}\}.$$

The set $[x]$ is a 2-sphere of radius $x_1 = |\underline{x}|$ centered at the real point x_0, which reduces to the point $x = x_0$ if $|\underline{x}| = 0$. In particular, when we refer to a 2-sphere in the following, we always include the degenerated case of a single point.

Definition 3.13. *A set $\Omega \subset \mathbb{H}$ is called* axially symmetric *if, for any $x \in \Omega$, the entire 2-sphere $[x]$ is contained in Ω.*

It is an important fact, that any slice domain can be extended to a larger slice domain that is axially symmetric.

Definition 3.14. *Let $\Omega \subset \mathbb{H}$. We call $[\Omega] = \bigcup_{x \in \Omega}[x]$ the* axially symmetric hull *of Ω.*

Example 3.15. Let $x = x_0 + I_x x_1 \in \mathbb{H}$ and let $\varepsilon > 0$. We determine the axially symmetric hull of the ball $B_\varepsilon(x) \subset \mathbb{H}$. Note that $B_\varepsilon(x) \cap \mathbb{C}_{I_x} = \{y_0 + I_x y_1 : (y_0, y_1) \in B_\varepsilon(x_0, x_1)\}$, where $B_\varepsilon(x_0, x_1)$ is the ball of radius ε centered at (x_0, x_1) in $\mathbb{R}^2$. Hence,

$$[B_\varepsilon(x)] \supset \{y_0 + Iy_1 : (y_0, y_1) \in B_\varepsilon(x_0, x_1)\}. \tag{3.1}$$

For $I \in \mathbb{S}$, set $x_I = x_0 + Ix_1 \in [x]$. If $y = y_0 + I_y y_1 \in \mathbb{H}$, then we can choose $J \in \mathbb{S}$ with $I_y \perp J$ such that $I \in \operatorname{span}\{I_y, J\}$. Then $x_I = x_0 + \tilde{x}_1 I_y + \tilde{x}_2 J$, where $x_1 = |\underline{x_I}| = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}$ by Corollary 3.11. Hence,

$$\begin{aligned}\|(x_0, x_1) - (y_0, y_1)\|^2 = (x_0 - y_0)^2 + (x_1 - y_1)^2 = (x_0 - y_0)^2 + x_1^2 - 2x_1 y_1 + y_1^2 = \\ = (x_0 - y_0)^2 + \tilde{x}_1^2 + \tilde{x}_2^2 - 2\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}\, y_1 + y_1^2 \leq \\ \leq (x_0 - y_0)^2 + \tilde{x}_1^2 + \tilde{x}_2^2 - 2\tilde{x}_1 y_1 + y_1^2 = (x_0 - y_0)^2 + (\tilde{x}_1 - y_1)^2 + \tilde{x}_2^2 = |y - x_I|^2.\end{aligned}$$

If $I = I_y$, then $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = 0$ and we obtain a chain of equalities. Hence,

$$\|(x_0, x_1) - (y_0, y_1)\| \leq \operatorname{dist}(y, [x]) = \inf_{x_I \in [x]} |y - x_I| \leq |y - x_{I_y}| = \|(x_0, x_1) - (y_0, y_1)\|,$$

and in turn

$$\operatorname{dist}(y, [x]) = |y - x_{I_y}| = \|(x_0, x_1) - (y_0, y_1)\|. \tag{3.2}$$

This implies $\|(x_0, x_1) - (y_0, y_1)\| = \operatorname{dist}(y, [x]) < |y - x| < \varepsilon$ for any $y = y_0 + I_y y_1 \in B_\varepsilon(x)$. Hence, in (3.1), the reverse inclusion also holds true. Altogether, we obtain

$$[B_\varepsilon(x)] = \{y_0 + Iy_1 : (y_0, y_1) \in B_\varepsilon(x_0, x_1)\} = B_\varepsilon([x]),$$

where $B_\varepsilon([x]) = \{y \in \mathbb{H} : \operatorname{dist}(y, [x]) < \varepsilon\}$.

We denote the closed upper half plane in $\mathbb{R}^2$ by $\mathbb{R}^2_+$, that is $\mathbb{R}^2_+ = \{(x_0, x_1) \in \mathbb{R}^2 : x_1 \geq 0\}$, and we define the function

$$\Psi : \begin{cases} \mathbb{H} & \to \mathbb{R}^2_+ \\ x = x_0 + I_x x_1 & \mapsto (x_0, x_1) \end{cases}, \tag{3.3}$$

which is a very useful tool for investigating the relation between the topological properties of a a set Ω and its axially symmetric hull $[\Omega]$.

Corollary 3.16. *Let $\Omega \subset \mathbb{H}$. Then $[\Omega] = \Psi^{-1}(\Psi(\Omega))$.*

Proof. We have $\Psi(\Omega) = \{(x_0, x_1) \in \mathbb{R}^2_+ : x = x_0 + I_x x_1 \in \Omega \text{ for some } I_x \in \mathbb{S}\}$ and $\Psi^{-1}(x_0, x_1) = \{x_0 + Ix_1 : I \in \mathbb{S}\} = [x_0 + I_x x_1]$. Hence,

$$\Psi^{-1}(\Psi(\Omega)) = \bigcup_{(x_0, x_1) \in \Psi(\Omega)} [x_0 + I_x x_1] = \bigcup_{x \in \Omega} [x] = [\Omega].$$

□

Example 3.17. Let us again consider the case of a ball $B_\varepsilon(x)$, where $x = x_0 + I_x x_1$ as in Corollary 3.11. Since

$$B_\varepsilon(x) \cap \mathbb{C}_{I_x} = \{y_0 + I_x y_1 : (y_0, y_1) \in B_\varepsilon(x_0, x_1)\}, \tag{3.4}$$

we have $B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+ \subset \Psi(B_\varepsilon(x))$. If on the other hand we write $y \in B_\varepsilon(x)$ as $y = y_0 + I_y y_1$ according to Corollary 3.11, then $\|(x_0, x_1) - (y_0, y_1)\| < \varepsilon$ as we have seen in Example 3.15. Since $y_1 = |\underline{y}| \geq 0$, this implies $(y_0, y_1) \in B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+$. Hence,

$$\Psi(B_\varepsilon(x)) = B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+.$$

The equation (3.4) implies that even $\Psi(B_\varepsilon(x) \cap \mathbb{C}_{I_x}) = B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+$. As a consequence, we obtain a more precise description of the axially symmetric hull of $B_\varepsilon(x)$ from Corollary 3.16, namely

$$[B_\varepsilon(x)] = \Psi^{-1}(\Psi(B_\varepsilon(x))) = \{y_0 + Iy_1 : (y_0, y_1) \in B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+\}.$$

Moreover, for the imaginary unit I_x, we have

$$[B_\varepsilon(x) \cap \mathbb{C}_{I_x}] = \Psi^1(\Psi(B_\varepsilon(x) \cap \mathbb{C}_{I_x})) = \{y_0 + Iy_1 : (y_0, y_1) \in B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+\} = [B_\varepsilon(x)].$$

Lemma 3.18. *The function Ψ defined in* (3.3) *is continuous, open and closed. Moreover, for any $I \in \mathbb{S}$, the restriction of Ψ to the plane $\mathbb{C}_I$ is continuous, open and closed, too.*

Proof. By Corollary 3.11, we have $\Psi(x) = (\operatorname{Re}[x], |\underline{x}|)$ for $x \in \mathbb{H}$. Hence, Ψ is continous. Consequently, its restriction to a complex plane $\mathbb{C}_I$ is continuous, too.

Let $O \subset \mathbb{H}$ be open. For $(x_0, x_1) \in \Psi(O) \subset \mathbb{R}^2_+$, there exists $I_x \in \mathbb{S}$ such that $x = x_0 + I_x x_1$ belongs to O. Since O is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset O$. In Example 3.17 we have seen that $\Psi(B_\varepsilon(x)) = B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+$, which is a neighborhood of (x_0, x_1) in $\mathbb{R}^2_+$. Since $\Psi(B_\varepsilon(x)) \subset \Psi(O)$, the set $\Psi(O)$ is open, and in turn, Ψ is an open map. We also have $\Psi(B_\varepsilon(x) \cap \mathbb{C}_I) = B_\varepsilon(x_0, x_1) \cap \mathbb{R}^2_+$ for any $x = x_0 + I x_1 \in \mathbb{C}_I$. Hence, the same argument shows that the restriction of Ψ to a complex plane $\mathbb{C}_I$ is open, too.

Finally, assume that $C \subset \mathbb{H}$ is closed and let $(x_{0,n}, x_{1,n}), n \in \mathbb{N}$, be a sequence in $\Psi(C)$ such that $(x_0, x_1) = \lim_{n\to\infty}(x_{0,n}, x_{1,n})$ exists. Then there exists a sequence of imaginary units $I_n, n \in \mathbb{N}$ such that $x_n = x_{0,n} + I_n x_{1,n}$ belongs to C for any $n \in \mathbb{N}$. Since the sequence $(x_{0,n}, x_{1,n})$ is convergent, it is bounded in $\mathbb{R}^2$. The identity $|x_n|^2 = x_{0,n}^2 + x_{1,n}^2 = \|(x_{0,n}, x_{1,n})\|^2$, implies that the sequence $x_n, n \in \mathbb{N}$ is bounded in $\mathbb{H}$. Hence, it contains a convergent subsequence $x_{n_k}, k \in \mathbb{N}$. The limit $x = \lim_{k\to\infty} x_{n,k}$ belongs to C because C is closed, and the continuity of Ψ implies that $(x_0, x_1) = \lim_{k\to\infty}(x_{0,n_k}, x_{1,n_k}) = \lim_{k\to\infty} \Psi(x_{n_k}) = \Psi(x)$ belongs to $\Psi(C)$. Hence, $\Psi(C)$ is closed and Ψ is a closed map. Since the restriction of a closed map to a closed set is closed again, $\Psi|_{\mathbb{C}_I}$ is closed for any $I \in \mathbb{S}$.

□

Lemma 3.19. *If $\Omega \subset \mathbb{H}$ is open (closed, compact), then $[\Omega]$ is open (closed, compact). Moreover,* $\sup\{|x| : x \in \Omega\} = \sup\{|x| : x \in [\Omega]\}$.

If $I \in \mathbb{S}$ and $\Omega_I \subset \mathbb{C}_I$ is open (closed, compact) in $\mathbb{C}_I$, then $[\Omega_I]$ is open (closed, compact).

Proof. The function Ψ defined in (3.3) is open, closed and continuous by Lemma 3.18. Hence, the set $\Psi(\Omega)$ is open (closed), if Ω is open (closed), and in turn, by Corollary 3.16, $[\Omega] = \Psi^{-1}(\Psi(\Omega))$ is open (closed).

By the Heine-Borel-Theorem a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. The identity $|x| = \sqrt{x_0^2 + x_1^2} = \|(x_0, x_1)\|$ for $x = x_0 + I_x x_1 \in \mathbb{H}$ implies

$$\sup\{|x| : x \in \Omega\} = \sup\{\|(x_0, x_1)\| : (x_0, x_1) \in \Psi(\Omega)\} = \sup\{|x| : x \in \Psi^{-1}(\Psi(\Omega))\} = \sup\{|x| : x \in [\Omega]\}.$$

Hence, if Ω is compact, then it is closed and bounded, and in turn, the set $[\Omega]$ is closed and bounded, too. Therefore, it is compact.

Since $\Psi|_{\mathbb{C}_I}$ is open, closed and continuous by Lemma 3.18, the same arguments show that Ω_I is open (closed, compact) if $[\Omega_I]$ is open (closed, compact) in $\mathbb{C}_I$.

□

Lemma 3.20. *Let $U \subset \mathbb{H}$ be a domain in $\mathbb{H}$ with $U \cap \mathbb{R} \neq \emptyset$ or let $U \subset \mathbb{C}_J$ be a domain in $\mathbb{C}_J$ with $U \cap \mathbb{R} \neq \emptyset$ for some imaginary unit J. Then $[U]$ is a slice domain and there exists a domain $D_{[U]}$ in $\mathbb{R}^2$ that is symmetric with respect to the x_0-axis such that $x_0 + I x_1 \in [U]$ if and only if $(x_0, x_1) \in D_{[U]}$. In particular, $[U] = \{x_0 + I x_1 : (x_0, x_1) \in D_{[U]}, I \in \mathbb{S}\}$.*

Proof. The axially symmetric hull $[U]$ of U is open by Lemma 3.19 because U is open in $\mathbb{H}$ (resp. $\mathbb{C}_J$). Consequently, the set $[U] \cap \mathbb{C}_I$ is open in $\mathbb{C}_I$ for any $I \in \mathbb{S}$.

As $[U] = \Psi^{-1}(\Psi(U)) = \{x_0 + I x_1 : (x_0, x_1) \in \Psi(U), I \in \mathbb{S}\}$, a quaternion x belongs to $[U] \cap \mathbb{C}_I$ if and only if $x = x_0 + I x_1$ or $x = x_0 + (-I)x_1$ with $(x_0, x_1) \in \Psi(U)$. Since $x_0 + (-I)x_1 = x_0 + I(-x_1)$, this is equivalent to $x = x_0 + I x_1$ with $(x_0, x_1) \in D_{[U]}$, where we set $D_{[U]} = \Psi(U) \cup -\Psi(U)$ with $-\Psi(U) = \{(x_0, -x_1) : (x_0, x_1) \in \Psi(U)\}$.

The set $\Psi(U)$ is open in $\mathbb{R}^2_+$ because U is open and because Ψ resp. $\Psi|_{\mathbb{C}_J}$ are open mappings by Lemma 3.18. Consequently, $D_{[U]}$ is open in $\mathbb{R}^2$ because it is the preimage of $\Psi(U)$ under the continuous mapping $(x_0, x_1) \mapsto (x_0, |x_1|)$ from $\mathbb{R}^2$ to $\mathbb{R}^2_+$. Since U is connected and Ψ is continuous, the set $\Psi(U)$ is connected, and in turn, $-\Psi(U)$ is connected, too. The set $\Psi(U) \cap -\Psi(U) = \{(x_0, x_1) \in \Psi(U) : x_1 = 0\} = \Psi(U \cap \mathbb{R})$ is nonempty. Therefore, $D_{[U]} = \Psi(U) \cup -\Psi(U)$ is connected because the union of connected sets with nonempty intersection is again connected. Altogether, we obtain that $D_{[U]}$ is a domain in $\mathbb{R}^2$ that is symmetric with respect to the x_0-axis such that $[U] = \{x_0 + I x_1 : (x_0, x_1) \in D_{[U]}, I \in \mathbb{S}\}$.

Finally, since $\tau : (x_0, x_1) \mapsto x_0 + Ix_1$ is a homeomorphism from $\mathbb{R}^2$ to $\mathbb{C}_I$, the set $[U] \cap \mathbb{C}_I = \{x_0 + Ix_1 : (x_0, x_1) \in D_{[U]}\} = \tau(D_{[U]})$ is a domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$ and $[U] \cap \mathbb{R} \supset U \cap \mathbb{R} \neq \emptyset$. Hence, $[U]$ is a slice domain.

□

Theorem 3.21 (Representation Formula)**.** *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let f be a left slice regular function on U. If we write $x \in U$ as $x = x_0 + I_x x_1$ according to Corollary 3.11, then the identity*

$$\begin{aligned} f(x) &= \frac{1}{2}\left[1 - I_x I\right] f(x_0 + Ix_1) + \frac{1}{2}\left[1 + I_x I\right] f(x_0 - Ix_1) = \\ &= \frac{1}{2}\Big[f(x_0 + Ix_1) + f(x_0 - Ix_1) + I_x I[f(x_0 - Ix_1) - f(x_0 + Ix_1)]\Big] \end{aligned} \tag{3.5}$$

holds true for any $I \in \mathbb{S}$ and any $x \in U$. Moreover, the quantities

$$\alpha(x_0, x_1) = \frac{1}{2}[f(x_0 + Ix_1) + f(x_0 - Ix_1)] \quad \text{and} \quad \beta(x_0, x_1) = \frac{1}{2} I[f(x_0 - Ix_1) - f(x_0 + Ix_1)] \tag{3.6}$$

do not depend on the imaginary unit I.
If on the other hand $g : U \to \mathbb{H}$ is right slice regular on U, then the corresponding identity

$$\begin{aligned} g(x) &= \frac{1}{2} g(x_0 + Ix_1)\left[1 - II_x\right] + \frac{1}{2} g(x_0 - Ix_1)\left[1 + II_x\right] = \\ &= \frac{1}{2}\Big[g(x_0 + Ix_1) + g(x_0 - Ix_1) + [g(x_0 - Ix_1) - g(x_0 + Ix_1)]II_x\Big] \end{aligned} \tag{3.7}$$

holds for any $I \in \mathbb{S}$ and any $x \in U$. Moreover, the quantities

$$\widehat{\alpha}(x_0, x_1) = \frac{1}{2}[g(x_0 + Ix_1) + g(x_0 - Ix_1)] \quad \text{and} \quad \widehat{\beta}(x_0, x_1) = \frac{1}{2}[g(x_0 - Ix_1) - g(x_0 + Ix_1)]I$$

do not depend on the imaginary unit I.

Proof. Let $I \in \mathbb{S}$. Writing $x = x_0 + I_x x_1$ according to Corollary 3.11, we define the function

$$\varphi(x) = \frac{1}{2}\left[1 - I_x I\right] f(x_0 + Ix_1) + \frac{1}{2}\left[1 + I_x I\right] f(x_0 - Ix_1)$$

for $x \in U$. Since U is axially symmetric, this function is well defined. Moreover, it is left slice regular because

$$\begin{aligned} 2\bar{\partial}_{I_x}\varphi(x) = &\frac{1}{2}\left[1 - I_x I\right] \frac{\partial}{\partial x_0} f(x_0 + Ix_1) + \frac{1}{2}\left[1 + I_x I\right] \frac{\partial}{\partial x_0} f(x_0 - Ix_1) + \\ &+ \frac{1}{2} I_x \left[1 - I_x I\right] \frac{\partial}{\partial x_1} f(x_0 + Ix_1) + \frac{1}{2} I_x \left[1 + I_x I\right] \frac{\partial}{\partial x_1} f(x_0 - Ix_1). \end{aligned}$$

Since f is left slice regular, we have $\frac{\partial}{\partial x_0} f(x_0 + Ix_1) = -I \frac{\partial}{\partial x_1} f(x_0 + Ix_1)$ and $\frac{\partial}{\partial x_0} f(x_0 - Ix_1) = I \frac{\partial}{\partial x_1} f(x_0 - Ix_1)$. Hence,

$$\begin{aligned} 2\bar{\partial}_{I_x}\varphi(x) = &\frac{1}{2}\left[1 - I_x I\right](-I) \frac{\partial}{\partial x_1} f(x_0 + Ix_1) + \frac{1}{2}\left[1 + I_x I\right] I \frac{\partial}{\partial x_1} f(x_0 - Ix_1) + \\ &+ \frac{1}{2} I_x \left[1 - I_x I\right] \frac{\partial}{\partial x_1} f(x_0 + Ix_1) + \frac{1}{2} I_x \left[1 + I_x I\right] \frac{\partial}{\partial x_1} f(x_0 - Ix_1) = \\ = &\frac{1}{2}\left[-I - I_x\right] \frac{\partial}{\partial x_1} f(x_0 + Ix_1) + \frac{1}{2}\left[I - I_x\right] \frac{\partial}{\partial x_1} f(x_0 - Ix_1) + \\ &+ \frac{1}{2}\left[I_x + I\right] \frac{\partial}{\partial x_1} f(x_0 + Ix_1) + \frac{1}{2}\left[I_x - I\right] \frac{\partial}{\partial x_1} f(x_0 - Ix_1) = 0. \end{aligned}$$

Furthermore, for $x \in U \cap \mathbb{C}_I$, we have either $I_x = I$, and in turn

$$\varphi(x) = \frac{1}{2}\left[1 - I^2\right] f(x_0 + Ix_1) + \frac{1}{2}\left[1 + I^2\right] f(x_0 - Ix_1) = f(x_0 + Ix_1) = f(x),$$

or we have $I_x = -I$, which yields

$$\varphi(x) = \frac{1}{2}\left[1 + I^2\right] f(x_0 + Ix_1) + \frac{1}{2}\left[1 - I^2\right] f(x_0 - Ix_1) = f(x_0 - Ix_1) = f(x).$$

Thus, from the Identity Principle, Theorem 3.8, it follows that $\varphi \equiv f$.

To show that α and β do not depend on the imaginary unit I, let $J \in \mathbb{S}$ be an arbitrary imaginary unit. Applying (3.5), we obtain

$$\begin{aligned}\alpha(x_0, x_1) =& \frac{1}{2}\left[f(x_0 + Ix_1) + f(x_0 - Ix_1)\right] = \\ =& \frac{1}{2}\left[\frac{1}{2}\left[f(x_0 + Jx_1) + f(x_0 - Jx_1)\right] + I\frac{1}{2}J\left[f(x_0 - Jx_1) - f(x_0 + Jx_1)\right] + \right. \\ & \left. + \frac{1}{2}\left[f(x_0 - Jx_1) + f(x_0 + Jx_1)\right] + I\frac{1}{2}J\left[f(x_0 + Jx_1) - f(x_0 - Jx_1)\right]\right] = \\ =& \frac{1}{2}\left[f(x_0 + Jx_1) + f(x_0 - Jx_1)\right]\end{aligned}$$

and similarly

$$\begin{aligned}\beta(x_0, x_1) =& \frac{1}{2}I\left[f(x_0 - Ix_1) - f(x_0 + Ix_1)\right] = \\ =& \frac{1}{2}I\left[\frac{1}{2}\left[f(x_0 - Jx_1) + f(x_0 + Jx_1)\right] + I\frac{1}{2}J\left[f(x_0 + Jx_1) - f(x_0 - Jx_1)\right] - \right. \\ & \left. - \frac{1}{2}\left[f(x_0 + Jx_1) + f(x_0 - Jx_1)\right] - I\frac{1}{2}J\left[f(x_0 - Jx_1) - f(x_0 + Jx_1)\right]\right] = \\ =& \frac{1}{2}J\left[f(x_0 - Jx_1) - f(x_0 + Jx_1)\right].\end{aligned}$$

□

Corollary 3.22 (Representation Formula II). *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let D_U be the domain in $\mathbb{R}^2$ such that $U = \{x_0 + Ix_1 : (x_0, x_1) \in D_U, I \in \mathbb{S}\}$ as in Lemma 3.20 with $[U] = U$. A function $f : U \to \mathbb{H}$ is left slice regular if and only if there exist two differentiable functions $\alpha, \beta : D \to \mathbb{H}$ that satisfy the conditions*

$$\alpha(x_0, x_1) = \alpha(x_0, -x_1) \qquad \text{and} \qquad \beta(x_0, x_1) = -\beta(x_0, -x_1) \tag{3.8}$$

and the Cauchy-Riemann system

$$\begin{cases} \frac{\partial}{\partial x_0}\alpha = \frac{\partial}{\partial x_1}\beta \\ \frac{\partial}{\partial x_1}\alpha = -\frac{\partial}{\partial x_0}\beta \end{cases} \tag{3.9}$$

such that

$$f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1) \tag{3.10}$$

for all $x = x_0 + Ix_1 \in U$.

The function f is right slice regular if and only if there exist two differentiable functions $\widehat{\alpha}, \widehat{\beta} : D_U \to \mathbb{H}$ satisfying the conditions (3.8) *and* (3.9) *such that*

$$f(x) = \widehat{\alpha}(x_0, x_1) + \widehat{\beta}(x_0, x_1)I$$

for all $x = x_0 + Ix_1 \in U$.

Proof. If f is left slice regular, we can apply Theorem 3.21 and define $\alpha(x_0, x_1)$ and $\beta(x_0, x_1)$ as in (3.6). Obviously, these functions satisfy (3.8). Furthermore, as f is left slice regular, we have $\frac{\partial}{\partial x_0}f(x_0 + Ix_1) = -I\frac{\partial}{\partial x_1}f(x_0 + Ix_1)$ and $\frac{\partial}{\partial x_0}f(x_0 - Ix_1) = I\frac{\partial}{\partial x_1}f(x_0 - Ix_1)$. Hence,

$$\begin{aligned}\frac{\partial}{\partial x_0}\alpha(x_0, x_1) &= \frac{1}{2}\left[\frac{\partial}{\partial x_0}f(x_0 + Ix_1) + \frac{\partial}{\partial x_0}f(x_0 - Ix_1)\right] = \\ &= \frac{1}{2}I\left[-\frac{\partial}{\partial x_1}f(x_0 + Ix_1) + \frac{\partial}{\partial x_1}f(x_0 - Ix_1)\right] = \frac{\partial}{\partial x_1}\beta(x_0, x_1)\end{aligned}$$

and

$$\frac{\partial}{\partial x_1}\alpha(x_0,x_1) = \frac{1}{2}\left[\frac{\partial}{\partial x_1}f(x_0+Ix_1) + \frac{\partial}{\partial x_1}f(x_0-Ix_1)\right] =$$
$$= -\frac{1}{2}I\left[-\frac{\partial}{\partial x_0}f(x_0+Ix_1) + \frac{\partial}{\partial x_0}f(x_0-Ix_1)\right] = -\frac{\partial}{\partial x_0}\beta(x_0,x_1).$$

If on the other hand α and β satisfy (3.8), the function $f(x) = \alpha(x_0,x_1)+I\beta(x_0,x_1)$ for $x = x_0+Ix_1 \in U$ is well defined on the open set U. In fact, if $x = x_0+Ix_1 \in U$ with $(x_0,x_1) \in D_{[U]}$, then $(x_0,-x_1) \in D_{[U]}$ because $D_{[U]}$ is symmetric with respect to the x_0-axis, and $x = x_0 - I(-x_1)$ where $-I \in \mathbb{S}$. However, the value $f(x)$ does not depend on the chosen representation as

$$f(x_0 - I(-x_1)) = \alpha(x_0,-x_1) - I\beta(x_0,-x_1) = \alpha(x_0,x_1) + I\beta(x_0,x_1) = f(x_1+Ix_1).$$

Finally, because of (3.9), f is left slice regular as

$$\bar{\partial}_I f(x_0+Ix_1) = \frac{\partial}{\partial x_0}\alpha(x_0,x_1) + I\frac{\partial}{\partial x_0}\beta(x_0,x_1) + I\frac{\partial}{\partial x_1}\alpha(x_0,x_1) + I^2\frac{\partial}{\partial x_1}\beta(x_0,x_1) =$$
$$= \frac{\partial}{\partial x_0}\alpha(x_0,x_1) - \frac{\partial}{\partial x_1}\beta(x_0,x_1) + I\left[\frac{\partial}{\partial x_0}\beta(x_0,x_1) + \frac{\partial}{\partial x_1}\alpha(x_0,x_1)\right] = 0.$$

□

Corollary 3.23. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain, let $f : U \to \mathbb{H}$ be left slice regular and let $x = x_0 + Ix_1 \in U$. Then there exist $a, b \in \mathbb{H}$ such that*

$$f(x_0+Ix_1) = a + Ib,$$

for all $I \in \mathbb{S}$. In particular, the image of the 2-sphere $[x]$ under f is the set $\{a + Ib : I \in \mathbb{S}\}$. Similarly, if f is right slice regular, then there exist $\widehat{a}, \widehat{b} \in \mathbb{H}$ such that

$$f(x_0+Ix_1) = \widehat{a} + \widehat{b}I,$$

for all $I \in \mathbb{S}$. In particular, the image of the 2-sphere $[x]$ under f is the set $\{\widehat{a} + \widehat{b}I : I \in \mathbb{S}\}$.

An important consequence of Theorem 3.21, the Representation Formula, is the fact that any holomorphic function defined on a suitable domain can be uniquely extended to a slice regular function.

Lemma 3.24 (Extension Lemma)**.** *Let $I \in \mathbb{S}$ and let D be a domain in $\mathbb{C}_I$ that is symmetric with respect to the real axis and such that $D \cap \mathbb{R} \neq \emptyset$. Then, the axially symmetric hull $[D]$ of D is a slice domain. Moreover, if $f : D \to \mathbb{H}$ is left holomorphic in the sense of Definition 3.1, then the function $\operatorname{ext}_L(f)$, which is defined by*

$$\operatorname{ext}_L(f)(x) = \frac{1}{2}\big[f(x_0+Ix_1) + f(x_0-Ix_1)\big] + I_x\frac{1}{2}I\big[f(x_0-Ix_1) - f(x_0+Ix_1)\big]$$

for $x \in [D]$, where $x = x_0 + I_x x_1$ as in Corollary 3.11, is the unique left slice regular extension of f to $[D]$. Similarly, if $g : D \to \mathbb{H}$ is right holomorphic, then the function $\operatorname{ext}_R(g)$, which is defined by

$$\operatorname{ext}_R(g)(x) = \frac{1}{2}\big[g(x_0+Ix_1) + g(x_0-Ix_1)\big] + \frac{1}{2}\big[g(x_0-Ix_1) - g(x_0+Ix_1)\big]II_x$$

for $x = x_0 + I_x x_1 \in [D]$, is the unique right slice regular extension of g to $[D]$.

Proof. By Lemma 3.20, the axially symmetric hull $[D]$ of D is a slice domain. The fact that $\operatorname{ext}_L(f)$ is left slice regular, follows in the same way as the left slice regularity of the function φ at the beginning of the proof of Lemma 3.5. Moreover, if $x \in \mathbb{C}_I$, we have

$$\operatorname{ext}_L(f)(x) = \frac{1}{2}\big[f(x_0+Ix_1) + f(x_0-Ix_1)\big] - \frac{1}{2}\big[f(x_0-Ix_1) - f(x_0+Ix_1)\big] = f(x_0+Ix_1) = f(x)$$

if $I_x = I$ and

$$\operatorname{ext}_L(f)(x) = \frac{1}{2}\big[f(x_0 + Ix_1) + f(x_0 - Ix_1)\big] + \frac{1}{2}\big[f(x_0 - Ix_1) - f(x_0 + Ix_1)\big] = f(x_0 - Ix_1) = f(x)$$

if $I_x = -I$. Therefore, $\operatorname{ext}_L(f)$ extends f. Furthermore, for any other left slice regular extension $\widetilde{f}$ of f, we have $\operatorname{ext}_L(f)|_D = f = \widetilde{f}|_D$. Theorem 3.8, the Identity Principle, implies $\operatorname{ext}_L(f) = \widetilde{f}$. Therefore, $\operatorname{ext}_L(f)$ is the unique left slice regular extension of f to $[D]$.

□

Remark 3.25. Note that in particular any holomorphic function $f : D \subset \mathbb{C}_I \to \mathbb{C}_I$ is both left and right holomorphic as a function from D to $\mathbb{H}$. Thus, there exist a left and a right slice regular extension of f. However, it is possible, that these two extensions do not coincide. For instance take $I \in \mathbb{S}$ and consider the function $f(z) = Iz$, which is holomorphic on $\mathbb{C}_I$. For $x = x_0 + I_x x_1 \in \mathbb{H}$, we have

$$\operatorname{ext}_L(f)(x) = \frac{1}{2}\big[I(x_0 + Ix_1) + I(x_0 - Ix_1)\big] + I_x\frac{1}{2}I\big[I(x_0 - Ix_1) - I(x_0 + Ix_1)\big] = Ix_0 + I_x I x_1 = xI,$$

but

$$\operatorname{ext}_R(f)(x) = \frac{1}{2}\big[I(x_0 + Ix_1) + I(x_0 - Ix_1)\big] + \frac{1}{2}\big[[I(x_0 - Ix_1) - I(x_0 + Ix_1)]I\big]I_x = Ix_0 + Ix_1 I_x = Ix.$$

If we consider $J \in \mathbb{S}$ with $J \perp I$, then $\operatorname{ext}_L(f)(J) = JI \neq IJ = \operatorname{ext}_R(f)(J)$. Hence, $\operatorname{ext}_L(f) \neq \operatorname{ext}_R(f)$.

In the following, we will only consider slice regular functions that are defined on axially symmetric slice domains. As we will see in the next lemma, this is no significant restriction because any slice regular function on a slice domain can be uniquely extended to the axially symmetric hull of this domain. To prove this lemma, we need the following generalization of the Representation Formula.

Corollary 3.26. *Let $U \cap \mathbb{H}$ be an axially symmetric slice domain, let $I, J \in \mathbb{S}$ with $I \perp J$ and let us write $x \in U$ as $x = x_0 + I_x x_1$ according to Corollary 3.11. If $f \in \mathcal{M}^L(U)$ and $g \in \mathcal{M}^R(U)$, then the following identities hold true for any $x \in U$:*

$$f(x) = (I - J)^{-1}[If(x_0 + Ix_1) - Jf(x_0 + Jx_1)] + I_x(I - J)^{-1}[f(x_0 + Ix_1) - f(x_0 + Jx_1)] \quad (3.11)$$
$$g(x) = [g(x_0 + Ix_1)I - f(x_0 + Jx_1)J](I - J)^{-1} + [g(x_0 + Ix_1) - g(x_0 + Jx_1)](I - J)^{-1}I_x. \quad (3.12)$$

Proof. If we apply Corollary 3.22, we obtain

$$f(x_0 + Ix_1) = \alpha(x_0, x_1) + I\beta(x_0, x_1) \qquad \text{and} \qquad f(x_0 + Jx_1) = \alpha(x_0, x_1) + J\beta(x_0, x_1). \quad (3.13)$$

If we subtract these equations and multiply the result by $(I - J)^{-1}$ from the left, we obtain

$$\beta(x_0, x_1) = (I - J)^{-1}[f(x_0 + x_1 I) - f(x_0 + Jx_1)].$$

If we multiply the equations in (3.13) by I and J from the left, subtract them and multiply the result by $(I - J)^{-1}$ from the left, we obtain

$$\alpha(x_0, x_1) = (I - J)^{-1}[If(x_0 + x_1 I) - Jf(x_0 + Jx_1)].$$

Plugging these expressions back into (3.10), we obtain the desired representation.

□

Lemma 3.27 (Extension Lemma, II). *Let U be a slice domain in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be left slice regular. Then there exists a unique left slice regular extension of f to the axially symmetric hull $[U]$ of U. Similarly, for any $g \in \mathcal{M}^R(U)$, there exists an unique right slice regular extension of g to $[U]$.*

Proof. Since U is a slice domain, there exists a point $c \in U \cap \mathbb{R}$. Furthermore, as U is open, there exists an open Ball $B_r(c) \subset U$. If f is left slice regular on U, then the restriction of f to $B_r(c)$ is a left slice regular function defined on an axially symmetric slice domain.

Let us consider the set $\mathcal{F}$ of all left slice regular functions ξ defined on an axially symmetric slice domain V such that $B_r(c) \subset V \subset [U]$ and $\xi|_{V\cap U} = f|_{V\cap U}$. This set is nonempty because

$(B_r(c), f|_{B_r(c)}) \in \mathcal{F}$. It is partially ordered by the set inclusion, that is, $(V_1, \xi_1) \preceq (V_2, \xi_2)$ if $V_1 \subset V_2$. For a chain $(V_k, \xi_k)_{k\in K}$ in $\mathcal{F}$, we can define

$$V^* = \bigcup_{k\in K} V_k \qquad \text{and} \qquad \xi^*(x) = \xi_{k_x}(x) \ \text{ for } x \in V^*, \tag{3.14}$$

where k_x is an arbitrary $k \in K$ such that $x \in K$. The Identity Principle, Theorem 3.8, implies $\xi_2|_{V_1} = \xi_1$ if $V_1 \subset V_2$. Hence, ξ is a well defined left slice regular function on V^* that extends f. Moreover, it is easy to check that V^* is an axially symmetric slice domain such that $B_r(x) \subset V^* \subset [U]$. Therefore, (V^*, ξ^*) belongs to $\mathcal{F}$ and it is an upper bound of the chain $(V_k, \xi_k)_{k\in K}$. By Zorn's lemma, there exits a maximal element (U^*, f^*) in $\mathcal{F}$.

In order to show that $U^* = [U]$, we assume the converse, $U^* \subsetneq [U]$. In this case, there exists $y = y_0 + I_y y_1 \in [U] \cap \partial U^*$. Since $y \in [U]$, there exists $I \in \mathbb{S}$ such that $y_I = y_0 + I y_1 \in U$. Since U is open, there exists a ball $B_{r_0}(y_I) \subset U$. Thus, we can find another imaginary unit J and $\varepsilon < r_0$ such that $B_\varepsilon(y_J) \subset U$, where $y_J = y_0 + J y_1$. In particular, the discs $B_\varepsilon(y_I) \cap \mathbb{C}_I$ and $B_\varepsilon(y_J) \cap \mathbb{C}_J$ are contained in U.

As an easy consequence of Example 3.15, we get $[B_\varepsilon(y)] = [B_\varepsilon(y_I) \cap \mathbb{C}_I] = [B_\varepsilon(y_J) \cap \mathbb{C}_J]$. Thus, for $x = x_0 + I_x x_1 \in [B_\varepsilon(y)]$, we can define

$$\phi(x) = (I-J)^{-1}[If(x_I) - Jf(x_J)] + I_x(I-J)^{-1}[f(x_I) - f(x_J)],$$

where we set $x_I = x_0 + I x_1$ and $x_J = x_0 + J x_1$. Then

$$\begin{aligned} 2\bar{\partial}_{I_x}\phi(x) =& (I-J)^{-1}\left[I\frac{\partial}{\partial x_0}f(x_I) - J\frac{\partial}{\partial x_0}f(x_J)\right] + I_x(I-J)^{-1}\left[\frac{\partial}{\partial x_0}f(x_I) - \frac{\partial}{\partial x_0}f(x_J)\right] + \\ &+ I_x(I-J)^{-1}\left[I\frac{\partial}{\partial x_1}f(x_I) - J\frac{\partial}{\partial x_1}f(x_J)\right] - (I-J)^{-1}\left[\frac{\partial}{\partial x_1}f(x_I) - \frac{\partial}{\partial x_1}f(x_J)\right]. \end{aligned}$$

Since f is left slice regular, we have $\frac{\partial}{\partial x_0}f(x_I) = -I\frac{\partial}{\partial x_1}f(x_I)$ and $\frac{\partial}{\partial x_0}f(x_J) = -J\frac{\partial}{\partial x_1}f(x_J)$ and in turn

$$\begin{aligned} 2\bar{\partial}_{I}\phi(x) =& (I-J)^{-1}\left[\frac{\partial}{\partial x_1}f(x_I) - \frac{\partial}{\partial x_1}f(x_J)\right] + I_x(I-J)^{-1}\left[-I\frac{\partial}{\partial x_1}f(x_I) + J\frac{\partial}{\partial x_1}f(x_J)\right] + \\ &+ I_x(I-J)^{-1}\left[I\frac{\partial}{\partial x_1}f(x_I) - J\frac{\partial}{\partial x_1}f(x_J)\right] - (I-J)^{-1}\left[\frac{\partial}{\partial x_1}f(x_I) - \frac{\partial}{\partial x_1}f(x_J)\right] = 0. \end{aligned}$$

Hence, ϕ is left slice regular. Moreover, $\phi(x) = f^*(x)$ for all $x \in U^* \cap [B_\varepsilon(y)]$ by Corollary 3.26. Therefore, the function

$$h(x) = \begin{cases} f^*(x) & \text{if } x \in U^* \\ \phi(x) & \text{if } x \in [B_\varepsilon(y)] \setminus U^* \end{cases}$$

is a well defined left slice regular extension of f to $U^* \cup [B_\varepsilon(y)]$.

The set $U^* \cup [B_\varepsilon(y)]$ is obviously axially symmetric. Moreover, the union of two open and connected sets is open and connected, if their intersection is nonempty. Hence, $U^* \cup [B_\varepsilon(y)]$ is open and connected because U^* and $[B_\varepsilon(y)]$ are open and connected and $[y] \subset U^* \cap [B_\varepsilon(y_I)]$. Furthermore, for any $K \in \mathbb{S}$, we have $\{y_K, \overline{y_K}\} \subset U^* \cap \mathbb{C}_K$, where $y_K = y_0 + K y_1$. The sets $(U^* \cap \mathbb{C}_K)$ and $(B_\varepsilon(y_K) \cap \mathbb{C}_K)$ are open and connected subsets of $\mathbb{C}_K$ whose intersection is nonempty because it contains y_K. Hence, $(U^* \cap \mathbb{C}_K) \cup (B_\varepsilon(y_K) \cap \mathbb{C}_K)$ is open and connected in $\mathbb{C}_K$. The intersection of $B_\varepsilon(\overline{y_K}) \cap \mathbb{C}_K$ and $(U^* \cup \mathbb{C}_K) \cup (B_\varepsilon(y_K) \cap \mathbb{C}_K)$ contains $\overline{y_K}$ and is therefore not empty either. Consequently, the set

$$(U^* \cup [B_\varepsilon(y)]) \cap \mathbb{C}_K = ((B_\varepsilon(\overline{y_K}) \cap \mathbb{C}_K) \cup [(U^* \cup \mathbb{C}_K) \cup (B_\varepsilon(y_K) \cap \mathbb{C}_K)]$$

is open and connected in $\mathbb{C}_K$ and $[U] \cup B_\varepsilon(y)$ is an axially symmetric slice domain.

Since $B_r(c) \subset U^* \cup [B_\varepsilon(y)] \subset [U]$, we have $(U^* \cup [B_\varepsilon(y)], h) \in \mathcal{F}$ which contradicts the maximality of U^* because U^* is a proper subset of $U^* \cup [B_\varepsilon(y)]$. Therefore, $[U] \cap \partial U^* = \emptyset$ and, in turn, $[U] = U^*$. Hence, f^* is a left slice regular extension of f to $[U]$.

Finally, it follows from Theorem 3.8, the Identity Principle, that the slice regular extension of f to $[U]$ is unique.

□

3.3 Power Series

A *power series in the quaternion variable* centered at $c \in \mathbb{H}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n \qquad \text{or} \qquad \sum_{n=0}^{\infty} (x-c)^n a_n \tag{3.15}$$

with coefficients $a_n \in \mathbb{H}$. Power series in the quaternion variable are more complicated than complex power series because of the noncommutativity of the quaternionic multiplication. Nevertheless, some of the classical theory holds also in the quaternionic case. In particular, the same arguments as in the complex case show that for any series of the form (3.15), there exists a unique $R \in [0,\infty]$, its *radius of convergence*, such that the series converges uniformly on any closed ball $\overline{B_r(c)}$ with $0 < r < R$ and diverges for any $x \in \mathbb{H}$ with $|x-c| > R$. Moreover, the classical formula

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

also holds true in the quaternionic case.

Before we consider power series expansions of slice regular functions, we need to introduce a new type of derivative for slice regular functions. It is the analogue of the usual derivative φ' in complex analysis.

Definition 3.28. *Let $U \subset \mathbb{H}$ be an open set and let f be a left slice regular function on U. Then we define its* slice derivative $\partial_s f$ *as*

$$\partial_s f(x) = \partial_I f(x) \qquad \text{if } x = x_0 + I x_1 \in U.$$

If f is a right slice regular function on U, then we define its slice derivative $\partial_s f$ *as*

$$\partial_s f(x) = (f \partial_I)(x) \qquad \text{if } x = x_0 + I x_1 \in U.$$

Remark 3.29. Note that the slice derivative is well defined, because it is only applied to slice regular functions. Indeed, if f is left slice regular, we have $\frac{\partial}{\partial x_0} f_I(x) = -I \frac{\partial}{\partial x_1} f_I(x)$ for $x = x_0 + I x_1 \in U$ and therefore

$$\partial_s f(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) - I \frac{\partial}{\partial x_1} f_I(x)\right) = \frac{\partial}{\partial x_0} f_I(x) = \frac{\partial}{\partial x_0} f(x). \tag{3.16}$$

Since the derivative with respect to x_0 does not depend on the imaginary unit I, the slice derivative is independent of the representation $x = x_0 + I x_1 = x_0 - I(-x_1)$ if $x \notin \mathbb{R}$ and of the imaginary unit I if $x \in \mathbb{R}$.

The identity (3.16) corresponds to the fact that $\varphi'(z) = \frac{\partial}{\partial z_0}\varphi(z)$ for any holomorphic function φ and $z = z_0 + i z_1 \in \mathbb{C}$.

Recall that the derivative φ' of a holomorphic function φ is holomorphic itself. The analogue result is true for the slice derivative of a slice regular function.

Corollary 3.30. *The slice derivative $\partial_s f$ of a left slice regular function f is left slice regular and the slice derivative $\partial_s g$ of a right slice regular function g is right slice regular.*

Proof. Let f be left slice regular on U and let $x = x_0 + I x_1 \in U$. Then we have $\partial_s f(x) = \frac{\partial}{\partial x_0} f_I(x)$ by (3.16), which implies

$$\bar{\partial}_I \partial_s f(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_0} f_I(x) + I \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_0} f_I(x)\right) = \frac{1}{2}\frac{\partial}{\partial x_0}\left(\frac{\partial}{\partial x_0} f_I(x) + I \frac{\partial}{\partial x_1} f_I(x)\right) = 0.$$

□

Remark 3.31. As we have seen in Example 3.4, polynomials of the the form $\sum_{n=0}^{N} x^n a_n$ with $a_n \in \mathbb{H}$ are left slice regular. If we consider a power series of the form $p(x) = \sum_{n=0}^{\infty} x^n a_n$ with radius of convergence $R > 0$, then the series converges uniformly on any closed ball $\overline{B_r(0)}$ with $0 < r < R$. In particular, the restriction $p_I(x) = \sum_{n=0}^{\infty} x^n a_n,\ x \in \mathbb{C}_I$, of p to $\mathbb{C}_I$ converges uniformly on $\overline{B_r(0)} \cap \mathbb{C}_I$ for any imaginary unit $I \in \mathbb{S}$ if $0 < r < R$. Therefore, if $x = x_0 + I x_1 \in B_R(0)$, we can choose r with $|x| < r < R$, and since the series p_I converges uniformly on $\overline{B_r(0)} \cap \mathbb{C}_I$, we can exchange summation and differentiation and obtain

$$\partial_I p_I(x) = \partial_I \sum_{n=0}^{\infty} x^n a_n = \sum_{n=0}^{\infty} \partial_I x^n a_n = 0.$$

Hence, p is left slice regular on $B_R(0)$. Similarly, any power series of the form $\sum_{n=0}^{\infty} a_n x^n$ is right slice regular.

However, a polynomial or a power series centered at an arbitrary point $c \in \mathbb{H}$ is in general not slice regular. Consider for instance the monomial x^2. If we center it at $c \in \mathbb{H}$, we obtain $(x-c)^2 = x^2 - xc - cx + c^2$. For $x = x_0 + Ix_1 \in \mathbb{H}$, we have

$$\bar{\partial}_I (x-c)^2 = \bar{\partial}_I x^2 - \bar{\partial}_I xc - \bar{\partial}_I cx + \bar{\partial}_I c^2 = \bar{\partial}_I cx.$$

Thus, $(x-c)^2$ is left slice regular if and only if $\bar{\partial}_I$ and c commute for all $I \in \mathbb{S}$. By Lemma 2.5, this implies $c \in \mathbb{R}$.

Therefore, we can not wish to find a power series expansion $\sum_{n=0}^{\infty}(x-c)^n a_n$ of a left slice regular function at an arbitrary point c. Nevertheless, at least at any real point, a slice regular function can be expanded into a power series.

We start with the case $c = 0$.

Theorem 3.32 (Power Series Expansion)**.** *Let $B_r(0) \subset \mathbb{H}$ be the open ball with radius r centered at 0. A function $f : B_r(0) \to \mathbb{H}$ is left slice regular if and only if it has a power series expansion of the form*

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(0) \tag{3.17}$$

converging on $B_r(0)$. In particular, if $f(x) = \sum_{n=0}^{\infty} x^n a_n$, then $a_n = \frac{1}{n!}\frac{\partial^n}{\partial x_0^n} f(0)$.

It is right slice regular if and only if it has a power series expansion of the form

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(0) \right) x^n$$

converging on $B_r(0)$. In particular, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $a_n = \frac{1}{n!}\frac{\partial^n}{\partial x_0^n} f(0)$.

Proof. If a function admits a series expansion as in (3.17), it is left slice regular on its ball of convergence as we have seen in Remark 3.31. To prove the converse, we use Lemma 3.5, the Splitting Lemma. Let $I, J \in \mathbb{S}$ with $I \perp J$ and let $f_1, f_2 : B_r(0) \cap \mathbb{C}_I \to \mathbb{C}_I$ be holomorphic functions such that $f_I = f_1 + f_2 J$. Since the functions f_1 and f_2 are holomorphic, they permit a Taylor series expansion on $B_r(0) \cap \mathbb{C}_I$. Thus, for $x \in \mathbb{C}_I$, we have

$$f_I(x) = \sum_{n=0}^{\infty} \frac{f_1^{(n)}(0)}{n!} x^n + \sum_{n=0}^{\infty} \frac{f_2^{(n)}(0)}{n!} x^n J = \sum_{n=0}^{\infty} x^n \left(\frac{f_1^{(n)}(0)}{n!} + \frac{f_2^{(n)}(0)}{n!} J \right).$$

Furthermore, for any holomorphic function φ on $\mathbb{C}_I$, the identity $\varphi'(x) = \frac{\partial}{\partial x_0}\varphi(x)$ holds true. Hence, we obtain

$$f_I(x) = \sum_{n=0}^{\infty} x^n \left(\frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f_1(0) + \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f_2(0) J \right) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f_I(0) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(0).$$

In particular, if $f(x) = \sum_{n=0}^{\infty} x^n a_n$ there exist $a_{n,1}, a_{n,2} \in \mathbb{C}_I$ for any $n \in \mathbb{N}_0$ such that $a_n = a_{n,1} + a_{n,2} J$, cf. Corollary 2.10. Since

$$f_1(x) + f_2(x) J = f_I(x) = \sum_{n=0}^{\infty} x^n a_n = \sum_{n=0}^{\infty} x^n a_{n,1} + \sum_{n=0}^{\infty} x^n a_{n,2} J$$

for $x \in \mathbb{C}_I$ and since 1 and J are left linearly independent over $\mathbb{C}_I$, we obtain $f_k(x) = \sum_{n=0}^{\infty} x^n a_{n,k}$, $k = 1, 2$. Hence, $a_{n,k} = \frac{1}{n!} f_k^{(n)}(0) = \frac{1}{n!}\frac{\partial^n}{\partial x_0^n} f_k^{(n)}(0)$, $k = 1, 2$, and

$$a_n = a_{n,1} + a_{n,2} J = \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f_1(0) + \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f_2(0) J = \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(0).$$

□

Corollary 3.33. *Let $U \subset \mathbb{H}$ be a slice domain and let f be a function on U with values in $\mathbb{H}$. Let $c \in U$ be a point on the real axis and let $B_r(c)$ be the largest ball centered at c that is contained in U. If f is left or right slice regular, then it allows the power series representation*

$$f(x) = \sum_{n=0}^{\infty} (x-c)^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(c) \qquad or \qquad f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(c) \right) (x-c)^n \tag{3.18}$$

on $B_r(c)$, respectively.

Proof. Let f be left slice regular and let $t_c(x) = x + c$. Then x and $t_c(x)$ lie in the same complex plane because c is real. Hence,

$$\partial_I f(t_c(x)) = \frac{\partial}{\partial x_0} f_I(x+c) + I \frac{\partial}{\partial x_1} f_I(x+c) = 0$$

for any $x = x_0 + Ix_1 \in \mathbb{H}$ with $t_x(c) \in U$. In particular, $f \circ t_c$ is left slice regular on $B_r(0)$. Thus, from Theorem 3.32, we get

$$f(x+c) = f \circ t_c(x) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} (f \circ t_c)(0) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(c),$$

or

$$f(x) = \sum_{n=0}^{\infty} (x-c)^n \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} f(c).$$

□

Corollary 3.34. *Let $f(x) = \sum_{n=0}^{\infty} (x-c)^n a_n$ and $g(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be a left and a right slice regular power series, respectively. Then*

$$\partial_s f(x) = \sum_{n=1}^{\infty} (x-c)^{n-1} n a_n \qquad and \qquad \partial_s g(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

and $\partial_s f$ and $\partial_s g$ have the same radius of convergence as f and g, respectively.

Proof. The proof follows the lines of the complex case. Let $f(x) = \sum_{n=0}^{\infty} (x-c)^n a_n$. The radius of convergence of this series is $R = (\limsup_{n\to\infty} \sqrt[n]{|a_n|})^{-1}$. For $x = x_0 + Ix_1 \in B_R(x)$, we can choose r such that $|x - c| < r < R$. Then f converges uniformly on $\overline{B_r(c)}$. Therefore, we can exchange summation and differentiation and obtain

$$\partial_s f(x) = \sum_{n=0}^{\infty} \frac{\partial}{\partial x_0} x^n a_n = \sum_{n=1}^{\infty} x^{n-1} n a_n$$

because $\partial_s f(x) = \frac{\partial}{\partial x_0} f(x)$ by (3.16).

For the radius of convergence ρ of $\partial_s f$, we have

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|(n+1)a_{n+1}|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n+1}} \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_{n+1}|}} = R.$$

□

We continue this section with some properties of a privileged class of slice regular functions. The following corollary will be useful to characterize them.

Corollary 3.35. *Let $U \subset \mathbb{H}$ be a slice domain and let $f : U \to \mathbb{H}$ be left or right slice regular. If there exists an imaginary unit $I \in \mathbb{S}$ such that $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$, then the power series expansion of f at any point $c \in U \cap \mathbb{R}$ has all its coefficients in $\mathbb{C}_I$. In particular, if there exist two different imaginary units $I, J \in \mathbb{S}, J \neq \pm I$, such that $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ and $f(U \cap \mathbb{C}_J) \subset \mathbb{C}_J$, then the coefficients are real.*

Proof. Let $f \in \mathcal{M}^L(U)$. If $I \in \mathbb{S}$ such that $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$, then we have $f(x) = f_I(x) \in \mathbb{C}_I$ for any $x \in U \cap \mathbb{R}$. Therefore, $\frac{\partial^n}{\partial x_0^n} f(c) \in \mathbb{C}_I$ for any $n \in \mathbb{N}_0$ and any $c \in U \cap \mathbb{R}$ and the conclusion follows from Corollary 3.33.

If $J \in \mathbb{S}$ is another imaginary unit with $f(U \cap \mathbb{C}_J) \subset \mathbb{C}_J$, then the coefficients belong to $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$. □

Note that for power series with real coefficients $\sum_{n=0}^{\infty} a_n x_n = \sum_{n=0}^{\infty} x_n a_n$ holds true because the coefficients a_n commute with the quaternionic variable x. Therefore, they are left and right slice regular. Moreover, recall that the compositions and the products of slice regular functions are in general not slice regular as we have seen in Example 3.4. However, for power series with real coefficients, the situation is different.

Lemma 3.36. *Let $f, g, h : B_r(0) \to \mathbb{H}$ be the power series*

$$f(x) = \sum_{n=0}^{\infty} x^n a_n, \qquad g(x) = \sum_{n=0}^{\infty} x^n b_n \qquad \text{and} \qquad h(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{3.19}$$

where $a_n \in \mathbb{R}$ and $b_n, c_n \in \mathbb{H}$ for $n \in \mathbb{N}_0$. Then the product fg is left slice regular and the product hf is right slice regular.

Proof. Since the coefficients a_n are real, they commute with the quaternion variable x and we obtain

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} x^n a_n\right)\left(\sum_{n=0}^{\infty} x^n b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} x^k a_k x^{n-k} b_{n-k} = \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_k b_{n-k}\right),$$

which is again a left slice regular power series. □

Lemma 3.37. *Let $f : B_r(0) \to \mathbb{H}$ and $g, h : B_R(0) \to \mathbb{H}$ be power series as in* (3.19) *such that $a_n \in \mathbb{R}$ and $b_n, c_n \in \mathbb{H}$ for $n \in \mathbb{N}_0$ and such that $f(B_r(0)) \subset B_R(0)$. Then $g \circ f$ is a left slice regular power series and $h \circ f$ is a right slice regular power series on $B_r(0)$.*

Proof. Let $I, J \in \mathbb{S}$ with $I \perp J$. By Corollary 2.10, there exist $b_{n,1}, b_{n,2} \in \mathbb{C}_I$ such that $b_n = b_{n,1} + b_{n,2}J$ for any $n \in \mathbb{N}_0$. Hence, $g_I(z) = \sum_{n=0}^{\infty} z^n a_{n,1} + \sum_{n=0}^{\infty} z^n a_{n,2} J$, where $g_1(z) = \sum_{n=0}^{\infty} z^n a_{n,1}$ and $g_2(z) = \sum_{n=0}^{\infty} z^n a_{n,2}$ are complex power series that converge on $B_R(0) \cap \mathbb{C}_I$. Moreover, the restriction f_I of f to $\mathbb{C}_I$ is also a complex power series on $\mathbb{C}_I$ because its coefficients are real. It converges on $B_r(0) \cap \mathbb{C}_I$ and satisfies $f_I(B_r(0) \cap \mathbb{C}_I) \subset B_R(0) \cap \mathbb{C}_I$. Hence, the compositions $g_1 \circ f_I$ and $g_2 \circ f_I$ are complex power series that converge on $B_r(0) \cap \mathbb{C}_I$. Let $d_{n,1} \in \mathbb{C}_I$ and $d_{n,2} \in \mathbb{C}_I$ be the coefficients of $g_1 \circ f_I$ and $g_2 \circ f_I$, respectively, that is, $g_k \circ f_I(z) = \sum_{n=0}^{\infty} d_{n,k} z^n$ for any $z \in B_r(0) \cap \mathbb{C}_I$ and $k = 1, 2$. If we set $d_n = d_{n,1} + d_{n,2}J$, then

$$g_I \circ f_I(x) = g_1 \circ f_I(x) + g_2 \circ f_2(x) J = \sum_{n=0}^{\infty} d_{n,1} x^n + \sum_{n=0}^{\infty} d_{n,2} x^n J = \sum_{n=0}^{\infty} x^n d_n.$$

In particular, $g_I \circ f_I(x)$ is left holomorphic on $B_r(0) \cap \mathbb{C}_I$, and since I was arbitrary, $g \circ f$ is left slice regular. Moreover, the left slice regular extension of a left holomorphic function is unique by Lemma 3.24, and hence,

$$g \circ f = \operatorname{ext}_L(g_I \circ f_I) = \operatorname{ext}_L\left(\sum_{n=0}^{\infty} z^n d_n\right) = \sum_{n=0}^{\infty} x^n d_n.$$

□

The preceding results motivate the following definition.

Definition 3.38. *Let $U \subset \mathbb{H}$ be a slice domain. By $\mathcal{N}(U)$ we denote the set of all left slice regular functions f such that $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for every $I \in \mathbb{S}$.*

Remark 3.39. By Corollary 3.35, the power series representation of a function $f \in \mathcal{N}(U)$ at a point $c \in \mathbb{R}$ has real coefficients. Conversely, if f is a left slice regular function on a slice domain U and there exists a point $c \in U \cap \mathbb{R}$ such that the power series representation of f at c has real coefficients, then $f \in \mathcal{N}(U)$. Indeed, if $f(x) = \sum_{n=0}^{\infty}(x-c)^n a_n$ on $B_r(c)$ with $a_n \in \mathbb{R}$, then, for any $I \in \mathbb{S}$, we can consider $J \in \mathbb{S}$ with $I \perp J$ and write $a_n = a_{n,1} + a_{n,2}J$ with $a_{n,1}, a_{n,2} \in \mathbb{C}_I$. We can apply Lemma 3.5, the Splitting Lemma, to obtain holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = f_1 + f_2 J$, where $f_1(z) = \sum_{n=0}^{\infty}(z-c)^n a_{n,1}$ and $f_2(z) = \sum_{n=0}^{\infty}(z-c)^n a_{n,2}$ on $\mathbb{C}_I \cap B_r(c)$. But since the coefficients a_n are real, we have $a_{n,1} = a_n$ and $a_{n,2} = 0$. The identity theorem for holomorphic functions implies $f_2 \equiv 0$. Therefore, $f_I = f_1$ and $f(U \cap \mathbb{C}_I) = f_1(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$.

Corollary 3.40. *Let U be a slice domain and let $f \in \mathcal{N}(U)$.*

(i) The function f is left and right slice regular. Moreover, if U is axially symmetric and we write $f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1)$ for $x = x_0 + Ix_1$ according to Corollary 3.22, then the functions α and β have values in $\mathbb{R}$.

(ii) If $g \in \mathcal{M}^L(U)$, then fg is left slice regular. If $h \in \mathcal{M}^R(U)$, then hf is right slice regular.

(iii) If $g \in \mathcal{M}^L(O)$, where O an open set with $f(U) \subset O$, then $g \circ f$ is left slice regular. Similarly, if $h \in \mathcal{M}^R(O)$, then $h \circ f$ is right slice regular.

Proof. By definition, f is left slice regular and satisfies $f_I(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for every $I \in \mathbb{S}$. Therefore,

$$(f\bar{\partial}_I)(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + \frac{\partial}{\partial x_1} f_I(x) I\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + I\frac{\partial}{\partial x_1} f_I(x)\right) = \bar{\partial}_I f(x) = 0$$

for evey $x = x_0 + Ix_1 \in U$, and in turn, f is right slice regular. Moreover, if $x_0, x_1 \in \mathbb{R}$ are such that $x = x_0 + Ix_1 \in U$ for $I \in \mathbb{S}$, then we have $\alpha(x_0, x_1) = \frac{1}{2}(f(x_0 + Ix_1) + f(x_0 - Ix_1)) \in \mathbb{C}_I$ and $\beta(x_0, x_1) = I\frac{1}{2}(f(x_0 - Ix_1) - f(x_0 + Ix_1)) \in \mathbb{C}_I$ for any $I \in \mathbb{S}$. Consequently, $\alpha(x_0, x_1)$ and $\beta(x_0, x_1)$ are real and *(i)* holds true.

Let $g \in \mathcal{M}^L(U)$ and let $x = x_0 + Ix_1 \in U$. Since I and $f_I(x)$ lie in $\mathbb{C}_I$, they commute and we obtain

$$\begin{aligned}
\bar{\partial}_I f_I(x) g_I(x) &= \frac{1}{2}\left(\frac{\partial}{\partial x_0}[f_I(x)g_I(x)] + I\frac{\partial}{\partial x_1}[f_I(x)g_I(x)]\right) = \\
&= \frac{1}{2}\left(\left(\frac{\partial}{\partial x_0} f_I(x)\right) g_I(x) + f_I(x)\frac{\partial}{\partial x_0} g_I(x) + \left(I\frac{\partial}{\partial x_1} f_I(x)\right) g_I(x) + f_I(x) I \frac{\partial}{\partial x_0} g_I(x)\right) = \\
&= (\bar{\partial}_I f_I)(x) g_I(x) + f_I(x)(\bar{\partial}_I g_I)(x) = 0.
\end{aligned}$$

To show (*iii*), we consider $g \in \mathcal{M}^L(O)$, an arbitrary point $x = x_0 + Ix_1 \in U$ and $J \in \mathbb{S}$ with $I \perp J$. By applying Lemma 3.5, the Splitting Lemma, we obtain holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ and $g_1, g_2 : f(U) \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = f_1 + f_2 J$ and $g_I = g_1 + g_2 J$. The fact that $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ implies $f_2 \equiv 0$. Hence, $f_I = f_1$ and

$$\bar{\partial}_I(g_I \circ f_I)(x) = \bar{\partial}_I(g_1 \circ f_1)(x) + \bar{\partial}_I(g_2 \circ f_1)(x) J.$$

Since g_1, g_2 and f_1 are holomorphic, the functions $g_1 \circ f_1$ and $g_2 \circ f_1$ are holomorphic, too. Hence, $\bar{\partial}_I(g_1 \circ f_1) = 0$ and $\bar{\partial}_I(g_2 \circ f_1) = 0$ and, in turn, $\bar{\partial}(g_I \circ f_I) = 0$. Therefore, $g \circ f$ is left slice regular. □

Corollary 3.41. *Let U be an axially symmetric slice domain and let $f \in \mathcal{N}(U)$. Then $f(\overline{x}) = \overline{f(x)}$ for any $x \in U$.*

Proof. Let $f \in \mathcal{N}(U)$ and let us write $f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1)$ according to Corollary 3.22. Then Corollary 3.40 implies that α and β are real-valued, and hence,

$$f(\overline{x}) = \alpha(x_0, -x_1) + I\beta(x_0, -x_1) = \alpha(x_0, x_1) - I\beta(x_0, x_1) = \overline{f(x)}.$$

□

Note that, although $f \in \mathcal{N}(U)$ implies that f is left and right slice regular, the converse is not true as the easy example $f(x) = x - a$ with $a \in \mathbb{H} \setminus \mathbb{R}$ shows. Obviously, f is left and right slice regular, but if x and a do not lie in the same complex plane, then $f(x)$ and x do not lie in the same complex plane either. Nevertheless, any function that is left and right slice regular can be characterized by means of a function in $\mathcal{N}(U)$.

Lemma 3.42. *Let $U \subset \mathbb{H}$ be a slice domain and let $f : U \to \mathbb{H}$ be left and right slice regular. Then there exists a constant $a \in \mathbb{H}$ and a function $\tilde{f} \in \mathcal{N}(U)$ such that $f = a + \tilde{f}$.*

Proof. Let $I \in \mathbb{S}$ and let $x = x_0 + Ix_1 \in U \cap \mathbb{C}_I$. Since $\bar{\partial}_I f(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + I \frac{\partial}{\partial x_1} f_I(x)\right) = 0$ and $(f\bar{\partial}_I)(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_0} f_I(x) + \frac{\partial}{\partial x_1} f_I(x) I\right) = 0$, we have

$$I \frac{\partial}{\partial x_1} f(x) = -\frac{\partial}{\partial x_0} f(x) = \frac{\partial}{\partial x_1} f(x) I.$$

Thus, $\frac{\partial}{\partial x_1} f(x) \in \mathbb{C}_I$ by Lemma 2.12 and $\frac{\partial}{\partial x_0} f(x) = -I \frac{\partial}{\partial x_1} f(x) \in \mathbb{C}_I$, too. Since the imaginary unit I was arbitrary, the slice derivative $\partial_s f$ satisfies $\partial_s f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for all $I \in \mathbb{S}$. Moreover, it is left slice regular by Corollary 3.30. Thus, $\partial_s f$ belongs to $\mathcal{N}(U)$.

For $c \in U \cap \mathbb{R}$, set $a = f(c)$ and $\tilde{f} = f - a$. On a ball $B_r(c)$, the function $\partial_s f$ allows the power series representation $\partial_s f(x) = \sum_{n=0}^{\infty} (x - c)^n a_n$ with $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$. Since $a_n = \frac{1}{n} \partial_s^n \partial_s f(c)$ by Corollary 3.33, we obtain

$$\tilde{f}(x) = \sum_{n=1}^{\infty} x^n \frac{1}{n!} \partial_s^n f(c) = \sum_{n=1}^{\infty} x^n \frac{1}{n!} \partial_s^{n-1} \partial_s f(c) = \sum_{n=1}^{\infty} \partial_s x^n \frac{1}{n} a_{n-1}$$

for $x \in B_r(c)$. Thus, the power series expansion of $\tilde{f}$ at c has real coefficients and hence, by Remark 3.39, the function $\tilde{f}$ belongs to $\mathcal{N}(U)$.

□

3.4 The slice regular product and Runge's Theorem

We are now going to generalize Runge's Theorem to the slice regular setting. In the classical case, it reads as follows (see Theorem 13.6 in [26]).

Theorem 3.43 (Runge's Theorem, complex)**.** *Let $K \subset \mathbb{C}$ be a compact set and let $\mathfrak{A}$ be a set that contains one point in each connected component of $(\mathbb{C} \cup \{\infty\}) \setminus K$. If f is holomorphic on an open set Ω with $K \subset \Omega$, then, for any $\varepsilon > 0$, there exists a rational function $r(z)$ whose poles lie in the set $\mathfrak{A}$ such that*

$$\sup\{|f(z) - r(z)| : z \in K\} < \varepsilon.$$

As we pointed out before, in general, the pointwise product of two left slice regular functions is not left slice regular. Therefore, it is not clear how to define a slice regular rational function because the pointwise quotients $p^{-1}(x)q(x)$ or $q(x)p^{-1}(x)$ of two left slice regular polynomials $p(x)$ and $q(x)$ are not necessarily slice regular. Indeed, we need a different product to give a meaningful definition of rational slice regular functions.

We start with an observation. The standard multiplication of polynomials over a skew field

$$(\sum_{n=0}^{N} x^n a_n)(\sum_{n=0}^{N} x^n b_n) := \sum_{n=0}^{N} x^n \sum_{k=0}^{n} a_k b_{n-k},$$

as it is discussed for instance in [22], extends naturally to power series. If we apply it to the left slice regular power series $\sum_{n=0}^{\infty} x^n a_n$ and $\sum_{n=0}^{\infty} x^n b_n$ with $a_n, b_n \in \mathbb{H}$, we define

$$\left(\sum_{n=0}^{\infty} x^n a_n\right) \cdot \left(\sum_{n=0}^{\infty} x^n b_n\right) := \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_k b_{n-k}\right), \tag{3.20}$$

which is again a left slice regular power series. To generalize this product, we express it in terms of the complex component functions obtained by Lemma 3.5, the Splitting Lemma.

Let $I, J \in \mathbb{S}$ with $J \perp I$ and let $p(x) = \sum_{n=0}^{\infty} x^n a_n$ and $q(x) = \sum_{n=0}^{\infty} x^n b_n$ with $a_n, b_n \in \mathbb{H}$. By Corollary 2.10, we can write $a_n = a_{n,1} + a_{n,2}J$ with $a_{n,1}, a_{n,2} \in \mathbb{C}_I$ and $b_n = b_{n,1} + b_{n,2}J$ with $b_{n,1}, b_{n,2} \in \mathbb{C}_I$ for $n \in \mathbb{N}_0$. Moreover, if we apply Lemma 3.5, we obtain holomorphic functions p_1, p_2 and q_1, q_2 on $\mathbb{C}_I$ such that $p_I(x) = p_1(x) + p_2(x)J$ and $q_I(x) = q_1(x) + q_2(x)J$ if $x \in \mathbb{C}_I$. Since

$$p_1(x) + p_2(x)J = p_I(x) = \sum_{n=0}^{\infty} x^n a_n = \sum_{n=0}^{\infty} x^n a_{n,1} + \sum_{n=0}^{\infty} x^n a_{n,2}J$$

and

$$q_1(x) + q_2(x)J = q_I(x) = \sum_{n=0}^{\infty} x^n b_n = \sum_{n=0}^{\infty} x^n b_{n,1} + \sum_{n=0}^{\infty} x^n b_{n,2}J$$

and since 1 and J are left linearly independent over $\mathbb{C}_I$, the component functions are nothing but the complex power series $p_\ell(x) = \sum_{n=0}^{\infty} x^n a_{n,\ell}$ and $q_\ell(x) = \sum_{n=0}^{\infty} x^n b_{n,\ell}$ for $\ell = 1, 2$. Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k b_{n-k} &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} (a_{k,1}b_{n-k,1} + a_{k,1}b_{n-k,2}J + a_{k,2}Jb_{n-k,1} + a_{k,2}Jb_{n-k,2}J) = \\ &= \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,1}b_{n-k,1}\right) + \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,2}Jb_{n-k,1}\right) + \\ &\quad + \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,1}b_{n-k,2}J\right) + \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,2}Jb_{n-k,2}J\right).\end{aligned}$$

Note that x and the coefficients $a_{n,1}$, $a_{n,2}$, $b_{n,1}$ and $b_{n,2}$ commute because they lie in the same complex plane $\mathbb{C}_I$. Moreover, for any $x \in \mathbb{C}_I$, we have $Jx = \overline{x}J$ by Corollary 2.11. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k b_{n-k} &= \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,1}b_{n-k,1}\right) - \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,2}\overline{b_{n-k,2}}\right) + \\ &\quad + \left[\sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,1}b_{n-k,2}\right) + \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_{k,2}\overline{b_{n-k,1}}\right)\right] J = \\ &= \left(\sum_{n=0}^{\infty} x^n a_{n,1}\right)\left(\sum_{n=0}^{\infty} x^n b_{n,1}\right) - \left(\sum_{n=0}^{\infty} x^n a_{n,2}\right)\left(\sum_{n=0}^{\infty} x^n \overline{b_{n,2}}\right) + \\ &\quad + \left[\left(\sum_{n=0}^{\infty} x^n a_{n,1}\right)\left(\sum_{n=0}^{\infty} x^n b_{n,2}\right) + \left(\sum_{n=0}^{\infty} x^n a_{k,2}\right)\left(\sum_{n=0}^{\infty} a_{k,2}\overline{b_{n,1}}\right)\right] J = \\ &= p_1(x)q_1(x) - p_2(x)\overline{q_2(\overline{x})} + \left(p_1(x)q_2(x) + p_2(x)\overline{q_1(\overline{x})}\right) J.\end{aligned}$$

Definition 3.44. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $I, J \in \mathbb{S}$ with $I \perp J$. Let $f, g : U \to \mathbb{H}$ be left slice regular and let $f_1, f_2, g_1, g_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ be the holomorphic functions such that $f_I = f_1 + f_2J$ and $g_I = g_1 + g_2J$ obtained by Lemma 3.5. Then we define*

$$f_I \circledast g_I(x) = f_1(x)g_1(x) - f_2(x)\overline{g_2(\overline{x})} + \left(f_1(x)g_2(x) + f_2(x)\overline{g_1(\overline{x})}\right) J \quad \text{for all } x \in U \cap \mathbb{C}_I.$$

This function is left holomorphic on $U \cap \mathbb{C}_I$ with values in $\mathbb{H}$. Hence, by Lemma 3.24, we can define

$$f \circledast g = \operatorname{ext}_L(f_I \circledast g_I)$$

as the unique left slice regular extension of $f_I \circledast g_I$. The function $f \circledast g$ is called the left slice regular product *of f and g.*

On the other hand, if $f,g : U \to \mathbb{H}$ are right slice regular, let $f_1, f_2, g_1, g_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ be the holomorphic functions such that $f_I = f_1 + Jf_2$ and $g_I = g_1 + Jg_2$ obtained by Lemma 3.5. Then we set

$$f_I \circledR g_I(x) = f_1(x)g_1(x) - \overline{f_2(\overline{x})}g_2(x) + J\left(f_1(x)g_2(x) + \overline{f_2(\overline{x})}g_1(x)\right) \quad \text{for all } x \in U \cap \mathbb{C}_I.$$

This function is right holomorphic on $U \cap \mathbb{C}_I$ with values in $\mathbb{H}$. Hence, by Lemma 3.24, we can define

$$f \circledR g = \operatorname{ext}_R(f_I \circledR g_I)$$

as the unique right slice regular extension of $f_I \circledR g_I$. The function $f \circledR g$ is called the right slice regular product *of f and g.*

Note that at the first glance $f \circledL g$ and $f \circledR g$ depend on the chosen $I, J \in \mathbb{S}$. We will see in Lemma 3.46 that $f \circledL g$ and $f \circledR g$ are in fact independent of I and J.

Remark 3.45. By the considerations before Definition 3.44, the slice regular product is consistent with (3.20). Moreover, if $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$, then $f_I = f_1$ and $f_2 = 0$. Therefore, if g is left slice regular, we have

$$f_I \circledL g_I(x) = f_1(x)g_1(x) + f_1(x)g_2(x)J = f_I(x)g_I(x).$$

In particular, if $f \in \mathcal{N}(U)$, the product fg is left slice regular by Corollary 3.40 and therefore $f \circledL g = \operatorname{ext}_L(f_I g_I) = fg$, that is, the slice regular product of f and g is nothing but the their pointwise product. Similarly, if $f \in \mathcal{M}^R(U)$ and $g \in \mathcal{N}(U)$, then $f \circledR g = fg$.

Nevertheless, if f and g are both left and right regular but $f, g \notin \mathcal{N}(U)$, then their left and right slice regular products do not coincide. Consider for instance the functions $f(x) = x + a$ and $g(x) = x + b$ with $a, b, \in \mathbb{H} \setminus \mathbb{R}$. Then f and g are obviously left and right slice regular but if $a + b \notin \mathbb{R}$, then there exists $x \in \mathbb{H}$ such that $(a + b)x \neq x(a + b)$ because of Lemma 2.5. Hence, by (3.20),

$$f \circledL g(x) = x^2 + x(a + b) + ab \neq x^2 + (a + b)x + ab = f \circledR g(x).$$

Lemma 3.46. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let f and g be left slice regular functions on U. Moreover, let α and β resp. γ and δ be functions as in Corollary 3.22, such that $f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1)$ and $g(x) = \gamma(x_0, x_1) + I\delta(x_0, x_1)$ for all $x = x_0 + Ix_1 \in U$. Then*

$$f \circledL g = \alpha\gamma - \beta\delta + I(\alpha\delta + \beta\gamma). \tag{3.21}$$

If on the other hand f and g are right slice regular functions on U and α and β resp. γ and δ are functions on D as in Corollary 3.22, such that $f(x) = \alpha(x_0, x_1) + \beta(x_0, x_1)I$ and $g(x) = \gamma(x_0, x_1) + \delta(x_0, x_1)I$ for all $x = x_0 + Ix_1 \in U$. Then

$$f \circledR g = \alpha\gamma - \beta\delta + (\alpha\delta + \beta\gamma)I.$$

In particular, the left and the right slice regular product are well defined as they do not depend on the elements $I, J \in \mathbb{S}$.

Proof. We consider again the left slice regular case. First of all note that $\eta = \alpha\gamma - \beta\delta$ and $\mu = \alpha\delta + \beta\gamma$ satisfy

$$\begin{aligned}\eta(x_0, -x_1) &= \alpha(x_0, -x_1)\gamma(x_0, -x_1) - \beta(x_0, -x_1)\delta(x_0, -x_1) = \\ &= \alpha(x_0, x_1)\gamma(x_0, x_1) - (-\beta(x_0, x_1))(-\delta(x_0, x_1)) = \eta(x_0, x_1)\end{aligned}$$

and

$$\begin{aligned}\mu(x_0, -x_1) &= \alpha(x_0, -x_1)\delta(x_0, -x_1) + \beta(x_0, -x_1)\gamma(x_0, -x_1) = \\ &= \alpha(x_0, x_1)(-\delta(x_0, x_1)) - \beta(x_0, x_1)\gamma(x_0, x_1) = -\mu(x_0, x_1).\end{aligned}$$

Since the functions α and β and the functions γ and δ satisfy the Cauchy-Riemann system (3.9),

$$\frac{\partial\eta}{\partial x_0} = \frac{\partial\alpha}{\partial x_0}\gamma + \alpha\frac{\partial\gamma}{\partial x_0} - \frac{\partial\beta}{\partial x_0}\delta - \beta\frac{\partial\delta}{\partial x_0} = \frac{\partial\beta}{\partial x_1}\gamma + \alpha\frac{\partial\delta}{\partial x_1} + \frac{\partial\alpha}{\partial x_1}\delta + \beta\frac{\partial\gamma}{\partial x_1} = \frac{\partial\mu}{\partial x_1}$$

and

$$\frac{\partial\mu}{\partial x_0} = \frac{\partial\alpha}{\partial x_0}\delta + \alpha\frac{\partial\delta}{\partial x_0} + \frac{\partial\beta}{\partial x_0}\gamma + \beta\frac{\partial\gamma}{\partial x_0} = \frac{\partial\beta}{\partial x_1}\delta - \alpha\frac{\partial\gamma}{\partial x_1} - \frac{\partial\alpha}{\partial x_1}\gamma + \beta\frac{\partial\delta}{\partial x_1} = -\frac{\partial\eta}{\partial x_1}.$$

Therefore, by Corollary 3.22, the right-hand side of (3.21), that is, the function

$$\xi(x) = \eta(x_0, x_1) + I\mu(x_0, x_1)$$

for $x = x_0 + Ix_1$, is a left slice regular function on U .

Let $I \in \mathbb{S}$ be an arbitrary imaginary unit and let $x \in U \cap \mathbb{C}_I$. Moreover, let $J \in \mathbb{S}$ with $J \perp I$ and let $f_1, f_2, g_1, g_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ be holomorphic functions such that $f_I = f_1 + f_2J$ and $g_I = g_1 + g_2J$. Since $zJ = J\overline{z}$ for any $z \in \mathbb{C}_I$, we have

$$\begin{aligned} f_I \circledcirc g_I(x) &= f_1(x)g_1(x) - f_2(x)\overline{g_2(\overline{x})} + \Big(f_1(x)g_2(x) + f_2(x)\overline{g_1(\overline{x})}\Big) J \\ &= f_1(x)g_1(x) + f_2(x)Jg_2(\overline{x})J + f_1(x)g_2(x)J + f_2(x)Jg_1(\overline{x}). \end{aligned}$$

By adding and subtracting the terms $\frac{1}{2}f_1(x)g_1(\overline{x})$ and $\frac{1}{2}f_1(x)g_2(\overline{x})J$ and the terms $\frac{1}{2}f_2(x)Jg_1(x)$ and $\frac{1}{2}f_2(x)Jg_2(x)J$, we obtain

$$\begin{aligned} f_I \circledcirc g_I(x) =& \frac{1}{2}f_1(x)g_1(x) + \frac{1}{2}f_1(x)g_2(x)J + \frac{1}{2}f_2(x)Jg_1(x) + \frac{1}{2}f_2(x)Jg_2(x)J + \\ &+ \frac{1}{2}f_1(x)g_1(\overline{x}) + \frac{1}{2}f_1(x)g_2(\overline{x})J + \frac{1}{2}f_2(x)Jg_2(\overline{x})J + \frac{1}{2}f_2(x)Jg_1(\overline{x}) + \\ &- \frac{1}{2}f_1(x)g_1(\overline{x}) - \frac{1}{2}f_1(x)g_2(\overline{x})J + \frac{1}{2}f_2(x)Jg_2(\overline{x})J + \frac{1}{2}f_2(x)Jg_1(\overline{x}) - \\ &+ \frac{1}{2}f_1(x)g_1(x) + \frac{1}{2}f_1(x)g_2(x)J - \frac{1}{2}f_2(x)Jg_1(x) - \frac{1}{2}f_2(x)Jg_2(x)J. \end{aligned}$$

If we group the terms in each line, we get

$$\begin{aligned} f_I \circledcirc g_I(x) =& \frac{1}{2}\big(f_1(x) + f_2(x)J\big)\big(g_1(x) + g_2(x)J\big) + \\ &+ \frac{1}{2}\big(f_1(x) + f_2(x)J\big)\big(g_1(\overline{x}) + g_2(\overline{x})J\big) + \\ &+ \frac{1}{2}\big(- f_1(x) + f_2(x)J\big)\big(g_1(\overline{x}) + g_2(\overline{x})J\big) + \\ &+ \frac{1}{2}\big(f_1(x) - f_2(x)J\big)\big(g_1(x) + g_2(x)J\big) = \\ =& f(x)\frac{1}{2}\big(g(x) + g(\overline{x})\big) + \big(- f_1(x) + f_2(x)J\big)\frac{1}{2}\big(g(\overline{x}) - g(x)\big) \end{aligned}$$

Due to (3.6), we have $f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1)$, $\gamma(x_0, x_1) = \frac{1}{2}\big(g(x) + g(\overline{x})\big)$ and $\delta(x_0, x_1) = \frac{1}{2}I\big(g(\overline{x}) - g(x)\big)$. Hence,

$$\begin{aligned} f_I \circledcirc g_I(x) &= f(x)\gamma(x_0, x_1) + (f_1(x) - f_2(x)J)I\delta(x_0, x_1) = \\ &= f(x)\gamma(x_0, x_1) + I(f_1(x) + f_2(x)J)\delta(x_0, x_1) = \\ &= f(x)\gamma(x_0, x_1) + If(x)\delta(x_0, x_1) = \\ &= \alpha(x_0, x_1)\gamma(x_0, x_1) - \beta(x_0, x_1)\delta(x_0, x_1) + I\big(\beta(x_0, x_1)\gamma(x_0, x_1) + \alpha(x_0, x_1)\delta(x_0, x_1)\big). \end{aligned}$$

Therefore, $\xi_I = f_I \circledcirc g_I$ and, since the left slice regular extension is unique, we obtain $\xi = f \circledcirc g$. Moreover, as the imaginary unit I was arbitrary and α, β, γ and δ do not depend on I by Theorem 3.21, the slice regular product is independent of the imaginary units I and J used in its definition.

□

Corollary 3.47. *The left and the right slice regular product are associative and distributive over the pointwise addition.*

Proof. Let $f = \alpha + I\beta$, let $g = \gamma + I\delta$ and $h = \eta + I\mu$ be left slice regular functions on an axially symmetric slice domain U. Then

$$\begin{aligned} f{\circledast_L}(g+h) &= \alpha(\gamma+\eta) - \beta(\delta+\mu) + I(\alpha(\delta+\mu) + \beta(\gamma+\eta)) = \\ &= \alpha\gamma - \beta\delta + I(\alpha\delta+\beta\gamma) + \alpha\eta - \beta\mu + I(\alpha\mu+\beta\eta) = f{\circledast_L}g + f{\circledast_L}h \end{aligned}$$

and

$$\begin{aligned} (f+g){\circledast_L}h &= (\alpha+\gamma)\eta - (\beta+\delta)\mu + I((\alpha+\gamma)\mu + (\beta+\delta)\eta) = \\ &= \alpha\eta - \beta\mu + I(\alpha\mu+\beta\eta) + \gamma\eta - \delta\mu + I(\gamma\mu+\delta\eta) = f{\circledast_L}g + f{\circledast_L}h. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (f{\circledast_L}g){\circledast_L}h &= (\alpha\gamma - \beta\delta + I(\alpha\delta+\beta\gamma){\circledast_L}h = \\ &= \alpha\gamma\eta - \beta\delta\eta - \alpha\delta\mu - \beta\gamma\mu + I(\alpha\gamma\mu - \beta\delta\mu + \alpha\delta\eta + \beta\gamma\eta) = \\ &= f{\circledast_L}(\gamma\eta - \delta\mu + I(\gamma\mu+\delta\eta) = f{\circledast_L}(g{\circledast_L}h). \end{aligned}$$

□

Note that in general the left and the right slice regular product are not commutative—except if one of the functions belongs to $\mathcal{N}(U)$.

Corollary 3.48. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $f \in \mathcal{N}(U)$ and $g \in \mathcal{M}^L(U)$. Then*

$$fg = f{\circledast_L}g = g{\circledast_L}f.$$

Similarly, if $f \in \mathcal{M}^R(U)$ and $g \in \mathcal{N}(U)$, then

$$fg = f{\circledast_R}g = g{\circledast_R}f.$$

Proof. Let $f \in \mathcal{N}(U)$ and $g \in \mathcal{M}^L(U)$. By Remark 3.45, $fg = f{\circledast_L}g$. For $c \in U \cap \mathbb{R}$, let $f(x) = \sum_{n=0}^{\infty}(x-c)^n a_n$ and $g(x) = \sum_{n=0}^{\infty}(x-c)^n b_n$ be the power series representation of f and g on a ball $B_r(c)$ centered at c. Then $a_n \in \mathbb{R}$ because of $f \in \mathcal{N}(U)$. By the considerations before Definition 3.44, we obtain

$$f{\circledast_L}g(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k b_{n-k} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} b_{n-k} a_k = g{\circledast_L}f(x)$$

for any $x \in B_r(c)$. By Theorem 3.8, the Identity Principle, we obtain $f{\circledast_L}g = g{\circledast_L}f$.

□

The slice regular product is the proper tool to define slice regular rational functions and to prove Runge's Theorem in a slice regular setting.

Definition 3.49. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $f = \alpha + I\beta : U \to \mathbb{H}$ be a left slice regular function as in Corollary 3.22. We call the function*

$$f^c = \overline{\alpha} + I\overline{\beta}$$

the left slice regular conjugate *of f and the function $f^c{\circledast_L}f$ the* symmetrization *of f.*

If $f = \alpha + \beta I : U \to \mathbb{H}$ is right slice regular, then the function

$$f^c = \overline{\alpha} + \overline{\beta} I$$

is called the right slice regular conjugate *of f and the function $f{\circledast_R}f^c$ is called the* symmetrization *of f.*

Remark 3.50. Let $f = \alpha + I\beta : U \to \mathbb{H}$ be left slice regular. Then we have

$$\overline{\alpha(x_0,-x_1)} = \overline{\alpha(x_0,x_1)} \qquad \text{and} \qquad \overline{\beta(x_0,-x_1)} = \overline{-\beta(x_0,x_1)} = -\overline{\beta(x_0,x_1)}.$$

Moreover, we have

$$\frac{\partial}{\partial x_0}\overline{\alpha(x_0,x_1)} = \overline{\frac{\partial}{\partial x_0}\alpha(x_0,x_1)} = \overline{\frac{\partial}{\partial x_1}\beta(x_0,x_1)} = \frac{\partial}{\partial x_1}\overline{\beta(x_0,x_1)}$$

and

$$\frac{\partial}{\partial x_1}\overline{\alpha(x_0,x_1)} = \overline{\frac{\partial}{\partial x_1}\alpha(x_0,x_1)} = -\overline{\frac{\partial}{\partial x_0}\beta(x_0,x_1)} = -\frac{\partial}{\partial x_0}\overline{\beta(x_0,x_1)}.$$

Hence, according to Corollary 3.22, the function f^c is actually left slice regular.

Remark 3.51. Let again $f = \alpha + I\beta : U \to \mathbb{H}$ be left slice regular and let $x = x_0 + Ix_1 \in U$. Since $\alpha(x_0, x_1) = \frac{1}{2}(f(x) + f(\overline{x}))$ and $\beta(x_0, x_1) = \frac{1}{2}I(f(\overline{x}) - f(x))$, we have

$$f^c(x) = \frac{1}{2}\overline{(f(x) + f(\overline{x}))} + I\frac{1}{2}\overline{I(f(\overline{x}) - f(x))} = \frac{1}{2}\left(\overline{f(x)} + \overline{f(\overline{x})} - I\overline{f(\overline{x})}I + I\overline{f(x)}I\right).$$

If we choose $J \in \mathbb{S}$ with $J \perp I$ and apply Lemma 3.5, the Splitting Lemma, to obtain holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = f_1 + f_2 J$, then we have $\overline{f_2(z)J} = \overline{J}\,\overline{f_2(z)} = -f_2(z)J$ for $z \in U \cap \mathbb{C}_I$. Hence,

$$\begin{aligned} f^c(x) &= \frac{1}{2}\left(\overline{f_1(x)} - f_2(x)J + \overline{f_1(\overline{x})} - f_2(\overline{x})J - I\overline{f_1(\overline{x})}I + If_2(\overline{x})JI + I\overline{f_1(x)}I - If_2(x)JI\right) = \\ &= \frac{1}{2}\left(\overline{f_1(x)} - f_2(x)J + \overline{f_1(\overline{x})} - f_2(\overline{x})J + \overline{f_1(\overline{x})} + f_2(\overline{x})J - \overline{f_1(x)} - f_2(x)J\right). \end{aligned}$$

Therefore,

$$f^c(x) = \overline{f_1(\overline{x})} - f_2(x)J, \tag{3.22}$$

and in turn

$$f^s(x) = f^c \circledast_L f(x) = \overline{f_1(\overline{x})}f_1(x) + f_2(x)\overline{f_2(\overline{x})} + \left(\overline{f_1(\overline{x})}f_2(x) - f_2(x)\overline{f_1(\overline{x})}\right)J.$$

But since f_1 and f_2 both have their values in the complex plane $\mathbb{C}_I$, they commute and we obtain

$$f^s(x) = \overline{f_1(\overline{x})}f_1(x) + \overline{f_2(\overline{x})}f_2(x). \tag{3.23}$$

Furthermore, we also have

$$f \circledast_L f^c(x) = f_1(x)\overline{f_1(\overline{x})} + f_2(x)\overline{f_2(\overline{x})} + (-f_1(x)f_2(x) + f_2(x)f_1(x))J = \overline{f_1(\overline{x})}f_1(x) + \overline{f_2(\overline{x})}f_2(x).$$

Consequently,

$$f^s(x) = f^c \circledast_L f = f \circledast_L f^c. \tag{3.24}$$

Corollary 3.52. *Let $c \in \mathbb{R}$ and let $f(x) = \sum_{n=0}^{\infty}(x-c)^n a_n$ be left slice regular power series centered at c that converges on $B_r(c)$. Then $f^c(x) = \sum_{n=0}^{\infty}(x-c)^n \overline{a_n}$ for any $x \in B_r(c)$.*

Similarly, if $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a right slice regular power series centered at c that converges on $B_r(c)$, then $f^c(x) = \sum_{n=0}^{\infty}\overline{a_n}(x-c)^n$ for any $x \in B_r(c)$.

Proof. Let $f = \sum_{n=0}^{\infty}(x-c)^n a_n$ be a left slice regular power series centered at c and let $I, J \in \mathbb{S}$ with $I \perp J$. Then, for each coefficient a_n, there exist $a_{n,1}, a_{n,2} \in \mathbb{C}_I$ such that $a_n = a_{n,1} + a_{n,2}J$. Moreover, $f_1(x) = \sum_{n=0}^{\infty}(x-c)^n a_{n,1}$ and $f_2(x) = \sum_{n=0}^{\infty}(x-c)^n a_{n,2}$ for any $x \in B_r(c) \cap \mathbb{C}_I$, where f_1 and f_2 are the holomorphic component functions obtained by Lemma 3.5, the Splitting Lemma. By Corollary 2.11, we have $a_{n,2}J = J\overline{a_{n,2}} = -\overline{a_{n,2}J}$. Hence, because of (3.22), we obtain

$$\begin{aligned} f^c(x) = \overline{f_1(\overline{x})} - f_2(x)J &= \overline{\sum_{n=0}^{\infty}(\overline{x}-c)^n a_{n,1}} - \sum_{n=0}^{\infty}(x-c)^n a_{n,2}J = \\ &= \sum_{n=0}^{\infty}(x-c)^n\overline{a_{n,1}} + \sum_{n=0}^{\infty}(x-c)^n\overline{a_{n,2}J} = \sum_{n=0}^{\infty}(x-c)^n\overline{a_n} \end{aligned}$$

for any $x \in \mathbb{C}_I$. Since I was arbitrary, the statement is verfied.

□

Corollary 3.53. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let f be a left or right slice regular function on U. Then $f^s \in \mathcal{N}(U)$.*

Proof. Let f be left slice regular and let us write it in the form $f = \alpha + I\beta$, according to Corollary 3.22. Then

$$f^c \circledast_L f = \overline{\alpha}\alpha - \overline{\beta}\beta + I(\overline{\alpha}\beta + \overline{\beta}\alpha) = |\alpha|^2 - |\beta|^2 + I(\overline{\alpha}\beta + \overline{\overline{\alpha}\beta}).$$

But $|\alpha|^2 - |\beta|^2$ and $2\mathrm{Re}(\overline{\alpha}\beta) = \overline{\alpha}\beta + \overline{\overline{\alpha}\beta}$ are obviously real. Hence, $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for any $I \in \mathbb{S}$.

□

Definition 3.54. *A zero x of f is called* isolated*, if there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(x)$ contains no other zero of f. A two-sphere $[x]$ of zeros is called* isolated*, if there exists $\varepsilon > 0$ such that $[B_\varepsilon(x)] \setminus [x]$ contains no zero of f, where $[B_\varepsilon(x)]$ is the axially symmetric hull of $B_\varepsilon(x)$, cf. Example 3.15.*

Theorem 3.55. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be a left or right slice regular function.*

(i) If $f(x) = 0$, then $f^s(x) = 0$. Moreover, f^s vanishes identically if and only if f vanishes identically.

(ii) Let $f^s \not\equiv 0$. If the set of zeros of f^s is nonempty, it consists of the union of isolated 2-spheres of the form $[x] = \{x_0 + Ix_1 : I \in \mathbb{S}\}$.

(iii) Let $f \not\equiv 0$. If the set of zeros of f is nonempty, it consists of the union of isolated 2-spheres of the form $[x]$ and of isolated points.

Proof. Let f be left slice regular and let us write it in the form $f = \alpha + I\beta$ according to Corollary 3.22. Assume that

$$f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1) = 0$$

for some $x = x_0 + Ix_1 \in U$. If $\beta(x_0, x_1) = 0$, then $\alpha(x_0, x_1) = 0$, which implies that $f(\widetilde{x}) = \alpha(x_0, x_1) + \tilde{I}\beta(x_0, x_1) = 0$ for any $\widetilde{x} = x_0 + \tilde{I}x_1 \in [x]$. If on the other hand $\beta(x_0, x_1) \neq 0$, then $\alpha(x_0, x_1) \neq 0$ and therefore $I = -\alpha(x_0, x_1)\beta(x_0, x_1)^{-1}$. Recall from Theorem 3.21 that α and β do not depend on the imaginary unit I. Hence, $x = x_0 + Ix_1$ is the unique solution of $f(x) = 0$ in the 2-sphere $[x]$. Therefore, if $f(x) = 0$, then either $f(\widetilde{x}) = 0$ for any $\widetilde{x} \in [x]$ or $f(\widetilde{x}) \neq 0$ for all $\widetilde{x} \in [x] \setminus \{x\}$.

As we have seen in the proof of Corollary 3.53, $f^s = \alpha_s + I\beta_s$, where $\alpha_s = |\alpha|^2 - |\beta|^2$ and $\beta_s = \overline{\alpha}\beta + \alpha\overline{\beta}$ are real-valued. Therefore, $f^s(x) = 0$ implies $\alpha_s(x_0, x_1) = 0$ and $\beta_s(x_0, x_1) = 0$. Consequently, the set of zeros of f^s consists of 2-spheres of the form $[x]$. Moreover, if $f(x) = 0$ then either $\alpha(x_0, x_1) = \beta(x_0, x_1) = 0$, which implies $\alpha_s(x_0, x_1) = \beta_s(x_0, x_1) = 0$ and therefore also $f^s(x_0, x_1) = 0$, or $\alpha(x_0, x_1) = -I\beta(x_0, x_1)$. But in this second case we have

$$\alpha_s(x_0, x_1) = |\alpha(x_0, x_1)|^2 - |\beta(x_0, x_1)|^2 = | - I\beta(x_0, x_1)|^2 - |\beta(x_0, x_1)|^2 = 0$$

and

$$\beta_s(x_0, x_1) = \overline{\alpha(x_0, x_1)}\beta(x_0, x_1) + \overline{\beta(x_0, x_1)}\alpha(x_0, x_1) = \overline{\beta(x_0, x_1)}I\beta(x_0, x_1) - \overline{\beta(x_0, x_1)}I\beta(x_0, x_1) = 0,$$

because $\overline{\alpha_s(x_0, x_1)} = \overline{\beta_s(x_0, x_1)}\,\overline{(-I)} = \overline{\beta_s(x_0, x_1)}I$. Thus, $f(x) = 0$ always implies $f^s(x) = 0$.

Let us assume that $f^s \equiv 0$. Let $I, J \in \mathbb{S}$ with $I \perp J$ and let $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ be the holomorphic functions such that $f_I = f_1 + f_2J$ obtained by Lemma 3.5, the Splitting Lemma. As U is a slice domain, there exists a point $c \in U \cap \mathbb{R}$. By Corollary 3.18, f allows a power series representation of the form $f(x) = \sum_{n=0}^{\infty}(x - c)^n a_n$ on a ball $B_r(c)$. Moreover, by Corollary 2.10, there exist $a_{n,1}, a_{n,2} \in \mathbb{C}_I$ such that $a_n = a_{n,1} + a_{n,2}J$ for each $n \in \mathbb{N}_0$. For any $x \in B_r(c) \cap \mathbb{C}_I$, we have $f_1(x) = \sum_{n=0}^{\infty}(x - c)^n a_{n,1}$ and $f_2(x) = \sum_{n=0}^{\infty}(x - c)^n a_{n,2}$. According to (3.23),

$$\begin{aligned}
f^s(x) &= \overline{f_1(\overline{x})}f_1(x) + \overline{f_2(\overline{x})}f_2(x) = \\
&= \overline{\sum_{n=0}^{\infty}(\overline{x} - c)^n a_{n,1}} \sum_{n=0}^{\infty}(x - c)^n a_{n,1} + \overline{\sum_{n=0}^{\infty}(\overline{x} - c)^n a_{n,2}} \sum_{n=0}^{\infty}(x - c)^n a_{n,2} = \\
&= \sum_{n=0}^{\infty}(x - c)^n \overline{a_{n,1}} \sum_{n=0}^{\infty}(x - c)^n a_{n,1} + \sum_{n=0}^{\infty}(x - c)^n \overline{a_{n,2}} \sum_{n=0}^{\infty}(x - c)^n a_{n,2} = \\
&= \sum_{n=0}^{\infty}(x - c)^n \sum_{k=0}^{n} \left(\overline{a_{k,1}}\, a_{n-k,1} + \overline{a_{k,2}}\, a_{n-k,2}\right).
\end{aligned}$$

If we set $c_n = \sum_{k=0}^{n} (\overline{a_{k,1}}\, a_{n-k,1} + \overline{a_{k,2}}\, a_{n-k,2}) = 0$, then $f^s \equiv 0$ yields $c_n = 0$ for all $n \in \mathbb{N}_0$. For $n = 0$, $c_0 = \overline{a_{0,1}}\, a_{0,1} + \overline{a_{0,2}}\, a_{0,2} = |a_{0,1}|^2 + |a_{0,2}|^2 = 0$ yields $a_0 = 0$. Assume, that $a_k = 0$ for all $k = 0, \dots, n-1$. Then we have

$$0 = c_{2n} = \sum_{k=0}^{2n} (\overline{a_{k,1}}\, a_{2n-k,1} + \overline{a_{k,2}}\, a_{2n-k,2}) = \overline{a_{n,1}} a_{n,1} + \overline{a_{n,2}} a_{n,2} = |a_n|^2,$$

because $\overline{a_{k,1}} = \overline{a_{k,2}} = 0$ for $k = 0, \dots, n-1$ and $a_{2n-k,1} = a_{2n-k,2} = 0$ for $k = n+1, \dots, 2n$. Thus, we obtain $a_n = 0$ for all $n \in \mathbb{N}_0$. But $f \equiv 0$ on $B_r(c)$ yields $f \equiv 0$ by Theorem 3.8, the Identity Principle. Therefore, *(i)* holds true.

We have already seen, that the set of zeros of f^s is the union of 2-spheres of the form $[x]$. To prove that they are isolated, we assume the converse. As $[B_\varepsilon(x)] = B_\varepsilon([x])$ (see Example 3.15), there exists a sequence $x_n = x_{n,0} + I_n x_{n,1}, n \in \mathbb{N}$ with $\operatorname{dist}(x_n, [x]) = \inf\{|\widetilde{x} - x_n| : \widetilde{x} \in [x]\} \to 0$ as $n \to \infty$ such that $f(x_n) = 0$ for all $n \in \mathbb{N}$. But this implies $f^s(x_{n,0} + I x_{n,1}) = 0$ for any $I \in \mathbb{S}$ and any $n \in \mathbb{N}$. Thus, in each complex plane $\mathbb{C}_I$, we can find an accumulation point of zeros of f^s. Hence, $f^s \equiv 0$ by Theorem 3.8, the Identity Principle. Therefore, *(ii)* holds true.

Finally, we show *(iii)*. We have already seen that $f(x) = 0$ implies either $f(\widetilde{x}) = 0$ for all $\widetilde{x} \in [x]$ or $f(\widetilde{x}) \neq 0$ for all $\widetilde{x} \in [x] \setminus \{x\}$. Assume that x is the only zero of f in $[x]$ and that it is not isolated. Then there exists a sequence $x_n \in U \setminus [x], n \in \mathbb{N}$ of zeros of f with $\lim_{n\to\infty} x_n = x$. But $[x]$ and $[x_n]$ are 2-spheres of zeros of f^s and $[x]$ is not isolated. Hence, $f^s \equiv 0$ by *(ii)*, which implies $f \equiv 0$ by *(i)*.

If on the other hand $[x]$ is a 2-sphere of zeros of f and it is not isolated, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of zeros f with $\operatorname{dist}(x_n, [x]) = \inf\{|\widetilde{x} - x_n| : \widetilde{x} \in [x]\} \to 0$ as $n \to \infty$. Then $[x]$ and $[x_n]$ are again 2-spheres of zeros of f^s and $[x]$ is not isolated. Hence, $f^s \equiv 0$ by *(ii)*, which implies $f \equiv 0$ by *(i)*.

Thus, if $f \not\equiv 0$, then its zero set consists of isolated points and isolated 2-spheres.

□

Lemma 3.56. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ with $f \not\equiv 0$ be left or right slice regular. If we set $\mathcal{Z}_{f^s} = \{x \in U : f^s(x) = 0\}$, then the function $x \mapsto f^s(x)^{-1}$ belongs to $\mathcal{N}(U \setminus \mathcal{Z}_{f^s})$.*

Proof. Let $f \in \mathcal{M}^L(U)$, let $I, J \in \mathbb{S}$ with $I \perp J$ and let $f_1^s, f_2^s : U \cap \mathbb{C}_I \to \mathbb{C}_I$ be the holomorphic functions obtained by the Splitting Lemma, Lemma 3.5, such that $f_I^s = f_1^s + f_2^s J$. By Corollary 3.53, we have $f^s(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for $I \in \mathbb{S}$, which implies $f_2^s \equiv 0$. Therefore, $f_I^s(x) = f_1^s(x)$ is holomorphic on $U \cap \mathbb{C}_I$. Moreover, because of *(ii)* in Theorem 3.55, the zero set of f_I^s, that is, the set $\mathcal{Z}_{f^s} \cap \mathbb{C}_I = \{x \in U \cap \mathbb{C}_I : f_I^s(x) = 0\}$, consists of isolated points. Therefore, the function $\frac{1}{f_I^s}$ is holomorphic on $(U \setminus \mathcal{Z}_{f^s}) \cap \mathbb{C}_I$. Hence,

$$\bar{\partial}_I f_I^s(x)^{-1} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} \frac{1}{f_I^s(x)} + I \frac{\partial}{\partial x_1} \frac{1}{f_I^s(x)} \right) = 0$$

for all $x \in (U \setminus \mathcal{Z}_{f^s}) \cap \mathbb{C}_I$. Moreover, $f^s(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ implies $f^s((U \setminus \mathcal{Z}_{f^s}) \cap \mathbb{C}_I)^{-1} \subset \mathbb{C}_I$.

Since I was arbitrary, $x \mapsto f^s(x)^{-1}$ belongs to $\mathcal{N}(U \setminus \mathcal{Z}_{f^s})$.

□

Definition 3.57. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain. For any function $f : U \to \mathbb{H}$, we define $\mathcal{Z}_{f^s} = \{x \in U : f^s(x) = 0\}$. If f is left slice regular, then the function*

$$f^{-Ⓛ} = (f^s)^{-1} f^c,$$

which is defined on $U \setminus \mathcal{Z}_{f^s}$, is called the left slice regular inverse of f. *If f is right slice regular, then the function*

$$f^{-Ⓡ} = f^c (f^s)^{-1},$$

which is defined on $U \setminus \mathcal{Z}_{f^s}$, is called the right slice regular inverse of f.

Remark 3.58. Let $f \in \mathcal{N}(U)$. Then $f(\overline{x}) = \overline{f(x)}$ for $x \in U$ by Corollary 3.41. Moreover, if we apply Lemma 3.5, the Splitting Lemma, and write $f_I = f_1 + f_2 J$ with holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$, then we have $f_1(x) = f_I(x) = f(x)$ and $f_2(x) = 0$ for any $x \in U \cap \mathbb{C}_I$. From (3.22) and (3.23), we get

$$f^c(x) = \overline{f_1(\overline{x})} - f_2(x) J = f(x) \qquad \text{and} \qquad f^s(x) = \overline{f_1(\overline{x})} f_1(x) + \overline{f_2(\overline{x})} f_2(x) = f(x)^2.$$

Hence,

$$f^{-\circledL}(x) = f(x)^{-2}f(x) = f(x)^{-1}.$$

Similarly, we obtain $f^{-\circledR}(x) = f(x)^{-1}$.

Corollary 3.59. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let f be a left slice regular function on U. Then*

$$f^{-\circledL} \circledL f = f \circledL f^{-\circledL} = 1 \qquad \text{on } U \setminus \mathcal{Z}_{f^s}.$$

Similarly, if f is a right slice regular function on U, then

$$f^{-\circledR} \circledR f = f \circledR f^{-\circledR} = 1 \qquad \text{on } U \setminus \mathcal{Z}_{f^s}.$$

Proof. Since $(f^s)^{-1} \in \mathcal{N}(U \setminus \mathcal{Z}_{f^s})$, Corollary 3.47, Corollary 3.48 and (3.24) imply

$$f^{-\circledL} \circledL f = \left((f^s)^{-1} f^c\right) \circledL f = \left((f^s)^{-1} \circledL f^c\right) \circledL f = (f^s)^{-1} \circledL (f^c \circledL f) = (f^c \circledL f)^{-1}(f^c \circledL f) = 1$$

and

$$f \circledL f^{-\circledL} = f \circledL \left((f^s)^{-1} f^c\right) = f \circledL \left(f^c \circledL (f^s)^{-1}\right) = (f \circledL f^c) \circledL (f^s)^{-1} = (f^c \circledL f)(f^c \circledL f)^{-1} = 1.$$

□

Example 3.60. Let $a \in \mathbb{H}$ and let us consider $f(x) = x + a$ as a left slice regular function. If we write $f = \alpha + I\beta$ according to Corollary 3.22, we have $\alpha(x_0, x_1) = \frac{1}{2}(f(x) + f(\overline{x})) = x_0 + a$ and $\beta(x_0, x_1) = I\frac{1}{2}(f(\overline{x}) - f(x)) = x_1$ for $x = x_0 + Ix_1$. Hence,

$$f^c(x) = \overline{\alpha(x_0, x_1)} + I\overline{\beta(x_0, x_1)} = x + \overline{a}$$

and

$$\begin{aligned} f^s(x) &= f^c(x) \circledL f(x) = \\ &= \overline{\alpha(x_0,x_1)}\alpha(x_0,x_1) - \overline{\beta(x_0,x_1)}\beta(x_0,x_1) + I(\overline{\alpha(x_0,x_1)}\beta(x_0,x_1) + \overline{\beta(x_0,x_1)}\alpha(x_0,x_1)) = \\ &= (\overline{a} + x_0)(a + x_0) - x_1^2 + I((\overline{a} + x_0)x_1 + x_1(a + x_0)) = \\ &= |a|^2 + \overline{a}x_0 + x_0 a + x_0^2 + I\overline{a}x_1 + Ix_0x_1 + Ix_1\overline{a} + Ix_1x_0 = \\ &= |a|^2 + 2\mathrm{Re}[a](x_0 + Ix_1) + x_0^2 + 2x_0x_1I - x_0^2 = x^2 + 2\mathrm{Re}[a]x + |a|^2. \end{aligned}$$

and

$$(x + a)^{-\circledL} = (x^2 - 2\mathrm{Re}[a]x + |a|^2)^{-1}(x + \overline{a}).$$

On the other hand, if we consider $f(x) = x + a$ as a right slice regular function and if we write $f = \alpha + \beta I$ according to Corollary 3.22, then we obtain again $\alpha(x_0, x_1) = \frac{1}{2}(f(x) + f(\overline{x})) = x_0 + a$ and $\beta(x_0, x_1) = \frac{1}{2}(f(\overline{x}) - f(x))I = x_1$ for $x = x_0 + Ix_1$. Thus, as before,

$$f^c(x) = \overline{\alpha(x_0, x_1)} + \overline{\beta(x_0, x_1)}I = x + \overline{a}$$

and

$$\begin{aligned} f^s(x) &= f(x) \circledR f^c(x) = \\ &= \alpha(x_0,x_1)\overline{\alpha(x_0,x_1)} - \beta(x_0,x_1)\overline{\beta(x_0,x_1)} + (\alpha(x_0,x_1)\overline{\beta(x_0,x_1)} + \beta(x_0,x_1)\overline{\alpha(x_0,x_1)})I = \\ &= (a + x_0)(\overline{a} + x_0) - x_1^2 + ((\overline{a} + x_0)x_1 + x_1(a + x_0))I = \\ &= |a|^2 + x_0\overline{a} + ax_0 + x_0^2 + \overline{a}x_1I + x_0x_1I + x_1\overline{a}I + x_1x_0I = \\ &= |a|^2 + 2\mathrm{Re}[a](x_0 + Ix_1) + x_0^2 + 2x_0x_1I - x_0^2 = x^2 + 2\mathrm{Re}[a]x + |a|^2. \end{aligned}$$

Hence,

$$(x + a)^{-\circledR} = (x + \overline{a})(x^2 - 2\mathrm{Re}[a]x + |a|^2)^{-1}.$$

In particular, we see $(x + a)^{-\circledL} \neq (x + a)^{-\circledR}$ if $a \notin \mathbb{R}$. Hence, the left and the right slice regular inverse of a function that is left and right slice regular on an axially symmetric slice domain U but does not belong to $\mathcal{N}(U)$ do not coincide in general.

Definition 3.61. *A left slice regular function f is called* left rational *if there exist two left slice regular polynomials p and q such that $f = q^{-\circledL} \circledL p$.*

A right slice regular function f is called right rational *if there exist two right slice regular polynomials p and q such that $f = p \circledR q^{-\circledR}$.*

A left rational function defined on an axially symmetric slice domain U is called real rational *if it satisfies $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for all $I \in \mathbb{S}$.*

Theorem 3.62. *Let $U \subset \mathbb{H}$ be a slice domain. A left slice regular function $f : U \to \mathbb{H}$ is left rational if and only if there exist $I, J \in \mathbb{S}$ with $I \perp J$ and rational functions $r_1, r_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = r_1 + r_2 J$. In this case, the holomorphic component functions obtained by Lemma 3.5, the Splitting Lemma, are rational functions on $\mathbb{C}_I$ for any $I, J \in \mathbb{S}$.*

A right slice regular function $f : U \to \mathbb{H}$ is right rational if and only if there exist $I, J \in \mathbb{S}$ with $I \perp J$ and rational functions $r_1, r_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = r_1 + J r_2$. In this case, the holomorphic component functions obtained by Lemma 3.5, the Splitting Lemma, are rational functions on $\mathbb{C}_I$ for any $I, J \in \mathbb{S}$.

A function f is real rational if and only if there exist two polynomials p and q with real coefficients such that $f(x) = p(x)^{-1} q(x)$. This is equivalent to the fact that f is left slice regular and that f_I is a rational function with real coefficients on $\mathbb{C}_I$ for one (and therefore for any) imaginary unit $I \in \mathbb{S}$.

Proof. Let f be left rational. Then there exist left slice regular polynomials $p(x) = \sum_{x=0}^{N} x^n a_n$ and $q(x) = \sum_{n=0}^{M} x^n b_n$, such that $f = q^{-\circledL} \circledL p = (q^s)^{-1} q^c \circledL p = (q^s)^{-1}(q^c \circledL p)$. For $I, J \in \mathbb{S}$ with $I \perp J$, there exist $a_{n,1}, a_{n,2} \in \mathbb{C}_I$ such that $a_n = a_{n,1} + a_{n,2} J$ for $n = 0, \ldots, N$, and $b_{n,1}, b_{n,2} \in \mathbb{C}_I$ such that $b_n = b_{n,1} + b_{n,2} J$ for $n = 0, \ldots, M$. Moreover, the holomorphic component functions of p and q obtained by Lemma 3.5, the Splitting Lemma, satisfy $p_\ell(x) = \sum_{n=0}^{N} x^n a_{n,\ell}$ and $q_\ell(x) = \sum_{n=0}^{M} x^n b_{n,\ell}$ for $\ell = 1, 2$. Thus, they are complex polynomials on $\mathbb{C}_I$.

Because of (3.23), we have

$$q_I^s(x) = \overline{q_1(\overline{x})} q_1(x) + \overline{q_2(\overline{x})} q_2(x).$$

Hence, $q_s(x)$ is a complex polynomial on $\mathbb{C}_I$, too. Moreover, $q_I^c(x) = \sum_{n=0}^{M} x^n \, \overline{a_{n,1}} - \sum_{n=0}^{M} x^n \, \overline{a_{n,2}} J$, and therefore, its holomorphic component functions

$$q_1^c(x) = \sum_{n=0}^{M} x^n \, \overline{a_{n,1}} \qquad \text{and} \qquad q_2^c(x) = \sum_{n=0}^{M} x^n (-\overline{a_{n,2}})$$

are also complex polynomials on $\mathbb{C}_I$. Thus, for $x \in U \cap \mathbb{C}_I$, we obtain

$$f_I(x) = (q_I^s(x))^{-1}(q_I^c(x) \circledL p_I(x)) = \frac{q_1^c(x) p_1(x) - q_2^c(x) \, \overline{p_2(\overline{x})}}{q_I^s(x)} + \frac{q_1^c(x) p_2(x) + q_2^c(x) \, \overline{p_1(\overline{x})}}{q_I^s(x)} J,$$

with rational component functions

$$f_1(x) = \frac{q_1^c(x) p_1(x) - q_2^c(x) \, \overline{p_2(\overline{x})}}{q_I^s(x)} \qquad \text{and} \qquad f_2(x) = \frac{q_1^c(x) p_2(x) + q_2^c(x) \, \overline{p_1(\overline{x})}}{q_I^s(x)}.$$

Since $I, J \in \mathbb{S}$ were arbitrary, the holomorphic component functions obtained by Lemma 3.5, the Splitting Lemma, are rational functions for any $I, J \in \mathbb{S}$ with $I \perp J$.

Conversely, assume that $f_I(z) = \frac{a(z)}{b(z)} + \frac{c(z)}{d(z)} J$, where a, b, c and d are complex polynomials on $\mathbb{C}_I$. Then

$$f_I = \frac{a}{b} + \frac{c}{d} J = \frac{1}{bd}(da + bcJ) = \frac{1}{bd(bd)^c} (bd)^c (da + bcJ),$$

where $(bd)^c(x) = \overline{bd(\overline{x})}$. Since bd has values in $\mathbb{C}_I$,

$$f_I = \frac{1}{bd \circledL (bd)^c} \circledL (bd)^c \circledL (da + bcJ)$$

by Remark 3.45. Hence,

$$f = \left(\mathrm{ext}_L(bd) \circledL \mathrm{ext}_L(bd)^c\right)^{-1} \circledL \mathrm{ext}_L\left((bd)^c\right) \circledL \mathrm{ext}_L(da + bcJ).$$

If we define $p = \operatorname{ext}_L(da + bcJ)$ and $q = \operatorname{ext}_L(bd)$, then p and q are left slice regular polynomials and $q^c = \operatorname{ext}((bd)^c)$. Thus, we obtain

$$f = (q \circledL q^c)^{-1} \circledL q^c \circledL p = q^{-\circledL} \circledL p.$$

Finally, we consider the case of a real rational function. Let f be left slice regular and let $I \in \mathbb{S}$. If $f_I(x) = q_I(x)/p_I(x)$ is a rational function, where p_I and q_I are polynomials with real coefficients on $\mathbb{C}_I$, then their left slice regular extension $p = \operatorname{ext}_L(p_I)$ and $p = \operatorname{ext}_L(q_I)$ are polynomials with real coefficients on $\mathbb{H}$. In particular, they belong to $\mathcal{N}(\mathbb{H})$. Therefore, $p^{-1} = p^{-\circledL}$ is left slice regular by Remark 3.58. It even belongs to $\mathcal{N}(\mathbb{H} \setminus \mathcal{Z}_p)$, where $\mathcal{Z}_p$ denotes the zero set of p, because $x \in \mathbb{C}_I$ implies $p(x) \in \mathbb{C}_I$ and, in turn, $p(x)^{-1} \in \mathbb{C}_I$. Hence, the pointwise product of p^{-1} and q is left slice regular by Corollary 3.40. Since the slice regular extension is unique by Lemma 3.24, we obtain $f = \operatorname{ext}_L(f_I) = \operatorname{ext}_L(p_I^{-1} q_I) = p^{-1}q$.

If on the other hand $f(x) = p(x)^{-1}q(x)$, where p and q are polynomials with real coefficients then $x \in \mathbb{C}_I$ implies $p(x), q(x) \in \mathbb{C}_I$ and, in turn, also $f(x) = p(x)^{-1}q(x) \in \mathbb{C}_I$. For any $I \in \mathbb{S}$, the functions p_I and q_I are polynomials with real coefficients on $\mathbb{C}_I$. Hence, $f_I(x) = p_I(x)^{-1}q_I(x) = q_I(x)/p_I(x)$ is a rational function on $\mathbb{C}_I$ with real coefficients. Consequently, f_I is holomorphic for any $I \in \mathbb{S}$ and f is left slice regular. Since $x \in \mathbb{C}_I$ implies $p(x) \in \mathbb{C}_I$, we have $f(x) = p(x)^{-1}q(x) = p(x)^{-\circledL} \circledL q(x)$ by Remark 3.45 and Remark 3.58. Hence, f is real rational.

Finally, if $f = q^{-\circledL} \circledL p$ is a real rational function, then let $I, J \in \mathbb{S}$ with $I \perp J$. Since f is left rational, we have $f_I = r_1 + r_2 J$ where r_1 and r_2 are rational functions on $\mathbb{C}_I$. But $x \in \mathbb{C}_I$ implies $f_I(x) \in \mathbb{C}_I$. Hence, $r_2 \equiv 0$ and $f_I = r_1$ is a rational function on $\mathbb{C}_I$. Since it satisfies $f_I(\overline{x}) = \overline{f_I(x)}$ by Corollary 3.41, its coefficients are real.

□

Definition 3.63. *Let $I \in \mathbb{S}$ and let $r = r_1 + r_2 J$, where r_1 and r_2 are rational functions on $\mathbb{C}_I$. We call $x \in \mathbb{C}_I \cup \{\infty\}$ a* pole *of r, if x is a pole of r_1 or r_2.*

Theorem 3.64 (Runge's Theorem)**.** *Let $K \subset \mathbb{H}$ be an axially symmetric compact set and let A be an axially symmetric set such that $A \cap C \neq \emptyset$ for any connected component C of $(\mathbb{H} \cup \{\infty\}) \setminus K$. If f is left slice regular on an axially symmetric slice domain U with $K \subset U$, then, for any $\varepsilon > 0$, there exists a left rational function r such that the poles of r_I lie in $A \cap (\mathbb{C}_I \cup \{\infty\})$ for any $I \in \mathbb{S}$ and such that*

$$\sup\{|f(x) - r(x)| : x \in K\} < \varepsilon. \tag{3.25}$$

Similarly, if f is right slice regular on an axially symmetric slice domain U with $K \subset U$, then, for any $\varepsilon > 0$, there exists a right rational function r such that the poles of r_I lie in $A \cap (\mathbb{C}_I \cup \{\infty\})$ for any $I \in \mathbb{S}$ and such that (3.25) *holds true.*

Finally, if $f \in \mathcal{N}(U)$ for some axially symmetric slice domain U with $K \subset U$, then, for any $\varepsilon > 0$, there exists a real rational function r such that the poles of r_I lie in $A \cap (\mathbb{C}_I \cup \{\infty\})$ for any $I \in \mathbb{S}$ and such that (3.25) *holds true.*

Proof. Let $f \in \mathcal{M}^L(U)$ where U is an axially symmetric slice domain with $K \subset U$, let $\varepsilon > 0$ and let $I, J \in \mathbb{S}$ with $I \perp J$. By applying Lemma 3.5, the Splitting Lemma, we obtain holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = f_1 + f_2 J$. The complex Runge's Theorem implies the existence of two rational functions r_1 and r_2 with poles in $A \cap (\mathbb{C}_I \cup \{\infty\})$ such that

$$\sup\{|f_1(z) - r_1(z)| : z \in K \cap \mathbb{C}_I\} < \frac{\varepsilon}{4} \qquad \text{and} \qquad \sup\{|f_2(z) - r_2(z)| : z \in K \cap \mathbb{C}_I\} < \frac{\varepsilon}{4}.$$

We set $\mathcal{P} = \{x \in \mathbb{C}_I : x \text{ or } \overline{x} \text{ is a pole of } r_1 \text{ or } r_2\}$ and apply Lemma 3.24 to define the function $r = \operatorname{ext}_L(r_1 + r_2 J)$ on the axially symmetric slice domain $[\mathbb{C}_I \setminus \mathcal{P}] = \mathbb{H} \setminus [\mathcal{P}]$, where $[\,\cdot\,]$ denotes the axially symmetric hull as in Definiton 3.13. By Theorem 3.62, r is left rational. For any $\mathfrak{I} \in \mathbb{S}$, the restriction $r_{\mathfrak{I}}$ of r to the complex plane $\mathbb{C}_{\mathfrak{I}}$ is left holomorphic on $\mathbb{C}_{\mathfrak{I}} \setminus [\mathcal{P}]$. Therefore, any pole of $r_{\mathfrak{I}}$ that lies in $\mathbb{C}_{\mathfrak{I}}$ belongs to $\mathbb{C}_{\mathfrak{I}} \cap [\mathcal{P}]$. Since A is axially symmetric, this is a subset of $\mathbb{C}_{\mathfrak{I}} \cap A$ because $\mathcal{P} \subset A \cap \mathbb{C}_I$ implies $[\mathcal{P}] \subset [A \cap \mathbb{C}_I] = A$. Moreover, if $\infty \notin A$, then ∞ is no pole of r_I and the limit $a = \lim_{|z| \to \infty} r_I(z)$ exists. If we set $x_I = x_0 + I x_1$ for $x = x_0 + \mathfrak{I} x_1 \in \mathbb{C}_{\mathfrak{I}}$, the Representation Formula, Theorem 3.21, implies

$$\lim_{|x| \to \infty} r_{\mathfrak{I}}(x) = \frac{1}{2}(1 - \mathfrak{I} I) \lim_{|x| \to \infty} r_I(x_I) + \frac{1}{2}(1 + \mathfrak{I} I) \lim_{|x| \to \infty} r_I(\overline{x_I}) = \frac{1}{2}(1 - \mathfrak{I} I) a + \frac{1}{2}(1 + \mathfrak{I} I) a = a.$$

Hence, ∞ is no pole of $r_{\mathfrak{J}}$ if $\infty \notin A$, and in turn, the poles of $r_{\mathfrak{J}}$ belong to $A \cap (\mathbb{C}_{\mathfrak{J}} \cup \{\infty\})$ for any $\mathfrak{J} \in \mathbb{S}$. Moreover, for $x \in K \cap \mathbb{C}_I$, we have

$$|f_I(x) - r_I(x)| = |f_1(x) + f_2(x)J - r_1(x) - r_2(x)J| \leq |f_1(x) - r_1(x)| + |f_2(x) - r_2(x)| < \frac{\varepsilon}{2}.$$

For $x = x_0 + I_x x_1 \in K$, we set again $x_I = x_0 + I_x x_1$. Since K is axially symmetric x_I and $\overline{x_I}$ belong to K. From the Representation Formula, Theorem 3.21, we deduce

$$\begin{aligned}|f(x) - r(x)| &= \frac{1}{2}\left|(1 - I_x I) f_I(x_I) + (1 + I_x I) f_I(\overline{x_I}) - (1 - I_x I) r_I(x_I) - (1 + I_x I) r_I(\overline{x_I})\right| \leq \\ &\leq \frac{1}{2}|(1 - I_x I) f_I(x_I) - (1 - I_x I) r_I(x_I)| + \frac{1}{2}|(1 + I_x I) f_I(\overline{x_I}) - (1 + I_x I) r_I(\overline{x_I})| \leq \\ &\leq |f_I(x_I) - r_I(x_I)| + |f_I(\overline{x_I}) - r_I(\overline{x_I})| < \varepsilon.\end{aligned}$$

The case $f \in \mathcal{M}^R(U)$ works analogously.

If $f \in \mathcal{N}(U)$, then the holomorphic component functions obtained by Lemma 3.5, the Splitting Lemma, satisfy $f_1 = f_I$ and $f_2 = 0$. Hence, we can choose $r_2 = 0$ when we approximate f_1 and f_2 by rational functions r_1 and r_2 on $\mathbb{C}_I$, whose poles belong to $A \cap (\mathbb{C}_I \cup \{\infty\})$. The function $R_1(z) = \frac{1}{2}(r_1(z) + \overline{r_1(\overline{z})})$ is a rational function on $\mathbb{C}_I$, whose poles are contained in the set $\{x, \overline{x} : x \text{ is a pole of } r_1\}$. Since $A \cap (\mathbb{C}_I \cup \{\infty\})$ is symmetric with respect to the real line, this is a subset of $A \cap (\mathbb{C}_I \cup \{\infty\})$. Moreover, R_1 has real coefficients because $R_1(z) = \overline{R_1(\overline{z})}$, and hence, by Theorem 3.62, the function $R = \operatorname{ext}_L(R_1)$ is real rational. Lemma 3.41 implies $f(\overline{x}) = \overline{f(x)}$, and in turn,

$$|f_I(x) - R_1(x)| = \frac{1}{2}|f_1(x) + \overline{f_1(\overline{x})} - r_1(x) - \overline{r_1(\overline{x})}| \leq \frac{1}{2}|f(x) - r_1(x)| + \frac{1}{2}|\overline{f(\overline{x})} - \overline{r_1(\overline{x})}| < \frac{\varepsilon}{2}$$

for $x \in K \cap \mathbb{C}_I$. Thus, as before, we see that the poles of $R_{\mathfrak{J}}$ belong to $A \cap (\mathbb{C}_{\mathfrak{J}} \cup \{\infty\})$ for any $\mathfrak{J} \in \mathbb{S}$ and that the Representation Formula, Theorem 3.21, implies $|f(x) - R(x)| < \varepsilon$ for any $x \in K$.

□

3.5 The Cauchy formula

We develop now the analogue of the Cauchy integral formula in the slice regular setting. If we consider the classical Cauchy formula

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B_r(w)} \frac{f(\zeta)}{z - \zeta}\, d\xi \qquad \text{for } z \in B_r(w),$$

and try to generalize it to the quaternionic situation, we find that the quaternionic function $x \mapsto (x - \zeta)^{-1}$ for $x \in \mathbb{H} \setminus \{0\}$ and fixed $\zeta \in \mathbb{H}$ is in general neither left nor right slice regular. In fact, for a differentiable function $f : U \subset \mathbb{H} \to \mathbb{H}$ and a point $x \in U$ with $f(x) \neq 0$, the directional derivative of f^{-1} at x along $v \in \mathbb{H}$ is

$$\begin{aligned}\frac{\partial}{\partial v} f^{-1}(x) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h}\left(f^{-1}(x + hv) - f^{-1}(x)\right) = \\ &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} f^{-1}(x + hv) \frac{1}{h}\big(f(x) - f(x + hv)\big) f^{-1}(x) = -f^{-1}(x)\left(\frac{\partial}{\partial v} f(x)\right) f^{-1}(x),\end{aligned} \tag{3.26}$$

because the quaternionic multiplication is not commutative. Therefore, if $x = x_0 + I x_1$ and $\xi \notin \mathbb{C}_I$, then ξ and I do not commute by Corollary 2.12. Hence, we obtain

$$\bar{\partial}_I \frac{1}{x - \xi} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} \frac{1}{x - \xi} + I \frac{\partial}{\partial x_1} \frac{1}{x - \xi}\right) = \frac{1}{2}\left(-(x - \xi)^{-2} - I (x - \xi)^{-1} I (x - \xi)^{-1}\right) \neq 0$$

and

$$\frac{1}{x - \xi} \bar{\partial}_I = \frac{1}{2}\left(\frac{\partial}{\partial x_0} \frac{1}{x - \xi} + \frac{\partial}{\partial x_1} \frac{1}{x - \xi} I\right) = \frac{1}{2}\left(-(x - \xi)^{-2} - (x - \xi)^{-1} I (x - \xi)^{-1} I\right) \neq 0.$$

We will choose a different starting point for developing the Cauchy formula for slice regular functions. Recall that the complex Cauchy kernel allows the power series representation

$$\frac{1}{z-\zeta} = \sum_{n=0}^{\infty} \zeta^n z^{-n-1} \tag{3.27}$$

for $|z| > |\zeta|$ as we have seen in (1.2). In the classical case, this property of the complex Cauchy kernel was crucial when we showed in (1.4) that the Riesz–Dunford functional calculus is consistent with polynomials. This motivates the following definition.

Definition 3.65. *Let $x, s \in \mathbb{H}$. We call*

$$\sum_{n=0}^{\infty} x^n s^{-n-1} \qquad \text{and} \qquad \sum_{n=0}^{\infty} s^{-n-1} x^n$$

the left *and* right noncommutative Cauchy kernel series*, respectively.*

Remark 3.66. From

$$\left|x^n s^{-n-1}\right| = \left|s^{-n-1} x^n\right| = |x|^n \, |s|^{-n-1}$$

we get

$$\sum_{n=0}^{\infty} |x^n s^{-n-1}| \leq \frac{1}{|s|} \sum_{n=0}^{\infty} \left(\frac{|x|}{|s|}\right)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} \left|s^{-n-1} x^n\right| \leq \frac{1}{|s|} \sum_{n=0}^{\infty} \left(\frac{|x|}{|s|}\right)^n.$$

Therefore, the left and the right noncommutative Cauchy kernel series are convergent for $|x| < |s|$.

Theorem 3.67. *On the set $\{(s, x) \in \mathbb{H}^2 : |x| < |s|\}$, the functions*

$$S_L(s, x) = -(x - \overline{s})^{-1}(x^2 - 2\mathrm{Re}[s]x + |s|^2)$$

and

$$S_R(s, x) = -(x^2 - 2\mathrm{Re}[s]x + |s|^2)(x - \overline{s})^{-1}$$

are the multiplicative inverses of the left and right noncommutative Cauchy kernel series, respectively.

Proof. We want to show that

$$S_L(s, x) \sum_{n=0}^{\infty} x^n s^{-n-1} = -(x - \overline{s})^{-1}(x^2 - 2\mathrm{Re}[s]x + |s|^2) \sum_{n=0}^{\infty} x^n s^{-n-1} = 1,$$

which is equivalent to

$$(x^2 - 2\mathrm{Re}[s]x + |s|^2) \sum_{n=0}^{\infty} x^n s^{-n-1} = \overline{s} - x.$$

As $2\mathrm{Re}[s] = s + \overline{s}$ and $|s|^2 = \overline{s}s$ are real, they commute with x and we get

$$\begin{aligned}
(x^2 - 2\mathrm{Re}[s]x + |s|^2) \sum_{n=0}^{\infty} x^n s^{-n-1} &= \sum_{n=0}^{\infty} x^{n+2} s^{-n-1} - \sum_{n=0}^{\infty} x^{n+1}(s + \overline{s}) s^{-n-1} + \sum_{n=0}^{\infty} x^n \overline{s} s s^{-n-1} = \\
&= \sum_{n=0}^{\infty} x^{n+2} s^{-n-1} - \sum_{n=0}^{\infty} x^{n+1} s^{-n} - \sum_{n=0}^{\infty} x^{n+1} \overline{s} s^{-n-1} + \sum_{n=0}^{\infty} x^n \overline{s} s^{-n} = \\
&= \sum_{n=0}^{\infty} x^{n+2} s^{-n-1} - \sum_{n=0}^{\infty} x^{n+2} s^{-n-1} - x - \sum_{n=0}^{\infty} x^{n+1} \overline{s} s^{-n-1} + \sum_{n=0}^{\infty} x^{n+1} \overline{s} s^{-n-1} + \overline{s} = \overline{s} - x.
\end{aligned}$$

□

Corollary 3.68. *Let $x, s \in \mathbb{H}$ such that $|x| < |s|$. Then*

$$\sum_{n=0}^{\infty} x^n s^{-n-1} = -(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}(x - \overline{s})$$

and

$$\sum_{n=0}^{\infty} s^{-n-1} x^n = -(x - \overline{s})(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}.$$

Remark 3.69. Note that the functions defined by the closed forms of the left and right noncommutative Cauchy kernel series are defined on the set $\{(s,x) \in \mathbb{H}^2 : x^2 - 2\mathrm{Re}[s]x + |s|^2 \neq 0\}$, which is larger than the domain of convergence of the Cauchy kernel series. In order to determine this set, we observe that $x^2 - 2\mathrm{Re}[s]x + |s|^2$ depends only on the real part and the absolute value of s. Hence, $x^2 - 2\mathrm{Re}[s]x + |s|^2 = 0$ implies $x^2 - 2\mathrm{Re}[\tilde{s}]x + |\tilde{s}|^2 = 0$ for any $\tilde{s} \in \mathbb{H}$ with $\mathrm{Re}[\tilde{s}] = \mathrm{Re}[s]$ and $|\tilde{s}| = |s|$. Since $\tilde{s}$ satisfies $\mathrm{Re}[\tilde{s}] = \mathrm{Re}[s]$ and $|\tilde{s}| = |s|$ if and only if $\tilde{s} \in [s]$, the set D_x of all $s \in \mathbb{H}$ such that $x^2 - 2\mathrm{Re}[s]x + |s|^2 = 0$ is axially symmetric for any $x \in \mathbb{H}$.

If x and s lie in the same complex plane $\mathbb{C}_I$, they commute and we obtain

$$x^2 - 2\mathrm{Re}[s]x + |s|^2 = x^2 - (s + \overline{s})x + \overline{s}s = (x - s)x - \overline{s}(x - s) = (x - s)(x - \overline{s}),$$

which implies $D_x \cap \mathbb{C}_I = \{x, \overline{x}\} = [x] \cap \mathbb{C}_I$. Since D_x is axially symmetric, $D_x = [x]$. Therefore, $x^2 - 2\mathrm{Re}[s]x + |s|^2 = 0$ if and only if $s \in [x]$, which is equivalent to $x \in [s]$.

Definition 3.70. *We call the functions*

$$S_L^{-1}(s,x) = -(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}(x - \overline{s})$$

and

$$S_R^{-1}(s,x) = -(x - \overline{s})(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1},$$

which are defined on the set $\mathcal{D}_S = \{(s,x) \in \mathbb{H}^2 : x \notin [s]\}$ the left *and* right noncommutative Cauchy kernel*, respectively.*

Corollary 3.71. *Let $(s,x) \in \mathcal{D}_S$. Then the equation*

$$-(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}(x - \overline{s}) = (s - \overline{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-1} \tag{3.28}$$

holds true, that is,

$$S_L^{-1}(s,x) = -S_R^{-1}(x,s).$$

Proof. We have

$$\begin{aligned}(x^2 - 2\mathrm{Re}[s]x + |s|^2)(s - \overline{x}) &= (x^2 - xs - x\overline{s} + \overline{s}s)(s - \overline{x}) = \\ &= x^2 s - xs^2 - x\overline{s}s + \overline{s}s^2 - x^2\overline{x} + xs\overline{x} + x\overline{s}\,\overline{x} - \overline{s}s\overline{x} = \\ &= x(xs - s^2 - x\overline{x}) + \overline{s}(s^2 - s\overline{x}) - x\overline{s}s + xs\overline{x} + x\overline{s}\,\overline{x}.\end{aligned}$$

But from

$$x\overline{s}s = x\,|s|^2 = |s|^2\,x = \overline{s}sx$$

and

$$xs\overline{x} + x\overline{s}\,\overline{x} = x2\mathrm{Re}[s]\overline{x} = 2\mathrm{Re}[s]x\overline{x} = 2\mathrm{Re}[s]\,|x|^2 = |x|^2\,s + \overline{s}\,|x|^2 = x\overline{x}s + \overline{s}\,\overline{x}x,$$

it follows that

$$\begin{aligned}(x^2 - 2\mathrm{Re}[s]x + |s|^2)(s - \overline{x}) &= x(xs - s^2 - x\overline{x}) + \overline{s}(s^2 - s\overline{x}) - \overline{s}sx + x\overline{x}s + \overline{s}\,\overline{x}x = \\ &= x(xs - s^2 - x\overline{x} + \overline{x}s) + \overline{s}(s^2 - s\overline{x} - sx + \overline{x}x) = \\ &= x(-s^2 + 2\mathrm{Re}[x]s - x\overline{x}) + \overline{s}(s^2 - 2\mathrm{Re}[x]s + \overline{x}x) = \\ &= -(x - \overline{s})(s^2 - 2\mathrm{Re}[x]s + |x|^2).\end{aligned}$$

If we multiply this equation by $(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-1}$ from the right and by $(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}$ from the left, we obtain (3.28).

□

Remark 3.72. If we compare Definition 3.70 to the formulas in Example 3.60, then we find that $S_L^{-1}(s,x)$ and $S_R^{-1}(s,x)$ are nothing but the left and right slice regular inverse of the function $x \mapsto s - x$, respectively.

Moreover, if x and s are elements of the same complex plane $\mathbb{C}_I$, they commute and the left and right noncommutative Cauchy kernels reduce to the standard Cauchy kernel. In this case, we have

$$(x^2 - \mathrm{Re}[s]x + |s|^2)^{-1} = (x^2 - (s+\overline{s})x + \overline{s}s)^{-1} = ((x-s)x - \overline{s}(x-s))^{-1} = (x-\overline{s})^{-1}(x-s)^{-1}.$$

Therefore, we get

$$S_L^{-1}(s,x) = -(x-s)^{-1}(x-\overline{s})^{-1}(x-\overline{s}) = (s-x)^{-1}$$

and

$$S_R^{-1}(s,x) = -(x-\overline{s})(x-\overline{s})^{-1}(x-s)^{-1} = (s-x)^{-1}.$$

One could think that the left and the right noncommutative Cauchy kernel can always be simplified. The next theorem shows that this is not possible.

Theorem 3.73. *The left and the right noncommutative Cauchy kernel $S_L^{-1}(s,x)$ and $S_R^{-1}(s,x)$ are irreducible. In case of the left kernel $S_L^{-1}(s,x)$, this means that, for any $s \in \mathbb{H} \setminus \mathbb{R}$, it is impossible to find a polynomial $P_s(x)$ such that*

$$x^2 - 2\mathrm{Re}[s]x + |s|^2 = (x-\overline{s})P_s(x), \tag{3.29}$$

which would allow the simplification

$$S_L^{-1}(s,x) = P_s^{-1}(x)(x-\overline{s})^{-1}(x-\overline{s}) = P_s^{-1}(x).$$

Proof. Assume that a there exists a polynomial $P_s(x)$ such that (3.29) holds true. Comparing the degree of the highest power, we see that $P_s(x)$ has to be a monic polynomial of degree one, i.e.,

$$P_s(x) = x - r \tag{3.30}$$

with $r \in \mathbb{H}$. Then the equation (3.29) turns into

$$x^2 - 2\mathrm{Re}[s]x + |s|^2 = (x-\overline{s})(x-r)$$

which gives

$$x^2 - sx - \overline{s}x + s\overline{s} = x^2 - \overline{s}x - xr + \overline{s}r.$$

Hence,

$$-s(x-\overline{s}) = -(x-\overline{s})r.$$

Solving for r, we finally obtain

$$r = (x-\overline{s})^{-1}s(x-\overline{s}). \tag{3.31}$$

If we chonsider $x \in \mathbb{H}$ that belongs to the same complex plane as s, then x and s commute. Therefore,

$$r = (x-\overline{s})^{-1}s(x-\overline{s}) = (x-\overline{s})^{-1}(x-\overline{s})s = s. \tag{3.32}$$

But, as $s \notin \mathbb{R}$, there exist elements $x \in \mathbb{H}$ that do not commute with s, which implies

$$r = (x-\overline{s})^{-1}s(x-\overline{s}) \neq (x-\overline{s})^{-1}(x-\overline{s})s = s. \tag{3.33}$$

Hence, a polynomial $P_s(x)$ satisfying (3.29) cannot exists.

□

Proposition 3.74. *The left noncommutative Cauchy kernel $S_L^{-1}(s,x)$ is left slice regular in the variable x and right slice regular in the variable s on its domain of definition.*

The right noncommutative Cauchy kernel $S_R^{-1}(s,x)$ is right slice regular in the variable x and left slice regular in the variable s on its domain of definition.

Proof. By Remark 3.72, $x \to S_L^{-1}(s,x)$ is the left slice regular inverse and $x \to S_R^{-1}(s,x)$ is the right slice regular inverse of the function $x \to s - x$. Hence, $S_L^{-1}(s,x)$ is left slice regular and $S_R^{-1}(s,x)$ is right slice regular in the second variable. Consequently, since $S_L^{-1}(s,x) = -S_R^{-1}(x,s)$ by Corollary 3.71, $S_L^{-1}(s,x)$ is right slice regular and $S_R^{-1}(s,x)$ is left slice regular in the first variable.

□

Lemma 3.75. *Let $s = s_0 + I_s s_1 \in \mathbb{H} \setminus \mathbb{R}$ and $I \in \mathbb{S}$. Then the functions*

$$\begin{cases} \mathbb{C}_I & \to \mathbb{H} \\ x & \mapsto S_L^{-1}(s,x) \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{C}_I & \to \mathbb{H} \\ x & \mapsto S_R^{-1}(s,x) \end{cases}$$

have two singularities at the points $s_0 \pm I s_1$ if $I \neq \pm I_s$. If $I = I_s$ or $I = -I_s$, they have only one singularity at the point s.

Proof. The singularities of the function $x \mapsto (x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}(x - \overline{s})$ are given by the roots of

$$x^2 - 2\mathrm{Re}[s]x + |s|^2 = 0. \tag{3.34}$$

As we have seen in Remark 3.69, these are exactly the points in $\mathbb{C}_I \cap [s]$, i.e., $x = s_0 \pm I s_1$.

If $I = I_s$ or $I = -I_s$, then x and s commute and the singularity $\overline{s} = s_0 - I s_1$ is removable as we have seen in Remark 3.72.

□

Although the restrictions of $S_L^{-1}(s,\cdot)$ and $S_R^{-1}(s,\cdot)$ to the plane $\mathbb{C}_{I_s}$ have a removable singularity at $\overline{s}$, the kernels themselves can not be extended continuously to $\overline{s}$.

Lemma 3.76. *For any $s \in \mathbb{H}$ the limits $\lim_{x\to\overline{s}} S_L^{-1}(s,x)$ and $\lim_{x\to\overline{s}} S_R^{-1}(s,x)$ do not exist.*

Proof. To prove that the limit $\lim_{x\to\overline{s}} S_L^{-1}(s,x)$ does not exist, we consider $S_L^{-1}(s,\overline{s}+\varepsilon)$ with $\varepsilon = \varepsilon_0 + \sum_{j=1}^3 \varepsilon_j e_j \in \mathbb{H}$. Since $2\mathrm{Re}[s] = s + \overline{s}$ and $|s|^2 = \overline{s}s$, we have

$$\begin{aligned} S_L^{-1}(s,\overline{s}+\varepsilon) &= ((\overline{s}+\varepsilon)^2 - 2(\overline{s}+\varepsilon)\mathrm{Re}[s] + |s|^2)^{-1}(\overline{s}+\varepsilon-\overline{s}) = \\ &= (\overline{s}^2 + \overline{s}\varepsilon + \varepsilon\overline{s} + \varepsilon^2 - \overline{s}s - \overline{s}^2 - \varepsilon s - \varepsilon\overline{s} + \overline{s}s)^{-1}\varepsilon = \\ &= (\overline{s}\varepsilon - \varepsilon s + \varepsilon^2)^{-1}\varepsilon = \\ &= \left(\varepsilon^{-1}(\overline{s}\varepsilon - \varepsilon\overline{s} + \varepsilon^2)\right)^{-1} = \\ &= \left(\varepsilon^{-1}\overline{s}\varepsilon - s + \varepsilon\right)^{-1}. \end{aligned}$$

For $s \in \mathbb{R}$ this expression simplifies to

$$S_L^{-1}(s,\overline{s}+\varepsilon) = \left(\varepsilon^{-1}\varepsilon s - s + \varepsilon\right)^{-1} = \varepsilon^{-1}$$

and, in turn, the limit $\lim_{x\to\overline{s}} S_L^{-1}(s,x)$ does not exist.

If we have $s \notin \mathbb{R}$ the expression does not converge either because the term $\varepsilon^{-1}\overline{s}\varepsilon$ has no limit. In fact, choosing $\varepsilon = \varepsilon_0 \in \mathbb{R}$, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon_0^{-1}\overline{s}\varepsilon_0 = \lim_{\varepsilon\to 0} \varepsilon_0^{-1}\varepsilon_0\overline{s} = \overline{s}.$$

On the other hand, if $\varepsilon = \varepsilon_i e_i$ with $e_i \neq I_s$, then s and e_i do not commute and

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\overline{s}\varepsilon = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon_i}(-e_i)\overline{s}\varepsilon_i e_i = \lim_{\varepsilon\to 0} -e_i\overline{s}e_i \neq -e_i e_i \overline{s} = \overline{s}.$$

□

Finally, we prove the Cauchy formula for slice regular functions. We start with the slice regular version of the Cauchy integral theorem.

Recall that any quaternionic Banach space V is a real Banach space by Corollary 2.38. In particular, the quaternions themselves are a real Banach space. Hence, if $\phi : [a,b] \to V$ is piecewise continuous, then $\int_a^b \phi(t)\,dt$ is defined in the sense of the integral of a function with values in a real Banach space.

Recall also that a function $\phi : [a,b] \to V$ is called *piecewise continuously differentiable*, if

(i) ϕ is continuous and

(ii) there exist $a = t_0 < t_1 < \ldots < t_n = b$ such that $\phi|_{(t_i,t_i+1)}$ is continuously differentiable and the one-sided limits $\lim_{t\to t_{i-1}^+} \phi'(t)$ and $\lim_{t\to t_i^-} \phi'(t)$ exist for any $i = 1,\ldots,n$.

Definition 3.77. *Let V be a quaternionic Banach space, let $U \subset \mathbb{H}$ and let $f : U \to V$ and $g : U \to \mathbb{H}$ or $f : U \to \mathbb{H}$ and $g : U \to V$ be continuous. For a piecewise continuously differentiable path $\gamma : [a,b] \to U$, we define the* quaternionic path integral $\int_\gamma f(s)\,ds\,g(s)$ *as*

$$\int_\gamma f(s)\,ds\,g(s) = \int_a^b f(\gamma(t))\,\gamma'(t)\,g(\gamma(t))\,dt.$$

If $I \in \mathbb{S}$ and γ has values in $\mathbb{C}_I$, then we define $ds_I = -I\,ds$, that is,

$$\int_\gamma f(s)\,ds_I\,g(s) = -\int_\gamma f(s)I\,ds\,g(s).$$

Corollary 3.78. *Let V be a quaternionic Banach space, let $U_1, U_2 \subset \mathbb{H}$ and let f, g and h be continuous functions on $U_1 \times U_2 \subset \mathbb{H}^2$ such that one of them has values in V and the other two have values in $\mathbb{H}$. If $\gamma_s : [a,b] \to U_1$ and $\gamma_p : [c,d] \to U_2$ are piecewise continuously differentiable paths, then*

$$\int_{\gamma_s}\int_{\gamma_p} f(s,p)\,ds\,g(s,p)\,dp\,h(s,p) = \int_{\gamma_p}\int_{\gamma_s} f(s,p)\,ds\,g(s,p)\,dp\,h(s,p),$$

where the indices s and p of γ_s and γ_p indicate the respective variable of integration.

Proof. To avoid case analysis, we denote the absolute value of a quaternion and the norm on V both by $\|.\|$. The definition of the quaternionic path integral implies

$$\int_{\gamma_s}\int_{\gamma_p} f(s,p)\,ds\,g(s,p)\,dp\,h(s,p) =$$
$$= \int_a^b \left[\int_c^d f(\gamma_s(t),\gamma_p(u))\,\gamma_s'(t)\,g(\gamma_s(t),\gamma_p(u))\,\gamma_p'(u)\,h(\gamma_s(t),\gamma_p(u))\,dt\right] du.$$

The functions $\|f\|$, $\|g\|$ and $\|h\|$ are bounded on $\gamma_s([a,b]) \times \gamma_p([c,d])$ because they are continuous and $\gamma_s([a,b]) \times \gamma_p([a,b])$ is compact. Since γ_s and γ_p are piecewise continuously differentiable, the absolute values of the derivatives $\|\gamma_s'\|$ and $\|\gamma_p'\|$ are bounded on $[a,b]$ resp. $[c,d]$, too. Consequently,

$$\int_a^b \int_c^d \|f(\gamma_s(t),\gamma_p(u))\,\gamma_s'(t)\,g(\gamma_s(t),\gamma_p(u))\,\gamma_p'(u)\,h(\gamma_s(t),\gamma_p(u))\|\,dt\,du < \infty$$

and we can apply Fubini's theorem, which yields

$$\int_{\gamma_s}\int_{\gamma_p} f(s,p)\,ds\,g(s,p)\,dp\,h(s,p) =$$
$$= \int_c^d \left[\int_a^b f(\gamma_s(t),\gamma_p(u))\,\gamma_s'(t)\,g(\gamma_s(t),\gamma_p(u))\,\gamma_p'(u)\,h(\gamma_s(t),\gamma_p(u))\,du\right] dt =$$
$$= \int_{\gamma_p}\int_{\gamma_s} f(s,p)\,ds\,g(s,p)\,dp\,h(s,p).$$

□

Remark 3.79. Note that, although we can exchange the order of integration, in general

$$\int_{\gamma_s}\int_{\gamma_p} f(s,p)\,ds\,g(s,p)\,dp\,h(s,p) \neq \int_{\gamma_s}\left[\int_{\gamma_p} f(s,p)\,g(s,p)\,h(s,p)\,dp\right] ds \neq$$
$$\neq \int_{\gamma_p}\left[\int_{\gamma_s} f(s,p)\,g(s,p)\,h(s,p)\,ds\right] dp,$$

because ds and dp do not commute with the function f, g and h.

Theorem 3.80 (Cauchy integral theorem)**.** *Let $U \subset \mathbb{H}$ be open, let $I \in \mathbb{S}$ and let $f \in \mathcal{M}^L(U)$ and $g \in \mathcal{M}^R(U)$. Moreover, let $D_I \subset U \cap \mathbb{C}_I$ be an open and bounded subset of the complex plane $\mathbb{C}_I$ with $\overline{D_I} \subset U \cap \mathbb{C}_I$ such that ∂D_I is a finite union of piecewise continuously differentiable Jordan curves. Then*

$$\int_{\partial D_I} g(s)\, ds_I\, f(s) = 0.$$

Proof. Let $J \in \mathbb{S}$ with $I \perp J$. By applying Lemma 3.5, the Splitting Lemma, we obtain holomorphic functions $f_1, f_2, g_1, g_2 : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = f_1 + f_2 J$ and $g_I = g_1 + J g_2$. Then we have

$$\begin{aligned}\int_{\partial D_I} g(s)\, ds_I\, f(s) = \int_{\partial D_I} g_1(s)\, ds_I\, f_1(s) + \left(\int_{\partial D_I} g_1(s)\, ds_I\, f_2(s)\right) J +& \\ + J\left(\int_{\partial D_I} g_2(s)\, ds_I\, f_1(s)\right) + J\left(\int_{\partial D_I} g_2(s)\, ds_I\, f_2(s)\right) J.&\end{aligned}$$

Let $\gamma : [a,b] \to \mathbb{C}_I$ be a parametrization of a Jordan curve, that belongs to ∂D_I. Then $f(\gamma(t))$, $g(\gamma(t))$, $\gamma'(t)$ and I belong to $\mathbb{C}_I$. Therefore, they commute and we get

$$\int_\gamma g_i(s)\, ds_I\, f_j(s) = -\int_a^b g_i(\gamma(t))\, I\, \gamma'(t)\, f_j(\gamma(t))\, dt = -I \int_a^b g_i(\gamma(t))\, f_j(\gamma(t))\, \gamma'(t)\, dt.$$

Hence, $\int_\gamma g_i(s)\, ds_I\, f_j(s)$ is nothing but the complex path integral of $g_i(s) f_j(s)$ along γ multiplied by $-I$ and, in turn, $\int_{\partial D_I} g_i(s)\, ds_I\, f_j(s)$ is nothing but the complex path integral of $g_i(s) f_j(s)$ along ∂D_I multiplied by $-I$.

For $i, j \in \{1, 2\}$, the function $g_i(s) f_j(s)$ is holomorphic because it is the product of two holomorphic functions. Hence, the usual complex Cauchy integral theorem implies $\int_{\partial D_I} g_i(s)\, ds_I\, f_j(s) = 0$ and we obtain

$$\int_{\partial D_I} g(s) ds_I f(s) = 0.$$

□

Theorem 3.81 (Cauchy formula)**.** *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $O \subset \mathbb{H}$ be an axially symmetric open set such that $\overline{O} \subset U$ and such that $\partial(O \cap \mathbb{C}_I)$ is the finite union of piecewise continuously differentiable Jordan curves for every $I \in \mathbb{S}$.*

If $f \in \mathcal{M}^L(U)$ and $x \in O$, then the identity

$$f(x) = \frac{1}{2\pi} \int_{\partial(O \cap \mathbb{C}_I)} S_L^{-1}(s, x)\, ds_I\, f(s) \tag{3.35}$$

holds true for any $I \in \mathbb{S}$.

If $g \in \mathcal{M}^R(U)$ and $x \in O$, then, for any $I \in \mathbb{S}$, we have

$$g(x) = \frac{1}{2\pi} \int_{\partial(O \cap \mathbb{C}_I)} g(s)\, ds_I\, S_R^{-1}(s, x).$$

Proof. Let $I \in \mathbb{S}$. If $x \in \mathbb{C}_I$, then $S_L^{-1}(s, x) = (s - x)^{-1}$ is nothing but the usual complex Cauchy kernel for $s \in \mathbb{C}_I$. We can choose $J \in \mathbb{S}$ with $I \perp J$ and apply Lemma 3.5, the Splitting Lemma, to obtain holomorphic functions $f_1, f_2 : O \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $f_I = f_1 + f_2 J$. Therefore, by applying the usual complex Cauchy formula, we obtain

$$\begin{aligned}\frac{1}{2\pi} \int_{\partial(O \cap \mathbb{C}_I)} S_L^{-1}(s, x)\, ds_I\, f(s) = \frac{1}{2\pi} \int_{\partial(O \cap \mathbb{C}_I)} (x - s)^{-1}\, ds_I\, f_1(s) + \frac{1}{2\pi} \int_{\partial(O \cap \mathbb{C}_I)} (x - s)^{-1}\, ds_I\, f_2(s) J =& \\ = \frac{1}{2\pi I} \int_{\partial(O \cap \mathbb{C}_I)} \frac{f_1(s)}{x - s}\, ds + \left(\frac{1}{2\pi I} \int_{\partial(O \cap \mathbb{C}_I)} \frac{f_2(s)}{x - s}\, ds\right) J = f_1(x) + f_2(x) J = f(x).&\end{aligned}$$

For $x = x_0 + I_x x_1$ with $I \neq I_x$, the function $s \mapsto S_L^{-1}(s,x)$ has the two singularities $x_I = x_0 + Ix_1$ and $\overline{x_I} = x_0 - Ix_1$ on $\mathbb{C}_I$, cf. Lemma 3.75. If $\varepsilon > 0$ is small enough, then $\overline{B_\varepsilon(x_I)} \subset O$ and $\overline{B_\varepsilon(\overline{x_I})} \subset O$. Thus, with $O_\varepsilon = O \setminus (\overline{B_\varepsilon(x_I)} \cup \overline{B_\varepsilon(\overline{x_I})})$, we get

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial(O_\varepsilon\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s)+ \\ + \frac{1}{2\pi}\int_{\partial(B_\varepsilon(x_I)\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s) + \frac{1}{2\pi}\int_{\partial(B_\varepsilon(\overline{x_I})\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s).$$

Since $f(s)$ and $S_L^{-1}(s,x)$ are left and right slice regular on O_ε in the variable s, respectively, it follows from Theorem 3.80 that the integral over $\partial(O_\varepsilon \cap \mathbb{C}_I)$ equals zero. Hence,

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} S_L^{-1}(s,x)\;ds_I\;f(s) = \\ = \underbrace{\frac{1}{2\pi}\int_{\partial(B_\varepsilon(x_I)\cap\mathbb{C}_I)} S_L^{-1}(s,x)\;ds_I\;f(s)}_{=\mathcal{I}_+^\varepsilon} + \underbrace{\frac{1}{2\pi}\int_{\partial(B_\varepsilon(\overline{x_I})\cap\mathbb{C}_I)} S_L^{-1}(s,x)\;ds_I\;f(s)}_{=\mathcal{I}_-^\varepsilon}.$$

If we parametrize $\partial B_\varepsilon(x_I)$ by $s(\theta) = x_0 + Ix_1 + \varepsilon e^{I\theta}$ with $\theta \in [0, 2\pi]$, we obtain

$$\begin{aligned} \mathrm{Re}[s] &= x_0 + \varepsilon\cos\theta \\ \overline{s} &= x_0 - Ix_1 + \varepsilon e^{-I\theta} \\ ds_I &= (\varepsilon I e^{I\theta} d\theta)(-I) = \varepsilon e^{I\theta} d\theta \end{aligned}$$

and

$$|s|^2 = x_0^2 + 2x_0\varepsilon\cos\theta + \varepsilon^2 + x_1^2 + 2\varepsilon x_1\sin\theta.$$

Hence,

$$\begin{aligned} 2\pi\mathcal{I}_+^\varepsilon &= \int_{\partial(B_\varepsilon(s_{+,I}))} -(x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}(x-\overline{s})\,ds_I\,f(s) = \\ &= \int_0^{2\pi} -(x^2 - 2(x_0+\varepsilon\cos\theta)x + x_0^2 + 2x_0\varepsilon\cos\theta + \varepsilon^2 + x_1^2 + 2\varepsilon x_1\sin\theta)^{-1}\cdot \\ &\qquad \cdot(x - x_0 + Ix_1 - \varepsilon e^{-I\theta})\varepsilon e^{I\theta} f(x_0 + Ix_1 + \varepsilon e^{I\theta})\,d\theta. \end{aligned}$$

As

$$\begin{aligned} &x^2 - 2(x_0+\varepsilon\cos\theta)x + x_0^2 + 2x_0\varepsilon\cos\theta + \varepsilon^2 + x_1^2 + 2\varepsilon x_1\sin\theta = \\ &= x_0^2 + 2x_0x_1I - x_1^2 - 2x_0^2 - 2x_0x_1I - 2\varepsilon\cos\theta x + x_0^2 + 2x_0\varepsilon\cos\theta + \varepsilon^2 + x_1^2 + 2\varepsilon x_1\sin\theta = \\ &= -2\varepsilon x\cos\theta + 2x_0\varepsilon\cos\theta + \varepsilon^2 + 2\varepsilon x_1\sin\theta, \end{aligned}$$

we get

$$\begin{aligned} 2\pi\mathcal{I}_+^\varepsilon &= \int_0^{2\pi} -(-2\varepsilon x\cos\theta + 2x_0\varepsilon\cos\theta + \varepsilon^2 + 2\varepsilon x_1\sin\theta)^{-1}\cdot \\ &\qquad \cdot(x - x_0 + Ix_1 + \varepsilon e^{-I\theta})\varepsilon e^{I\theta} f(x_0 + Ix_1 + \varepsilon e^{I\theta})\,d\theta = \\ &= \int_0^{2\pi} -(-2x\cos\theta + 2x_0\cos\theta + \varepsilon + 2x_1\sin\theta)^{-1}\cdot \\ &\qquad \cdot(x - x_0 + Ix_1 + \varepsilon e^{-I\theta}) e^{I\theta} f(x_0 + Ix_1 + \varepsilon e^{I\theta})\,d\theta. \end{aligned}$$

For $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0} \mathcal{I}_+^\varepsilon = \int_0^{2\pi} -(-2x\cos\theta + 2x_0\cos\theta + 2x_1\sin\theta)^{-1}(x - x_0 + Ix_1)e^{I\theta} f(x_0 + Ix_1)\, d\theta =$$
$$= \int_0^{2\pi} (2I_x x_1 \cos\theta - 2x_1\sin\theta)^{-1}(x - x_0 + Ix_1)e^{I\theta} f(x_0 + Ix_1)\, d\theta.$$

Since $p^{-1} = \frac{1}{|p|^2}\overline{p}$ for any quaternion $p \in \mathbb{H}$, we have

$$(2I_x x_1\cos\theta - 2x_1\sin\theta)^{-1} = \frac{1}{2}(I_x x_1\cos\theta - x_1\sin\theta)^{-1} = \frac{1}{2x_1^2}(-I_x x_1\cos\theta - x_1\sin\theta),$$

and in turn

$$(2I_x x_1\cos\theta - 2\sin\theta x_1)^{-1}(x - x_0 + Ix_1) = \frac{1}{2x_1^2}(-I_x x_1\cos\theta - \sin\theta x_1)(I_x x_1 + Ix_1) =$$
$$= \frac{1}{2x_1^2}\left(-I_x x_1\cos\theta I_x x_1 - \sin\theta x_1 I_x x_1 - I_x x_1 \cos\theta I x_1 - \sin\theta x_1 I x_1\right) =$$
$$= \frac{1}{2x_1^2}\left(x_1^2\cos\theta - x_1^2\sin\theta I_x - x_1^2\cos\theta I_x I - x_1^2 \sin\theta I\right) =$$
$$= \frac{1}{2x_1^2}x_1^2\left(\cos\theta(1 - I_x I) - \sin\theta(I_x + I)\right) = \frac{1}{2}\left[\cos\theta(1 - I_x I) - \sin\theta(I_x I - 1)(-I)\right] =$$
$$= \frac{1}{2}(1 - I_x I)(\cos\theta - I\sin\theta).$$

Therefore,

$$\lim_{\varepsilon\to 0}\mathcal{I}_+^\varepsilon = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(1 - I_x I)(\cos\theta - I\sin\theta)e^{I\theta} f(x_0 + Ix_1)\, d\theta =$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(1 - I_x I)(\cos\theta - I\sin\theta)(\cos\theta + I\sin\theta) f(x_I)\, d\theta =$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(1 - I_x I)(\cos^2\theta + \sin^2\theta) f(x_I)\, d\theta =$$
$$= \frac{1}{2}(1 - I_x I) f(x_I).$$

With analogous computations, we get

$$\lim_{\varepsilon\to 0}\mathcal{I}_-^\varepsilon = \frac{1}{2}(1 + I_x I) f(\overline{x_I}).$$

By Theorem 3.21, the Representations Formula, we finally obtain

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} S_L^{-1}(s,x) ds_I f(s) = \lim_{\varepsilon\to 0}\mathcal{I}_+^\varepsilon + \lim_{\varepsilon\to 0}\mathcal{I}_-^\varepsilon = \frac{1}{2}(1 - I_x I) f(x_I) + \frac{1}{2}(1 + I_x I) f(\overline{x_I}) = f(x).$$

□

Corollary 3.82. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain such that $\partial(U\cap\mathbb{C}_I)$ is a finite union of piecewise continuously differentiable Jordan curves for every $I \in \mathbb{S}$.*

If $f \in \mathcal{M}^L(\overline{U})$, then, for any $I \in \mathbb{S}$, we have

$$f(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,x)\, ds_I\, f(s) \qquad \text{for any } x \in U. \tag{3.36}$$

If $g \in \mathcal{M}^R(\overline{U})$, then, for any $I \in \mathbb{S}$, we have

$$f(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g(s)\, ds_I\, S_R^{-1}(s,x) \qquad \text{for any } x \in U.$$

Proof. We obtain the statement, if we apply Theorem 3.81 with $O = U$. $\square$

Theorem 3.83 (Cauchy formula outside an axially symmetric slice domain)**.** *Let $U \subset \mathbb{H}$ be a bounded slice domain such that $\overline{U}^c$ is connected and such that the set $\partial(U \cap \mathbb{C}_I)$ is the union of a finite number of piecewise continuously differentiable Jordan curves for any $I \in \mathbb{S}$. If $f \in \mathcal{M}^L(\overline{U^c})$ such that $f(\infty) = \lim_{|x|\to\infty} f(x)$ exists, then, for any $I \in \mathbb{S}$, we have*

$$f(x) = f(\infty) - \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s) \qquad \text{for any } x \in \overline{U}^c.$$

If $f \in \mathcal{M}^R(\overline{U^c})$ such that $f(\infty) = \lim_{|x|\to\infty} f(x)$ exists, then, for any $I \in \mathbb{S}$, we have

$$f(x) = f(\infty) - \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,x) \qquad \text{for any } x \in \overline{U}^c.$$

Proof. Let $f \in \mathcal{M}^L(\overline{U^c})$ and let $x \in \overline{U}^c$. Since U is bounded, x and U are contained in the ball $B_r(0)$ for r large enough. Then $V_r = B_r(0) \setminus \overline{U}$ is an axially symmetric slice domain such that $\partial(V_r \cap \mathbb{C}_I)$ is the union of a finite number of piecewise continuously differentiable Jordan curves for any $I \in \mathbb{S}$ and f is left slice regular on V_r. Thus, we can apply Corollary 3.82 and obtain for $I \in \mathbb{S}$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\partial(V_r\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s) = \\ &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s) - \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s). \end{aligned}$$

For $s = re^{I\theta}$, we get

$$\begin{aligned} \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s) &= \frac{1}{2\pi}\int_0^{2\pi} S_L^{-1}(re^{I\theta},x)(-I)Ire^{I\theta}f(re^{-I\theta})\,d\theta = \\ &= \frac{1}{2\pi}\int_0^{2\pi} -(x^2 - 2r\cos\theta x + r^2)^{-1}(x - re^{-I\theta})re^{I\theta}f(re^{-I\theta})\,d\theta = \end{aligned}$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}(x^2 - 2r\cos\theta x + r^2)^{-1}xre^{I\theta}f(re^{-I\theta})\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}(x^2 - 2r\cos\theta x + r^2)^{-1}r^2 f(re^{-I\theta})\,d\theta.$$

Since $\lim_{r\to\infty}(x^2 - 2r\cos\theta x + r^2)^{-1}xr = 0$ and $\lim_{r\to\infty}(x^2 - 2r\cos\theta x + r^2)^{-1}r^2 = 1$ uniformly and since $f(\infty) = \lim_{|x|\to\infty} f(x)$ exists, the integrands converge uniformly. Hence, we can exchange limit and integration and obtain

$$\lim_{r\to\infty} -\frac{1}{2\pi}\int_0^{2\pi}(x^2 - 2r\cos\theta x + r^2)^{-1}xre^{I\theta}f(re^{-I\theta})\,d\theta = 0$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}(x^2 - 2r\cos\theta x + r^2)^{-1}r^2 f(re^{-I\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\lim_{r\to\infty} f(re^{-I\theta})\,d\theta = f(\infty).$$

Thus,

$$f(x) = f(\infty) - \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,x)\,ds_I\,f(s).$$

$\square$

Chapter 4

The S-resolvent operator and the S-spectrum

In the following, we consider right linear operators on a quaternionic Banach space V. Nevertheless, since we can identify $\mathcal{B}_L(V)$ and $\mathcal{B}_R(V)$ the theory of left linear operators on V coincides with the one for right linear operators, if we interpret the formulas according to Definition 2.36. However, it is important to keep in mind that although the obtained resolvent operators etc. are formally the same, they act in different ways when they are interpreted as left or as right linear operators.

We state all results for both, the left and the right slice regular case, but again we only give the proofs for the left slice regular one because the proofs for the right slice regular case are similar with obvious modifications.

The presented results can be found in Chapter 3 and Chapter 4 of [12], except for Theorem 4.16, which is going to appear in the paper [2] that was recently accepted for publication.

4.1 The S-resolvent operator and the S-spectrum

As in the case of the classical Riesz-Dunford-calculus, we define $f(T)$ for an operator T by formally replacing the variable x by the operator T in the Cauchy formula. The following discussion shows that this is actually possible. We start by replacing the variable x in the series expansion of the Cauchy kernel.

Definition 4.1. *Let $T \in \mathcal{B}_R(V)$ and let $s \in \mathbb{H}$. We call the series*

$$\sum_{n=0}^{\infty} T^n s^{-1-n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} s^{-1-n} T^n$$

the left and right Cauchy kernel operator series, respectively.

Remark 4.2. As for the scalar Cauchy kernel series, we have

$$\sum_{n=0}^{\infty} \|T^n s^{-1-n}\| \leq |s|^{-1} \sum_{n=0}^{\infty} \left(\|T\| |s|^{-1}\right)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} \|s^{-1-n} T^n\| \leq |s|^{-1} \sum_{n=0}^{\infty} \left(|s|^{-1} \|T\|\right)^n.$$

Thus, the left and the right Cauchy kernel operator series converge if $\|T\| < |s|$.

Recall that we used the fact that $x - \overline{s}$ is invertible when we determined the inverse of the scalar noncommutative Cauchy kernel in the proof of Theorem 3.67. To determine the inverse of the Cauchy kernel operator series, we need the corresponding result for operators.

Theorem 4.3. *Let $T \in \mathcal{B}_R(V)$ and let $\|T\| < |s|$. Then the series*

$$\sum_{n=0}^{\infty} (s^{-1} T)^n s^{-1}$$

converges in the operator norm and it is the inverse of $s\mathcal{I} - T$, where $\mathcal{I}$ denotes the identity operator on V.

Proof. Since $\sum_{n=0}^{\infty} \|(s^{-1}T)^n s^{-1}\| = |s|^{-1} \sum_{n=0}^{\infty} (\|T\||s|^{-1})^n < \infty$, the series converges in the operator norm. Moreover, we have

$$\begin{aligned}
(s\mathcal{I} - T) \sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1} &= s \sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1} - T \sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1} = \\
&= s \sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1} - ss^{-1}T \sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1} = \\
&= s \sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1} - s \sum_{n=1}^{\infty} (s^{-1}T)^n s^{-1} = s\mathcal{I}s^{-1} = \mathcal{I}
\end{aligned}$$

and

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1}\right)(s\mathcal{I} - T) &= \left(\sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1}\right) s\mathcal{I} - \left(\sum_{n=0}^{\infty} (s^{-1}T)^n s^{-1}\right) T = \\
&= \sum_{n=0}^{\infty} (s^{-1}T)^n - \sum_{n=0}^{\infty} (s^{-1}T)^{n+1} = \sum_{n=0}^{\infty} (s^{-1}T)^n - \sum_{n=1}^{\infty} (s^{-1}T)^n = \mathcal{I}.
\end{aligned}$$

□

Theorem 4.4. *Let $T \in \mathcal{B}_R(V)$ and let $s \in \mathbb{H}$ with $\|T\| < |s|$. Then*

(i) the operator

$$S_L(s,T) = -(T - \overline{s}\mathcal{I})^{-1}(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})$$

is the left inverse of the left Cauchy kernel series

(ii) the operator

$$S_R(s,T) = -(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})(T - \overline{s}\mathcal{I})^{-1}$$

is the right inverse of the right Cauchy kernel series.

Proof. We proceed as in the proof of Theorem 3.67 in order to show

$$\mathcal{I} = -(T - \overline{s}\mathcal{I})^{-1}(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) \sum_{n=0}^{\infty} T^n s^{-1-n},$$

which is equivalent to

$$\overline{s}\mathcal{I} - T = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) \sum_{n=0}^{\infty} T^n s^{-1-n}.$$

Since $2\mathrm{Re}[s] = s + \overline{s}$ and $|s|^2 = s\,\overline{s} = \overline{s}\,s$ are real, they commute with the operator T, and hence,

$$\begin{aligned}
(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) \sum_{n=0}^{\infty} T^n s^{-n-1} &= \sum_{n=0}^{\infty} T^{n+2} s^{-n-1} - \sum_{n=0}^{\infty} T^{n+1}(s + \overline{s}) s^{-n-1} + \sum_{n=0}^{\infty} T^n \overline{s} s s^{-n-1} = \\
&= \sum_{n=0}^{\infty} T^{n+2} s^{-n-1} - \sum_{n=0}^{\infty} T^{n+1} s^{-n} - \sum_{n=0}^{\infty} T^{n+1} \overline{s} s^{-n-1} + \sum_{n=0}^{\infty} T^n \overline{s} s^{-n} = \\
&= \sum_{n=0}^{\infty} T^{n+2} s^{-n-1} - \sum_{n=0}^{\infty} T^{n+2} s^{-n-1} - T - \sum_{n=0}^{\infty} T^{n+1} \overline{s} s^{-n-1} + \sum_{n=0}^{\infty} T^{n+1} \overline{s} s^{-n-1} + \overline{s}\mathcal{I} = \overline{s}\mathcal{I} - T.
\end{aligned}$$

□

The previous result motivates the following definitions.

Definition 4.5. *Let $T \in \mathcal{B}_R(V)$. We define the* S-resolvent set $\rho_S(T)$ *of T as*

$$\rho_S(T) = \{s \in \mathbb{H} : T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I} \text{ is invertible}\}$$

and we define the S-spectrum $\sigma_S(T)$ *of T as*

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

Definition 4.6. *Let $T \in \mathcal{B}_R(V)$. For $s \in \rho_S(T)$, we define the* left S-resolvent operator *as*

$$S_L^{-1}(s,T) = -(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\,\mathcal{I}),$$

and the right S-resolvent operator *as*

$$S_R^{-1}(s,T) = -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}.$$

In analogy to Proposition 3.74, we obtain the following result.

Lemma 4.7. *The left S-resolvent operator $S_L^{-1}(s,T)$ is a $\mathcal{B}_R(V)$-valued right-slice regular function of the variable s on $\rho_S(T)$, that is, $S_L^{-1}(s,T)\bar{\partial}_I = 0$ for all $I \in \mathbb{S}$ and all $s \in \rho_S(T)$.*

The right S-resolvent operator $S_R^{-1}(s,T)$ is a $\mathcal{B}_R(V)$-valued left-slice regular function of the variable s on $\rho_S(T)$, that is, $\bar{\partial}_I S_R^{-1}(s,T) = 0$ for all $I \in \mathbb{S}$ and all $s \in \rho_S(T)$.

Proof. Let $s = s_0 + Is_1 \in \rho_s(T)$ and let $Q(s) = T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}$. Then $\frac{\partial}{\partial s_0}Q(s) = -2T + 2s_0\mathcal{I}$ and $\frac{\partial}{\partial s_1}Q(s) = 2s_1\mathcal{I}$. Hence, $Q(s)$ and $\frac{\partial}{\partial s_j}Q(s)$ commute and a computation as in (3.26) yields $\frac{\partial}{\partial s_j}Q(s)^{-1} = -Q(s)^{-2}\frac{\partial}{\partial s_j}Q(s)$ for $j = 1,2$. Since $S_L^{-1}(s,T) = -Q^{-1}(s)(T - \overline{s}\mathcal{I})$, we obtain

$$\frac{\partial}{\partial s_0}S_L^{-1}(s,T) = \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-2}(-2T + 2s_0\mathcal{I})(T - \overline{s}\mathcal{I}) + \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-1}$$

and

$$\frac{\partial}{\partial s_1}S_L^{-1}(s,T) = \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-2}2s_1(T - \overline{s}\mathcal{I}) - \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-1}I.$$

Therefore, if we apply the operator $\bar{\partial}_I$ in the variable s from the right, we obtain

$$\begin{aligned} S_L^{-1}(s,T)\bar{\partial}_I &= \frac{1}{2}\left(\frac{\partial}{\partial s_0}S_L^{-1}(s,T) + \frac{\partial}{\partial s_1}S_L^{-1}(s,T)I\right) = \\ &= \frac{1}{2}\Bigg(\left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-2}(-2T + 2s_0\mathcal{I})(T - \overline{s}\mathcal{I}) + \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-1} + \\ &\qquad + \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-2}2s_1(T - \overline{s}\mathcal{I})I - \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-1}I^2\Bigg). \end{aligned}$$

Since $2s_0$ and $T - \overline{s}\mathcal{I}$ commute, we finally get

$$\begin{aligned} S_L^{-1}(s,T)\bar{\partial}_I &= \frac{1}{2}\Bigg(\left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-2}\big(-2T(T - \overline{s}\mathcal{I}) + (T - \overline{s}\mathcal{I})(2s_0 + 2Is_1)\big) + \\ &\qquad + 2\left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-1}\Bigg) = \\ &= -\left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-2}\left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right) + \left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)^{-1} = 0. \end{aligned}$$

□

4.2 Properties of the S-spectrum

Many results that hold for the spectrum of an operator in classical functional analysis can be generalized to the case of the S-spectrum. The first important result is the fact that the S-spectrum is bounded by the norm of T.

Lemma 4.8. *Let $T \in \mathcal{B}_R(V)$ and let $s \in \mathbb{H}$ with $\|T\| < |s|$. Then the series*

$$L(s,T) = \sum_{n=0}^{\infty} T^n |s|^{-2n-2} \sum_{k=0}^{n} \overline{s}^k s^{n-k} \tag{4.1}$$

converges in $\mathcal{B}_R(V)$ and it is the inverse of the operator $T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}$.

Proof. We have

$$\sum_{n=0}^{\infty} \left\| T^n |s|^{-2n-2} \sum_{k=0}^{n} \overline{s}^k s^{n-k} \right\| \leq \sum_{n=0}^{\infty} \|T\|^n |s|^{-2n-2} \sum_{k=0}^{n} |s|^n = \sum_{n=0}^{\infty} \|T\|^n |s|^{-n-2}(n+1).$$

From $\|T\| < |s|$, we conclude

$$\lim_{n\to\infty} \frac{\|T\|^{n+1}|s|^{-n-3}(n+2)}{\|T\|^n|s|^{-n-2}(n+1)} = \lim_{n\to\infty} \frac{(n+2)\|T\|}{(n+1)|s|} = \frac{\|T\|}{|s|} < 1.$$

Hence, the series above converges absolutely by the ratio test. Moreover

$$\begin{aligned}
(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})L(s,T) &= (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) \sum_{n=0}^{\infty} T^n |s|^{-2n-2} \sum_{k=0}^{n} \overline{s}^k s^{n-k} = \\
&= \sum_{n=0}^{\infty} T^{n+2} |s|^{-2n-2} \sum_{k=0}^{n} \overline{s}^k s^{n-k} - \sum_{n=0}^{\infty} T^{n+1} |s|^{-2n-2} \sum_{k=0}^{n} \overline{s}^k (s+\overline{s}) s^{n-k} + \sum_{n=0}^{\infty} T^n |s|^{-2n} \sum_{k=0}^{n} \overline{s}^k s^{n-k} = \\
&= \sum_{n=2}^{\infty} T^n |s|^{-2n} \sum_{k=0}^{n-2} \overline{s}^k |s|^2 s^{n-2-k} - \sum_{n=1}^{\infty} T^n |s|^{-2n} \sum_{k=0}^{n-1} \overline{s}^k (s+\overline{s}) s^{n-1-k} + \sum_{n=0}^{\infty} T^n |s|^{-2n} \sum_{k=0}^{n} \overline{s}^k s^{n-k} = \\
&= \sum_{n=2}^{\infty} T^n |s|^{-2n} \left(\sum_{k=0}^{n-2} \overline{s}^k |s|^2 s^{n-2-k} - \sum_{k=0}^{n-1} \overline{s}^k (s+\overline{s}) s^{n-1-k} + \sum_{k=0}^{n} \overline{s}^k s^{n-k} \right) - \\
&\quad - T|s|^{-2}(s+\overline{s}) + \mathcal{I} + T|s|^{-2}(\overline{s}+s).
\end{aligned}$$

As

$$\begin{aligned}
&\sum_{k=0}^{n-2} \overline{s}^k |s|^2 s^{n-2-k} - \sum_{k=0}^{n-1} \overline{s}^k (s+\overline{s}) s^{n-1-k} + \sum_{k=0}^{n} \overline{s}^k s^{n-k} = \\
&= \sum_{k=0}^{n-2} \overline{s}^{k+1} s^{n-1-k} - \sum_{k=0}^{n-1} \overline{s}^k s^{n-k} - \sum_{k=0}^{n-1} \overline{s}^{k+1} s^{n-1-k} + \sum_{k=0}^{n} \overline{s}^k s^{n-k} = \\
&= \sum_{k=1}^{n-1} \overline{s}^k s^{n-k} - \sum_{k=0}^{n-1} \overline{s}^k s^{n-k} - \sum_{k=1}^{n} \overline{s}^k s^{n-k} + \sum_{k=0}^{n} \overline{s}^k s^{n-k} = 0,
\end{aligned}$$

we obtain

$$(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})L(s,T) = \mathcal{I}.$$

Since

$$\overline{\sum_{k=0}^{n} \overline{s}^k s^{n-k}} = \sum_{k=0}^{n} \overline{\overline{s}^k s^{n-k}} = \sum_{k=0}^{n} \overline{s}^{n-k} s^k = \sum_{k=0}^{n} \overline{s}^k s^{n-k},$$

Corollary 2.3 implies that $\sum_{k=0}^{n} \overline{s}^k s^{n-k}$ is real. Thus, the series (4.1) has real coefficients, and therefore, it commutes with T, which gives

$$L(s,T)(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})L(s,T) = \mathcal{I}.$$

□

Theorem 4.9. *Let $T \in \mathcal{B}_R(V)$. The S-spectrum $\sigma_S(T)$ of T is a nonempty, compact set contained in the closed ball $\overline{B_{\|T\|}(0)}$.*

Proof. For $\|T\| < r$, the series $S_L^{-1}(s,T) = \sum_{n=0}^{\infty} T^n s^{-n-1}$ converges uniformly on $\partial B_r(0)$. Thus, for $I \in \mathbb{S}$, we obtain

$$\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds = \sum_{n=0}^{\infty} T^n \int_{\partial(B_r(0)\cap\mathbb{C}_I)} s^{-n-1} ds = 2\pi I \mathcal{I},$$

because $\int_{\partial(B_r(0)\cap\mathbb{C}_I)} s^{-n-1}ds$ equals $2\pi I$ if $n = 0$ and 0 otherwise.

Recall from Corollary 2.38 that we can consider any quaternionic Banach space as a complex Banach space if we restrict the right scalar multiplication to a complex plane $\mathbb{C}_I$. Hence, by Lemma 4.7 the map $s \mapsto S_L^{-1}(s,T)$ is holomorphic on $\rho_s(T) \cap \mathbb{C}_I$. We conclude $(B_r(0) \cap \mathbb{C}_I) \nsubseteq (\rho_S(T) \cap \mathbb{C}_I)$ because otherwise the vector-valued version of Cauchy's Theorem would imply $\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,T)ds = 0$. Thus, $\sigma_S(T) = \rho_S(T)^c \neq \emptyset$.

We can also consider $\mathcal{B}_R(V)$ as a real Banach algebra if we restrict the scalar multiplication to $\mathbb{R}$. If we define the map $\tau : s \mapsto T^2 + 2\mathrm{Re}[s]T + |s|^2\mathcal{I}$, then $s \in \rho_S(T)$ if and only if $\tau(s)$ is invertible. But the set $\mathrm{Inv}(\mathcal{B}_R(V))$ of invertible elements of a Banach algebra is open (see for instance Theorem 10.12 in [25]). Since τ is continuous, $\rho_S(T) = \tau^{-1}(\mathrm{Inv}(\mathcal{B}_R(V)))$ is open, and in turn, $\sigma_S(T)$ is closed.

Finally, Lemma 4.8 implies $|s| \leq \|T\|$ for any $s \in \sigma_S(T)$. Thus, $\sigma_S(T)$ is closed and bounded, and therefore, compact.

□

As the following result shows, the S-spectrum has a structure that is compatible with the structure of slice regular functions.

Proposition 4.10. *Let $T \in \mathcal{B}_R(V)$. Then $\sigma_S(T)$ is axially symmetric.*

Proof. Let $s = s_0 + Is_1 \in \sigma_S(T)$ and let $\tilde{s} \in [s]$. Then $\mathrm{Re}[s] = \mathrm{Re}[\tilde{s}]$ and $|s|^2 = |\tilde{s}|^2$, and hence,

$$T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I} = T^2 - 2\mathrm{Re}[\tilde{s}]T + |\tilde{s}|^2\mathcal{I}.$$

Thus, $s \in \sigma_S(T)$ if and only if $\tilde{s} \in \sigma_S(T)$.

□

Recall from the introduction that there are two different types of eigenvalues in the quaternionic case, namely left and right eigenvalues which satisfy $T\boldsymbol{v} = \lambda\boldsymbol{v}$ and $T\boldsymbol{v} = \boldsymbol{v}\lambda$, respectively, for some $\boldsymbol{v} \in V$. Moreover, recall that it is the set of right eigenvalues $\sigma_R(T)$ that is meaningful in applications and that allows to prove the spectral theorem for quaternionic matrices.

Before we discuss the relation of the $\sigma_R(T)$ and $\sigma_S(T)$, we need an auxiliary lemma (see [8, Section 5]).

Lemma 4.11. *Let $s \in \mathbb{H}$ and let $p \in [s]$. Then there exists $\omega \in \mathbb{H} \setminus \{0\}$ such that $p = \omega^{-1}s\omega$.*

Proof. Let $I_1, I_2 \in \mathbb{S}$ with $I_1 \perp I_2$ and set $I_0 = 1$ and $I_3 = I_1I_2$. By Lemma 2.9, the set $\{I_0, I_1, I_2, I_3\}$ is an orthogonal basis of $\mathbb{H}$ that satisfies (2.1) and (2.3), the defining relations of the quaternionic product, just as $\{e_0, e_1, e_2, e_3\}$. Thus, we can write $x, y \in \mathbb{H}$ as $x = \sum_{i=0}^{3} x_iI_i$ and $y = \sum_{i=0}^{3} y_iI_i$ with $x_i, y_i \in \mathbb{R}$. For their product, we obtain

$$\begin{aligned} xy =& x_0y_0I_0I_0 + x_0y_1I_0I_1 + x_0y_2I_0I_2 + x_0y_3I_0I_3 + \\ &+ x_1y_0I_1I_0 + x_1y_1I_1I_1 + x_1y_2I_1I_2 + x_1y_3I_1I_3 + \\ &+ x_2y_0I_2I_0 + x_2y_1I_2I_1 + x_2y_2I_2I_2 + x_2y_3I_2I_3 + \\ &+ x_2y_0I_3I_0 + x_3y_1I_3I_1 + x_3y_2I_3I_2 + x_3y_3I_3I_3. \end{aligned}$$

By (2.1) and (2.3), this equals

$$\begin{aligned} xy =& x_0y_0I_0 + x_0y_1I_1 + x_0y_2I_2 + x_0y_3I_3 + \\ &+ x_1y_0I_1 - x_1y_1I_0 + x_1y_2I_3 - x_1y_3I_2 + \\ &+ x_2y_0I_2 - x_2y_1I_3 - x_2y_2I_0 + x_2y_3I_1 + \\ &+ x_2y_0I_3 + x_3y_1I_2 - x_3y_2I_1 - x_3y_3I_0. \end{aligned}$$

If we sort the terms, we finally get

$$\begin{aligned} xy =& (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3)I_0 + \\ &+ (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)I_1 + \\ &+ (x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1)I_2 + \\ &+ (x_0y_3 + x_1y_2 - x_2y_1 + x_2y_0)I_3. \end{aligned}$$

Let $s \in \mathbb{H}$ and let $p \in [s]$. The equation $p = \omega^{-1}s\omega$ is equivalent to $\omega p = s\omega$. By the preceding calculation, this is equivalent to the system

$$\begin{aligned} \omega_0p_0 - \omega_1p_1 - \omega_2p_2 - \omega_3p_3 &= s_0\omega_0 - s_1\omega_1 - s_2\omega_2 - s_3\omega_3 \\ \omega_0p_1 + \omega_1p_0 + \omega_2p_3 - \omega_3p_2 &= s_0\omega_1 + s_1\omega_0 + s_2\omega_3 - s_3\omega_2 \\ \omega_0p_2 - \omega_1p_3 + \omega_2p_0 + \omega_3p_1 &= s_0\omega_2 - s_1\omega_3 + s_2\omega_0 + s_3\omega_1 \\ \omega_0p_3 + \omega_1p_2 - \omega_2p_1 + \omega_3p_0 &= s_0\omega_3 + s_1\omega_2 - s_2\omega_1 + s_3\omega_0 \end{aligned}$$

respectively to

$$\begin{pmatrix} p_0 - s_0 & -p_1 + s_1 & -p_2 + s_2 & -p_3 + s_3 \\ p_1 - s_1 & p_0 - s_0 & p_3 + s_3 & -p_2 - s_2 \\ p_2 - s_2 & -p_3 - s_3 & p_0 - s_0 & p_1 + s_1 \\ p_3 - s_3 & p_2 + s_2 & -p_1 - s_1 & p_0 - s_0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

If we choose $I_1 = I_s$, then $s = s_0 + I_1s_1$ and $s_2 = s_3 = 0$. Moreover, we can choose $I_2 \perp I_1$ such that $p \in \operatorname{span}\{1, I_1, I_2\}$. Then $p = p_0 + I_1p_1 + I_2p_2$ and $p_3 = 0$. Since $p \in [s]$, we also have $s_0 = p_0$. Hence, the system (4.2) simplifies to

$$\begin{pmatrix} 0 & -p_1 + s_1 & -p_2 & 0 \\ p_1 - s_1 & 0 & 0 & -p_2 \\ p_2 & 0 & 0 & p_1 + s_1 \\ 0 & p_2 & -p_1 - s_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.3}$$

The determinant of the matrix M of this system is

$$\det M = p_1^4 - 2p_1^2s_1^2 + 2p_1^2p_2^2 + s_1^4 - 2s_1^2p_2^2 + p_2^4 = (p_1^2 + p_2^2)^2 + s_1^4 - 2s_1^2(p_1^2 + p_2^2).$$

Since $s \in [p]$, the vector parts of s and p have the same absolute value. Hence, $s_1^2 = |\underline{s}|^2 = |\underline{p}|^2 = p_1^2 + p_2^2$ and

$$\det M = s_1^4 + s_1^4 - 2s_1^2s_1^2 = 0.$$

Therefore, the system (4.3) has a nontrivial solution $(\omega_0, \omega_1, \omega_2, \omega_3)^T$ and $\omega = \omega_0 + \sum_{i=1}^3 \omega_iI_i$ satisfies $\omega p = s\omega$, which implies $p = \omega^{-1}s\omega$.

□

The set of right eigenvalues has a structure that is analogue to the one of the S-spectrum.

Lemma 4.12. *Let $T \in \mathcal{B}_R(T)$ and let $s \in \sigma_R(T)$. Then the whole 2-sphere $[s]$ belongs to $\sigma_R(T)$.*

Proof. Since s is a right eigenvalue, there exists a vector $\boldsymbol{v} \in V$ such that $T\boldsymbol{v} = \boldsymbol{v}s$. Let $\widetilde{s} \in [s]$. By Lemma 4.11 there exists $\omega \in \mathbb{H} \setminus \{0\}$ such that $\widetilde{s} = \omega^{-1}s\omega$. If we consider the vector $\boldsymbol{v}\omega$, we obtain

$$T(\boldsymbol{v}\omega) = T(\boldsymbol{v})\omega = \boldsymbol{v}s\omega = (\boldsymbol{v}\omega)\omega^{-1}s\omega = (\boldsymbol{v}\omega)\widetilde{s}.$$

Hence, $\widetilde{s}$ is a right eigenvalue of T, too.

□

Definition 4.13. *Let $T \in \mathcal{B}(V)$. A quaternion s is called an S-eigenvalue of T if there exists a vector $\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}$ such that*

$$\left(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}\right)\boldsymbol{v} = \boldsymbol{0}. \tag{4.4}$$

The following Lemma shows that the S-spectrum can be considered as a generalization of the set of right eigenvalues, just in the same way as the spectrum of an operator on a complex Banach space can be considered as a generalization of the set of eigenvalues in classical functional analysis; see [9].

Lemma 4.14. *Let $T \in \mathcal{B}(V)$. The set of right eigenvalues $\sigma_R(T)$ coincides with the set of S-eigenvalues of T. Therefore, $\sigma_R(T) \subset \sigma_S(T)$. If V has finite dimension, then we even have $\sigma_S(T) = \sigma_R(T)$.*

Proof. Let s be an S-eigenvalue. Then there exists a vector $\boldsymbol{v} \neq \boldsymbol{0}$ such that $T^2\boldsymbol{v} - 2\mathrm{Re}[s]T\boldsymbol{v} + |s|^2\boldsymbol{v} = \boldsymbol{0}$. If $T\boldsymbol{v} = \boldsymbol{v}s$, then $s \in \sigma_R(T)$. Otherwise, $\boldsymbol{w} = T\boldsymbol{v} - \boldsymbol{v}s \neq \boldsymbol{0}$. Since $2\mathrm{Re}[s] = s + \overline{s}$ commutes with T and $\boldsymbol{v}$ and since $|s|^2 = s\overline{s}$, we obtain

$$\boldsymbol{0} = T^2\boldsymbol{v} - 2\mathrm{Re}[s]T\boldsymbol{v} + |s|^2\boldsymbol{v} = T^2\boldsymbol{v} - T\boldsymbol{v}s - T\boldsymbol{v}\overline{s} + \boldsymbol{v}s\overline{s} = T(T\boldsymbol{v} - \boldsymbol{v}s) - (T\boldsymbol{v} - \boldsymbol{v}s)\overline{s} = T\boldsymbol{w} - \boldsymbol{w}\overline{s}.$$

Thus, $T\boldsymbol{w} = \boldsymbol{w}\overline{s}$ and $\overline{s}$ is a right eigenvalue of T. By Lemma 4.12, the entire 2-sphere $[\overline{s}]$ belongs to $\sigma_R(T)$. In particular, s is a right eigenvalue of T itself.

If on the other hand s is a right eigenvalue, then there exists a vector $\boldsymbol{v} \neq \boldsymbol{0}$ such that $T\boldsymbol{v} = \boldsymbol{v}s$. Since $2\mathrm{Re}[s] = s + \overline{s}$ commutes with $\boldsymbol{v}$ and since $|s|^2 = s\overline{s}$, we have

$$T^2\boldsymbol{v} - 2\mathrm{Re}[s]T\boldsymbol{v} + |s|^2\boldsymbol{v} = T\boldsymbol{v}s - 2\mathrm{Re}[s]\boldsymbol{v}s + \boldsymbol{v}|s|^2 = \boldsymbol{v}s^2 - \boldsymbol{v}s^2 - \boldsymbol{v}s\overline{s} + \boldsymbol{v}s\overline{s} = \boldsymbol{0}.$$

Hence, s is an S-eigenvalue.

A quaternion s is an S-eigenvalue of T if and only if the operator $L(s,T) = T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}$ is not injective because in this case $L(s,T)\boldsymbol{v} = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})\boldsymbol{v} = \boldsymbol{0}$ for some $\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}$. In particular, $L(s,T)$ is not invertible, and hence, s belongs to the $\sigma_S(T)$. Since the set of S-eigenvalues coincides with the set of right eigenvalues, we obtain $\sigma_R(T) \subset \sigma_S(T)$. Moreover, if V has finite dimension, then $L(s,T)$ is invertible if and only if it is injective by Corollary 2.33. Hence, in this case, $\sigma_R(T) = \sigma_S(T)$.

□

4.3 Resolvent equations

In contrast to the classical case, the quaternionic S-functional calculus involves two different important types of resolvent equations. The first one substitutes the fact that in the classical Riesz-Dunford calculus an operator commutes with its resolvent, which is in general not true for the S-resolvent operators.

Theorem 4.15 (Left and right S-resolvent equation)**.** *Let $T \in \mathcal{B}_R(V)$ and let $s \in \rho_S(T)$. Then, the left S-resolvent operator satisfies the* left S-resolvent equation

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I} \tag{4.5}$$

and the right S-resolvent operator satisfies the right S-resolvent equation

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{4.6}$$

Proof. Since $2\mathrm{Re}[s]$ and $|s|^2$ are real, they commute with the operator T. Hence,

$$T(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})T$$

and in turn,

$$(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}T = T(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}.$$

Thus,

$$\begin{aligned} &S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \\ &\quad = -(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I})s + T(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}) = \\ &\quad = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(-T + \overline{s}\mathcal{I})s + (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}T(T - \overline{s}\mathcal{I}) = \\ &\quad = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})^{-1}(T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}) = \mathcal{I}. \end{aligned}$$

□

The left and the right S-resolvent equation cannot be considered as generalizations of the classical resolvent equation

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda)R_\lambda(T)R_\mu(T), \tag{4.7}$$

where $R_\lambda(T) = (\lambda\mathcal{I} - T)^{-1}$ is the classical resolvent operator and λ and μ lie in the resolvent set of T (see Lemma 6 in [16, Chapter VII.3.5]). In fact, the classical resolvent equation provides a possibility to split the product of two resolvent operators into a sum of the factors, whereas the left and the right S-resolvent equation provide a possibility to split the product of T and the left and right S-resolvent operator, respectively.

The next result, Theorem 3.8 in [2], can be regarded as the generalization of the classical resolvent equation, which preserves its philosophy. However, it is remarkable that this equation involves both, the left and the right resolvent operator and that no generalization of (4.7) including just one of them, either the left or the right resolvent operator, has been found yet.

Theorem 4.16. *Let $T \in \mathcal{B}_R(V)$ and let $s, p \in \rho_S(T)$. Then the equation*

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = \left[\left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right)p - \overline{s}\left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right)\right](p^2 - 2\mathrm{Re}[s]p + |s|^2)^{-1} \tag{4.8}$$

holds true. Equivalently, it can also be written as

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = (s^2 - 2\mathrm{Re}[p]s + |p|^2)^{-1}\left[\left(S_L^{-1}(p,T) - S_R^{-1}(s,T)\right)\overline{p} - s\left(S_L^{-1}(p,T) - S_R^{-1}(s,T)\right)\right]. \tag{4.9}$$

Proof. We prove the first identity (4.8). The second identity (4.9) follows by analogous computations. We show that for $s, p \in \rho_S(T)$

$$S_R^{-1}(s,T)S_L^{-1}(p,T)\left(p^2 - 2\mathrm{Re}[s]p + |s|^2\right) = \left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right)p - \overline{s}\left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right). \tag{4.10}$$

To abbreviate the formulas, we denote

$$\mathfrak{S}(s,p,T) = S_R^{-1}(s,T)S_L^{-1}(p,T)\left(p^2 - 2\mathrm{Re}[s]p + |s|^2\right).$$

The left S-resolvent equation (4.5) implies $S_L^{-1}(p,T)p = TS_L^{-1}(p,T) + \mathcal{I}$. Hence,

$$\begin{aligned}
\mathfrak{S}(s,p,T) &= S_R^{-1}(s,T)S_L^{-1}(p,T)p^2 - 2\mathrm{Re}[s]S_R^{-1}(s,T)S_L^{-1}(p,T)p + |s|^2S_R^{-1}(s,T)S_L^{-1}(p,T) = \\
&= S_R^{-1}(s,T)[TS_L^{-1}(p,T) + \mathcal{I}]p - 2\mathrm{Re}[s]S_R^{-1}(s,T)[TS_L^{-1}(p,T) + \mathcal{I}] + |s|^2S_R^{-1}(s,T)S_L^{-1}(p,T) = \\
&= S_R^{-1}(s,T)T[TS_L^{-1}(p,T) + \mathcal{I}] + S_R^{-1}(s,T)p - \\
&\qquad - 2\mathrm{Re}[s]S_R^{-1}(s,T)[TS_L^{-1}(p,T) + \mathcal{I}] + |s|^2S_R^{-1}(s,T)S_L^{-1}(p,T) = \\
&= S_R^{-1}(s,T)T^2S_L^{-1}(p,T) + S_R^{-1}(s,T)T + S_R^{-1}(s,T)p - \\
&\qquad - 2\mathrm{Re}[s][S_R^{-1}(s,T)TS_L^{-1}(p,T) + S_R^{-1}(s,T)] + |s|^2S_R^{-1}(s,T)S_L^{-1}(p,T).
\end{aligned}$$

On the other hand, the right S-resolvent equation (4.6) implies $S_R^{-1}(s,T)T = sS_R^{-1}(s,T) - \mathcal{I}$. Therefore,

$$\begin{aligned}
\mathfrak{S}(s,p,T) &= [sS_R^{-1}(s,T) - \mathcal{I}]TS_L^{-1}(p,T) + sS_R^{-1}(s,T) - \mathcal{I} + S_R^{-1}(s,T)p - \\
&\qquad - 2\mathrm{Re}[s]\left[[sS_R^{-1}(s,T) - \mathcal{I}]S_L^{-1}(p,T) + S_R^{-1}(s,T)\right] + |s|^2S_R^{-1}(s,T)S_L^{-1}(p,T) = \\
&= s[sS_R^{-1}(s,T) - \mathcal{I}] - T]S_L^{-1}(p,T) + sS_R^{-1}(s,T) - \mathcal{I} + S_R^{-1}(s,T)p - \\
&\qquad - 2\mathrm{Re}[s]\left[[sS_R^{-1}(s,T)S_L^{-1}(p,T) - S_L^{-1}(p,T)] + S_R^{-1}(s,T)\right] + |s|^2S_R^{-1}(s,T)S_L^{-1}(p,T) = \\
&= (s^2 - 2\mathrm{Re}[s]s + |s|^2)S_R^{-1}(s,T)S_L^{-1}(p,T) + [S_R^{-1}(s,T) - S_L^{-1}(p,T)]p - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)].
\end{aligned}$$

Since $s^2 - 2\mathrm{Re}[s]s + |s|^2 = 0$, we obtain (4.10).

□

Chapter 5

The S-functional calculus

The most obvious advantage of the notion of slice regularity over the notion of Cauchy-Fueter-regularity is the fact that polynomials of a quaternionic variable are slice regular. It is this fact that makes this class of functions extremely useful for a functional calculus in a quaternionic setting. When we finally define this functional calculus, we follow again Chapters 3 and 4 of [12], except for the proofs of the product rule and the existence of the Riesz-projectors, which can be found in [2].

5.1 The definition of the S-functional calculus

We consider again a right linear operator T on a quaternionic Banach space V. Before we can define the S-functional calculus, we have to specify the underlying class of functions.

Definition 5.1 (T-admissible slice domain)**.** *Let $T \in \mathcal{B}_R(V)$. A bounded axially symmetric slice domain $U \subset \mathbb{H}$ is called* T-admissible *if $\sigma_S(T) \subset U$ and $\partial(U \cap \mathbb{C}_I)$ is the union of a finite number of piecewise continuously differentiable Jordan curves for any $I \in \mathbb{S}$.*

Definition 5.2. *Let $T \in \mathcal{B}_R(V)$.*

(i) A function f is called locally left slice regular *on $\sigma_S(T)$, if there exists a T-admissible slice domain $U \subset \mathbb{H}$ such that $f \in \mathcal{M}^L(\overline{U})$. We denote the set of all locally left slice regular functions on $\sigma_S(T)$ by $\mathcal{M}^L(\sigma_S(T))$.*

(ii) A function f is called locally right slice regular *on $\sigma_S(T)$, if there exists a T-admissible slice domain $U \subset \mathbb{H}$ such that $f \in \mathcal{M}^R(\overline{U})$. We denote the set of all locally left slice regular functions on $\sigma_S(T)$ by $\mathcal{M}^R(\sigma_S(T))$.*

(iii) By $\mathcal{N}(\sigma_S(T))$ we denote the set of all functions $f \in \mathcal{M}^L(\sigma_S(T))$ such that there exists a T-admissible slice domain U with $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for all $I \in \mathbb{S}$.

To show that T-admissible slice domains do actually exist, we need the following result, which is well known from complex analysis; see Proposition 1.1 in [14, Chapter VIII].

Lemma 5.3. *Let $G \subset \mathbb{C}$ be a domain and let K be a compact subset of G. There exists an open set O such that $K \subset O$ and $\overline{O} \subset G$ and such that ∂O consists of a finite number of piecewise continuously differentiable Jordan curves.*

Remark 5.4. Since the result is well known, we omit the proof because it is quite technical. The basic idea is to cover K with sufficiently small rectangles $R_i, i \in I$. Since K is compact, there exists a finite subset $R_i, i = 1, \dots, n$ that covers K. Then $O = \bigcup_{i=1}^n R_i$ has the desired properties. However, if K and G are symmetric with respect to the real axis, then we can chose the covering $R_i, i \in I$, and in turn also the set O, symmetric with respect to the real axis. Adding additional rectangles $R_j, j = 1, \dots, m$, we can even chose the set O such that it is connected because G is connected.

Lemma 5.5. *Let G be an axially symmetric slice domain and let K be a compact, axially symmetric subset of G. Then there exists an axially symmetric slice domain U such that $K \subset U$ and $\overline{U} \subset G$ and such that $\partial(U \cap \mathbb{C}_I)$ consists of a finite number of piecewise continuously differentiable Jordan curves for any $I \in \mathbb{S}$.*

Proof. Let $I \in \mathbb{S}$. We apply Lemma 5.3 with the compact set $K \cap \mathbb{C}_I$ and the domain $G \cap \mathbb{C}_I$ and obtain a domain $U_I \subset \mathbb{C}_I$ that is symmetric with respect to the real axis such that $(K \cap \mathbb{C}_I) \subset U_I$ and $\overline{U}_I \subset (G \cap \mathbb{C}_I)$ and such that ∂U_I consists of a finite number of piecewise continuously differentiable Jordan curves. We set $U = [U_I]$, where $[U_I]$ denotes the axially symmetric hull of U_I. Since U_I is symmetric with respect to the real axis, we have $U \cap \mathbb{C}_I = U_I$. Since U_I is also connected, $U_I \cap \mathbb{R} \neq \emptyset$, and hence, U is a slice domain by Lemma 3.20. Finally, $\partial(U \cap \mathbb{C}_I) = \partial U_I$ consists of a finite number of piecewise continuously differentiable Jordan curves. For $J \in \mathbb{S}$, Lemma 3.20 implies $U \cap \mathbb{C}_J = \{x_0 + Jx_1 : x_0 + Ix_1 \in U\}$. Therefore, also $\partial(U \cap \mathbb{C}_J)$ consists of a finite number of piecewise continuously differentiable Jordan curves.

□

Corollary 5.6. *Let $T \in \mathcal{B}_R(V)$. If f is left or slice regular on an axially symmetric slice domain U with $\sigma_S(T) \subset U$, then there exists a T-admissible slice domain U' wit $\overline{U'} \subset U$.*

Proof. We obtain the U' by applying Lemma 5.5 with $G = U$ and $K = \sigma_S(T)$.

□

Lemma 5.7. *Let $O \subset \mathbb{H}$ be an axially symmetric open set and let K be an axially symmetric compact subset of O. Then there exists an axially symmetric open set A with $K \subset A$ and $\overline{A} \subset O$ such that, for any $I \in \mathbb{S}$, the boundary $\partial(A \cap \mathbb{C}_I)$ consists of a finite union of piecewise continuously differentiable Jordan curves.*

Proof. Let $O_i, i = 1, \dots, n$ be the connected components of O with $K \cap O \neq \emptyset$. We assume that they are ordered such that $O_i \cap \mathbb{R} \neq \emptyset$ if $1 \leq i \leq m$ and $O_i \cap \mathbb{R} = \emptyset$ if $m+1 \leq i \leq n$ for some $m \in \{0, \dots, n\}$.

If $x \in O_i$ with $i \in \{1, \dots, n\}$, then $[x] \cap O_j, j = 1, \dots, n$ is a decomposition of $[x]$ into open subsets whose intersection is empty. $x \in O_i$ implies $[x] \subset O_i$ and $[x] \cap O_j = \emptyset$ for $j \neq i$ because $[x]$ is connected. Therefore, the sets O_i are axially symmetric domains in $\mathbb{H}$. Since K is axially symmetric and compact and since $O_i \cap O_j = \emptyset$ if $i \neq j$, the sets $K_i = K \cap O_i$ are axially symmetric and compact.

For $i = 1, \dots, m$, Lemma 3.20 implies that O_i is an axially symmetric slice domain because $O_i \cap \mathbb{R} \neq \emptyset$ and $[O_i] = O_i$. If we apply Lemma 5.5 with $K = K_i$ and $G = O_i$, we obtain an axially symmetric slice domain A_i such that $K_i \subset A_i$ and $\overline{A_i} \subset O_i$ and such that $\partial(A_i \cap \mathbb{C}_I)$ consists of a finite union of piecewise continuously differentiable Jordan curves.

For $i = m+1, \dots, n$, we chose $I \in \mathbb{S}$ and set $\mathbb{C}_I^< = \{x = x_0 + Ix_1 \in \mathbb{C}_I : 0 < x_1\}$. The set $O_i \cap \mathbb{C}_I^<$ is open in $\mathbb{C}_I$. If there exists two open subsets B_1 and B_2 of $\mathbb{C}_I^<$ with $O_i \cap \mathbb{C}_I = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$, then, by Lemma 3.19, their axially symmetric hulls $[B_1]$ and $[B_2]$ are open and satisfy $[B_1] \cup [B_2] = O_i$ and $[B_1] \cap [B_2] = \emptyset$. Since O_i is connected, either $[B_1] = \emptyset$ or $[B_2] = \emptyset$, and in turn, $B_1 = \emptyset$ or $B_2 = \emptyset$. Hence, $O_i \cap \mathbb{C}_I^<$ is connected. Moreover, the set $K_i \cap \mathbb{C}_I^< = K_i \cap \overline{\mathbb{C}_I^<}$ is compact in $\mathbb{C}_I$. Therefore, we can apply Lemma 5.3 with $G = O_i \cap \mathbb{C}_I^<$ and $K = K_i \cap \mathbb{C}_I^<$. We obtain an open set $A_{i,I}$ such that $K_i \cap \mathbb{C}_I^< \subset A_{i,I}$ and $\overline{A_{i,I}} \subset O_i \cap \mathbb{C}_I^<$ and such that $\partial A_{i,I}$ consists of the finite union of piecewise continuously differentiable Jordan curves. If we set $A_i = [A_{i,I}]$, then A_i is open by Lemma 3.19 and satisfies $K_i \subset A_i$ and $\overline{A_i} \subset O_i$. For any $J \in \mathbb{S}$, the boundary $A_i \cap \mathbb{C}_J$ consists of the union of the disjoint sets $\{x_0 + Jx_1 : x_0 + Ix_1 \in \partial A_{i,I}\}$ and $\{x_0 - Jx_1 : x_0 + Ix_1 \in \partial A_{i,I}\}$. Hence, it is the finite union of piecewise continuously differentiable Jordan curves.

Finally, the set $A = \bigcup_{i=1}^n A_i$ has the desired properties.

□

Recall the definition of the integral with respect to ds_I in Definition 3.77. The following theorem motivates the S-functional calculus and shows that it is well defined in the sense that it is compatible with polynomials of the quaternionic variable.

Theorem 5.8. *Let $T \in \mathcal{B}_R(V)$, let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $U \subset \mathbb{H}$ be a T-admissible slice domain. Then*

$$T^m = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, s^m = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} s^m\, ds_I\, S_R^{-1}(s,T)$$

for any imaginary unit $I \in \mathbb{S}$.

Proof. Let us first consider the case that U is a ball $B_r(0)$ with $\|T\| < r$. Then $S_L^{-1}(s,T) = \sum_{n=0}^{\infty} T^n s^{-n-1}$ for any $s \in \partial B_r(0)$; see Definition 4.1. Moreover, the series converges uniformly on $\partial B_r(0)$, and hence

$$\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m = \frac{1}{2\pi}\sum_{n=0}^{\infty} T^n \int_{\partial(B_r(0)\cap\mathbb{C}_I)} s^{-1-n+m} ds_I.$$

The fact that

$$\int_{\partial(B_r(0)\cap\mathbb{C}_I)} s^{-1-n+m}\,ds_I = \begin{cases} 0 & \text{if } n \neq m \\ 2\pi & \text{if } n = m \end{cases}$$

gives

$$\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m = T^m.$$

Now let U be an arbitrary T-admissible slice domain. Then there exists a radius r such that $\overline{U} \subset B_r(0)$. Moreover, if we restrict the right scalar multiplication to the complex plane $\mathbb{C}_I$, then $\mathcal{B}_R(V)$ is a complex Banach space by Corollary 2.38. By Lemma 4.7, the mapping $s \mapsto S_L^{-1}(s,T)s^m$ is a $\mathcal{B}_R(V)$-valued holomorphic function on $\mathbb{C}_I \setminus \sigma_S(T)$. Thus, the vector-valued Cauchy theorem implies

$$\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m - \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m =$$
$$= \frac{1}{2\pi}\int_{\partial((B_r(0)\setminus U)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m = 0$$

and further

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m = \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^m = T^m.$$

The second identity, which involves the right S-resolvent operator $S_R^{-1}(s,T)$, follows analogously from the corresponding series expansion of the right S-resolvent operator.

□

Corollary 5.9. *Let $T \in \mathcal{B}_R(V)$, let U be a T-admissible slice domain and let $p(x) = \sum_{n=0}^{N} x^n a_n$ with $a_n \in \mathbb{H}$ be a left slice regular polynomial. If we set $p(T) = \sum_{n=0}^{N} T^n a_n$, then*

$$p(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,p(s) \tag{5.1}$$

for any imaginary unit $I \in \mathbb{S}$. Similarly, if $p(x) = \sum_{n=0}^{N} a_n x^n$ with $a_n \in \mathbb{H}$ is a right slice regular polynomial and we set $p(T) = \sum_{n=0}^{N} a_n T^n$, then

$$p(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I} p(s)\,ds_I\,S_R^{-1}(s,T) \tag{5.2}$$

for any imaginary unit $I \in \mathbb{S}$.

In particular, if the polynomial p has real coefficients, the integrals (5.1) *and* (5.2) *define the same operator.*

Proof. Let $p(x) = \sum_{n=0}^{N} x^n a_n$. Then Theorem 5.8 implies

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,p(s) = \sum_{n=0}^{N}\left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s^n\right] a_n = \sum_{n=0}^{N} T^n a_n = p(T).$$

The case of a right slice regular polynomial follows with analogous computations. Moreover, if $p(x) = \sum_{n=0}^{N} a_n x^n$ has real coefficients, then

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,p(s) = \sum_{n=0}^{N} T^n a_n = \sum_{n=0}^{N} a_n T^n = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} p(s)\,ds_I\,S_R^{-1}(s,T).$$

□

Theorem 5.10. *Let $T \in \mathcal{B}_R(V)$, let $U \subset \mathbb{H}$ be a T-admissible slice domain and let $I \in \mathbb{S}$. Then the integrals*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s), \quad \text{for } f \in \mathcal{M}^L(\sigma_S(T)) \tag{5.3}$$

and

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,T), \qquad \text{for } f \in \mathcal{M}^R(\sigma_S(T)),$$

do neither depend on the choice of the imaginary unit I nor on the set U.

Proof. Let be $V_{\mathbb{R}}$ be the real part of V as in Lemma 2.17. For any vector $\boldsymbol{v} \in V$ there exist vectors $\boldsymbol{v}_i \in V_{\mathbb{R}}, i = 0,\dots,3$ such that $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i$. If a quaternionic right linear operator $\widetilde{T}$ satisfies $\widetilde{T}(\boldsymbol{v}) = T(\boldsymbol{v})$ for any $\boldsymbol{v} \in V_{\mathbb{R}}$, then

$$\widetilde{T}(\boldsymbol{v}) = \sum_{i=0}^{3} \widetilde{T}(\boldsymbol{v}_i)e_i = \sum_{i=0}^{3} T(\boldsymbol{v}_i)e_i = T(\boldsymbol{v})$$

for any $\boldsymbol{v} = \sum_{i=0}^{3} \boldsymbol{v}_i e_i \in V$. Hence, two quaternionic right linear operators are equal if and only if they coincide on $V_{\mathbb{R}}$.

For $\boldsymbol{v} \in V_{\mathbb{R}}$ and for any left linear functional $\phi \in V_R'$, we define

$$g_{\boldsymbol{v},\phi}(s) = \langle \phi, S_L^{-1}(s,T)\boldsymbol{v}\rangle \qquad \text{for } s \in \rho_s(T).$$

Then $g_{\boldsymbol{v},\phi}$ is right slice regular on $\rho_S(T)$ because of Lemma 4.7. Indeed, for $s = s_0 + Is_1 \in \rho_S(T)$, we have

$$\begin{aligned} g_{\boldsymbol{v},\phi}(s)\bar{\partial}_I &= \frac{1}{2}\left(\left\langle \phi, \frac{\partial}{\partial s_0} S_L^{-1}(s,T)\boldsymbol{v}\right\rangle + \left\langle \phi, \frac{\partial}{\partial s_1} S_L^{-1}(s,T)\boldsymbol{v}\right\rangle I\right) = \\ &= \frac{1}{2}\left(\left\langle \phi, \frac{\partial}{\partial s_0} S_L^{-1}(s,T)\boldsymbol{v}\right\rangle + \left\langle \phi, \frac{\partial}{\partial s_1} S_L^{-1}(s,T)I\boldsymbol{v}\right\rangle\right) = \left\langle \phi, \left(S_L^{-1}(s,T)\bar{\partial}_I\right)\boldsymbol{v}\right\rangle = 0 \end{aligned}$$

because the vector $\boldsymbol{v} \in V_{\mathbb{R}}$ commutes with any imaginary unit I.

Moreover, for any $\phi \in V_R'$ and any $\boldsymbol{v} \in V_{\mathbb{R}}$, we obtain

$$\begin{aligned} \left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}\right\rangle &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \langle \phi, S_L^{-1}(s,T)\boldsymbol{v}\rangle\,ds_I\,f(s) = \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g_{\boldsymbol{v},\phi}(s)\,ds_I\,f(s). \end{aligned}$$

We first show that the integral (5.3) does not depend on the slice domain U. Let U' be another T-admissible slice domain. Applying Lemma 5.7 with the axially symmetric open set $U \cap U'$ and the axially symmetric compact set $\sigma_S(T)$, we obtain an axially symmetric open set O with $\sigma_S(T) \subset O$ and $\overline{O} \subset U$ such that $\partial(O \cap \mathbb{C}_I)$ consists of a finite union of piecewise continuously differentiable Jordan curves for any $I \in \mathbb{S}$.

Applying Cauchy's Integral Theorem, Theorem 3.80, with $D_i = (U \cap \mathbb{C}_I) \setminus (O \cap \mathbb{C}_I)$, we obtain

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g_{\phi,\boldsymbol{v}}(s)\,ds_I\,f(s) - \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} g_{\phi,\boldsymbol{v}}(s)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial((U\cap\mathbb{C}_I)\setminus(O\cap\mathbb{C}_I)} g_{\phi,\boldsymbol{v}}(s)\,ds_I\,f(s) = 0.$$

Hence,

$$\left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}\right\rangle = \left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}\right\rangle \tag{5.4}$$

for all $\phi \in V_R'$ and all $\boldsymbol{v} \in V_{\mathbb{R}}$. We know from Corollary 2.43 that V_R' separates the points of V. Thus, (5.4) implies

$$\left[\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v} = \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}$$

for all $\boldsymbol{v} \in V_{\mathbb{R}}$ and, in turn, for all $\boldsymbol{v} \in V$. The same calculation holds true if we replace U by U'. Therefore,

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s).$$

To show that (5.3) does not depend on the imaginary unit, we consider $I, J \in \mathbb{S}$ with $I \neq J$ and a T-admissible slice domain U such that $f \in \mathcal{M}^L(\overline{U})$. The series $\sum_{n=0}^{\infty} T^n s^{-n-1}$ converges uniformly on the set $\{s \in \mathbb{H} : 2\|T\| < |s|\}$. Hence,

$$\lim_{|s|\to\infty} S_L^{-1}(s,T) = \lim_{|s|\to\infty}\sum_{n=0}^{\infty} T^n s^{-1-n} = \sum_{n=0}^{\infty}\lim_{|s|\to\infty} T^n s^{-1-n} = 0.$$

By Lemma 5.5, we can choose another T-admissible slice domain U' such that $\overline{U'} \subset U$ and apply Theorem 3.83, the Cauchy formula outside an axially symmetric slice domain, to calculate the values of $g_{\phi,\boldsymbol{v}}(x)$ for $x \in \mathbb{H}\setminus\overline{U'}$. For all $\phi \in V_R'$ and all $\boldsymbol{v} \in V_{\mathbb{R}}$, we obtain

$$g_{\phi,\boldsymbol{v}}(x) = -\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)} g_{\phi,\boldsymbol{v}}(s)\,ds_J\,S_R^{-1}(s,x) \qquad \text{for } x \in \mathbb{H}\setminus\overline{U'},$$

and

$$\begin{aligned}\left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}\right\rangle &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g_{\boldsymbol{v},\phi}(s)\,ds_I\,f(s) = \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)}\left[-\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)} g_{\phi,\boldsymbol{v}}(p)\,dp_J\,S_R^{-1}(p,s)\right] ds_I f(s) = \\ &= \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)} g_{\phi,\boldsymbol{v}}(p)\,dp_J\left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} -S_R^{-1}(p,s)\,ds_I\,f(s)\right],\end{aligned}$$

where the last equality follows from Lemma 3.78. Since $-S_R^{-1}(p,s) = S_L^{-1}(s,p)$ by Corollary 3.71 and since f is left slice regular on $\overline{U}$, we obtain from Corollary 3.82

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} -S_R^{-1}(p,s)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,p)\,ds_I\,f(s) = f(p).$$

As (5.3) does not depend on the set U, we finally derive

$$\begin{aligned}\left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}\right\rangle &= \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)} g_{\phi,\boldsymbol{v}}(p)\,dp_J\,f(p) = \\ = \left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)} S_L^{-1}(p,T)\,dp_J\,f(p)\right]\boldsymbol{v}\right\rangle &= \left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} S_L^{-1}(p,T)\,dp_J\,f(p)\right]\boldsymbol{v}\right\rangle\end{aligned}$$

for all $\phi \in V_R'$ and all $\boldsymbol{v} \in V_{\mathbb{R}}$. By Corollary 2.43, V_R' separates the points of V. It follows that $\left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v} = \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} S_L^{-1}(s,T)\,ds_I\,f(s)\right]\boldsymbol{v}$ for all $\boldsymbol{v} \in V_{\mathbb{R}}$, and therefore, for any $\boldsymbol{v} \in V$. Thus, (5.3) does not depend on the choice of the imaginary unit $I \in \mathbb{S}$.

□

Definition 5.11 (S-functional calculus)**.** *Let $T \in \mathcal{B}_R(V)$. For any $f \in \mathcal{M}^L(\sigma_S(T))$, we define*

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s), \tag{5.5}$$

where I is an arbitrary imaginary unit and U is an arbitrary T-admissible slice domain such that f is left slice regular on $\overline{U}$.

For any $f \in \mathcal{M}^R(\sigma_S(T))$, we define

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,T), \tag{5.6}$$

where I is an arbitrary imaginary unit and U is an arbitrary T-admissible slice domain such that f is right slice regular on $\overline{U}$.

Theorems 5.8 and 5.10 show that the S-functional calculus is well defined for any left or right slice regular function. However, if f is left and right slice regular it is not yet clear that (5.5) and (5.6) give the same operator. To show this, we need the fact, that the S-functional calculus is consistent with the limits of uniformly convergent sequences of slice regular functions.

Theorem 5.12. *Let $T \in \mathcal{B}_R(V)$. Let $f_n, f \in \mathcal{M}^L(\sigma_S(T))$ or let $f_n, f \in \mathcal{M}^R(\sigma_S(T))$ for $n \in \mathbb{N}$. If there exists a T-admissible slice domain U such that $f_n \to f$ uniformly on $\overline{U}$, then $f_n(T)$ converges to $f(T)$ in $\mathcal{B}_R(V)$.*

Proof. Since $f_n \to f$ uniformly on $\overline{U}$, we can exchange limit and integration and obtain

$$\lim_{n\to\infty} f_n(T) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T) ds_I f_n(s) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T) ds_I f(s) = f(T).$$

□

The following Theorem 5.13 shows that the representations (5.5) and (5.6) are equivalent, if f belongs to $\mathcal{N}(\sigma_S(T))$. However, the proof of this theorem uses the product rule for the S-functional calculus in the special case that f is a polynomial with real coefficients. We prove the product rule later in Theorem 5.17. The reason why we postpone the proof of the product rule is that it uses the fact that (5.5) and (5.6) are equivalent for any function in $\mathcal{N}(U)$, which is exactly the statement of the following Theorem 5.13. Nevertheless, this is not circular reasoning because for the special case of a polynomial with real coefficients, the equivalence of (5.5) and (5.6) has already been shown in Corollary 5.9. Thus, strictly speaking, we have to proceed as follows:

Step 1) We show that the product rule $(fg)(T) = f(T)g(T)$ holds in the special case that either f is a polynomial with real coefficients and $g \in \mathcal{M}^L(\sigma_S(T))$ or that $f \in \mathcal{M}^R(U)$ and g is a polynomial with real coefficients. Thereby, we use that the representations (5.5) and (5.6) coincide for any polynomial with real coefficients by Corollary 5.9.

Step 2) We show Theorem 5.13, where we use Step 1).

Step 3) We show the general product rule, where we use Theorem 5.13, that is, that (5.5) and (5.6) coincide for $f \in \mathcal{N}(U)$.

But since the proofs of Step 1) and Step 3) are exactly the same, we only write them down for the general case in Theorem 5.17, keeping in mind this remark.

Theorem 5.13. *Let $T \in \mathcal{B}_R(V)$ and let $f \in \mathcal{N}(\sigma_S(T))$. Then*

$$\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s)\, ds_I\, S_R^{-1}(s,T). \tag{5.7}$$

Proof. Let $p \neq 0 \in \mathcal{N}(\mathbb{H})$ be a polynomial with real coefficients such that $p^{-1} \in \mathcal{N}(\sigma_S(T))$. Then $p(T)$ is invertible and we have $p(T)^{-1} = p^{-1}(T)$, where

$$p^{-1}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, p^{-1}(s) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} p^{-1}(s)\, ds_I\, S_R^{-1}(s,T).$$

Indeed, by applying Theorem 5.8 and the product rule for the S-functional calculus, we obtain

$$\mathcal{I} = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, 1 =$$
$$= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, p^{-1}(s)p(s) = \left[\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, p^{-1}(s)\right] p(T)$$

and

$$\mathcal{I} = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} 1\, ds_I\, S_R^{-1}(s,T) =$$
$$= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} p(s)p^{-1}(s)\, ds_I\, S_R^{-1}(s,T) = p(T) \left[\frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} p^{-1}(s)\, ds_I\, S_R^{-1}(s,T)\right].$$

Therefore,

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,p^{-1}(s) = p(T)^{-1} = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} p^{-1}(s)\,ds_I\,S_R^{-1}(s,T).$$

In particular, $p^{-1}(T)$ is well defined. Moreover, if $r = q^{-1}p$ is a real rational function, then p and q have real coefficients, and we can apply the product rule to obtain $r(T) = q^{-1}(T)p(T)$. Hence, (5.7) holds also for r.

Let $f \in \mathcal{N}(\sigma_S(T))$ and let U be a T-admissible slice domain such that $f \in \mathcal{N}(\overline{U})$. Then $\overline{U}$ is compact and therefore Runge's Theorem, Theorem 3.64, implies the existence of a sequence r_n of real rational functions in $\mathcal{N}(\overline{U})$ such that $f = \lim_{n\to\infty} r_n$ uniformly on $\overline{U}$. Theorem 5.12 then gives

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) &= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,r_n(s) = \\ &= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} r_n(s)\,ds_I\,S_R^{-1}(s,T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,T).\end{aligned}$$

□

Corollary 5.14. *For $T \in \mathcal{B}_R(V)$ and $f \in \mathcal{M}^L(\sigma_S(T)) \cap \mathcal{M}^R(\sigma_S(T))$, we have*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,T).$$

Proof. Let U be a T-admissible slice domain such that $f \in \mathcal{M}^L(\overline{U}) \cap \mathcal{M}^R(\overline{U})$. By Lemma 3.42, there exist a constant $a \in \mathbb{H}$ and a function $\tilde{f} \in \mathcal{N}(\overline{U})$ such that $f = a + \tilde{f}$. From Theorem 5.8, we know that

$$\mathcal{I} = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} ds_I\,S_R^{-1}(s,T).$$

Hence,

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) = \\ &\quad= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,\tilde{f}(s) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,a = \\ &\quad= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,\tilde{f}(s) + \left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\right)a = \tilde{f}(T) + \mathcal{I}a.\end{aligned}$$

Since by (2.14) the identity operator $\mathcal{I}$ commutes with the scalar a, this equals $\tilde{f}(T) + a\mathcal{I}$. Theorem 5.13 implies

$$\begin{aligned}\tilde{f}(T) + a\mathcal{I} &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \tilde{f}(s)\,ds_I\,S_R^{-1}(s,T) + a\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} ds_I\,S_R^{-1}(s,T)\right) = \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \tilde{f}(s)\,ds_I\,S_R^{-1}(s,T) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} a\,ds_I\,S_R^{-1}(s,T) = \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,T).\end{aligned}$$

□

5.2 Algebraic properties

An immediate consequence of Definition 5.11 is that the S-functional calculus for left slice regular functions is quaternionic right linear and that the S-functional calculus for right slice regular functions is quaternionic left linear.

Lemma 5.15. *Let $T \in \mathcal{B}_R(V)$.*

(i) If $f, g \in \mathcal{M}^L(U)$ and $a \in \mathbb{H}$, then

$$(f+g)(T) = f(T) + g(T) \qquad \text{and} \qquad (fa)(T) = f(T)a.$$

(ii) If $f, g \in \mathcal{M}^R(U)$ and $a \in \mathbb{H}$, then

$$(f+g)(T) = f(T) + g(T) \qquad \text{and} \qquad (af)(T) = af(T).$$

Proof. If $f, g \in \mathcal{M}^L(U)$ and $a \in \mathbb{H}$, then we have

$$\begin{aligned}(f+g)(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,(f(s)+g(s)) = \\ = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,g(s) = f(T)+g(T)\end{aligned}$$

and

$$(fa)(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)a = \left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,f(s)\right)a = f(T)a.$$

□

Since the product of two slice regular functions is not necessarily slice regular, we cannot expect to obtain a product rule for arbitrary slice regular functions. However, at least if $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{M}^L(\sigma_S(T))$ or if $f \in \mathcal{M}^R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T)$, then $fg \in \mathcal{M}^L(U)$ resp. $fg \in \mathcal{M}^R(U)$. To show that the S-functional calculus is compatible with the product in these cases, we need the following lemma. Note that in this lemma, we do not assume that O is a slice domain.

Lemma 5.16. *Let $B \in \mathcal{B}_R(V)$, let O be an axially symmetric open set such that $\partial(O\cap\mathbb{C}_I)$ consists of the finite union of piecewise continuously differentiable Jordan curves and let U be an axially symmetric slice domain such that $\overline{O} \subset U$. If $f \in \mathcal{N}(U)$, then for any $I \in \mathbb{S}$*

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(\overline{s}B - Bp)(p^2 - 2\mathrm{Re}[s]p + |s|^2)^{-1} = Bf(p), \qquad p \in O.$$

Proof. Since $s\overline{s} = |s|^2$ and $s + \overline{s} = 2\mathrm{Re}[s]$ are real, they commute with the operator B. Hence,

$$\begin{aligned}(s^2 - 2\mathrm{Re}[p]s + |p|^2)(\overline{s}B - Bp) = s|s|^2B - 2\mathrm{Re}[p]|s|^2B + |p|^2\overline{s}B - s^2Bp + 2\mathrm{Re}[p]sBp - |p|^2Bp = \\ = sB|s|^2 - B|s|^2 2\mathrm{Re}[p] + \overline{s}B|p|^2 - s^2Bp + sB2\mathrm{Re}[p]p - B|p|^2p = \\ = sB|s|^2 - B\overline{p}|s|^2 - |s|^2Bp + \overline{s}B|p|^2 - s^2Bp + sBp^2 + sB|p|^2B|p|^2p = \\ = (sB - B\overline{p})|s|^2 - s(s+\overline{s})Bp + (s+\overline{s})B\overline{p}p + (sB - Bp)p^2 = \\ = (sB - B\overline{p})|s|^2 - (sB - B\overline{p})2\mathrm{Re}[s]p + (sB - Bp)p^2 = (sB - B\overline{p})(p^2 - 2\mathrm{Re}[s]p + |s|^2).\end{aligned}$$

Thus,

$$(\overline{s}B - Bp)(p^2 - 2\mathrm{Re}[s]p + |s|^2)^{-1} = (s^2 - 2\mathrm{Re}[p]s + |p|^2)^{-1}(sB - B\overline{p}).$$

We calculate

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(\overline{s}B - Bp)(p^2 - 2\mathrm{Re}[s]p + |s|^2)^{-1} = \\ = \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(s^2 - 2\mathrm{Re}[p]s + |p|^2)^{-1}(sB - B\overline{p}) = \\ = \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(s^2 - 2\mathrm{Re}[p]s + |p|^2)^{-1}(s - \overline{p})B + \\ + \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(s^2 - 2\mathrm{Re}[p]s + |p|^2)^{-1}(\overline{p}B - B\overline{p}).\end{aligned} \tag{5.8}$$

For the first integral, we have

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(s^2-2\mathrm{Re}[p]s+|p|^2)^{-1}(s-\overline{p})B = \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,p)B = f(p)B, \quad (5.9)$$

where we use $-S_L^{-1}(p,s) = S_R^{-1}(s,p)$, cf. Corollary 3.71.

Let us consider the second integral. If we write $s = s_0 + Is_1$ and $p = p_0 + I_p p_1$ according to Corollary 3.11, then the solutions of the equation $s^2 - 2p_0 s + |p|^2 = 0$ in the plane $\mathbb{C}_I$ are $p_I = p_0 + Ip_1$ and $\overline{p_I} = p_0 - Ip_1$. $f \in \mathcal{N}(U)$ gives $f(O\cap\mathbb{C}_I) \subset \mathbb{C}_I$. Hence, the function f_I is holomorphic on $O\cap\mathbb{C}_I$ and we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} & f(s)\,ds_I\,(s^2-2p_0 s+|p|^2)^{-1}(\overline{p}B - B\overline{p}) = \\ &= \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} \frac{f_I(s)}{(s-p_I)(s-\overline{p_I})}\,ds_I\,(\overline{p}B - B\overline{p}).\end{aligned}$$

If we denote $F_I(s) = \frac{f_I(s)}{(s-p_I)(s-\overline{p_I})}$, the residue theorem implies

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(s^2-2\mathrm{Re}[p]s+|p|^2)^{-1}(\overline{p}B - B\overline{p}) = (\mathrm{Res}(F_I,p_I)+\mathrm{Res}(F_I,\overline{p_I}))\,(\overline{p}B - B\overline{p}),$$

where

$$\mathrm{Res}(F_I,p_I) = \lim_{z\to p_I\in\mathbb{C}_I}(z-p_I)F_I(z) = \frac{-I}{2p_1}f_I(p_I)$$

and

$$\mathrm{Res}(F_I,\overline{p_I}) = \lim_{z\to \overline{p_I}\in\mathbb{C}_I}(z-\overline{p_I})F_I(z) = \frac{I}{2p_1}f_I(\overline{p_I}).$$

Therefore,

$$\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(s^2-2\mathrm{Re}[p]s+|p|^2)^{-1}(\overline{p}B - B\overline{p}) = \frac{I}{2p_1}[f(\overline{p_I})-f(p_I)](\overline{p}B - B\overline{p}).$$

Recall that by Theorem 3.21 the slice regular function f can be written as $f(p) = \alpha(p_0,p_1) + I_p\beta(p_0,p_1)$, where $\alpha(p_0,p_1) = \frac{1}{2}(f(p_I)+f(\overline{p_I}))$ and $\beta(p_0,p_1) = I\frac{1}{2}(f(\overline{p_I})-f(p_I))$. Moreover, since $f\in\mathcal{N}(U)$, the functions α and β are real-valued by Corollary 3.40. Therefore, if we plug the values of the first and the second integral into (5.8), we finally obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_I)} f(s)\,ds_I\,(\overline{s}B - Bp)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1} &= f(p)B + \frac{I}{2p_1}[f(\overline{p_I})-f(p_I)](\overline{p}B - B\overline{p}) = \\ &= \alpha(p_0,p_1)B + I_p\beta(p_0,p_1)B + \frac{1}{p_1}\beta(p_0,p_1)(\overline{p}B - B\overline{p}) = \\ &= \alpha(p_0,p_1)B + I_p\beta(p_0,p_1)B + \frac{\beta(p_0,p_1)}{p_1}((p_0-I_pp_1)B - B(p_0-I_pp_1)) = \\ &= B(\alpha(p_0,p_1)+I_p\beta(p_0,p_1)) = \\ &= Bf(p).\end{aligned}$$

□

Theorem 5.17. *Let $T\in\mathcal{B}_R(V)$ and let $f\in\mathcal{N}(\sigma_S(T))$ and $g\in\mathcal{M}^L(\sigma_S(T))$ or let $f\in\mathcal{M}^R(\sigma_S(T))$ and $g\in\mathcal{N}(\sigma_S(T))$. Then*

$$(fg)(T) = f(T)g(T).$$

Proof. Let U_p and U_s be T-admissible slice domains such that $\overline{U_p}\subset U_s$ and such that $f\in\mathcal{N}(\overline{U_s})$ and $g\in\mathcal{M}^L(\overline{U_s})$. The subscripts p and s are chosen in order to indicate the respective variable of integration in the following computation. For $I\in\mathbb{S}$, $p\in\partial(U_p\cap\mathbb{C}_I)$ and $s\in\partial(U_s\cap\mathbb{C}_I)$, we have

$$\begin{aligned}f(T)g(T) =& \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I\,S_R^{-1}(s,T)\,\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)\,dp_I\,g(p) = \\ =& \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I\left[\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_I)} S_R^{-1}(s,T)S_L^{-1}(p,T)\,dp_I\,g(p)\right].\end{aligned}$$

Applying (4.8), we obtain

$$\begin{aligned}
f(T)g(T) =& \frac{1}{(2\pi)^2}\left[\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \int_{\partial(U_p\cap\mathbb{C}_I)} S_R^{-1}(s,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right] -\\
&- \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right] -\\
&- \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} \overline{s}S_R^{-1}(s,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right] +\\
&+ \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} \overline{s}S_L^{-1}(p,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} S_R^{-1}(s,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right] =\\
=&\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I S_R^{-1}(s,T)\left[-\int_{\partial(U_p\cap\mathbb{C}_I)} dp\,p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}I\,g(p)\right] = 0
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} \overline{s}\,S_R^{-1}(s,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right] =\\
=&-\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I\,\overline{s}\,S_R^{-1}(s,T)\left[-\int_{\partial(U_p\cap\mathbb{C}_I)} dp\,(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}Ig(p)\right] = 0
\end{aligned}$$

by the complex Cauchy theorem for vector-valued holomorphic functions, because the functions $p\mapsto p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}Ig(p)$ and $p\mapsto(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}Ig(p)$ are left holomorphic on $\overline{U_p}\cap\mathbb{C}_I$ for $s\in\partial U_s$ as $\overline{U_p}\subset U_s$. Therefore,

$$\begin{aligned}
f(T)g(T) =& -\frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right]\\
&+ \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I \left[\int_{\partial(U_p\cap\mathbb{C}_I)} \overline{s}S_L^{-1}(p,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right] =\\
=& \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_I)} \left[\int_{\partial(U_p\cap\mathbb{C}_I)} f(s)\,ds_I[\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p]\cdot\right.\\
&\left.\cdot(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}dp_I\,g(p)\right]
\end{aligned}$$

Since $\partial U_p\subset U_s$ and since $\partial U_p,\partial U_s\subset\rho_S(T)$, the integrand in the last integral is continuous on $(\partial U_p\cap\mathbb{C}_I)\times(\partial U_s\cap\mathbb{C}_I)$. Hence, we can apply Fubini's theorem to change the order of integration and obtain

$$\begin{aligned}
&f(T)g(T) =\\
=&\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_I)}\left[\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} f(s)\,ds_I[\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p](p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\right]dp_I\,g(p).
\end{aligned}$$

Applying Lemma 5.16 with $B=S_L^{-1}(p,T)$, we get

$$f(T)g(T) = \frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)\,dp_I\,f(p)g(p) = (fg)(T).$$

□

Corollary 5.18. *Let $T \in \mathcal{B}_R(V)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $f^{-1} \in \mathcal{N}(\sigma_S(T))$, then $f(T)$ is invertible and $f(T)^{-1} = f^{-1}(T)$.*

Proof. Let U be a T-admissible slice domain such that f and f^{-1} are left slice regular on $\overline{U}$. The product rule implies

$$\mathcal{I} = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, 1 = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s)f^{-1}(s) =$$
$$= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s)\, \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f^{-1}(s) = f(T)f^{-1}(T)$$

and

$$\mathcal{I} = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T) ds_I 1 = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f^{-1}(s)f(s) =$$
$$= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f^{-1}(s)\, \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, f(s) = f^{-1}(T)f(T).$$

□

Finally, also the existence of Riesz-projectors onto invariant subspaces generalizes to the case of the S-functional calculus.

Theorem 5.19. *Let $T \in \mathcal{B}_R(V)$ and let $\sigma_S(T) = \sigma_{S,1}(T) \cup \sigma_{S,2}(T)$ with $\operatorname{dist}(\sigma_{S,1}(T), \sigma_{S,2}(T)) > 0$. Let O_1 and O_2 be two axially symmetric, bounded open sets with $\overline{O_1} \cap \overline{O_2} = \emptyset$ such that $\sigma_{S,1}(T) \subset O_1$ and $\sigma_{S,2}(T) \subset O_2$, such that $\partial(O_j \cap \mathbb{C}_I)$ is the finite union of piecewise continuously differentiable Jordan curves for any $I \in \mathbb{S}$ and $j = 1, 2$. If we define*

$$P_j = \frac{1}{2\pi}\int_{\partial(O_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I, \qquad j = 1,2, \tag{5.10}$$
$$T_j = \frac{1}{2\pi}\int_{\partial(O_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, s, \qquad j = 1,2, \tag{5.11}$$

then P_j is a projector, that is $P_j^2 = P_j$, and satisfies $T_j = TP_j = P_jT$ for $j = 1, 2$.

Proof. Note that the functions $\mathbb{1}_{O_j}$ and $x\mathbb{1}_{O_j}$ satisfy the Representation formulas (3.5) and (3.7) although they are left and right slice regular only on an axially symmetric open set, but not necessarily on an axially symmetric slice domain. Following the lines of the proof of Theorem 3.81, we deduce easily that they satisfy the Cauchy integral formula. Hence, we can repeat the arguments in the proof of Theorem 5.10 to see that the integrals in (5.10) and (5.11) do neither depend on the open set O_j nor on the imaginary unit I, see also Remark 5.20. Moreover, we can repeat the arguments in Theorem 3.64 to approximate $\mathbb{1}_{O_j}$ and $x\mathbb{1}_{O_j}$ by real rational functions and, as in the proof of Theorem 5.12, we obtain

$$P_j = \frac{1}{2\pi}\int_{\partial(O_j\cap\mathbb{C}_I)} ds_I\, S_R^{-1}(s,T) = \frac{1}{2\pi}\int_{\partial(O_j\cap\mathbb{C}_I)} S_L^{-1}(p,T)\, dp_I. \tag{5.12}$$

and

$$T_j = \frac{1}{2\pi}\int_{\partial(O_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\ ds_I\, s = \frac{1}{2\pi}\int_{\partial(O_j\cap\mathbb{C}_I)} s\, ds_I\ S_R^{-1}(s,T). \tag{5.13}$$

The S-spectrum $\sigma_S(T) = \sigma_{S,1}(T) \cup \sigma_{S,2}(T)$ is compact and axially symmetric by Theorem 4.9 and Lemma 4.10. Since $\operatorname{dist}(\sigma_{S,1}(T), \sigma_{S,2}(T)) > 0$, the sets $\sigma_{S,1}(T)$ and $\sigma_{S,2}(T)$ are compact. If $x \in \sigma_{S,1}(T)$, then $\operatorname{dist}(\sigma_{S,1}(T), \sigma_{S,2}(T)) > 0$ implies $[x] \cap \sigma_{S,2}(T) = \emptyset$, and in turn $[x] \subset \sigma_{S,1}(T)$, because $[x]$ is connected. Hence, $\sigma_{S,1}(T)$ is axially symmetric. Similarly, $\sigma_{S,2}(T)$ is axially symmetric.

Let us fix $j \in \{1, 2\}$ and let G_p and G_s be two axially symmetric open sets such that $\sigma_{S,j}(T) \subset G_p$, $\overline{G_p} \subset G_s$ and $\overline{G_s} \subset U_j$ and such that $\partial(G_p \cap \mathbb{C}_I)$ an $\partial(G_s \cap \mathbb{C}_I)$ consist of a finite union of continuously differentiable Jordan curves. We can, for instance, apply Lemma 5.7 with $K = \sigma_{S,j}(T)$ and $O = O_j$ to obtain G_s and then apply Lemma 5.7 again with $K = \sigma_{S,j}(T)$ and $O = G_s$ to obtain G_p. The subscripts p and s are again chosen in order to indicate the respective variable of integration in following computation.

By (5.12), we can consider P_j^2 written as

$$P_j^2 = \frac{1}{2\pi}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\partial(G_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)\,dp_I = \\ = \frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} S_R^{-1}(s,T)S_L^{-1}(p,T)\,dp_I.$$

Applying (4.8), we obtain

$$\begin{aligned} P_j^2 =& \frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} S_R^{-1}(s,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I- \\ &- \frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I \\ &- \frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} \overline{s}S_R^{-1}(s,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I+ \\ &+ \frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} \overline{s}S_L^{-1}(p,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I. \end{aligned}$$

Hereby,

$$\frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I\, S_R^{-1}(s,T)\int_{\partial(G_p\cap\mathbb{C}_I)} p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I\ = 0$$

and

$$-\frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I\,\overline{s}S_R^{-1}(s,T)\int_{\partial(G_p\cap\mathbb{C}_I)} (p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I\ = 0$$

by the complex Cauchy theorem, because $\overline{G_p}\subset G_s$ and the functions $p\mapsto p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}$ and $p\mapsto(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}$ are holomorphic on $\overline{G_p\cap\mathbb{C}_I}$ for $s\in\partial G_s$. We obtain

$$\begin{aligned} P_j^2 =& -\frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)p(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I \\ &+ \frac{1}{(2\pi)^2}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I \int_{\partial(G_p\cap\mathbb{C}_I)} \overline{s}S_L^{-1}(p,T)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\,dp_I\,, \\ =& \frac{1}{2\pi}\int_{\partial(G_p\cap\mathbb{C}_I)} \left[\frac{1}{2\pi}\int_{\partial(G_s\cap\mathbb{C}_I)} ds_I\,(\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p)(p^2-2\mathrm{Re}[s]p+|s|^2)^{-1}\right] dp_I. \end{aligned}$$

Finally, since $\partial G_p\subset G_s$, we can apply Lemma 5.16 with $B=S_L^{-1}(p,T)$ and $f\equiv 1$ and we get

$$P_j^2 = \frac{1}{2\pi}\int_{\partial(G_p\cap\mathbb{C}_I)} S_L^{-1}(p,T)\,dp_I\ = P_j.$$

In order to prove $T_j = TP_j = P_jT$, we apply the left S-resolvent equation (4.5) to obtain

$$\begin{aligned} TP_j =& \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} TS_L^{-1}(s,T)\,ds_I = \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} [S_L^{-1}(s,T)\ s-\mathcal{I}]\,ds_I = \\ =& \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,s\,ds_I - \mathcal{I}\,\frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} ds_I = \\ =& \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,s = T_j. \end{aligned}$$

On the other hand, by (5.12) and (5.13), the operators P_j and T_j also allow an integral representation based on the right S-resolvent operator. The right S-resolvent equation (4.6) yields

$$\begin{aligned} P_jT &= \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} ds_I\, S_R^{-1}(s,T)T = \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} ds_I\,[s\,S_R^{-1}(s,T)-\mathcal{I}] = \\ &= \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} ds_I\, s\, S_R^{-1}(s,T) - \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} ds_I\,\mathcal{I} = \\ &= \frac{1}{2\pi}\int_{\partial(U_j\cap\mathbb{C}_I)} s\, ds_I S_R^{-1}(s,T) = T_j. \end{aligned}$$

Hence, $P_jT = T_j = TP_j$.

□

Remark 5.20. We recall the proof of the existence of Riesz projectors in the complex case. Let us assume that the spectrum $\sigma(T)$ of an operator T on a complex Banach space satisfies $\sigma(T) = \sigma_1(T)\cup\sigma_2(T)$ with $\operatorname{dist}(\sigma_1(T),\sigma_2(T)) > 0$. Then we can apply the functional calculus to the indicator functions and define $P_i = \mathbb{1}_{U_i}, i = 1,2$, for two open sets U_1 and U_2 with $\sigma_i(T)\subset U_i, i = 1,2$, and $\operatorname{dist}(U_1,U_2) > 0$. The properties $P_i^2 = P_i$ and $P_iT = TP_i$ follow immediately from the product rule as we have seen in (1.5) and (1.6).

This approach is not possible in the case of the S-functional calculus because it is only defined for functions that are slice regular on an axially symmetric *slice domain* . Hence, the proof of the existence of Riesz-projectors is a lot more complicated in the quaternionic setting.

One may wonder whether it is possible to enlarge the class of functions that are admissible for a quaternionic right linear operator T, such that it also contains functions that are defined on more general sets. Due to the fact that the S-spectrum is axially symmetric, it is clear that there is no sense in weakening the condition that the domain of definition of an admissible function must be axially symmetric. On the other hand, Theorem 3.8, the Identity Principle, holds only for functions, which are slice regular on a slice domain. This theorem is fundamental in the proof of Theorem 3.21, the Representation Formula, but it is actually the Representation Formula and not the Identity Principle, that is the crucial argument in the proofs of Theorem 3.81 and Theorem 5.10, the Cauchy Formula and the well definedness of the S-functional calculus.

In principle, one could therefore consider a notion of *strong slice regularity* that applies to functions that are slice regular on an arbitrary axially symmetric open set and satisfy the Representation Formula. Indeed, the Cauchy formula, and therefore also Theorem 5.10, hold true for these functions. Hence, if we plug a strongly slice regular function that is defined on an axially symmetric open set U, which is not necessarily a slice domain, into (5.5) or (5.6), the formulas, which define the S-functional calculus, then the integral still does neither depend on the axially symmetric open set U nor on the choice of the imaginary unit.

If f is strongly left and right slice regular on an axially symmetric open set O and satisfies $f(O\cap\mathbb{C}_I)\subset\mathbb{C}_I$ for all $I\in\mathbb{S}$, that is $f\in\mathcal{N}^s(O)$, then we can approximate it by real rational functions and obtain that (5.5) and (5.6), the formulations of the S-functional calculus for strongly left and right slice regular functions, yield the same operator $f(T)$. However, since Lemma 3.42 is based on the Identity Principle, it does not hold for strongly slice regular functions. Thus, for an arbitrary strongly left and right slice regular function, we do not obtain a decomposition of the form $f = \tilde{f} + a$ with $\tilde{f}\in\mathcal{N}^s(U)$ and $a\in\mathbb{H}$ as in Lemma 3.42. Hence, we cannot follow the proof of Lemma 5.14 and it is not clear that both formulations of the S-functional calculus yield the same operator.

If $\sigma_S(T) = \sigma_{S,1}(T)\cup\sigma_{S,2}(T)$ with $\operatorname{dist}(\sigma_{S,1}(T),\sigma_{S,2}(T)) > 0$, then we apply Lemma 5.7 to obtain axially symmetric open sets U_i with $\sigma_{S,i}(T)\subset U_i$ for $i = 1,2$ and $\overline{U_1}\cap\overline{U_2} = \emptyset$ such that $\partial(U_1\cap\mathbb{C}_I)$ and $\partial(U_2\cap\mathbb{C}_I)$ consist of a finite union of continuously differentiable Jordan curves for any $I\in\mathbb{S}$. Obviously, $\mathbb{1}_{U_i}(T) = P_i$ for $i = 1,2$. But for the function $a\mathbb{1}_{U_i}$ with $a\in\mathbb{H}\setminus\mathbb{R}$, we get

$$a\mathbb{1}_{U_i}(T) = \frac{1}{2\pi}\int_{\partial(U_i\cap\mathbb{C}_I)} S_L^{-1}(s,T)\, ds_I\, a = P_i a$$

if we consider it as a strongly left slice regular function and

$$a\mathbb{1}_{U_i}(T) = \frac{1}{2\pi}\int_{\partial(U_i\cap\mathbb{C}_I)} a\, ds_I S_R^{-1}(s,T) = aP_i$$

if we consider it as a strongly right slice regular function. In contrast to the case of the identity operator $\mathcal{I}$, it is not clear that the projections P_i commute with the scalar a. Indeed, since P_i is quaternionic right linear, the invariant subspace $V_i = P_i(V)$ is a right linear subspace of V. But if P_i commutes with any scalar $a \in \mathbb{H}$, then

$$a\boldsymbol{v} = aP_i\boldsymbol{v} = P_ia\boldsymbol{v} \in V_i,$$

for $\boldsymbol{v} \in V_i$. Hence, V_i is also a left linear, and therefore a two-sided subspace of V.

We see that, in the case of strongly slice regular functions, which are not necessarily defined on a slice domain, the equality of both of formulation of the S-functional calculus is not immediate at all. If it holds true, it will allow new insights into the structure of quaternionic linear operators and it will allow to generalize further properties of the Riesz-Dunford-functional calculus. The question whether and how the S-functional calculus can be extended to functions that are not necessarily defined on a slice domain is currently under investigation.

5.3 The Spectral Mapping Theorem

We conclude with the Spectral Mapping Theorem in the slice regular case and two important consequences. However, in contrast to the complex case, the Spectral Mapping Theorem does not hold for arbitrary left or right slice regular functions. In fact, it is clear that it can only be true for slice regular functions that preserve the fundamental geometry of the S-spectrum, that is, the axially symmetry. The functions, that preserve this property are exactly the functions in $\mathcal{N}(\sigma_S(T))$.

Theorem 5.21 (The Spectral Mapping Theorem)**.** *Let $T \in \mathcal{B}_R(V)$ and let $f \in \mathcal{N}(\sigma_S(T))$. Then*

$$\sigma_S(f(T)) = f(\sigma_S(T)) = \{f(s) : s \in \sigma_S(T)\}.$$

Proof. Let U be a T-admissible open slice domain such that $f \in \mathcal{N}(\overline{U})$, let U' be an axially symmetric slice domain with $\overline{U} \subset U'$ and $f \in \mathcal{N}(U')$ and let $s = s_0 + I_s s_1 \in \sigma_S(T)$. For $x \in U' \setminus [s]$, we define

$$\widetilde{g}(x) = (x^2 - 2\mathrm{Re}[s]x + |s|^2)^{-1}(f(x)^2 - 2\mathrm{Re}[f(s)]f(x) + |f(s)|^2).$$

By Theorem 3.62 and Corollary 3.40, the function $\widetilde{g}$ belongs to $\mathcal{N}(U') \setminus [s]$. Moreover, we can extend $\widetilde{g}$ to a function $g \in \mathcal{N}(U')$. Indeed, if $s \notin \mathbb{R}$ and $I \in \mathbb{S}$, then the function $\widetilde{g}_I$ has the singularities $s_I = s_0 + Is_1$ and $\overline{s_I} = s_0 - Is_1$ in $U \cap \mathbb{C}_I$. Moreover, if we write $f(x) = \alpha(x_0, x_1) + I\beta(x_0, x_1)$ according to Corollary 3.22, then α and β are real valued by Corollary 3.40. Hence, $\mathrm{Re}f(s) = \alpha(s_0, s_1) = \mathrm{Re}f(s_I)$ and $|f(s)|^2 = |\alpha(s_0, s_1)|^2 + |\beta(s_0, s_1)|^2 = |f(s_I)|^2$ for any $I \in \mathbb{S}$. Therefore,

$$\begin{aligned}\lim_{z\to s_I} \widetilde{g}_I(z) &= \lim_{z\to s_I} (z^2 - 2\mathrm{Re}[s_I]z + |s_I|^2)^{-1}(f(z)^2 - 2\mathrm{Re}[f(s_I)]f(z) + |f(s_I)|^2) = \\ &= \lim_{z\to s_I} \frac{(f(z) - f(s_I))(f(z) - \overline{f(s_I)})}{(z - s_I)(z - \overline{s_I})} = f_I'(s_I)\frac{f(s) - \overline{f(s_I)}}{s_I - \overline{s_I}}.\end{aligned}$$

As $f(\overline{s_I}) = \overline{f(s_I)}$ by Corollary 3.41, we also have

$$\begin{aligned}\lim_{z\to \overline{s_I}} \widetilde{g}_I(z) &= \lim_{z\to \overline{s_I}} (z^2 - 2\mathrm{Re}[s_I]z + |s_I|^2)^{-1}(f(z)^2 - 2\mathrm{Re}[f(s_I)]f(z) + |f(s_I)|^2) = \\ &= \lim_{z\to \overline{s_I}} \frac{(f(z) - \overline{f(s_I)})(f(z) - f(s_I))}{(z - \overline{s_I})(z - s_I)} = f_I'(\overline{s_I})\frac{f(\overline{s_I}) - \overline{f(\overline{s_I})}}{\overline{s_I} - s_I}.\end{aligned}$$

Thus, s_I and $\overline{s_I}$ are removable singularities of $\widetilde{g}_I$ and since $\overline{s_I} = s_{-I}$, the function

$$g(x) = \begin{cases} \widetilde{g}(x) & \text{if } x \in U' \setminus [s], \\ \frac{\partial f}{\partial s}(x)\frac{f(x) - \overline{f(x)}}{x - \overline{x}} & \text{if } x \in [s] \end{cases}$$

is well defined. Obviously, its restriction g_I to the plane $\mathbb{C}_I$ is holomorphic and satisfies $g_I(U' \cap \mathbb{C}_I) \subset \mathbb{C}_I$ for any $I \in \mathbb{S}$. Moreover, g is real differentiable at any point $x \in U' \setminus [s]$. It satisfies the Representation Formula (3.5)

$$g(x_0 + I_x x_1) = \frac{1}{2}(1 - I_x I)g_I(x_0 + Ix_1) + \frac{1}{2}(1 + I_x I)g_I(x_0 - Ix_1)$$

for $x \in U' \setminus [s]$ and by continuity even for $x \in [s]$. The function $g_I(x_0 + Ix_1)$ is a real differentiable function of x_0 and x_1 and $x \mapsto I_x = \underline{x}/|\underline{x}|$ is real differentiable for $x \notin \mathbb{R}$, where $\underline{x}$ is the vector part of x. Hence, g is also real differentiable at any point $x \in [s]$. Therefore, it is left slice regular and belongs to $\mathcal{N}(U')$.

If on the other hand $s \in \mathbb{R}$, then the point s is the only singularity of the function $\widetilde{g}_I$ for any $I \in \mathbb{S}$. Since $\overline{f(s)} = f(\overline{s}) = f(s)$ by Corollary 3.41, we also have $f(s) \in \mathbb{R}$. Hence, $\mathrm{Re}[s] = s$ and $\mathrm{Re}[f(s)] = f(s)$, which implies

$$\lim_{z\to s} \widetilde{g}_I(z) = \lim_{z\to s_I} (z^2 - 2sz + s^2)^{-1}(f(z)^2 - 2f(s)f(z) + f(s)^2) =$$

$$= \lim_{z\to\infty} \left(\frac{f(z) - f(s)}{z - s} \right)^2 = (f_I'(s))^2 = \left(\frac{\partial}{\partial s} f(s)) \right)^2 .$$

Therefore, the singularity s of $\widetilde{g}_I$ is removable for any $I \in \mathbb{S}$. Since $\left(\frac{\partial}{\partial s} f(s)\right)^2$ does not depend on the imaginary unit I, the function

$$g(x) = \begin{cases} \widetilde{g}(x) & \text{if } x \in U' \setminus \{s\}, \\ \left(\frac{\partial}{\partial s} f(s)\right)^2 & \text{if } x = s \end{cases}$$

is well defined. It is real differentiable on $U' \setminus \{s\}$ and any restriction g_I of g to a complex plane $\mathbb{C}_I$ is holomorphic on $U' \cap \mathbb{C}_I$ and satisfies $g_I(U' \cap \mathbb{C}_I) \subset \mathbb{C}_I$. Since U' is open, there exists an open ball $B_r(s)$ with $\overline{B_r(s)} \subset U'$. The Taylor series expansion $g_I(s + h) = \sum_{n=0}^{\infty} \frac{1}{n!} g_I^{(n)}(s)\, h^n$ of g_I at s converges absolutely and uniformly for $h \in \overline{B_r(0)} \cap \mathbb{C}_I$.

Any restriction g_I of g to a complex plane $\mathbb{C}_I$ is holomorphic and therefore infinitely differentiable as a function of two real variables. Consequently, g is infinitely differentiable at s with respect to x_0 because $g|_{U'\cap\mathbb{R}} = g_I|_{U'\cap\mathbb{R}}$. For arbitrary $h = h_0 + I_h h_1 \in \mathbb{H}$ with $|h| \leq r$, the equality $g_{I_h}^{(n)}(s) = \frac{\partial^n}{\partial x_0^n} g_{I_h}(s) = \frac{\partial^n}{\partial x_0^n} g(s)$ implies

$$g(s + h) = g_{I_h}(s + h) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} g(s)\, h^n = g(s) + \frac{\partial}{\partial x_0} g(s)\, h + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} g(s)\, h^n,$$

where the uniform convergence of the series implies

$$\lim_{h\to 0} \left| \frac{1}{|h|} \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} g(s) h^n \right| \leq \lim_{h\to 0} \sum_{n=2}^{\infty} \left| \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} g(s) \right| |h|^{n-1} = \sum_{n=2}^{\infty} \lim_{h\to 0} \left| \frac{1}{n!} \frac{\partial^n}{\partial x_0^n} g(s) \right| |h|^{n-1} = 0.$$

Hence, the function g is also real differentiable at s because

$$g(s + h) = g(s) + \frac{\partial}{\partial x_0} g(s)\, h + o(|h|),$$

where the mapping $h \mapsto \frac{\partial}{\partial x_0} g(s)\, h$ is $\mathbb{R}$-linear. Therefore, it is left slice regular on U' and belongs to the class $\mathcal{N}(U')$.

The product rule implies

$$f(T)^2 - 2\mathrm{Re}[f(s)]f(T) + |f(s)|^2\mathcal{I} = (T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I})g(T).$$

Thus, if $f(T)^2 - 2\mathrm{Re}[f(s)]f(T) + |f(s)|^2\mathcal{I}$ were invertible, then $g(T)(f(T)^2 - 2\mathrm{Re}[f(s)]f(T) + |f(s)|^2\mathcal{I})^{-1}$ would be the inverse of $T^2 - 2\mathrm{Re}[s]T + |s|^2\mathcal{I}$. But this contradicts $s \in \sigma_S(T)$. Hence, $f(s) \in \sigma_S(f(T))$.

If on the other hand $s \notin f(\sigma_S(T))$, then we can define the function

$$h(x) = (f^2(x) - 2\mathrm{Re}[s]f(x) + |s|^2)^{-1} \in \mathcal{N}(\sigma_S(T)).$$

The set of singularities of h is exactly the set $\{x \in U' : f(x) \in [s]\}$.

Since $f \in \mathcal{N}(U)$, the component functions α and β obtained by Corollary 3.22 are real valued because of Corollary 3.40. Corollary 3.23 implies $f([x]) = [f(x)]$ for all $x \in U'$. Therefore, we have either $[p] \subset f(\sigma_S(T))$ or $[p] \cap f(\sigma_S(T)) = \emptyset$ for $p \in \mathbb{H}$ and $s \notin f(\sigma_S(T))$ implies $f(\sigma_s(T)) \cap [s] = \emptyset$. Thus, no point in $\sigma_S(T)$ is a singularity of h. Hence, h belongs to $\mathcal{N}(\sigma_S(T))$ and we can define $h(T)$. The product rule implies that $h(T)$ is the inverse of $f(T)^2 - 2\mathrm{Re}[s]f(T) + |s|^2$, that is, $s \notin \sigma_S(f(T))$.

□

The Spectral Mapping Theorem allows us to generalize the Spectral Radius Theorem, Theorem 1.7, to the slice regular case.

Definition 5.22. *Let $T \in \mathcal{B}_R(V)$. Then the S-spectral radius of T is defined to be the nonnegative real number*

$$r_S(T) = \sup\{|s| : s \in \sigma_S(T)\}.$$

Theorem 5.23. *For $T \in \mathcal{B}_R(V)$, we have*

$$r_S(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. By Theorem 4.9, we have that $r_S(T) \leq \|T\|$. Recall from Corollary 2.38 that $\mathcal{B}_R(V)$ is a complex Banach space, if we restrict the right scalar multiplication to a complex plane $\mathbb{C}_I$. Because of Lemma 4.7 and because of $\lim_{s\to\infty} S_L^{-1}(s,T) = 0$, the mapping $\tau_I : s \mapsto S_L^{-1}(s,T)$ is a Banach space-valued holomorphic function on $(\rho_s(T) \cap \mathbb{C}_I) \cup \{\infty\}$. We know that it allows the series representation $\tau_I(s) = \sum_{n=0}^{\infty} T^n s^{-1-n}$ for $\|T\| < |s|$. Since it is holomorphic not only on $\{s \in \mathbb{C}_I : \|T\| < s\}$ but even on $\{s \in \mathbb{C}_I : r_S(T) < s\}$, this series representation holds for $r_S(T) < |s|$. Thus, the series $S_L^{-1}(s,T) = \sum_{n=0}^{\infty} T^n s^{-1-n}$ converges with respect to the norm of $\mathcal{B}_R(V)$ not only for $\|T\| < |s|$ but even for any s with $r_S(T) < |s|$. In particular, $\|T^n s^{-n-1}\|, n \in \mathbb{N}_0$, is bounded for any s with $|s| > r_S(T)$.

Let $s \in \mathbb{H}$ with $|s| > r_S(T)$ and set

$$C_s = \sup_{n\in\mathbb{N}_0} \|T^n s^{-n-1}\| < \infty.$$

Then,

$$\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \frac{1}{|s|} = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} |s|^{-\frac{n+1}{n}} = \limsup_{n\to\infty} \|T^n s^{-n-1}\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} C_s^{\frac{1}{n}} = 1,$$

and hence, $\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq |s|$. Since s was arbitrary with $|s| > r_S(T)$, we obtain

$$\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq r_S(T).$$

Moreover, the Spectral Mapping Theorem, Theorem 5.21, implies $\sigma_S(T^n) = \sigma_S(T)^n$. By Theorem 4.9, we obtain

$$r_S(T)^n = \sup\{|s|^n : s \in \sigma_S(T)\} = \sup\{|s| : s \in \sigma_S(T^n)\} = r_S(T^n) \leq \|T^n\|.$$

for any $n \in \mathbb{N}$. Therefore, we get

$$r_S(T) \leq \liminf_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq r_S(T), \tag{5.14}$$

and in turn $r_S(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$, where (5.14) also implies the existence of the limit. □

Finally, the Spectral Mapping Theorem, Theorem 5.21 also allows us to generalize the composition rule corresponding to *(iii)* in Corollary 3.40.

Theorem 5.24. *Let $T \in \mathcal{B}_R(V)$, let U be a T-admissible slice domain and let $f \in \mathcal{N}(\overline{U})$. Moreover, let $g \in \mathcal{M}^L(\overline{W})$ or $g \in \mathcal{M}^R(\overline{W})$, where W is an axially symmetric slice domain with $\overline{f(U)} \subset W$. Then $g \in \mathcal{M}^L(\sigma_S(f(T)))$ resp. $g \in \mathcal{M}^R(\sigma_S(f(T)))$ and*

$$g(f(T)) = (g \circ f)(T).$$

Proof. Let $g \in \mathcal{M}^L(\overline{W})$. Since $\sigma_S(f(T)) = \sigma_S(f(T))$ by Theorem 5.21, we have $f(\sigma_S(T)) \subset f(U)$. By Corollary 3.40, we have $f(x_0 + Ix_1) = \alpha(x_0, x_1) + I\beta(x_0, x_1)$ where α and β are real-valued because $f \in \mathcal{N}(U)$. Hence, $f([x]) = [f(x)]$, and in turn $f(U)$ is axially symmetric. Therefore, we can assume W to be an $f(T)$-admissible slice domain. Otherwise, we can apply Lemma 5.5 with $K = \overline{f(T)}$ and $G = W$ and switch to an $f(T)$-admissible slice domain W' with $\overline{f(T)} \subset W'$ and $\overline{W'} \subset W$. Consequently, $g \in \mathcal{M}^L(\sigma_S(f(T)))$.

The mapping $s \mapsto S_L^{-1}(p, f(s))$ is left slice regular on $\{s : f(s) \notin [p]\} = \{s : p \notin [f(s)]\}$ by Corollary 3.40. Hence, $s \mapsto S_L^{-1}(p, f(s))$ is left slice regular on $\sigma_S(T)$ if $p \notin \sigma_S(f(T))$, because $f(\sigma_S(T)) = \sigma_S(f(T))$ by Theorem 5.21. By Corollary 5.15, Lemma 5.17 and Corollary 5.18, the S-functional calculus is compatible with algebraic operations, which implies

$$\begin{aligned} S_L^{-1}(p, f(T)) &= -(f(T)^2 - 2\mathrm{Re}[p]f(T) + |p|^2\mathcal{I})^{-1}(f(T) - \overline{p}\mathcal{I}) = \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,\left[-(f(s)^2 - 2\mathrm{Re}[p]f(s) + |p|^2)^{-1}(f(s) - \overline{p})\right] = \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_L^{-1}(p, f(s)). \end{aligned}$$

Therefore,

$$\begin{aligned} g(f(T)) &= \frac{1}{2\pi}\int_{\partial(W_p\cap\mathbb{C}_I)} S_L^{-1}(p, f(T))\,dp_I\,g(p) = \\ &= \frac{1}{2\pi}\int_{\partial(W_p\cap\mathbb{C}_I)} \left[\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_L^{-1}(p, f(s))\right] dp_I\,g(p) = \\ &= \frac{1}{2\pi}\int_{\partial(W_p\cap\mathbb{C}_I)} \left[\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,S_L^{-1}(p, f(s))\,dp_I\,g(p)\right], \end{aligned}$$

where $W_p = W$ and $U_s = U$ and where the subscripts s and p indicate the respective variable of integration. Since the integrand in the last integral is continuous on $\partial(W_p \cap \mathbb{C}_I) \times \partial(U_s \cap \mathbb{C}_I)$, we can apply Corollary 3.78 to change the order of integration and obtain

$$\begin{aligned} g(f(T)) &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,\left[\frac{1}{2\pi}\int_{\partial(W_p\cap\mathbb{C}_I)} S_L^{-1}(p, f(s))\,dp_I\,g(p)\right] = \\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,g(f(s)) = \\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} S_L^{-1}(s,T)\,ds_I\,(g\circ f)(s) = (g\circ f)(T). \end{aligned}$$

□

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