Masterarbeit

Abstract scattering theory and wave operators

ausgeführt am Institut für
Analysis und Scientific Computing
der Technischen Universität Wien

unter der Anleitung von
Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Harald Woracek

durch

Florent Barjon
137, rue Saint-Alban
42153 Riorges, Frankreich

Datum

Unterschrift

Preface

Scattering theory has two sides. One is the direct investigation of scattering for Schrödinger equations intensively developed in quantum physics. The other side is called abstract scattering theory and is developed in an operator-theoretic framework. In this frame the Schrödinger operator is replaced by an arbitrary selfadjoint operator. Abstract scattering methods rely mainly on the concept of wave operators and scattering operators.
The first objective of the present work is to give a systematic construction of abstract scattering by presenting conditions for the existence and completeness of wave operators. Once this is achieved, our aim is to obtain explicit formula for the scattering operator in the general case, similar to the one used in quantum mechanics.
We have divided this thesis in five chapters. After a brief introduction dedicated to physical motivations and definition of wave and scattering operators, we present in a second chapter preliminary facts about spectral theory and perturbation theory. The third chapter is dedicated to the study of abstract scattering in a single Hilbert space. There, we introduce simple wave operators and discuss their properties which we illustrate via practical examples taken from quantum scattering. In the fourth chapter, we set the stage for the extension of wave operators to scattering on two Hilbert spaces. In this view, we introduce identification operators in a very natural and physically relevant way known as algebraic scattering. The last chapter constitutes the main part of this thesis and is devoted to the study of wave operators in the frame of two-space scattering. After some general considerations, we focus on the investigation of existence and completeness of wave operators. We develop first time-dependent methods in which we deal essentially with time limits before turning to stationary techniques. Thanks to stationary representations, we derive in the very last part explicit formulas for scattering operators and scattering matrix.
Comprehensive accounts on abstract scattering theory in a single-volume book are rather rare in the litterature. Among the sources we used, the book [11] and the third volume of the course [14] cover the issue of simple scattering. The book [1] and the monograph [18] give a systematic presentation of two-space scattering.

I would like to thank Prof. Harald Woracek for his support and helpful suggestions.

Vienna, October 2009

Contents

1 Introduction

As an introduction we shall first consider the case of quantum scattering by the one-dimensional potential barrier. The time-dependent Schrödinger equation describing the system reads :

$$i\frac{\partial\varphi}{\partial t}(x,t) = H\varphi(x,t) \tag{1}$$

where H is the Hamiltonian operator of the system,

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$$

the reduced Planck's constant $\hbar$ and the mass m of the particle described by the wave function φ were arbitrarily set to 1. $V(x)$ is the scattering potential

$$V(x) = \begin{cases} V_0 \in \mathbb{R}^+, & |x| < a \\ 0, & |x| > a \end{cases}$$

Assuming we are looking for solutions $\varphi(x,t) = u(t)\psi(x)$, $\psi \in C^1(\mathbb{R})$, with $H\psi(x) = E\psi(x)$, $E \in \mathbb{R}^+$ representing the energy of the wave function, the time-independent Schrödinger equation reads :

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi(x) = E\psi(x) \tag{2}$$

We choose for instance to examine the case $E < V_0$ corresponding to a tunnelling effect. The general solution is given by a linear combination of fundamental solutions :

$$\psi(x) = \begin{cases} Ae^{-ikx} + Be^{ikx}, & x < -a \\ Ce^{-\kappa x} + De^{\kappa x}, & |x| < a \\ Fe^{-ikx} + Ge^{ikx}, & x > a \end{cases}$$

where we have introduced $k = \sqrt{E}$ and $\kappa = \sqrt{V_0 - E}$. The plane wave solutions in the domains $|x| > a$ are not normalisable - for instance, square integrable - but may be physically interpreted as a superposition of a right-oriented and left-oriented particle flux.

With the conditions at the rands issued from $\psi \in C^1(\mathbb{R})$:

$$\begin{cases} \lim\limits_{\substack{x\to a \\ x>a}} \psi(\pm x) = \lim\limits_{\substack{x\to a \\ x<a}} \psi(\pm x) \\ \lim\limits_{\substack{x\to a \\ x>a}} \psi'(\pm x) = \lim\limits_{\substack{x\to a \\ x<a}} \psi'(\pm x) \end{cases}$$

we are able to determine $M \in M^2(\mathbb{C})$ so that

$$\begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} F \\ G \end{pmatrix}$$

It appears natural to express the out-going amplitudes in term of the incoming ones with $S \in M^2(\mathbb{C})$

$$\begin{pmatrix} B \\ F \end{pmatrix} = S(E) \begin{pmatrix} A \\ G \end{pmatrix}.$$

We call $S(E)$ the *scattering matrix.* This S-matrix correlates the coordinate asymptotics of incoming and out-going particle flux. Here we have directly investigated the problem (2) not depending on time and have obtained explicit expression of S. This is the explicit approach of scattering theory carried out in quantum physics. There is an other approach to scattering theory which does not involve the solving of Schrödinger equations.
In this frame, we are interested in examining the time evolution of a particle coming in a region where it interacts with a pertubation potential and then leaves this region again. Considering that the potential is negligible out of the region, we shall expect that the particle looks asymptotically free for $t \to \pm\infty$.
Let be H_0 and H selfadjoint operators respectively in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively describing the free and interactive systems. As shown by Stone's theorem, they generate unitary evolution groups $U_0(t) = e^{-iH_0 t}$ and $U(t) = e^{-iHt}$.
Now we define a quantum state φ of the system $(H_0, \mathcal{H}_0)$ as one-dimensional subspace of $\mathcal{H}_0$. That is, a normalized quantum state is represented by a unique vector $\varphi \in \{u \in \mathcal{H}_0 : \|u\| = 1\}$ of the unit sphere of $\mathcal{H}_0$ up to phase factors. If φ solves (1) with the initial condition $\varphi(0) = f$, then $\varphi(t) = e^{-iHt} f$.
Further, we suppose there exists some linear bounded operator $J \in \mathcal{L}(\mathcal{H}_0, \mathcal{H})$ connecting the states of the free system $\mathcal{H}_0$ and the perturbed system $\mathcal{H}$. J is denoted as the identification operator.

That $\varphi(t) \in \mathcal{H}$ looks asymptotically free as $t \to \infty$ means that we expect to have :

$$\exists \varphi_0{}^+(t) \in \mathcal{H}_0 : \lim_{t\to\infty} \left\| \varphi(t) - J\varphi_0{}^+(t) \right\|_{\mathcal{H}} = 0, \quad \varphi_0{}^+(t) = e^{-iH_0 t} f_0{}^+ \qquad (3)$$

that is,

$$\lim_{t\to\infty} \left\| e^{-iHt} \big(f - e^{iHt} J e^{-iH_0 t} f_0{}^+ \big) \right\| = 0 \qquad (4)$$

$$\lim_{t\to\infty} \left\| f - e^{iHt} J e^{-iH_0 t} f_0{}^+ \right\| = 0 \qquad (5)$$

since e^{-iHt} is unitary.
It seems now meaningful to define the so-called *wave operator* $\Omega_+ : \mathcal{H}_0 \to \mathcal{H}$ to describe this behaviour :

$$\mathcal{D}(\Omega_+) = \left\{ \varphi \in \mathcal{H}_0 \colon \lim_{t\to\infty} e^{iHt} J e^{-iH_0 t} \varphi \text{ exists in} \mathcal{H} \right\}$$

$$\forall \varphi \in \mathcal{D}(\Omega_+), \quad \Omega_+ \varphi = \lim_{t\to\infty} e^{iHt} J e^{-iH_0 t} \varphi$$

That is, a state looks asymptotically free when its initial data is the image by Ω^+ of the initial data of some free state. Similarly, we define :

$$\mathcal{D}(\Omega_-) = \left\{ \varphi \in \mathcal{H}_0 \colon \lim_{t\to-\infty} e^{iHt} J e^{-iH_0 t} \varphi \text{ exists in} \mathcal{H} \right\}$$

$$\forall \varphi \in \mathcal{D}(\Omega_-), \quad \Omega_- \varphi = \lim_{t\to-\infty} e^{iHt} J e^{-iH_0 t} \varphi$$

Furthermore, it is convenient to consider the action of wave operators only onto certain subspaces of the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$. Let $f \in \mathcal{H}$ and let $\mu_f = (E_H(\cdot)f, f)$ be its associated spectral measure on the spectrum $\sigma(H)$. As the Lebesgue decomposition theorem provides a unique decomposition of μ_f in its absolutely continuous and singular parts, we gain the representation $\mathcal{H} = \mathcal{H}^{ac} \oplus \mathcal{H}^s$, where $\mathcal{H}^{ac}$ ($\mathcal{H}^s$) is the subspace of vectors whose spectral measures are absolutely continuous (singular) with respect to the Lebesgue measure. Physically, the singular subspaces shall be interpreted as the set of bound or physically irrelevant states whereas the absolutely continuous subspaces relates to the scattering states. One can convince oneself of this fact by refining the Lebesgue decomposition of the singular part $\mu_s = \mu_{sc} + \mu_{pp}$ into a singular continuous and a pure point part, whose associated subspace $\mathcal{H}^{pp}$ corresponds obviously to quantificated bound states. Moreover, for practical purposes of scattering, the singular continuous part will often be absent. A really demonstrative picture of this is the hydrogen atom where the spectrum splits in a discrete negative part describing electron orbits and an absolutely continuous one $\sigma_{ac} = \big[0, \infty\big)$ related to ionized atom.
Consequently, we define P_0^{ac} the (orthogonal) projection onto the absolutely continuous subspace $\mathcal{H}_0^{ac}$. The proposition for the existence of wave operators now is

$$\forall \varphi \in \mathcal{D}(\Omega_\pm), \quad \Omega_\pm \varphi = \lim_{t \to \pm\infty} e^{iHt} J e^{-iH_0 t} P_0^{ac} \varphi$$

along with $Ran(\Omega_\pm) \subset \mathcal{H}^{ac}$. A further assumption is that each scattering state $f \in \mathcal{H}^{ac}$ looks asymptotically free, that is $Ran(\Omega_\pm) = \mathcal{H}^{ac}$. Then the wave operators are said to be complete.
Investigation of the existence and completeness of wave operators is the main content of scattering theory.

The other object of interest is the scattering operator

$$\mathcal{D}(S) = \Big\{ \varphi \in \mathcal{D}(\Omega_-) : \Omega_- \varphi \in Ran(\Omega_+) \Big\}$$

$$S = {\Omega_+}^* \Omega_-$$

which puts directly the ${f_0}^\pm$, namely the incoming and outcoming waves, in relation without computing the perturbed wave function.
The main issue for physics in the study of S is to obtain explicit formulas in order to determine its behaviour according to the energy of incoming wave functions. As there holds for wave operators a so-called intertwining property $H\Omega_\pm = \Omega_\pm H_0$, S and H_0 commute. The spectral theorem for selfadjoint operators provides through a unitary equivalence a representation of $\mathcal{H}_0^{ac}$ in which H_0 is diagonal and acts as a multiplication by the variable λ taking values in $\sigma_{ac}(H_0)$. As S commutes with H_0, S acts in this representation as multiplication by some function $S(\lambda)$. The scattering matrix $S(\lambda)$ can be interpreted as the projection of S onto an energy shell $h_k \subset \mathcal{H}_0^{ac}$ like in the example above.

2 Preliminaries

2.1 Measures

In scattering theory, we require a classification of the spectrum of selfadjoint operators based on properties of the spectral measure. We are interested in properties regarding Borel measures on $\mathbb{R}$.

2.1.1 Definitions

Borel sets on $\mathbb{R}$ are obtained by countable unions and intersections of open and closed sets and form the Borel-σ-algebra $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$. A non-negative countably additive set function μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is called a (Borel) measure μ. If for some Borel set Δ, $\mu(\Delta) = 0$ then Δ is called a μ-nullset. When needed one can add every subset of the μ-nullsets to the Borel-σ-algebra on which μ is defined, which forms a new σ-algebra. In this case, the extension of the measure is called a complete measure. The Lebesgue measure denoted by $|\cdot|$ is the completion of a Borel measure.
Any set X of full μ-measure, that is $\mu(\mathbb{R} \setminus X) = 0$, is called a Borel support. There exists a smallest closed set of full measure, called the support of μ and denoted by supp μ. The restriction of a measure μ_1 onto a Borel set X is denoted by $\mu_{1|X}(\Delta) = \mu_1(\Delta \cap X)$.
A measure μ is said to be finite if $\mu(X) < \infty$ for any Borel set X and σ-finite if each Borel set is a countable union of finite sets for μ. Borel measures are σ-finite.

2.1.2 Lebesgue decomposition

A measure μ_1 on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is called absolutely continuous with respect to the measure μ_2, in symbols $\mu_1 \ll \mu_2$, if for any Borel set Δ, $\mu_2(\Delta) = 0$ implies that $\mu_1(\Delta) = 0$. By the Radon-Nikodym theorem, μ_1 is absolutely continuous with respect to μ_2 if and only if there is a non-negative μ_2-measurable function f so that for Borel sets Δ

$$\mu_1(\Delta) = \int_{\Delta} f(\lambda) d\mu_2(\lambda) \tag{6}$$

f is called the Radon-Nikodym derivative, is uniquely defined μ_2-almost everywhere. If μ_1 is finite, then $f \in L_1(\mathbb{R}, d\mu_2)$. Two measures which are absolutely continuous with respect to one another are called equivalent. The equivalence class with respect to mutual absolute continuity in the set of measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is called the type of the measure. Types together with the relation $\ll$ build a complete distributive lattice in which every countable subset is bounded.

μ_1 and μ_2 are called mutually singular if they have disjoint Borel supports. The Lebesgue decomposition theorem states that given two measures μ_1 and μ_2, μ_1 can be uniquely represented in the form

$$\mu_1 = \mu_1^{ac} + \mu_1^{s} \tag{7}$$

where μ_1^{ac} is absolutely continuous with respect to μ_2 and μ_1^s singular with respect to μ_2. Unless explicitely stipulated otherwise, absolute continuity and singularity are understood to be with respect to the Lebesgue measure. The support of a singular measure is a nullset. A measure μ_1 is called continuous if $\mu_1(\{\lambda\}) = 0$ for any singleton. If $\mu_1(\{\lambda\}) > 0$, λ is called a pure point.

2.1.3 Generating functions

Let F be a left-continuous non-decreasing function on $\mathbb{R}$ and for each interval $[\lambda_1, \lambda_2)$ define the non-negative set function $\mu([\lambda_1, \lambda_2)) = F(\lambda_2) - F(\lambda_1)$ which extends to a Borel measure on $\mathbb{R}$ by countable additivity. Conversely, given a measure μ, take $F(\lambda) = \mu([-\infty, \lambda))$ so that $\mu([\lambda_1, \lambda_2)) = F(\lambda_2) - F(\lambda_1)$. F is called the generating function of μ and is uniquely defined up to an additive constant.
F is almost everywhere differentiable with derivate $f \in L_1^{loc}(\mathbb{R})$ almost everywhere defined and $\mu(\Delta) = \int_\Delta dF(\lambda) = \int_\Delta f(\lambda) d\lambda$. As the Lebesgue decomposition also gives

$$\mu(\Delta) = \int_\Delta \rho(\lambda) d\lambda + \mu^s(\Delta) \tag{8}$$

we have that f coincides a.e. with the Radon-Nikodym derivative of the absolutely continuous part of μ. The generating functions of μ and μ^{ac} coincide a.e. up to additive constants.

2.1.4 Borel supports

A Borel support of μ^{ac} can be characterized in terms of $f(\lambda)$. Consider $f(\lambda)$ as the symmetric derivative of the generating function $f(\lambda) = \lim_{\varepsilon \to \infty} (2\varepsilon)^{-1}(F(\lambda + \varepsilon) - F(\lambda - \varepsilon))$. Then the singular part of μ is concentrated on the Lebesgue nullset where $f(\lambda)$ exists and is infinite. Further the set $\Delta_e = \{\lambda,\ f(\lambda)$ does not exist$\}$ is a nullset by definition, that is $\mu^{ac}(\Delta_e) = 0$. For the set $\Delta_0 = \{\lambda,\ f(\lambda) = 0\}$, we have $\mu^{ac}(\Delta_0) = \int_{\Delta_0} f(\lambda) d\lambda = 0$. Then μ^{ac} is concentrated on the Borel support $X_a = \{\lambda,\ f(\lambda)$ exists and $f(\lambda) \neq 0\}$.

Given a measure μ, a Borel support X_m of μ is called minimal if for any other Borel support X of μ, there holds $\left|X_m \setminus X\right| = 0$. A minimal Borel support always exists and might not be unique: given X_m a minimal Borel support and $|X_0| = 0$, $X_m \cup X_0$ is a minimal Borel support.
In the special case of an absolutely continuous measure, one can even substract any Lebesgue nullsets from a minimal Borel support X_m^{ac} of μ^{ac} since $\mu^{ac}(X) = 0$ for $|X| = 0$. In the case of singular measures, minimal Borel supports are nullsets. Hence, a minimal Borel support for μ is also a minimal Borel support for μ^{ac}.
The support of a measure μ may not be a minimal Borel support for this measure since it has to be closed. Nevertheless, a minimal Borel support X_m can be chosen to belong to the support of μ. Then the support is the closure of X_m.

These results extend to signed measures (Hahn decomposition) and to complex-valued measures. This can also be extended to measures taking values on a separable Hilbert space $\mathcal{H}$. In particular for a measure μ taking values on $\mathcal{H}$, the Radon-Nikodym property still holds and strong and weak μ-measurability are equivalent.

2.2 Spectral measures

2.2.1 Definition

Let be H a closed possibly unbounded selfadjoint operator with dense domain of definition $\mathcal{D}(H)$ in a separable Hilbert space $\mathcal{H}$. The spectral theorem grants a one-to-one correspondance between H and a partition of the unity $E(\cdot)$ so that

1. $\mathcal{D}(H) = \{f, \int_{\mathbb{R}} \lambda^2 d(E(\lambda)f, f) < \infty\}$
2. $(Hf, g) = \int_{\mathbb{R}} \lambda d(E(\lambda)f, g)$

where $E(\lambda) = E((-\infty, \lambda))$ denotes the generating function of the projection-valued measure $E(\cdot)$. The support of $E(\cdot)$ coincides with the spectrum of H $\sigma(H) = \mathbb{R}\backslash\rho(H)$, where $\rho(H) = \{\lambda, H - \lambda I$ bijective mapping from $\mathcal{D}(H)$ to $\mathcal{H}\}$ is the resolvent set of H. For $z \in \rho(H)$, the inverse mapping $R(z) = (H - zI)^{-1}$ is defined on $\mathcal{H}$ and is bounded by the closed graph theorem.
$E(\cdot)$ is called the spectral measure of H. For any $f, g \in \mathcal{H}$, $\mu_f = (E(\cdot)f, f)$ defines a non-negative measure with support $\sigma(H)$. Note for a restriction, there holds $\mu_{f|X} = \mu_{E(X)f}$. By the polarization identity, $\mu_{f,g} = (E(\cdot)f, g) = 1/4(\mu_{f+g} - \mu_{f-g} + i\mu_{f+ig} - i\mu_{f-ig})$ is a complex-valued measure with same support.

2.2.2 Classification of spectra

Definition $f \in \mathcal{H}$ is called absolutely continuous (respectively singular) with respect to H if μ_f is absolutely continuous (respectively singular). The set of absolutely continuous (respectively singular) elements is denoted by $\mathcal{H}^{ac}$ (respectively $\mathcal{H}^s$).

Lemma 2.1 *For any $f \in \mathcal{H}^{ac}$ (resp. $\mathcal{H}^s$) and arbitrary $g \in \mathcal{H}$, the measure $\mu_{f,g}$ is absolutely continuous (resp. singular).*

Proof By the Schwarz inequality and $E(\Delta)^2 = E(\Delta)$, we have for any Borel set Δ

$$\left|\mu_{f,g}(\Delta)\right|^2 = \left|(E(\Delta)f, g)\right|^2 \leq (E(\Delta)f, f)(E(\Delta)g, g) \leq \mu_f(\Delta)\|g\|^2 \tag{9}$$

If $f \in \mathcal{H}^{ac}$, then for $|\Delta| = 0$, $\mu_f(\Delta) = 0$ then $\mu_{f,g}(\Delta) = 0$ and $\mu_{f,g}$ is absolutely continuous. Conversely if $f \in \mathcal{H}^s$, there is a set $|X| = 0$ so that $\mu_f(\mathbb{R} \setminus X) = 0$. Hence by (9), $\mu_{f,g}(\mathbb{R} \setminus X) = 0$ and $\mu_{f,g}$ is singular. $\square$

Proposition 2.2 *$\mathcal{H}^{ac}$ and $\mathcal{H}^s$ form orthogonal subspaces and $\mathcal{H}^{ac} \oplus \mathcal{H}^s = \mathcal{H}$.*

Proof $\mu_{f+g} = \mu_f + \mu_g + \mu_{f,g} + \mu_{g,f}$ imply together with the preceeding lemma that the sets $\mathcal{H}^{ac}$ and $\mathcal{H}^s$ are linear. Taking $f \in \mathcal{H}^{ac}$ and $g \in \mathcal{H}^s$, the same lemma states that $\mu_{f,g}$ is absolutely continuous and singular, that is identically

zero. In particular, $\mu_{f,g}(\mathbb{R}) = (f, g) = 0$. Then $\mathcal{H}^{ac}$ and $\mathcal{H}^s$ are orthogonal.
For any $\varphi \in \mathcal{H}$, $\mu_\varphi = \mu_\varphi^{ac} + \mu_\varphi^s$. Let be X_s a Borel support of μ_φ^s, then μ_φ^s can be represented as the restriction $\mu_{\varphi|X_s} = \mu_{E(X_s)\varphi}$. By unicity of the Lebesgue decomposition, $\mu_\varphi^{ac} = \mu_{\varphi|\mathbb{R}\setminus X_s} = \mu_{E(\mathbb{R}\setminus X)\varphi}$. Note $f = E(\mathbb{R} \setminus X_s)\varphi$ and $g = E(X_s)\varphi$. Then we have the wanted decomposition $\varphi = f + g$ and $\mathcal{H}^{ac} \oplus \mathcal{H}^s = \mathcal{H}$. □

A subspace X of $\mathcal{H}$ is called a reducing subspace for H if X and its orthogonal $X^\perp$ are invariant subspaces for H. From the representation of (Hf, g) provided by the spectral theorem, this is fulfilled if X is invariant for the spectral measure $E(\cdot)$. If X is a reducing subspace, then $X^\perp$ also reduces H.

Proposition 2.3 *$\mathcal{H}^{ac}$ and $\mathcal{H}^s$ reduce the operator H.*

Proof If $f \in \mathcal{H}^{ac}$, then $\mu_{E(\Delta)f} = \mu_{f|\Delta}$ is absolutely continuous and $E(\Delta)f \in \mathcal{H}^{ac}$ for any Borel set Δ. Then $\mathcal{H}^{ac}$ reduces H and so does $\mathcal{H}^s$. □

Definition We denote by P^{ac} and P^s the projections respectively onto the subspaces $\mathcal{H}^{ac}$ and $\mathcal{H}^s$. The parts $H^{ac} = P^{ac}H$ and $H^s = P^sH$ are called the absolutely continuous and singular parts of H. The spectrum of H^{ac} is called the absolutely continuous spectrum of H, denoted by $\sigma^{ac}(H)$. Similarily the singular spectrum $\sigma^s(H)$ is the spectrum of H^s.

The subspace spanned by the eigenvectors of H is called the spectrally discrete subspace, denoted by $\mathcal{H}^p$. Its projection is denoted by P^p. The spectrum σ^p of the operator $H^p = P^pH$ coincides with the closure of the set of eigenvalues of H.

A minimal Borel support of the spectral measure $E(\cdot)$ of H is called a spectral core of the operator H and is denoted by $\hat{\sigma}(H)$. The spectral core inherits the properties of minimal Borel supports. The spectral core of H is also a spectral core for the absolutely continuous part of $E(\cdot)$, that is, a spectral core for H^{ac}, denoted $\hat{\sigma}^{ac}$ (the converse may not hold). The closure of the spectral core of H coincides with the spectrum of H.
Adding Lebesgue nullsets to a spectral core of H gives a spectral core of H. Substracting Lebesgue nullsets to a spectral core of H^{ac} gives a spectral core of H^{ac}. Therefore, $\hat{\sigma}^{ac}$ is defined up to Lebesgue nullsets.

2.2.3 Properties of the absolutely continuous subspace

If there is a $f \in \mathcal{H}$ such that μ_f is a representant for a type $[\mu]$ of measures on the space $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, then the type $[\mu]$ is called a spectral type for $E(\cdot)$. μ_f or f are said to have spectral type. If $\mathcal{H}$ is separable, one can show that the set of spectral types is at most countable, then bounded. There is then a maximal spectral type, that is there exists $g \in \mathcal{H}$ such that $\mu_f \ll \mu_g$ for any $f \in \mathcal{H}$. For the measure μ_g, $\mu_g(X) = 0$ implies $\mu_f(X) = 0$ for any $f \in \mathcal{H}$, then $E(X) = 0$. That is, μ_g has the type of the projection-valued measure E.

Proposition 2.4 *P^{ac} is a spectral projection, that is $P^{ac} = E(X_{ac})$ for some Borel set X_{ac}.*

Proof In proposition 1.2, we had that $P^{ac}f = E(X(f))f$. Thanks to an element of maximal spectral type g, we can get rid of the dependance in f. Let X_s be a Borel support of μ_g^s. Then $|X_s| = 0$ and for any $f \in \mathcal{H}$, the measure $\mu_{f|X_s} = \mu_{E(X_s)f}$ is singular. Then $E(X_s)f$ is singular. Note $X_{ac} = \mathbb{R}\backslash X_s$, then $\mu_{g|X_{ac}} = \mu_g^{ac}$. Then for $|\Delta| = 0$, and any $f \in \mathcal{H}$, $\mu_{E(X_{ac}f)}(\Delta) = \mu_f(\Delta \cap X_{ac}) = 0$ since $\mu_g(\Delta \cap X_{ac}) = \mu_g^{ac}(\Delta) = O$ and $\mu_f \ll \mu_g$. Then $E(X_{ac})f$ is absolutely continuous. We can now choose $P^{ac} = E(X_{ac})$. $\square$

As the Borel support of μ_g^s is not unique, X_{ac} is not unique.

Let f and g be arbitrary. The generating function of $\mu_{f,g}$ is $\lambda \mapsto (E((-\infty, \lambda))f, g)$ abbreviated in $(E(\lambda)f, g)$. Since the measure $\mu_{f,g}$ is finite $\left(\left|\mu_{f,g}\right|(\mathbb{R}) \leq \|f\|\|g\|\right)$, the derivative $\frac{\mathrm{d}(E(\lambda)f,g)}{\mathrm{d}\lambda}$ exists a.e. and belongs to $L_1(\mathbb{R})$.
From Cauchy-Schwarz, we have $\left|(E(\Delta)f, g)\right|^2 = \left|(E(\Delta)f, g)\right|^2 \leq (E(\Delta)f, f)(E(\Delta)g, g)$ for any Borel set Δ which implies that there holds almost everywhere

$$\left|\frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda}\right|^2 \leq \frac{\mathrm{d}(E(\lambda)f, f)}{\mathrm{d}\lambda}\frac{\mathrm{d}(E(\lambda)g, g)}{\mathrm{d}\lambda} \tag{10}$$

The Lebesgue decomposition of $\mu_{f,g}$ is given by $\mu_{f,g} = \mu_{P^{ac}f,g} + \mu_{P^s f,g}$. Since the Radon-Nikodym derivative for $\mu_{f,g}^{ac}$ and the derivative of the generating function of $\mu_{f,g}$ coincide, there holds

$$\frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda} = \frac{\mathrm{d}(E(\lambda)P^{ac}f, P^{ac}g)}{\mathrm{d}\lambda} \tag{11}$$

for any $f, g \in \mathcal{H}$. Moreover

$$(P^{ac}f, g) = \int_{\mathbb{R}} \frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda} d\lambda \tag{12}$$

More generally, for any $f \in \mathcal{H}^{ac}$, we note that there holds for any Borel sets Δ, X

$$\begin{aligned}(E(\Delta)E(X)f, g) = \int_{\Delta \cap X} \frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda} d\lambda &= \int_X \gamma_\Delta(\lambda)\frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda} d\lambda \\ &= \int_X \frac{\mathrm{d}(E(\lambda)E(\Delta)f, g)}{\mathrm{d}\lambda} d\lambda\end{aligned}$$

then we have almost everywhere

$$\frac{\mathrm{d}(E(\lambda)E(\Delta)f, g)}{\mathrm{d}\lambda} = \gamma_\Delta(\lambda)\frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda} \tag{13}$$

2.2.4 Functional calculus

The spectral theorem implies a functional calculus for selfadjoint operators. Namely, if φ is measurable and essentially bounded with respect to the spectral measure, we can construct the function $\varphi(H)$ on the domain $\mathcal{D}(\varphi(H))$, dense in $\mathcal{H}$, defined by

$$\mathcal{D}(\varphi(H)) = \left\{\lambda \int_{\mathbb{R}} \left|\varphi(\lambda)\right|^2 d(E(\lambda)f, f)\right\}$$

$\varphi(H)$ is defined by its sesquilinear form on $\mathcal{D}(\varphi(H)) \times \mathcal{H}$ by

$$(\varphi(H)f, g) = \int_{\mathbb{R}} \varphi(\lambda) d(E(\lambda)f, g) \tag{14}$$

Note that for real φ, $\varphi(H)$ is selfadjoint; for bounded φ, $\varphi(H)$ is bounded and extends to $\mathcal{D}(\varphi(H)) = \mathcal{H}$. A function of special importance is $\varphi(\lambda) = e^{-i\lambda t}$ defining the unitary group $U(t) = e^{-iHt}$. For any $f, g \in \mathcal{H}$,

$$(U(t)f, g) = \int_{\mathbb{R}} e^{-i\lambda t} d(E(\lambda)f, g) \tag{15}$$

is the Fourier transform of the spectral measure $\mu_{f,g}$. Another important function is the resolvent $R(z) = (H - zI)^{-1}$. For any $f, g \in \mathcal{H}$,

$$(R(z)f, g) = \int_{\mathbb{R}} (\lambda - z)^{-1} d(E(\lambda)f, g) \tag{16}$$

If the spectrum of H has two gaps, the spectral measure between the gaps can be directly retrieved by means of the resolvent. Let be $\Lambda \subseteq \sigma(H)$ a bounded interval and Γ a closed contour enclosing one time counterclockwise Λ so that the intersection of the interior of Γ with $\sigma(H)$ is Λ (that is also $\Gamma \cap \sigma(H) = \emptyset$). Then we have the following formula called the Riesz projection

$$E(\Lambda) = -(2\pi i)^{-1} \int_{\Gamma} R(z) dz \tag{17}$$

The next lemma is essential for scattering theory.

Lemma 2.5 *The relation w-* $\lim_{t \to \pm\infty} U(t)P^{ac} = 0$ *is valid. Moreover, if K is a compact operator, then*

$$s - \lim_{t \to \pm\infty} KU(t)P^{ac} = 0 \tag{18}$$

Proof For any $f, g \in \mathcal{H}$

$$(U(t)P^{ac}f, g) = \int_{\mathbb{R}} e^{-i\lambda t} d(E(\lambda)P^{ac}f, g) = \int_{\mathbb{R}} e^{-i\lambda t} \frac{\mathrm{d}(E(\lambda)f, g)}{\mathrm{d}\lambda} d\lambda$$

since $\mu_{P^{ac}f,g}$ is absolutely continuous, with $\frac{\mathrm{d}(E(\lambda)f,g)}{\mathrm{d}\lambda} \in L_1(\mathbb{R})$. By the Riemann-Lebesgue lemma, $(U(t)P^{ac}f, g) \to 0$ as $t \to \pm\infty$. Compact operators turn weak convergent sequences to strong convergent ones. So $KU(t)P^{ac}f \to 0$ strongly as $t \to \pm\infty$. $\square$

2.3 Analytic properties of resolvents

2.3.1 Cauchy-Stieltjes transforms

Given a complex function of bounded variation F, its Cauchy-Stieltjes transform is defined by

$$\mathscr{C}(z) = \int_{\mathbb{R}} (\nu - z)^{-1} dF(\nu) \tag{19}$$

which is an analytic function on the upper (lower) half-plane. We are interested in the boundary values of $\mathscr{C}(z)$ as z tends to $\lambda \in \mathbb{R}$, more precisely we consider radial limit values $\lim_{\varepsilon \to 0} \mathscr{C}(\lambda \pm i\varepsilon)$. The key result in this investigation is the Priwalow theorem (see [13], p.139)

Theorem 2.6 *Given a complex function of bounded variation F, the radial limit values of $\mathscr{C}(z)$ exists for almost every $\lambda \in \mathbb{R}$*

$$\lim_{\varepsilon\to 0} \mathscr{C}(\lambda \pm i\varepsilon) = \pm\pi i F'(\lambda) + p.v. \int_{-\infty}^{\infty} (\nu-\lambda)^{-1} dF(\nu) \tag{20}$$

where the integral on the right-hand side exists also for almost every $\lambda \in \mathbb{R}$ and is understood as the Cauchy principal value

$$\lim_{\delta\to 0} \left(\int_{-\infty}^{\lambda-\delta} (\nu-\lambda)^{-1} dF(\nu) + \int_{\lambda+\delta}^{\infty} (\nu-\lambda)^{-1} dF(\nu) \right)$$

The function F is the generating function of a finite complex-valued measure μ which can be retrieved by means of the Cauchy-Stieltjes inversion formula: given an open interval Λ on the real line

$$\mu(\Lambda) + \mu(\overline{\Lambda}) = (\pi i)^{-1} \lim_{\varepsilon\to 0} \int_{\Lambda} \big(\mathscr{C}(\lambda+i\varepsilon) - \mathscr{C}(\lambda-i\varepsilon)\big)\, d\lambda \tag{21}$$

2.3.2 Boundary values of resolvents

The resolvents can be expressed in terms of the unitary evolution groups. For any $z \in \rho(H)$, $z = \lambda \pm i\varepsilon$, $\varepsilon > 0$, there holds

$$R(\lambda \pm i\varepsilon) = \pm i \int_0^{\infty} e^{-\varepsilon t \pm i\lambda t} U(\pm t) dt \tag{22}$$

Indeed we have for any $f, g \in \mathcal{H}$

$$\begin{aligned} \pm i \int_0^{\infty} e^{-\varepsilon t \pm i\lambda t} (U(\pm t) f, g) dt &= \pm i \int_0^{\infty} \int_{\mathbb{R}} e^{-\varepsilon t \pm i(\lambda-\nu)t} d(E(\nu) f, g) dt \\ = \pm i \int_{\mathbb{R}} \frac{-1}{\pm i(\lambda-\nu) - \varepsilon} d(E(\nu) f, g) &= \int_{\mathbb{R}} (\nu - (\lambda \pm i\varepsilon))^{-1} d(E(\nu) f, g) \end{aligned}$$

where the interchange in the order of integration is justified by Fubini's theorem since the integral absolutely converges to $-\varepsilon^{-1}(f, g)$.
Further, we define the operator on $\mathcal{H}$

$$\delta(\lambda, \varepsilon) = (2\pi i)^{-1} \big(R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon)\big) \tag{23}$$

By (23), we obtain for any $f, g \in \mathcal{H}$

$$\begin{aligned} (\delta(\lambda,\varepsilon) f, g) = (2\pi i)^{-1} \int_{\mathbb{R}} \frac{2i\varepsilon}{(\nu - (\lambda+i\varepsilon))(\nu-(\lambda-i\varepsilon))} d(E(\nu) u f, g) \\ = \pi^{-1} \varepsilon \int_{\mathbb{R}} \Big((\nu-\lambda)^2 + \varepsilon^2 \Big)^{-1} d(E(\nu) f, g) \end{aligned} \tag{24}$$

We note that if f or g is absolutely continuous, $\delta(\lambda, \varepsilon)$ is the Poisson integral of the function $\frac{\mathrm{d}(E(\nu)f,g)}{\mathrm{d}\nu}$. (24) implies that $(\delta(\lambda,\varepsilon) f, f) \geq 0$, that is $\delta(\lambda, \varepsilon)$ is a positive operator. By the functional calculus, (24) also implies that

$$\delta(\lambda, \varepsilon) = \pi^{-1} \varepsilon R(\lambda+i\varepsilon) R(\lambda-i\varepsilon) \tag{25}$$

This implies immediately that for any $f \in \mathcal{H}$

$$\pi^{-1}\varepsilon\|R(\lambda \pm i\varepsilon)f\|^2 = (\delta(\lambda,\varepsilon)f, f) \tag{26}$$

Note further

$$\int_{\mathbb{R}} (\delta(\lambda,\varepsilon)f, g)d\lambda = \pi^{-1}\varepsilon \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} \left((\nu-\lambda)^2+\varepsilon^2\right)^{-1} d\lambda}_{\varepsilon^{-1}\pi} d(E(\nu)f, g) = (f, g) \tag{27}$$

That is $\int_{\mathbb{R}} \delta(\lambda,\varepsilon)d\lambda = I$. The interchange of the integrals in (27) is justified by Fubini's theorem, since the integrand is positive.
By the formula (16), given the resolvent $R(z)$ of a selfadjoint operator H, $(R(z)f, g)$ is the Cauchy-Stieltjes transform of the finite complex-valued measure $\mu_{f,g}$. Then the results of the first section hold. That is, for almost every $\lambda \in \mathbb{R}$, there exists

$$\lim_{\varepsilon\to 0}(R(\lambda \pm i\varepsilon f, g) = \pm\pi i \frac{d(E(\lambda)f, g)}{d\lambda} + p.v. \int_{\mathbb{R}} (\nu-\lambda)^{-1} d(E(\nu)f, g) \tag{28}$$

and in particular, (23) implies that there exists for almost every $\lambda \in \mathbb{R}$

$$\lim_{\varepsilon\to 0}(\delta(\lambda,\varepsilon)f, g) = \frac{d(E(\lambda)f, g)}{d\lambda} \tag{29}$$

By the Priwalow theorem, the sets of full Lebesgue measure on which the limits (28), (29) are defined depend on $\mu_{f,g}$. Then the existence of these limits do not imply the existence of weak limits for $R(\lambda \pm i\varepsilon)$ or $\delta(\lambda,\varepsilon)$.
From the existence of the limit (29) and from the general form of $(\delta(\lambda,\varepsilon)f, g)$ given in (24), $(\delta(\lambda,\varepsilon)f, f)$ has to be uniformly bounded in ε. For a.e. $\lambda \in \mathbb{R}$, there exists $C(\lambda, f)$ such that for any $\varepsilon > 0$

$$(\delta(\lambda,\varepsilon)f, f) \leq C(\lambda, f) \tag{30}$$

Lastly, we note that the Cauchy-Stieltjes inversion formula applied to $(R(\lambda \pm i\varepsilon)f, g)$ provides for any $f, g \in \mathcal{H}$ and open interval $\Lambda \subseteq \mathbb{R}$

$$\lim_{\varepsilon\to 0}\int_{\Lambda} (\delta(\lambda,\varepsilon)f, g)d\lambda = 1/2\left((E(\Lambda)f, g) + (E(\overline{\Lambda})f, g)\right)$$

2.4 Representation as multiplication operators

2.4.1 Commutants

We denote by $com(\mathcal{T})$ the algebra of bounded operators on $\mathcal{H}$ commuting with a set of operators $\mathcal{T}$. If $\mathcal{T}$ is a set of bounded operators, then $com(\mathcal{T})$ is a von Neumann algebra. Moreover, given a von Neumann algebra $\mathfrak{A}$, there holds $\mathfrak{A} \subseteq com(\mathfrak{A})$ and $\mathfrak{A} = com(com(\mathfrak{A}))$ (von Neumann bicommutant theorem).

2.4.2 Direct integrals

Let be a measure space (X, Σ, μ) and a family of Hilbert spaces $\mathfrak{h}_\lambda$, endowed with the scalar products $\langle\cdot,\cdot\rangle_\lambda$ and norms $\|\cdot\|_\lambda$. We note $|||\cdot|||_\lambda$ for the operator norms. The direct integral of Hilbert spaces $\mathfrak{h}_\lambda$, denoted by $\mathfrak{h} = \int_X \bigoplus \mathfrak{h}_\lambda d\mu(\lambda)$

is the Hilbert space of vector-valued functions h: $\lambda \mapsto h(\lambda) \in \mathfrak{h}_\lambda$ that are μ-a.e. defined and square integrable, supplied with the scalar product $\int_X \langle \cdot, \cdot \rangle_\lambda \, d\mu(\lambda)$. We call $N(\lambda) = \dim \mathfrak{h}_\lambda$ the dimension function, taking values in $\mathbb{N} \cup \infty$ and we assume that the level sets of $N(\lambda)$ are μ-measurable.
For any vector $f \in \mathcal{H}$, we denote by $\tilde{f}(\lambda)$ its image under the unitary transformation in $\mathfrak{h}_\lambda$. Important operators on $\mathfrak{h}$ are the multiplication operators by essentially bounded scalar functions with respect to μ. When $\alpha \in L_\infty(X, \mu)$, we denote by M_α the operator defined by the equation

$$(M_\alpha f)(\lambda) = \alpha(\lambda) f(\lambda)$$

for any $f \in \mathcal{H}$. The set of these operators is a von Neumann algebra, denoted $\mathfrak{M}$.
Another set of importance is the set of operators A on $\mathfrak{h}$ defined by the equation

$$(Af)(\lambda) = A(\lambda) f(\lambda)$$

where $A(\cdot)$ is an operator-valued function almost everywhere defined with $|||A(\lambda)|||_\lambda$ essentially bounded with respect to μ. It can be shown that the set of all these operators A is a von Neumann algebra $\mathfrak{A}$ and moreover there holds $\mathfrak{A} = com(\mathfrak{M})$.

2.4.3 Direct integral representation

Given a selfadjoint operator H on the separable Hilbert space $\mathcal{H}$, $\mathcal{H}$ is unitarily equivalent to a direct integral $\mathfrak{h}$ of Hilbert spaces $\mathfrak{h}_\lambda$ on the measure space $(\mathbb{R}, \mathcal{B}_R, \mu)$ in which μ has the spectral type of $E(\cdot)$ and $N(\lambda)$ coincides with the multiplicity of the spectrum of H at the point λ (zero when $\lambda \notin \sigma(H)$). We note $m(H) = \mu\text{-}\operatorname{esssup}_{\lambda \in \mathbb{R}} \{N(\lambda)\}$ the multiplicity of the spectrum of H. In this representation, H is unitarily equivalent to the multiplication operator by the variable λ. For any Borel set Δ, $E(\Delta)$ is unitarily equivalent to multiplication by the characteristic function γ_Δ. In other words, the von Neumann algebra $\mathfrak{W}(E(\Delta), \Delta \in \mathcal{B}_\mathbb{R}\})$ generated by the spectral measure $E(\cdot)$ is unitarily equivalent to $\mathfrak{M}$. Since $com(com(H)) = \mathfrak{W}(E(\Delta))$ and then $com(H) \subseteq com(\mathfrak{W}(E(\Delta))$, then for any $T \in com(com(H))$, T is unitarily equivalent to an operator M_t of $\mathfrak{M}$ and for any $S \in com(H)$, S is unitarily equivalent to an operator $\tilde{S}$ of $\mathfrak{A}$.

In scattering theory, it is sufficient to have a direct integral representation for the absolutely continuous part H^{ac} of H. We construct this direct integral on the basis of the absolutely continuous part E^{ac} of the spectral measure $E(\cdot)$. We recall that the spectral core $\hat{\sigma}$ of H is also a spectral core of E^{ac}. Hence, for any Borel set Δ with $E^{ac}(\Delta) = 0$, we have $|\Delta \cap \hat{\sigma}| = 0$ since $\hat{\sigma}$ is a minimal Borel support. This means that E^{ac} has the spectral type of $|\cdot|_{\hat{\sigma}}$, the restriction of the Lebesgue measure on the set $\hat{\sigma}$. Then a possible direct integral representation of H^{ac} is

$$\mathfrak{h}^{ac} = \int_{\hat{\sigma}} \bigoplus \mathfrak{h}_\lambda d\lambda \tag{31}$$

Moreover we save the following lemma for later use.

Lemma 2.7 *Suppose that $a(\cdot)$ is an operator-valued function defined for a.e. $\lambda \in \hat{\sigma}$ with $a(\lambda)$ bounded operator on $\mathfrak{h}_\lambda$. If for any $u_1, u_2 \in \mathfrak{D}$ dense in $\mathcal{H}$*

$$\langle a(\lambda)\tilde{u}_1, \tilde{u}_2 \rangle = 0 \tag{32}$$

for almost every $\lambda \in \hat{\sigma}$, then $a(\lambda) = 0$ for a.e. $\lambda \in \hat{\sigma}$

2.4.4 Simple spectrum

H is said to have simple spectrum if $N(\lambda) = 1$ μ-a.e. Note that by identifying each of the spaces $\mathfrak{h}_\lambda$ to spaces $\mathfrak{h}_{N(\lambda)}$ and considering the sets $\sigma_k = \{\lambda \in \mathbb{R}, N(\lambda) = k\}$, the direct integral representation turns to

$$\bigoplus_{k=1}^{m(H)} L_2(\sigma_k, \mu, \mathfrak{h}_k)$$

by identifying $L_2(\sigma_k, \mu_k, \mathfrak{h}_k)$ with the set $\{h \in \mathfrak{h},\ h(\lambda) = 0\ \mu\text{-a.e. on } \mathbb{R} \setminus \sigma_k\}$. If H has simple spectrum, then H is unitarily equivalent to multiplication by λ in the space $L_2(\mathbb{R}, d\mu)$.
Moreover, since the $\mathfrak{h}_\lambda$ are one-dimensional, $\mathfrak{M} = \mathfrak{A} = com(\mathfrak{M})$. Then $com(H) = \mathfrak{W}(\{E(\Delta),\ \Delta \in \mathcal{B}_\mathbb{R}\})$. This leads to the following technical lemma

Lemma 2.8 *Let H have simple spectrum and P_1, P_2 be projections commuting with H. Then $P_1 = P_2$ iff $\hat{\sigma}(HP_1) = \hat{\sigma}(HP_2)$.*

Proof The indirect way is obvious. Conversely, let $\hat{\sigma}(HP_1) = \hat{\sigma}(HP_2) = \Delta_0$, Borel set. Since $com(H) = \mathfrak{W}(E(\Delta))$ and P_1, P_2 are projections, there are Borel sets Δ_1, Δ_2 such that $P_1 = E(\Delta_1))$ and $P_2 = E(\Delta_2)$. Obviously Δ_1 and Δ_2 can be taken respectively as spectral cores of the restrictions HP_1 and HP_2 since spectral cores are sets of full spectral measure. Then $P_1 = E(\Delta_0) = P_2$. □

2.5 Classes of compact operators

The spectrum of a compact operator consists of real eigenvalues; non zero eigenvalues have finite multiplicity.

2.5.1 Schmidt representation

Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let K be a compact operator from $\mathcal{H}$ to $\mathcal{K}$ with polar decomposition $A = UP$. A has a Schmidt representation $K = \sum_{n=1}^{\infty} s_n(K)(\cdot, \varphi_n)\psi_n$ where s_n are the eigenvalues of the positive operator $P = (K^*K)^{1/2}$ listed with account of multiplicity in decreasing order, φ_n are the corresponding eigenvectors and $\psi_n = U\varphi_n$. s_n are called the singular numbers of K. For any $1 \leq p < \infty$, we denote by $\mathfrak{G}_p$ the set of all compacts operators K for which $\sum_{n=1}^{\infty} s_n(K)^p < \infty$. The sets $\mathfrak{G}_p$ equipped with $\|\cdot\|_p = \left(\sum_{n=1}^{\infty} s_n(\cdot)^p\right)^{1/p}$ are Banach spaces. Note that for $K \in \mathfrak{G}_p$ and A, B bounded operators, $AKB \in \mathfrak{G}_p$ and $\|AKB\|_p \leq |||A||||||B|||\|K\|_p$. Moreover, if $K \in \mathfrak{G}_p$, then $K^* \in \mathfrak{G}_p$ too. If $p^{-1} + q^{-1} = 1$ and $A \in \mathfrak{G}_p$, $B \in \mathfrak{G}_q$, then $AB \in \mathfrak{G}_1$ and $\|AB\|_1 \leq \|A\|_p\|B\|_q$.

2.5.2 Hilbert-Schmidt operators

The set $\mathfrak{S}_2$ is called the set of Hilbert-Schmidt operators. For any orthogonal system $\{u_n\}$, $\|A\|_2^2 = \sum_{n=1}^{\infty} \|Au_n\|^2 < \infty$. $\mathfrak{S}_2$ can be equivalently defined as the set of operators A for which there exists an orthonormal system $\{u_n\}$ such that $\sum_{n=1}^{\infty} \|Au_n\|^2 < \infty$.
When the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are realized as spaces $L_2(X, \mu)$ and $L_2(Y, \nu)$, it can be shown that the Hilbert-Schmidt operators are the integral operators of the form

$$(Af)(y) = \int_X a(x, y) f(x) d\mu(x)$$

with a kernel $a(x, y)$ defined $\mu \times \nu$-a.e. on $X \times Y$ satisfying

$$\|A\|_2^2 = \int_X \int_Y \left|a(x, y)\right|^2 d\mu(x) d\nu(y) < \infty$$

2.5.3 Trace class operators

The set $\mathfrak{S}_1$ is called the set of trace class operators. For $A \in \mathfrak{S}_1$, one can define the function Tr which generalizes matrix traces by $\mathrm{Tr}\ A = \sum_{n=1}^{\infty} (Au_n, u_n)$ independent from the choice of the orthonormal system $\{u_n\}$. There always holds $\mathrm{Tr}\ A \le \|A\|_1$, the equality being obtained for non-negative A. More generally, given two orthonormal system $\{u_n\}$, $\{v_n\}$, there holds $\sum_{n=1}^{\infty} \left|(Au_n, v_n)\right| \le \|A\|_1$.
There are sufficient criteria for an operator A to be trace class. In particular, if for any orthonormal systems $\{u_n\}$, $\{v_n\}$, $\sum_{n=1}^{\infty} (Au_n, v_n)$ converges, then $A \in \mathfrak{S}_1$.

We have already seen that, given $A, B \in \mathfrak{S}_2$, $AB \in \mathfrak{S}_1$. Moreover, for any $K \in \mathfrak{S}_1$, there exists a factorization $K = AB$ with A, B Hilbert-Schmidt operators. In the polar decomposition $K = UP$, it suffices to take $A = UP^{1/2}$, $B = P^{1/2}$. Indeed, $s_n(P) = \lambda_n(P) = s_n(K)$ which implies $P \in \mathfrak{S}_1$. Note that $s_n(P) = s_n(P^{1/2})^2$ and $P^{1/2} \in \mathfrak{S}_2$. U being bounded, $A \in \mathfrak{S}_2$.

2.6 Conditions for selfadjointness

2.6.1 Relative boundedness

Definition Given two operators A: $\mathcal{H} \to \mathcal{K}$ and B: $\mathcal{H} \to \mathcal{K}'$, B is called relatively bounded with respect to A, or A-bounded, if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and there exist a, b non-negative constants such that for any $u \in \mathcal{D}(A)$

$$\|Bu\| \le a\|Au\| + b\|u\| \tag{33}$$

The smallest non-negative constant b such that (33) holds is called the A-bound of B.

We already note that if B is a bounded operator with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, then B is A-bounded with A-bound zero. Note that, since $R_A(z) = (A - zI)^{-1}$ is a bounded bijective mapping from $\mathcal{K}$ to $\mathcal{D}(A)$ for $z \in \rho(A)$, this definitions implies for any $f \in \mathcal{K}$

$$\big\|BR_A(z)f\big\| \leq a\big\|AR_A(z)f\big\|+b\big\|R_A(z)f\big\| \leq a\|f\|+(a|z|+b)\big\|R_A(z)f\big\| \leq C\|f\|$$

that is, $BR_A(z)$ are bounded operators from $\mathcal{K}$ to $\mathcal{K}'$.
Note further that, if A and B are closed operators, the inclusion $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ already implies that B is A-bounded. Since A is closed, the set $\mathcal{D}(A)$ equipped with the norm $\|u\|_A = \|u\|+\|Au\|$ turns to a Banach space and the restriction of B to $\mathcal{D}(A)$ can be seen as an operator $\tilde{B}$ from $\mathcal{D}(A)$ to $\mathcal{K}'$. Since $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and B is closed, $\tilde{B}$ is closed and everywhere defined on $\mathcal{D}(A)$, hence bounded by the closed graph theorem. That is, for any $u \in \mathcal{D}(A)$

$$\|Bu\| \leq C\|u\|_A \leq C\|Au\| + C\|u\|$$

and B is A-bounded. This implies in particular that if B is a bounded operator with domain of definition $\mathcal{H}$, then B is A-bounded for any closed operator A on $\mathcal{H}$.

Note further that the condition (33) can be equivalently expressed as

$$\|Bu\|^2 \leq a'^2\|u\|^2 + b'^2\|Au\|^2 \tag{34}$$

Clearly (34) implies (33) with $a = a'$ and $b = b'$. Conversely (33) implies

$$\begin{aligned}\|Bu\|^2 \leq a^2|u|^2+b^2|Au|^2+2ab\|u\|\|Au\| &= a^2(1+\varepsilon^{-2})\|u\|+b^2(1+\varepsilon^2)\|Au\|^2 \\ -(a\varepsilon^{-1}\|u\| - b\varepsilon\|Au\|)^2 &\leq a^2(1+\varepsilon^{-2})\|u\|^2 + b^2(1+\varepsilon^2)\|Au\|^2\end{aligned}$$

for arbitrary $\varepsilon \neq 0$. Then the A-bound can be equivalently taken as the smallest b' such that (34) holds.

2.6.2 Kato-Rellich theorem

In abstract scattering theory, criteria for existence and completeness of wave operators are given with respect to selfadjoint operators H and H_0 representing respectively the real and free Hamiltonians of a system. Conversely, in applications, we know how the free Hamiltonian H_0 looks like and we apply a perturbation V to it. In order to apply abstract results, we have to know under which conditions the sum $H_0 + V$ is a selfadjoint operator. One can show that relative bounded perturbation preserve the closedness. The next theorem states that relatively bounded symmetric perturbations preserve selfadjointness.

Theorem 2.9 *Let H_0 and V be operators on $\mathcal{H}$. Suppose that H_0 is selfadjoint and V is symmetric and H_0-bounded with H_0-bound smaller than 1, then H_0+V is also selfadjoint.*

Proof From the assumptions, $H_0 + V$ has domain $\mathcal{D}(T)$ and is closed symmetric. Since H_0 is symmetric, there holds for any $u \in \mathcal{D}(V)$

$$\|Vu\|^2 \leq a^2\|u\|^2 + b^2\|H_0u\|^2 = \big\|(bH_0 \pm ia)u\big\|^2$$

with $a > 0$ and $0 < b < 1$. Since $R_0(z)$ is a bijective mapping from $\mathcal{H}$ to $\mathcal{D}(H_0)$, we obtain that for any $f \in \mathcal{H}$ $\|VR_0(\pm ic)f\| \leq b\|f\|$, noting $c = a/b$. Hence, $VR_0(\pm ic)$ are bounded operators on $\mathcal{H}$ with $\||VR_0(\pm ic)|\| \leq b < 1$. Then we can define $(I + VR_0(\pm ic))^{-1}$ by the series $\sum (VR_0(\pm ic))^n$ as a bounded mapping on $\mathcal{H}$. Then $I + VR_0(\pm ic)$ maps bijectively $\mathcal{H}$ on itself. It remains to note that $(H_0 + V) \pm ic = (I + VR_0(\pm ic))(H_0 \mp ic)$. Since $Ran(H_0 \mp ic) = \mathcal{H}$, $Ran(H_0 + V \pm ic) = \mathcal{H}$, that is the deficiency indices of $H_0 + V$ vanish and $H_0 + V$ is selfadjoint. □

Similarily it can be shown that if H_0 is selfadjoint and bounded below, then given a relatively bounded symmetric perturbation V with H_0-bound smaller that 1, $H_0 + V$ is selfadjoint and bounded below.

2.6.3 Schrödinger operators with relatively bounded potentials

We recall that the Laplacian operator $H_0 = -\Delta$ on $\mathcal{H} = L_2(\mathbb{R}^3)$ is a selfadjoint operator when considered on the set $\mathcal{D}(H_0) = \{f \in L_2(\mathbb{R}^3, dx),\ |\xi|^2 \hat{f}(\xi) \in L_2(\mathbb{R}^3, d\xi)\}$ where we note $\mathcal{F}f = \hat{f}$ the Fourier transform of f. The domain of the Laplacian coincides with the Sobolev space $\mathcal{D}(H_0) = H^2(\mathbb{R}^3) = W^{2,2}(\mathbb{R}^3)$. Let V be a maximal multiplication operator, that is defined on $\mathcal{D}(V) = \{f \in L_2(\mathbb{R}^3),\ Vf \in L_2(\mathbb{R}^3)\}$, by the real-valued function $v = v_1 + v_2$ with $v_1 \in L_2(\mathbb{R}^3)$ and $v_2 \in L_\infty(\mathbb{R}^3)$. Since, for any $u, \tilde{u} \in \mathcal{D}(V)$, $|(Vu, \tilde{u})| \leq \|u\|_2 \|Vu\|_2$ and v is real, we have $(Vu, \tilde{u}) = (u, V\tilde{u})$ and V is symmetric.
For any $u \in \mathcal{D}(H_0)$, $\|v_2 u\|_2 \leq \|v_2\|_\infty \|u\|_2$ and by Cauchy-Schwarz inequality

$$\left(\int_{\mathbb{R}^3} |\hat{u}(\xi)|\, d\xi\right)^2 \leq \int_{\mathbb{R}^3} (|\xi|^2 + \alpha^2)^{-2} d\xi \int_{\mathbb{R}^3} (|\xi|^2 + \alpha^2)^2 |u(\xi)|^2\, d\xi$$
$$= \pi^2 \alpha^{-1} \left\|(H_0 + \alpha^2) u\right\|^2 < \infty \quad (35)$$

for an arbitrary $\alpha > 0$. Then

$$|u(x)| \leq (2\pi)^{-3/2} \int |\hat{u}(\xi)|\, d\xi \leq C\alpha^{-1/2} \left\|(H_0 + \alpha^2) u\right\|$$

which implies $\|u\|_\infty < \infty$. Then $\|v_1 u\|_2 \leq \|v_1\|_2 \|u\|_\infty$. Hence, $\mathcal{D}(H_0) \subseteq \mathcal{D}(V)$ and for any $u \in \mathcal{D}(H_0)$, (33) is fulfilled with $a = \|v_2\|_\infty + C\alpha^{3/2} \|v_1\|_2$ and $b = C\alpha^{-1/2} \|v_1\|_2$. Choose α large enough so that $b < 1$, then we can apply the Kato-Rellich theorem.
$H_0 + V$ is then selfadjoint. Moreover it is well-known that H_0 is bounded below with bound zero, then $H_0 + V$ is bounded below.

3 Simple scattering

We already motivated in the introduction the definition of the following object.

Definition The (strong) wave operator for a pair of selfadjoint operators H, H_0 respectively in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_0$ and a bounded identification operator $J: \mathcal{H}_0 \to \mathcal{H}$ is defined by :

$$\mathcal{D}(\Omega_\pm) = \left\{ f \in \mathcal{H}_0 \text{: s-}\lim_{t\to\pm\infty} U(-t)JU_0(t)P_0^{ac}f \;\text{ exists} \right\}$$

$$\Omega_\pm(H, H_0, J) = \text{s-}\lim_{t\to\pm\infty} U(-t)JU_0(t)P_0^{ac} \tag{36}$$

In this section, we consider $\mathcal{H}_0 = \mathcal{H}$. The spaces $\mathcal{H}^{ac}$ and $\mathcal{H}_0^{ac}$ are subspaces of the same Hilbert space $\mathcal{H}$. One-space scattering theory constitutes the first frame in which the concepts of wave-operators were developed, along with the study of scattering of a single particle by a potential in $\mathcal{H} = L_2(\mathbb{R}^3)$. In this frame, we first expect that J turns to the identity operator. The goal of this section is to underline that the necessity of considering non-trivial J quickly occurs. We illustrate this by considering more and more spread out potentials, for which the Coulomb potential constitutes a limit case. We note $\Omega_\pm(H, H_0; I) = \Omega_\pm(H, H_0)$. Moreover, the investigation of $\Omega_\pm(H, H_0)$ can also be motivated by the following fact.

3.1 Reduction to one-space problem

Let H, H_0 be two selfadjoint, respectively on $\mathcal{H}$ and $\mathcal{H}_0$, and $J: \mathcal{H}_0 \to \mathcal{H}$ be unitary. Then $\widetilde{H}_0 = JH_0J^*$ is a selfadjoint operator on $\mathcal{H}$ and we define $\widetilde{U}_0(t) = e^{-i\widetilde{H}_0 t}$.
We introduce the pre-wave operator $\Omega(H, H_0, J)(t) = U(-t)JU_0(t)$. As J is unitary,

$$\Omega(H, H_0, J)(t) = e^{iHt}Je^{-iH_0t}J^*J = e^{iHt}e^{-i\widetilde{H}_0t}J = \Omega(H, \widetilde{H}_0)(t)J \tag{37}$$

Thus, the wave operator $\Omega_\pm(H, H_0, J)$ exists iff $\Omega_\pm(H, \widetilde{H}_0)$ exists and $\Omega_\pm(H, H_0; J) = \Omega_\pm(H, \widetilde{H}_0)J$. The problem is reduced to one-space scattering on $\mathcal{H}$ with a new free Hamiltonian $\widetilde{H}_0$.

3.2 Properties

We assume here the wave operator $\Omega_\pm(H, H_0)$ exists and we collect some useful properties. When no confusion may occur, the dependence on H, H_0 and J of $\Omega_\pm$ may be omitted.

Proposition 3.1 $\Omega_\pm(H, H_0)$ *are partial isometries with initial subspace* $\mathcal{H}_0^{ac}$.

Proof For any $t \in \mathbb{R}$ and $f \in \mathcal{H}_0$ we have

$$\|\Omega(t)P^{ac}f\| = \|U(-t)U_0(t)P_0^{ac}f\| = \|P_0^{ac}f\|$$

which implies that $\|\Omega_\pm f\| = \|P_0^{ac}f\|$. Then $\Omega_\pm$ are partial isometries with initial subspace $\mathcal{H}_0$. □

Proposition 3.2 $\Omega_\pm(H, H_0)$ *are intertwining operators for* H *and* H_0*, that is*

$$H\Omega_\pm(H, H_0) = \Omega_\pm(H, H_0)H_0 \tag{38}$$

Proof

$$\begin{aligned}\forall f \in \mathcal{H}_0\colon\ \Omega_\pm f &= \lim_{t\to\infty} U(-t)U_0(t)P_0^{ac} f \\ &= \lim_{t\to\infty} U(s-t)U_0(t-s)P_0^{ac} f \\ &= U(s)\Omega_\pm U_0(-s) f\end{aligned}$$

that is $U(s)\Omega_\pm = \Omega_\pm U_0(s)$.
From Stone's theorem, we have

$$\begin{aligned}\forall f \in \mathcal{D}(H_0)\colon\ -i\Omega_\pm H_0 f &= \lim_{s\to 0} \frac{\Omega_\pm U_0(s)f - \Omega_\pm f}{s} \\ &= \lim_{s\to 0} \frac{U(s)\Omega_\pm f - \Omega_\pm f}{s} = -iH\Omega_\pm f\end{aligned}$$

Thus, $\Omega_\pm(H, H_0)H_0 = H\Omega_\pm(H, H_0)$. □

The intertwining property has important consequences. It implies first that $\Omega_\pm$ maps the domain of definition $\mathcal{D}(H_0)$ into $\mathcal{D}(H)$. Moreover, $\forall f \in Ran(\Omega_\pm)$, $\exists g \in \mathcal{H}_0$, $f = \Omega_\pm g$ and 38 imply that $Hf = \Omega_\pm H_0 g \in Ran(\Omega_\pm)$. Conversely, if $f \in \ker(\Omega_\pm^*)$, $\Omega_\pm^* Hf = H_0\Omega_\pm^* f = 0$. Then $Ran(\Omega_\pm)$ is a reducing subspace for H.
We obtain then an interesting spectral property. The restriction of $\Omega_\pm$ on $\mathcal{H}_0^{ac}$ is an isometry and then the mapping $\Omega_\pm\colon \mathcal{H}_0^{ac} \to Ran(\Omega_\pm)$ is unitary. Explicitely, the initial projection of $\Omega_\pm$ yields $\Omega_\pm^*\Omega_\pm = P_0^{ac}$ and $\Omega_\pm^*$ is a left inverse of $\Omega_\pm$ on $\mathcal{H}_0^{ac}$. Clearly for any $f \in Ran(\Omega_\pm)$

$$\exists g \in \mathcal{H}_0^{ac}\colon\ ||\Omega_\pm^* f|| = ||\Omega_\pm^*\Omega_\pm g|| = ||P_0^{ac} g|| \ = ||g|| = ||\Omega_\pm g|| = ||f||$$

$$\forall g \in \mathcal{H}_0^{ac}\colon\ g = P_0^{ac} g = \Omega_\pm^*\Omega_\pm g \in Ran(\Omega_\pm^*)$$

and the restriction of $\Omega_\pm^*$ on $Ran(\Omega_\pm)$ onto $\mathcal{H}_0^{ac}$ is an unitary mapping. We recall that $\mathcal{H}_0^{ac}$ is a reducing subspace for H_0. Let H_0^{ac} and H^+ be respectively the restrictions of H_0 on $\mathcal{H}_0^{ac}$ and of H on $Ran(\Omega_\pm)$. Then 38 implies

$$H_0^{ac} = \Omega_\pm^* H^+ \Omega_\pm \tag{39}$$

and H_0^{ac} and H^+ are unitarily equivalent. Hence, H^+ is absolutely continuous. Then, if we let H^{ac} be the absolutely continuous part of H, H^+ is a part of H^{ac}. This is equivalent to the fact that the domain of definition $Ran(\Omega_\pm)$ of H^+ is a subspace of $\mathcal{H}^{ac}$.

H^+ is maximal when $H^+ = H^{ac}$, which motivates the following definition, together with the introduction:

Definition Suppose that the wave operator $\Omega_\pm(H, H_0)$ exists. We say that $\Omega_\pm(H, H_0)$ is *complete* if $Ran(\Omega_\pm) = \mathcal{H}^{ac}$.

If $\Omega_\pm$ is complete, then $H^+ = H^{ac}$ and the absolutely continuous parts of H and H_0 are unitarily equivalent.

We now see that the direct investigation of the completeness of $\Omega_\pm$ is not necessary for practical purposes. We need the following lemma

Lemma 3.3 (Chain rule) *Let H, H_0, and H_1 be self-adjoint operators on $\mathcal{H}$. If the wave operators $\Omega_\pm(H, H_1)$ and $\Omega_\pm(H_1, H_0)$ exist, then $\Omega_\pm(H, H_0)$ exists and*

$$\Omega_\pm(H, H_0) = \Omega_\pm(H, H_1)\Omega_\pm(H_1, H_0) \tag{40}$$

Proof We write P_1^{ac} for the orthogonal projection on the subspace of absolute continuity of H_1 and $U_1(t) = e^{-iH_1 t}$ for the unitary group generated by H_1. We can rewrite the prewave operator $\Omega(H, H_0)(t)$ as

$$\begin{aligned} \forall t \in \mathbb{R}, \qquad \Omega(H, H_0)(t) &= U(-t)U_1(t)\left[P_1^{ac} + \left(I - P_1^{ac}\right)\right]U_1(-t)U_0(t)P_0^{ac} \\ &= \Omega(H, H_1)(t)\Omega(H_1, H_0)(t) + U(-t)U_1(t)\left(I - P_1^{ac}\right)\Omega(H_1, H_0)(t) \end{aligned}$$

If $\Omega_\pm(H_1, H_0)$ exists, then $Ran(\Omega_\pm(H_1, H_0)) \subseteq H_1^{ac}$, that is $P_1^{ac}\Omega_\pm(H_1, H_0) = \Omega_\pm(H_1, H_0)$. Hence, we have the strong limit

$$\mathrm{s} - \lim_{t\to\infty}(I - P_1^{ac})\Omega(H_1, H_0)(t) = 0$$

Moreover, under the assumptions of existence, $\Omega(H, H_1)(t)$ and $\Omega(H_1, H_0)(t)$ converge strongly respectively to $\Omega_\pm(H, H_1)$ and $\Omega(H_1, H_0)$. Therefore their products and subsequently $\Omega(H, H_0)(t)$ converge strongly to the products of their limits, which implies that $\Omega_\pm(H, H_0)$ exists and is equal to
$\Omega_\pm(H, H_1)\Omega_\pm(H_1, H_0)$. □

Proposition 3.4 *Suppose that $\Omega_\pm(H, H_0)$ exists. Then $\Omega_\pm(H, H_0)$ is complete iff $\Omega_\pm(H_0, H)$ exists.*

Proof Suppose that $\Omega_\pm(H, H_0)$ and $\Omega_\pm(H_0, H)$ exist. The chain rule implies

$$\Omega_\pm(H, H_0)\Omega_\pm(H_0, H) = \Omega_\pm(H, H) = P^{ac} \tag{41}$$

Then, $\forall f \in \mathcal{H}^{ac},\ f = P^{ac}f = \Omega_\pm(H, H_0)\Omega_\pm(H_0, H)f \in Ran(\Omega_\pm(H, H_0))$. That is $\mathcal{H}^{ac} \subseteq Ran(\Omega_\pm(H, H_0))$. As $Ran(\Omega_\pm(H, H_0)) \subseteq \mathcal{H}^{ac}$ by existence of the wave operator, the two sets are equal and $\Omega_\pm(H, H_0)$ is complete.
Conversely, if $\Omega_\pm(H, H_0)$ exists and $Ran(\Omega_\pm(H, H_0)) = \mathcal{H}^{ac}$, then

$$\forall f \in \mathcal{H}^{ac},\ \exists g \in \mathcal{H}_0\colon\ \lim_{t\to\infty} ||f - U(-t)U_0(t)P_0^{ac}g|| = 0 \tag{42}$$

As the mapping $\Omega_\pm(H, H_0)$: $\mathcal{H}_0^{ac} \to Ran(\Omega_\pm(H, H_0))$ is one-to-one and onto, g can be chosen to be the unique preimage of f in $\mathcal{H}_0^{ac}$ and (7) is equivalent to

$$\forall f \in \mathcal{H}^{ac},\ \exists! g \in \mathcal{H}_0^{ac}\colon\ \lim_{t\to\infty} ||U_0(-t)U(t)P^{ac}f - g|| = 0 \tag{43}$$

Thus, the wave operator $\Omega_\pm(H_0, H)$ exists on $\mathcal{H}^{ac}$ and therefore on $\mathcal{H}$ with $\Omega_\pm(H_0, H) = 0$ elsewhere. □

Remark The chain rule has granted us (41) and conversely

$$\Omega_\pm(H_0, H)\Omega_\pm(H, H_0) = P_0^{ac} \tag{44}$$

Then, if they exist, the wave operators $\Omega_\pm(H, H_0)$ and $\Omega_\pm(H_0, H)$ are unitary, mutually inverse mappings between $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$. Moreover, this property shows that if one can find sufficient conditions for the existence of wave operator $\Omega_\pm(H, H_0)$ which are symmetric in H and H_0, then $\Omega_\pm(H, H_0)$ is already complete. We now discuss a sufficient condition for the existence.

3.3 First investigation of existence of wave operators

Our goal is to obtain the existence of $\Omega_\pm$ by showing the convergence in the strong topology of the pre-wave operator $\Omega(H, H_0)(t)$ for a set of elements dense in its support $\mathcal{H}_0^{ac}$.

Theorem 3.5 (Cook's criterion) *Let H_0 and H be self-adjoint operators. Suppose that there exists $\mathfrak{D} \subseteq \mathcal{D}(H_0)$, dense set in $\mathcal{H}_0^{ac}$, and $t_0 \in \mathbb{R}_\pm$*

1. *$\forall t \in [t_0, \pm\infty)$ and $f \in \mathfrak{D}$, $U_0(t)f \in \mathcal{D}(H)$*
2. $\displaystyle\int_{t_0}^{\pm\infty} ||(H - H_0)U_0(t)f||dt < \infty$

then $\Omega_\pm(H, H_0)$ exists.

Proof For any $f \in \mathfrak{D}$, we define the vector-valued function $\omega_f(t) = \Omega(H, H_0)(t)f$. As $\mathfrak{D} \subseteq \mathcal{D}(H_0)$ and condition (1) hold, $U(-t)U_0(t)f$ is differentiable in t, that is for any $f \in \mathfrak{D}$, ω_f is differentiable. $\forall g \in \mathcal{D}(H)$, $U(t)g$ is differentiable in t and $(g, \omega'_f(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(g, \omega_f)(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(g, \omega_f) = \frac{\mathrm{d}}{\mathrm{d}t}(U(t)g, U_0(t)f) = (-iHU(t)g, U_0(t)f) + (U(t)g, -iH_0U_0(t)f)$$
$$= (g, iU(-t)HU_0(t)f) - (g, iU(-t)H_0U_0(t)f) = (g, iU(-t)(H - H_0)U_0(t)f)$$

H is densely defined in $\mathcal{H}$, then $\omega'_f(t) = iU(-t)(H - H_0)U_0(t)f$. As $||\omega'_f(t)|| = ||(H - H_0)U_0(t)f||$, condition (2) implies that ω'_f is absolutely integrable on $\mathbb{R}_\pm$. Thus, $\lim_{t\to\pm\infty} ||\omega_f(t) - \omega_f(t_0)|| \leq \int_{t_0}^{\pm\infty} ||\omega'_f(t)||dt$. As the norm is strongly continuous, the wave operator $\Omega_\pm$ exists on $\mathfrak{D}$, dense subset in $\mathcal{H}_0^{ac}$, and the following estimate holds:

$$||(\Omega_\pm - \Omega(t_0))f|| \leq \int_{t_0}^{\pm\infty} ||(H - H_0)U_0(t)f||dt \tag{45}$$

Then $\forall f \in \mathcal{H}_0^{ac}$, $\forall \varepsilon > 0$, $\exists g \in \mathfrak{D}$: $||\Omega(t)f - \Omega(t)g|| = ||f - g|| < \varepsilon$ and $\Omega_\pm$ exists on whole $\mathcal{H}_0^{ac}$. $\square$

We also remark that the Cook's criterion is not symmetric in H and H_0. Moreover, in order to show the condition (2), we need explicit formula of the unitary group, which are much more accessible for the free Hamiltonian than for the interacting one.
In order to understand the practicability of this, let take the case of two interacting quantum particles described by wave functions in $L_2(\mathbb{R}^3)$, which equivalently reduces to the scattering of a particle in an external potential. Let be $\mathcal{H} = L_2(\mathbb{R}^3)$, $H_0 = -\Delta$, $\mathcal{D}(H_0) = H^2(\mathbb{R}^3)$, self-adjoint extension of the laplacian on $\mathcal{H}$ and $H = H_0 + V$. V is a multiplication operator by a real-valued function.

Application 3.6 *The scattering system is the one described above. If V is a multiplication operator by some potential v in $L_2(\mathbb{R}^3)$, then $\Omega_\pm(H, H_0)$ exists.*

Proof We treat the case of Ω_+. Ω_- is similar. As $v \in L_2(\mathbb{R}^3)$, H is self-adjoint with same domain of definition $\mathcal{D}(H) = \mathcal{D}(H_0)$ and the condition (1) is fulfilled. We want to estimate $||VU_0(t)f||_2$. Let S denote the Schwartz space. For $f \in \mathcal{S} \subseteq L_1(\mathbb{R}^3) \cap L_2(\mathbb{R}^3)$, we have the representation for the free unitary evolution group

$$U_0(t)f(x) = \frac{1}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{i\frac{|x-y|}{4t}^2} f(y)dy \tag{46}$$

and $\left|U_0(t)f(x)\right| \leq \frac{1}{(4\pi|t|)^{3/2}} \int_{\mathbb{R}^3} \left|f(y)\right| dy$. Then, for fixed t, $U_0(t)f$ is bounded and

$$||U_0(t)f||_\infty \leq \frac{1}{(4\pi|t|)^{3/2}} ||f||_1 \tag{47}$$

By Hölder's inequality,

$$||VU_0(t)f||_2 \leq ||v||_2 ||U_0(t)f||_\infty$$

Then, $\int\limits_1^\infty ||VU_0(t)f||_2 dt \leq ||v||_2||f||_1 \int\limits_1^\infty \frac{1}{(4\pi t)^{3/2}} dt = \frac{2||v||_2||f||_1}{5(4\pi)^{3/2}} < \infty.$

As S lies dense in $L_2(\mathbb{R}^3)$, the condition (2) is fulfilled and $\Omega_+(H, H_0)$ exists on $\mathcal{D}(H_0)$. $\square$

In fact, one can push the estimate of $||VU_0(t)f||_2$ a little further to obtain a slight refinement of the previous corollary, in order to allow a little more spread out potentials.

Application 3.7 *The scattering system is the one described above. If V is a multiplication operator by some potential $v = v_2 + v_r$ with $v_2 \in L_2(\mathbb{R}^3)$ and $v_r \in L_r(\mathbb{R}^3)$ for some $2 \leq r < 3$, then $\Omega_\pm(H, H_0)$ exists.*

Proof Again with the same estimate and Holder's inequality, we have $\forall f \in \mathcal{S}$, for $t > 0$,

$$||VU_0(t)f||_2 \leq ||v_2U_0(t)f||_2 + ||v_rU_0(t)f||_2 \leq ||v_2||_2||f||_1 t^{-\frac{3}{2}} + ||v_r||_r ||U_0(t)f||_{r'}$$

with $r' = (\frac{1}{2} - \frac{1}{r})^{-1}$. On $L_2(\mathbb{R}^3)$, $U_0(t)$ is a bounded operator with the norm estimate due to its unitarity $||U_0(t)f||_2 = ||f||_2$, for any $f \in L_2(\mathbb{R}^3)$. By the estimate (47), $U_0(t)$: $L_1(\mathbb{R}^3) \to L_\infty(\mathbb{R}^3)$ is also bounded, and the Riesz-Thorin interpolation holds. That is, for any $2 \leq q \leq \infty$ and $p = (1 - \frac{1}{q})^{-1}$,
$U_0(t)$: $L^p \to L^q$ is a bounded operator with the estimate

$$||U_0(t)f||_q \leq (t^{-\frac{3}{2}})^\theta ||f||_p$$

with $\theta = 1 - \frac{2}{q}$. As $r' = (\frac{1}{2} - \frac{1}{r})^{-1} \in [2, \infty)$, there holds

$$||U_0(t)f||_{r'} \leq t^{-\frac{3}{r}} ||f||_{(\frac{1}{2}+\frac{1}{r})^{-1}}$$

and

$$||VU_0(t)f||_2 \leq ||v_2||_2||f||_1 t^{-\frac{3}{2}} + ||v_r||_r ||f||_{(\frac{1}{2}+\frac{1}{r})^{-1}} t^{-\frac{3}{r}}$$

As $r < 3$, $||VU_0(t)f||_2$ lies in $L_1([1, \infty))$ and the condition (2) holds again. $\square$

Remark Let consider radial potentials $v(|x|) = \mathcal{O}(|x|^{-\alpha})$, then a necessary condition for v to be in $L_r(\mathbb{R}^3)$ is $\alpha > 3/r$. Since the condition $r < 3$ is crucial to ensure the integrability of $||VU_0(t)f||_2$ on $[1, \infty)$, this method covers potentials up to falloffs $\mathcal{O}(|x|^{-1-\varepsilon})$, $\varepsilon > 0$. The next theorem states that local singularities of the potential do not influence the existence of wave operators.

Theorem 3.8 (Kupsch-Sandhas) *Let H_0 and H be self-adjoint operators and $\mathfrak{D} \subseteq \mathcal{D}(H_0)$ dense in $\mathcal{H}_0^{ac}$. Let moreover Λ be a bounded operator and $t_0 \in \mathbb{R}$ such that*

1. *$\forall t \in [t_0, \pm\infty)$ and $f \in \mathfrak{D}$, $(I - \Lambda)U_0(t)f \in \mathcal{D}(H)$*
2. $\displaystyle\int_{t_0}^{\pm\infty} ||(H(I - \Lambda) - (I - \Lambda)H_0)U_0(t)f||dt < \infty$
3. *$\exists n \in \mathbb{N}$, $\Lambda(H_0 + i)^{-n}$ is compact and $\mathfrak{D} \subseteq \mathcal{D}(H_0^n)$*

then $\Omega_\pm(H, H_0)$ exists.

Proof The proof is more or less a reproduction of the original Cook's criterion proof but is interesting since it introduces an auxiliary identification operator $J = I - \Lambda$. We introduce $\Omega_J(t) = U(-t)JU_0(t)$ and we define $\forall f \in \mathfrak{D}$, $\widetilde{\omega}_f(t) = \Omega_J(t)f$, which is differentiable according to the assumptions and, similarly to the previous proof, we obtain $\widetilde{\omega}'_f(t) = iU(-t)(HJ - JH_0)U_0(t)$. By condition (2), $||\widetilde{\omega}'_f(t)||$ is integrable on $(t_0, \pm\infty)$; therefore the strong limit of $U(-t)(I - \Lambda)U_0(t)$ when $t \to \pm\infty$ exists on $\mathfrak{D}$, that is on $\mathcal{H}_0^{ac}$ by density. We recall that $\forall f \in \mathcal{H}_0^{ac}$, w-$\lim_{t\to\pm\infty} U_0(t)f = 0$ and that $Ran(H_0 \pm i) = \mathcal{H}$ since H_0 is self-adjoint. Then $\forall f \in \mathcal{H}_0^{ac}$

$$\forall g \in \mathcal{H}, \ \lim_{t\to\pm\infty}(U_0(t)f, g) = 0 \Rightarrow \forall \tilde{g} \in \mathcal{D}(H_0), \ \lim_{t\to\pm\infty}(U_0(t)f, ((H_0 - i)\tilde{g}) = 0$$

and by density and boundedness of $U_0(t)$ w-$\lim_{t\to\pm\infty}(H_0 + i)U_0(t)f = 0$. By iterating the same idea, we obtain $\forall m \in \mathbb{N}$, w-$\lim_{t\to\pm\infty}(H_0 + i)^m U_0(t)f = 0$. In particular, for $m = n$, $\Lambda(H_0 + i)^{-n}$ is compact. Then $\Lambda U_0(t)$, and therefore $U(-t)\Lambda U_0(t)$, converge strongly to 0 on $\mathcal{H}_0^{ac}$. Hence $U(-t)U_0(t)$ converges strongly on $\mathcal{H}_0^{ac}$ and $\Omega_\pm(H, H_0) = \Omega_\pm(H, H_0, I - \Lambda)$. □

Remark In application, one sets Λ to the identity on a neighbourhood of the singularities of V so that their contribution vanishes in $||(H(I - \Lambda) - (I - \Lambda)H_0)U_0(t)f||$. We give below an example.

Application 3.9 *Let $\mathcal{H} = L_2(\mathbb{R}^3)$, $H_0 = -\Delta$ with $\mathcal{D}(H_0) = H^2(\mathbb{R}^3)$ and V be a multiplication operator by a potential v, measurable and decreasing asymptotically faster than the Coulomb potential, that is $\exists R \in \mathbb{R}_+$, $C \in \mathbb{R}$, $\varepsilon > 0$ so that $\forall |x| > R$, $|v(x)| \leq C|x|^{-1-\varepsilon}$. In other words, local singularities are allowed on an arbitrary neighbourhood of 0. Then $\Omega_\pm(H, H_0)$ exists.*

Proof Condition (1) is fulfilled according to the results in the preliminaries. We define naturally Λ as a multiplication operator by the function $\phi \in C_0^\infty(\mathbb{R}^3)$

with $\forall|x| \leq R,\ \phi(x) = 1$ and $\phi(x) \leq 1$ elsewhere. We gain that

$$\int_{\mathbb{R}^3} \left|v(x)(1-\phi(x))\right|^r dx \leq 4\pi \int\limits_R^\infty \frac{1}{(s^{1+\varepsilon})^r} s^2 ds < \infty$$

and $v(1-\phi) \in L_r(\mathbb{R}^3)$ for some $2 \leq r < 3$. Moreover, $\forall f \in \mathcal{S}$, we denote $f(t) = U_0(t)f$ and compute

$$\begin{aligned}\big(H(I-\Lambda) - (I - \Lambda H_0)\big) U_0(t) f &= (-\Delta + v)\left[(1-\phi)f(t)\right] + (1-\phi)\Delta f(t) \\ &= v(1-\phi)f(t) + \Delta(\phi f(t)) - \phi \Delta f(t) \\ &= v(1-\phi)f(t) + f(t)\Delta\phi + 2\nabla\phi \cdot \nabla f(t)\end{aligned}$$

We already now that $||v(1-\phi)f(t) + f(t)\Delta\phi||_2 \leq ||f(t)||_{(\frac{1}{2}-\frac{1}{r})^{-1}} ||v(1-\phi)||_r + ||\Delta\phi||_2 ||f(t)||_\infty$ lies in $L_1\left([1,\infty)\right)$. As $\nabla\Delta f = \Delta \nabla f$, for $t > 0$

$$||\nabla\phi\nabla f(t)||_2 \leq ||\nabla\phi||_2 ||U_0(t)\nabla f||_\infty \leq \frac{1}{4\pi t^{\frac{3}{2}}} ||\nabla\phi||_2 ||\nabla f||_1$$

lies in $L_1\left([1,\infty)\right)$. Furthermore, we claim that $\Lambda(H_0+i)^{-1}$ is compact. Note that Λ is self-adjoint as bounded multiplication operator by real-valued ϕ. We compute

$$\mathcal{F}(H_0 - i)^{-1}\Lambda f = \mathcal{F}(H_0 - i)^{-1}\mathcal{F}^{-1}\mathcal{F}\Lambda f$$

with $\mathcal{F}(H_0 - i)^{-1}\mathcal{F}^{-1}$ being a multiplication operator by $\xi \mapsto \dfrac{1}{|\xi|^2 - i}$. That is,

$$\mathcal{F}(H_0 - i)^{-1}\Lambda f(\xi) = \int_{\mathbb{R}^3} \frac{e^{-ix\cdot\xi}}{|\xi|^2 - i} \phi(x) f(x) dx$$

is an integral operator with kernel $k(x,\xi) = \dfrac{e^{-ix\cdot\xi}\phi(x)}{|\xi|^2 - i}$ such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left|k(x,\xi)\right|^2 dx d\xi \leq ||\phi||_2^2 \int_{\mathbb{R}^3} \frac{1}{|\xi|^4 + 1} d\xi < \infty \tag{48}$$

Then $\mathcal{F}(H_0 - i)^{-1}\Lambda$ is Hilbert-Schmidt, and so is $\Lambda(H_0 + i)^{-1}$ by unitarity of $\mathcal{F}$ and adjointness. Hence, $\Lambda(H_0 + i)^{-1}$ is compact and one applies (3.8). □

We now want to extend our simple scattering theory to systems $(\mathcal{H}, H, H_0)$ for which $\Omega_\pm(H, H_0)$ do not exist, such as in the case of the Coulomb potential. That is, we want to find some bounded linear operator J in $\mathcal{H}$ such that the strong limit $\lim_{t\to\infty} U(-t)JU_0(t)$ exists.

4 Identification operators

We recall that H and H_0 are seen as Hamiltonian operators of a quantum mechanical system in some Hilbert space $\mathcal{H}$. Equations with $\pm$ are to be understood as couples of independent equations for $+$ and $-$. We recall that we denote by $com(H)$ the algebra of bounded operators which commute with H.

4.1 Basic ideas

In the introduction, we motivated the definition of the wave operator as a way to compare the evolutions of the free and interacting dynamics asymptotically. Physically one is often more interested in comparing the corresponding time evolutions of some physical observable X which corresponds to a measure of the scattering experiment. In the Heisenberg picture related to the dynamics $U(t)$, the observable X is naturally given by $X_H(t) = U(-t)XU(t)$. In particular, we are interested in observables which are compatible with the free motion, that is, X commutes with H_0. For such X (e.g. impulsion $\vec{P}$, momentum $\vec{P} \times \vec{X}$ or spin), we have $X_{H_0}(t) = X$ and then we expect that X is an asymptotic constant for H as the particle scatters in regions of negligeable perturbations. Namely, there exist $X_\pm$ such that we have $\lim\limits_{t\to\pm\infty} X_H(t)P^{ac} = X_\pm$ in the sense of strong limits. As it concerns only scattering states, we take the projection onto the absolutely continuous subspace with respect to the real Hamiltonian.

In order to characterize completely the scattering states, we turn to a set of observables $\mathcal{O} = \{X_1, X_2, \ldots, X_n, \ldots\}$ which commute with H_0 and are asymptotic constants of the $U(t)$ dynamics. This set generates a C*-algebra $\mathfrak{O} = \bigcap\limits_{\mathcal{O}\subseteq\mathcal{A}} \mathcal{A}$, $\mathcal{A}$ C*-algebras, such that $\mathfrak{O} \subseteq com(H_0)$ and, if we denote $\mu^{\pm H}$: $X \mapsto X_\pm$, $\mathfrak{O} \subseteq \mathcal{D}(\mu^{\pm H})$. In the scattering frame, it is convenient to consider only the absolutely continuous parts of observables with respect to H or H_0. An equivalent picture is to restrict directly our *-algebras of observables: given a *-algebra $\mathfrak{A}$, we define the restrictions $\mathfrak{A}^{ac} = \{X \in \mathfrak{A} : XP^{ac} = P^{ac}X = X\}$, similarly $\mathfrak{A}_0^{ac}$.

Proposition 4.1 *$\mu^{\pm H}$: $X \mapsto X_\pm$ are *-homomorphisms from $\mathfrak{O}$ to $\mu^{\pm H}(\mathfrak{O})$.*

Proof Clearly, $\forall X_1, X_2 \in \mathcal{D}(\mu^{\pm H})$, $\forall a_1, a_2 \in \mathbb{C}$: $\mu^{\pm H}(a_1X_1 + a_2X_2) = a_1\mu^{\pm H}(X_1) + a_2\mu^{\pm H}(X_2)$.

We claim that $Ran(\mu^{\pm H}) = com(H)^{ac}$. Indeed, if $X \in com(H)^{ac}$, then $\forall t \in \mathbb{R}$, $X = U(-t)XU(t)P^{ac}$ and $X = \mu^{\pm H}(X) \in Ran(\mu^{\pm H})$. Conversely, if $X \in Ran(\mu^{\pm H})$, then $\exists X_0$ such that $\text{s-}\lim\limits_{t\to\infty} U(-t)X_0U(t)P^{ac} = X$. Hence $X = XP^{ac}$ as P^{ac} is an orthogonal projection. Moreover, like in the proof of the intertwining property, we get $\forall s \in \mathbb{R}$, $XU(s) = U(s)X$, which implies $XH = HX$. As P^{ac} is a spectral projection of H, P^{ac} and X commute, and $X = XP^{ac} = P^{ac}X$. So $X \in com(H)^{ac}$.

Now, $\forall X_1, X_2 \in \mathcal{D}(\mu^{\pm H})$:

$$U(-t)X_1X_2U(t)P^{ac} = U(-t)X_1U(t)P^{ac}U(-t)X_2U(t)P^{ac} \\ +U(-t)X_1U(t)(I-P^{ac})U(-t)X_2U(t)P^{ac}$$

As $Ran(\mu^{\pm H}) = com(H)^{ac}$, s-$\lim_{t\to\infty}(I-P^{ac})U(-t)X_2U(t)P^{ac} = 0$ and the sum converges strongly to the product $\mu^{\pm H}(X_1)\mu^{\pm H}(X_2)) = \mu^{\pm H}(X_1X_2)$.

Let $X, X^* \in \mathcal{D}(\mu^{\pm H})$. As $Ran(\mu^{\pm H}) = com(H)^{ac}$, $\forall f, g \in \mathcal{H}$

$$\begin{aligned}(\mu^{\pm H}(X)f, g) &= \lim_{t\to\infty}(P^{ac}U(-t)XU(t)P^{ac}f, g) \\ &= \lim_{t\to\infty}(f, P^{ac}U(-t)X^*U(t)P^{ac}g) = (f, \mu^{\pm H}(X^*)g)\end{aligned}$$

Hence $\mu^{\pm H}$ are *-homomorphisms from $\mathcal{D}(\mu^{\pm H})$ to $Ran(\mu^{\pm H})$. □

Then, we would like to perform a scattering experiment on a state $f_\pm \in \mathcal{H}^{ac}$ and gather informations $X_\pm f_\pm$, $X \in \mathfrak{O}$, which determine completely $\mathcal{H}^{ac}$. This is fulfilled if $\mu^{\pm H}(\mathfrak{O})^{ac}$ is cyclic, that is, $f \in \mathcal{H}^{ac}$ is such that $\{X_\pm f,\ X_\pm \in \mu^{\pm H}(\mathfrak{O})^{ac}\}$ is dense in $\mathcal{H}^{ac}$. Similarly, we assume that $\mathfrak{O}_0^{ac}$ is cyclic for some preparated state $f \in \mathcal{H}_0^{ac}$ so that $\{Xf, X \in \mathfrak{O}_0^{ac}\}$ characterizes completely $\mathcal{H}_0^{ac}$. Then the fundamental scattering relation between $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$ can now be given by a relation between Xf and the $X_\pm f_\pm$.

Proposition 4.2 *Assume that $\mathfrak{O}_0^{ac}$ and $\mu^{\pm H}(\mathfrak{O})^{ac}$ are cyclic, respectively with generating vectors $f \in \mathcal{H}_0^{ac}$ and $f_\pm \in \mathcal{H}^{ac}$. Assume moreover that $\forall X \in \mathfrak{O}$, $(f, Xf) = (f_\pm, X_\pm f_\pm)$.*
*Then $\mu^{\pm H}$: $\mathfrak{O}_0^{ac} \to \mu^{\pm H}(\mathfrak{O})^{ac}$ are isometric *-isomorphisms. Moreover there exist isometries $W_\pm$: $\mathcal{H}_0^{ac} \to \mathcal{H}^{ac}$ such that $\mu^{\pm H}(X) = W_\pm X W_\pm^*$.*

Proof If we have the representation $\mu_\pm(X) = W_\pm X W_\pm^*$ for $X \in \mathfrak{O}_0^{ac}$, we extend $W_\pm$ to partial isometries with support $\mathcal{H}_0^{ac}$ and if $Ran(W_\pm^*)$ are dense in $\mathcal{H}_0^{ac}$, then

$$\begin{aligned}|||\mu^{\pm H}(X)||| = \sup_{f\in\mathcal{H}\setminus\{0\}} \frac{||W_\pm X W_\pm^* f||}{||f||} &= \sup_{f\in\mathcal{H}\setminus\{0\}} \frac{||XW_\pm^* f||}{||W_\pm^* f||} \\ &= \sup_{g\in\mathcal{H}_0^{ac}\setminus\{0\}} \frac{||Xg||}{||g||} = |||X|||\end{aligned}$$

Hence, $\mu^{\pm H}$ is isometric and then one-to-one. Then $\mu^{\pm H}$: $\mathfrak{O}_0^{ac} \to \mu^{\pm H}(\mathfrak{O})^{ac}$ is a *-isomorphism. It remains only to construct the operators $W_\pm$.

We define $W_\pm$ densely on $\mathcal{H}_0^{ac}$ by $W_\pm Xf = X_\pm f_\pm$ for all $X \in \mathfrak{O}$ and elsewhere by $W_\pm(\mathcal{H}_0^{ac\perp}) = 0$. As $\mu^{\pm H}$ is *-homomorphism and by assumption, $\forall X, Y \in \mathfrak{O}$,

$$(f, X^*Yf) = (f_\pm, X_\pm^* Y_\pm f_\pm)$$

which implies $(Xf, Yf) = (X_\pm f_\pm, Y_\pm f_\pm)$ and, using the definition of $W_\pm$, $(W_\pm Xf, W_\pm Yf) = (Xf, Yf)$. Then $W_\pm$ are isometries on $\{Xf;\ X \in \mathfrak{O}\}$, which we extend by continuity on $\mathcal{H}_0^{ac}$. Therefore $\forall u \in \mathcal{H}_0^{ac}$, $(u, W_\pm^* W_\pm u) = (u, u)$, that is

$$W_\pm^* W_\pm = P_0^{ac} \tag{49}$$

As $W_\pm Xf = X_\pm f_\pm$ and since the sets of vectors $X_\pm f_\pm$ lie dense in $\mathcal{H}^{ac}$, we have that $W_\pm(\{Xf,\ X \in \mathfrak{O}_0^{ac}\})$ is dense in $\mathcal{H}^{ac}$. The vectors Xf are dense in $\mathcal{H}_0^{ac}$ and $W_\pm$ are isometries on $\mathcal{H}_0^{ac}$, then it follows that $Ran(W_\pm)$ are dense in $\mathcal{H}^{ac}$.

$$\forall u \in \mathcal{H}_0^{ac}:\ (u, W_\pm^* W_\pm u) = (u, u) \Rightarrow (W_\pm u, W_\pm W_\pm^* W_\pm u) = (W_\pm u, W_\pm u)$$

Then, by density, $\forall v \in \mathcal{H}^{ac}$: $(v, W_\pm W_\pm^* v) = (v, v)$ and

$$W_\pm W_\pm^* = P^{ac} \tag{50}$$

(49) and (50) characterize $W_\pm$ as partial isometries with initial subspace $\mathcal{H}_0^{ac}$ and final subspace $\mathcal{H}^{ac}$.

We have $W_\pm XYf = X_\pm Y_\pm f_\pm$. Further, we have $P_0^{ac} Y = Y$ for all $Y \in \mathfrak{O}_0^{ac}$. Hence

$$Yf = W_\pm^* W_\pm Yf = W_\pm^* Y_\pm f_\pm \Rightarrow W_\pm X W_\pm^* Y_\pm f_\pm = X_\pm Y_\pm f_\pm$$

As the sets of vectors $Y_\pm f_\pm$ are dense in $\mathcal{H}^{ac}$, we obtain $W_\pm X W_\pm^* = \mu^{\pm H}(X)$ for all $X \in \mathfrak{O}$. □

Definition We denote the partial isometries $W_\pm$ as *generalized wave operators* with respect to $\{H, H_0, \mathfrak{O}\}$.

4.2 Algebraic scattering systems

Without considering the particular physical criteria of proposition 4.2, we now give the following definition.

Definition Let be H, H_0 self-adjoint operators on $\mathcal{H}$ and $\mathfrak{A}$ a C*-algebra. $\{H, H_0, \mathfrak{A}\}$ is called an *algebraic scatering system* if

1. $\mathfrak{A} \subseteq com(H_0)$ and $\mathfrak{A}_0^{ac} \subseteq \mathcal{D}(\mu^{\pm H})$
2. $\mu^{\pm H}$ are injective on $\mathfrak{A}$
3. $\mu^{\pm H}$: $\mathfrak{A}_0^{ac} \to \mu^{\pm H}(\mathfrak{A})^{ac}$ are isometric *-isomorphisms with the representations $\mu^{\pm H}(X) = W_\pm X W_\pm^*$, $W_\pm$ partial isometries with support $\mathcal{H}_0^{ac}$ and final subspace $\mathcal{H}^{ac}$.

Proposition 4.3 *Given an algebraic scattering system $\{H, H_0, \mathfrak{A}\}$, the generalized wave operators $W_\pm$ are unique up to multiplications from the right by partial isometries with initial and final subspace $\mathcal{H}_0^{ac}$, and commuting with $\mathfrak{A}$.*

Proof Let be a couple of partial isometries $U_\pm$ on $\mathcal{H}_0^{ac}$ commuting with $\mathfrak{A}$. Let be $\widetilde{W}_\pm = W_\pm U_\pm$. Then

$$\widetilde{W}_\pm \widetilde{W}_\pm^* = W_\pm U_\pm U_\pm^* W_\pm^* = W_\pm P_0^{ac} W_\pm^* = W_\pm W_\pm^* W_\pm W_\pm^* = P^{ac}$$

Similarly, $\widetilde{W}_\pm^* \widetilde{W}_\pm = U_\pm^* W_\pm^* W_\pm U_\pm = U_\pm^* P_0^{ac} U_\pm = U_\pm^* U_\pm U_\pm^* U_\pm = P_0^{ac}$. Moreover, $\forall A \in \mathfrak{A}_0^{ac}$, $\widetilde{W}_\pm A \widetilde{W}_\pm^* = W_\pm U_\pm U_\pm^* A W_\pm^* = W_\pm A W_\pm^* = \mu^{\pm H}(A)$.

Conversely, assume there are generalized wave operators $\widetilde{W}_\pm$. (50) implies that $\forall v \in \mathcal{H}^{ac}$, $\widetilde{W}_\pm \widetilde{W}_\pm^* v = v$ and $W_\pm W_\pm^* v = v$. Let be $U_\pm$ the mapping $\widetilde{W}_\pm^* v \mapsto W_\pm^* v$ for all $v \in \mathcal{H}^{ac}$. Then $||U_\pm \widetilde{W}_\pm^* v|| = ||v|| = ||\widetilde{W}_\pm^* v||$. Moreover (49) and (50) imply that $Ran(W_\pm^*) = Ran(\widetilde{W}_\pm^*) = \mathcal{H}_0^{ac}$. Then $U_\pm$ can be extended to a partial isometry with support and final subspace $\mathcal{H}_0^{ac}$. We have $\forall u \in \mathcal{H}_0^{ac}$, $\exists v \in \mathcal{H}^{ac}$:

$$W_\pm U u = W_\pm U_\pm \widetilde{W}_\pm^* v = W_\pm W_\pm^* v = \widetilde{W}_\pm \widetilde{W}_\pm^* v = \widetilde{W}_\pm u$$

Then $\widetilde{W}_\pm = W_\pm U_\pm$. Moreover, $\forall A \in \mathfrak{A}$, $\mu^{\pm H}(A) = W_\pm U_\pm A U_\pm^* W_\pm^* = W_\pm A W_\pm^*$. Then $U_\pm A U_\pm^* = P_0^{ac} A P_0^{ac} = A P_0^{ac}$, which implies $U_\pm A P_0^{ac} = A P_0^{ac} U_\pm$. Since A commutes with H_0 and $U_\pm$ are partial isometries on $\mathcal{H}_0^{ac}$, we have $U_\pm A = A U_\pm$. $\square$

Given an algebraic scattering system $\{H, H_0, \mathfrak{A}\}$ with generalized wave operators $W_\pm$, we recall that the time-evolution operator $U(t) = e^{-iHt}$ describes the real dynamics and we investigate the properties of the operators $V_\pm(t) = W_\pm^* U(t) W_\pm + (I - P_0^{ac})$. $V_\pm(t)$ are defined on $\mathcal{H}$. Moreover, as $W_\pm$ are partial isometries with support $\mathcal{H}_0^{ac}$, we have $W_\pm(I - P_0^{ac}) = 0$, so that $\forall t_1, t_2 \in \mathbb{R}$:

$$\begin{aligned} V_\pm(t_1) V_\pm(t_2) &= W_\pm^* U(t_1) W_\pm W_\pm^* U(t_2) W_\pm + (I - P_0^{ac})^2 \\ &= W_\pm^* U(t_1) P^{ac} U(t_2) W_\pm + (I - P_0^{ac}) = V_\pm(t_1 + t_2) \end{aligned}$$

where we used $HP^{ac} = P^{ac} H$ and $Ran(W_\pm) = \mathcal{H}^{ac}$ in the last equation. Moreover, $\forall f \in \mathcal{H}$:

$$\lim_{t \to 0} ||(W_\pm^* U(t) W_\pm + (I - P_0^{ac}))f - f|| = ||(W_\pm^* W_\pm + (I - P_0^{ac}))f - f|| = 0$$

As we further have $V_\pm(t)^* = V_\pm(-t) = V_\pm(t)^{-1}$, the following proposition holds.

Proposition 4.4 $V_\pm(t) = W_\pm^* U(t) W_\pm + (I - P_0^{ac})$ *are two strong continuous unitary groups. There exist two self-adjoint operators* $K_\pm$ *on* $\mathcal{H}$ *so that* $V_\pm(t) = e^{-iK_\pm t}$.

We now remark some interesting properties of $K_\pm$.

Proposition 4.5 *The following assertions hold:*

1. $K_\pm(\mathcal{H}_0^{ac\perp}) = 0$
2. $\mathfrak{A}_0^{ac} \subseteq com(K_\pm)$
3. $W_\pm$ *are intertwining operators for* $K_\pm$ *and* H.

Proof We have $\forall f \in \mathcal{H}_0^{ac\perp}$, $\forall t \in \mathbb{R}$, $V_\pm(t) f = f$. By definition of the infinitesimal generator,

$$-iK_\pm f = \lim_{h \to 0} \frac{V_\pm(h) f - f}{h} = 0, \text{ that is, } K_\pm(\mathcal{H}_0^{ac\perp}) = 0.$$

$\forall A \in \mathfrak{A}_0^{ac}$, $\mathcal{H}_0^{ac}$ and $\mathcal{H}_0^{ac\perp}$ are invariant subspaces for A. Together with $K_\pm(\mathcal{H}_0^{ac\perp}) = 0$, this implies that it is enough to show $AK_\pm = K_\pm A$ on $\mathcal{H}_0^{ac}$. As $W_\pm^* \mu^{\pm H}(A) W_\pm = AP_0^{ac}$, we have $\forall f \in \mathcal{H}_0^{ac}$, $\forall t \in \mathbb{R}$:

$$\begin{aligned} V_\pm(t)Af = W_\pm^* U(t) \underbrace{W_\pm W_\pm^*}_{P^{ac}} \mu^{\pm H}(A) W_\pm f &= W_\pm^* \mu^{\pm H}(A) W_\pm W_\pm^* U(t) W_\pm f \\ &= AV_\pm(t) f \end{aligned}$$

where we used $Ran(\mu^{\pm H}) = com(H)^{ac}$. Then A commutes also with $K_\pm$ and $\mathfrak{A}_0^{ac} \subseteq com(K_\pm)$.

Finally, according to the definition of $V_\pm(t)$, we have

$$W_\pm V_\pm(t) = P^{ac} U(t) W_\pm = U(t) W_\pm$$

Like in the proof of the intertwining relation, it implies that $W_\pm K_\pm = HW_\pm$. □

Because of the striking similarities between $K_\pm$ in the generalized frame and H_0 in the classical one, one calls $K_\pm$ *renormalized free Hamiltonians.*

4.3 Relation to wave operators

The simple scattering theory granted us $\Omega_+(H, H_0) = \Omega_-(-H, -H_0)$. In other words, the scattering of a particle in positive time direction was equivalent to the scattering of its antiparticle in negative time direction. For general $K_\pm$, this is obviously here not the case since we have different asymptotic dynamics. That's why we wish to have $K_\pm = K_0$ in physical applications. Indeed, under this assumption, the generalized wave operators can be given under a familiar form.

Proposition 4.6 *Let $\{H, H_0, \mathfrak{A}\}$ be an algebraic scattering system with generalized wave operators $W_\pm$. If the renormalized free Hamiltonians are given by $K_+ = K_- = K_0$, then there exists a bounded operator J on $\mathcal{H}$ such that $W_\pm = \Omega_\pm(H, K_0, J)$.*

We first turn to two technical lemmas which constitute the core of the proof of this proposition.

Lemma 4.7 *Let be a projection $P \in \mathcal{D}(\mu^{\pm H})$ such that $\mu^H(P) = P^{ac}$ and $\mu^{-H}(P) = 0$. Let further be a projection Q, commuting with H, with $Q \leq P^{ac}$. Then there is a projection $\widetilde{P} \leq Q$ with $\widetilde{P} \in \mathcal{D}(\mu^{\pm H})$, $\mu^H(\widetilde{P}) = P^{ac}$ and $\mu^{-H}(\widetilde{P}) = 0$*

Proof If we consider $\widetilde{P} = QPQ$, then $\widetilde{P}$ is a projection with $\widetilde{P}Q = Q\widetilde{P} = \widetilde{P}$. $\forall t \in \mathbb{R}$, $U(-t)QPQU(t)P^{ac} = QU(-t)PU(t)P^{ac}Q$. So $\widetilde{P} \in \mathcal{D}(\mu^{\pm H})$ and $\mu^{\pm H}(\widetilde{P}) = Q\mu^{\pm H}(P)Q = QP^{ac}Q = Q$. □

Lemma 4.8 *There exists a projection $P \in \mathcal{D}(\mu^{\pm H})$ with $P < P^{ac}$ such that $\mu^H(P) = P^{ac}$ and $\mu^{-H}(P) = 0$.*

Proof There is a representation of $\mathcal{H}^{ac}$ as a direct sum of subspaces $\mathcal{H}^{ac}_n$, reducing H, so that the restrictions of H on $\mathcal{H}^{ac}_n$ have simple spectrum and are absolutely continuous. We note P^{ac}_n the projections onto $\mathcal{H}^{ac}_n$. As $P < P^{ac}$, it suffices to construct the projection P separately in each $\mathcal{H}^{ac}_n$.
Moreover the spectral representation implies that $\mathcal{H}^{ac}_n$ is unitary equivalent to $L_2(\Delta_n, d\xi)$ with $\Delta_n \subseteq \sigma^{ac}$ Borel set. Let denote Υ: $\mathcal{H}^{ac}_n \to L_2(\Delta_n, d\xi)$ this unitary transformation, then $\Upsilon H \Upsilon^*$ acts as multiplication by ξ and $\Upsilon P^{ac}_n \Upsilon^*$ acts as multiplication by the characteristic function γ_{Δ_n}.
In the view of applying lemma (4.7), we note that $L_2(\Delta_n, d\xi)$ can be regarded as the subspace of the Hilbert space $L_2(\mathbb{R}, d\xi)$ constituted by functions in L_2 with support Δ_n. The associated projection Q onto $L_2(\Delta_n, d\xi)$ is the multiplicative operator by γ_{Δ_n}. Q commutes with $\Upsilon H \Upsilon^*$ and $Q \leq \Upsilon P^{ac} \Upsilon^*$ since $\Delta_n \subseteq \sigma^{ac}$. Further, via inverse Fourier transform $\mathcal{F}^*$: $L_2(\mathbb{R}, d\xi) \to L_2(\mathbb{R}, dx)$, we have $\check{H} = \mathcal{F}^* \Upsilon H \Upsilon^* \mathcal{F} = -i\frac{\mathrm{d}}{\mathrm{d}x}$. We now note $\gamma_{[0,\infty)}$ for the multiplication operator by the function $\gamma_{[0,\infty)}$ and remark that $\forall u \in L_2(\mathbb{R}, dx)$:

$$e^{i\check{H}t}\gamma_{[0,\infty)}(x)e^{-i\check{H}t}u(x) = \gamma_{[0,\infty)}(x+t)u(x) = \gamma_{[-t,\infty)}(x)u(x)$$

Then $\text{s-}\lim_{t\to\infty} e^{i\check{H}t}\gamma_{[0,\infty)}e^{-i\check{H}t} = I$ and $\text{s-}\lim_{t\to-\infty} e^{i\check{H}t}\gamma_{[0,\infty)}e^{-i\check{H}t} = 0$. By Fourier transform follow $\text{s-}\lim_{t\to\infty,\text{ resp. }-\infty} e^{i\Upsilon H\Upsilon^* t}\mathcal{F}\gamma_{[0,\infty)}\mathcal{F}^* e^{-i\Upsilon H \Upsilon^* t} = I$, resp. 0. As these equations hold in particular on $Ran(\Upsilon P^{ac}\Upsilon^*)$, we obtain

$$\mu^H(\mathcal{F}\gamma_{[0,\infty)}\mathcal{F}^*) = \Upsilon P^{ac}\Upsilon^*, \quad \mu^{-H}(\mathcal{F}\gamma_{[0,\infty)}\mathcal{F}^*) = 0$$

We can now apply 4.7 and turn back to $\mathcal{H}^{ac}_n$. P is given by

$$\sum_n \Phi^*\gamma_{\Delta_n}\mathcal{F}\gamma_{[0,\infty)}\mathcal{F}^*\gamma_{\Delta_n}\Phi$$

□

Proof *of 4.6* Let P be the projection of 4.8 with respect to K_0. We define the bounded operator $J = W_+P + W_-(1-P)$ and we note $V_0(t) = e^{-iK_0t}$, $P^{ac}_{K_0}$ the projection on the absolutely continuous subset of $\mathcal{H}$ with respect to K_0 . Thanks to the intertwining property of $W_\pm$, we obtain:

$$\begin{aligned} U(-t)JV_0(t)P^{ac}_{K_0} &= U(-t)W_+PV_0(t)P^{ac}_{K_0} + U(-t)W_-(1-P)V_0(t)P^{ac}_{K_0} \\ &= W_+V_0(-t)PV_0(t)P^{ac}_{K_0} + W_-P^{ac}_{K_0} - W_-V_0(-t)PV_0(t)P^{ac}_{K_0} \end{aligned}$$

which implies

$$\lim_{t\to\pm\infty} U(-t)JV_0(t)P^{ac}_{K_0} = W_+\mu^{\pm K_0}(P) + W_-P^{ac}_{K_0} - W_-\mu^{\pm K_0}(P)$$

According to the assumptions on P, $\Omega_\pm(H, K_0, J) = W_\pm P^{ac}_{K_0} = P^{ac}W_\pm = W_\pm$.
□

4.4 Application to Coulomb scattering

For practical purposes, our strategy in the investigation of the existence of generalized wave operators is the following:

1. Find a C*-algebra $\mathcal{A}$ such that $\mathcal{A} \subseteq com(H_0)$ and $\mathcal{A}^{ac}_0 \subseteq \mathcal{D}(\mu^{\pm H})$.

2. Show that $\{H, H_0, \mathcal{A}\}$ is an algebraic scattering system.

3. Eventually, investigate if the renormalized free Hamiltonians are equal.

Steps (2),(3) and the last part of (1) are equivalent to finding explicite partial isometries $W_\pm$ such that $\mu^{\pm H}(A) = W_\pm A W_\pm^*$. In this view, the next proposition states that admissible $W_\pm$ can be constructed as strong limits of operators. Equations with $\pm$ are understood as two independent equations of $+$ and $-$.

Proposition 4.9 *Let $\mathfrak{A}$ be a C^*-algebra with $\mathfrak{A} \subseteq com(H_0)$. If there is a strongly continuous family of operators $T_t\colon \mathbb{R}_\pm \to \mathcal{L}(\mathcal{H})$ such that:*

1. *$T_t P_0^{ac}$ is asymptotically unitary on $\mathcal{H}_0^{ac}$ as $t \to \pm\infty$. That is, $\forall u \in \mathcal{H}_0^{ac}$, $\lim\limits_{t\to\pm\infty} T_t P_0^{ac} T_t^* u = \lim\limits_{t\to\pm\infty} P_0^{ac} T_t^* T_t u = u$*

2. *$T_t \in com(\mathfrak{A})$*

3. *There exists $\forall u \in \mathcal{H}$: $\lim\limits_{t\to\pm\infty} U(-t) T_t P_0^{ac} u = V_\pm u$ and $V_\pm V_\pm^* = P^{ac}$.*

Then, $\forall A \in \mathfrak{A}_0^{ac}$: $A \in \mathcal{D}(\mu^{\pm H})$ and $\mu^{\pm H}(A) = V_\pm A V_\pm^*$.

Proof In the case $T_t\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{H})$: From (1) and (3), $\forall u \in \mathcal{H}_0^{ac}$, $||V_+ u|| = \lim\limits_{t\to\infty} ||T_t P_0^{ac} u|| = ||u||$. $\forall u \in \mathcal{H}_0^{ac\perp}$, $V_+ u = 0$. Then V_+ is a partial isometry with $V_+^* V_+ = P_0^{ac}$ and $V_+ V_+^* = P^{ac}$. Further, $\forall u \in \mathcal{H}$:

$$\begin{aligned} ||(P_0^{ac} T_t^* U(t) P^{ac} - V_+^*)u||^2 &= ||P_0^{ac} T_t^* U(t) P^{ac} u||^2 + ||V_+^* u||^2 \\ &\quad - (P_0^{ac} T_t^* U(t) P^{ac} u, V_+^* u) - (V_+^* u, P_0^{ac} T_t^* U(t) P^{ac} u) \end{aligned}$$

V_+^* being a partial isometry with support $\mathcal{H}^{ac}$ and $P_0^{ac} T_t^*$ being asymptotically unitary, it holds:

$$\begin{aligned} \lim_{t\to\infty} ||(P_0^{ac} T_t^* U(t) P^{ac} - V_+^*)u||^2 &\leq 2||P^{ac} u||^2 - \lim_{t\to\infty} 2\Re\{(P_0^{ac} T_t^* U(t) P^{ac} u, V_+^* u)\} \\ &\leq 2||P^{ac} u||^2 - \underbrace{\lim_{t\to\infty} 2\Re\{(P^{ac} u, U(-t) T_t P_0^{ac} V_+^* u)\}}_{2||P^{ac} u||^2} \end{aligned}$$

Hence, $V_+^* = \lim\limits_{t\to\infty} P_0^{ac} T_t^* U(t) P^{ac}$. Then, $\forall A \in \mathfrak{A}_0^{ac}$:

$$\begin{aligned} \lim_{t\to\infty} U(-t) A U(t) P^{ac} &= \lim_{t\to\infty} U(-t) T_t P_0^{ac} T_t^* A U(t) P^{ac} \\ &= \lim_{t\to\infty} U(-t) T_t P_0^{ac} A P_0^{ac} T_t^* U(t) P^{ac} = V_+ A V_+^*. \end{aligned}$$

Proceed similarly in the case $-$.

We can now turn to the investigation of the Coulomb case.

Application 4.10 *Let be $\mathcal{H} = L_2(\mathbb{R}^3)$, $H_0 = -\Delta$ on $H^2(\mathbb{R}^3)$ and $H = -\Delta + c|x|^{-1}$, $c \in \mathbb{R}$. Then $\{H, H_0, com(H_0)\}$ is an algebraic scattering system.*

Proof It remains to find two strong continuous families $T_t^\pm\colon \mathbb{R}_\pm \to \mathcal{H}$ fulfilling 4.9. We define $T_t^\pm = T_t = e^{-iX_t}$ with

$$X_t = tH_0 + \frac{1}{2}\int_0^t c(H_0^{1/2}|s|)^{-1}\gamma_{(0,\infty)}(4|s|\,H_0 - 1)ds \tag{51}$$

As $X_t = f(H_0)$, A commutes with X_t for any $A \in com(H_0)$. As $f(\cdot)$ is real-valued, X_t is self-adjoint and T_t is unitary. Thus our choice satisfies (1) and (2) of proposition 4.9. We obtain the last assumption by a Cook's criterion-like argument.

The characteristic function in (51) is here to avoid singularities of X_t when $t = 0$. In the momentum space $L_2(\mathbb{R}^3, d\xi)$, the Fourier transform $\widehat{X}_t$ is multiplication by

$\hat{x}_t(\xi) = t|\xi|^2 + \frac{1}{2}\int\limits_0^t c(|\xi||s|)^{-1}\gamma_{(0,\infty)}(4|s||\xi|^2) - 1)ds$. That is, for $|t| > \frac{1}{4|\xi|^2}$,

$$\hat{x}_t(\xi) = t|\xi|^2 + c(2|\xi|)^{-1}\left(\log(|t|) + \log(4|\xi|^2)\right) \tag{52}$$

$\hat{x}_t$ is singular in $\xi = 0$ so that $\widehat{X}_t$ can be defined on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. We note $\mathfrak{D} = \{u \in L_2(\mathbb{R}^3, dx);\ \hat{u} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})\}$. $\mathfrak{D}$ is dense in $\mathcal{H}$ and lies in $\mathcal{D}(H_0) = \mathcal{D}(H)$. Hence, $\forall u \in \mathfrak{D}$, and for sufficiently large $t_u > 0$, we have $\forall |t| > t_u$:

$$\begin{aligned} X_t u &= \left(tH_0 + c(2H_0^{1/2})^{-1}\left(\log(|t|) + \log(4H_0)\right)\right)u \\ T_t u &= e^{-i\left(tH_0 + c(2H_0^{1/2})^{-1}\left(\log(|t|)+\log(4H_0)\right)\right)}u \end{aligned} \tag{53}$$

Moreover, we have $\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_t(\xi) = |\xi|^2 + c(2|\xi||t|)^{-1}$. Then $C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \subseteq \mathcal{D}(\hat{X}_t')$ and $\mathfrak{D} \subseteq \mathcal{D}(X_t')$ for all $t > t_u$. Further the form (52) of the multiplicative operator implies that $X_t(\mathfrak{D}) \subseteq \mathfrak{D}$, and then $T_t(\mathfrak{D}) \subseteq \mathfrak{D}$. Then $U(-t)T_t$ is strongly differentiable on $\mathfrak{D}$ and $\forall t > t_u$:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(-t)T_t u = U(-t)\left(iH - i\frac{\mathrm{d}}{\mathrm{d}t}X_t\right)u = iU(-t)\left(H - H_0 - c(2H_0^{1/2}|t|)^{-1}\right)T_t u$$

We now restrict ourselves to the study of $t \to \infty$. As in Cook's method, we use the estimate $\forall t > s > t_u$

$$||U(-t)T_t u - U(-s)T_s u|| \le \int\limits_s^t ||(H - H_0 - c(2H_0^{1/2}t')^{-1})T_t u||dt' \tag{54}$$

T_t and $H_0^{-1/2}$ commute and we set $\forall u \in \mathfrak{D}$, $u_1 = H_0^{-1/2}u$, that is also $\hat{u}_1(\xi) = |\xi|^{-1}\hat{u}(\xi)$. Then it follows:

$$((H - H_0 - c(2H_0^{1/2}t)^{-1})T_t u)(x) = c|x|^{-1}T_t u(x) - c(2t)^{-1}T_t u_1(x) \tag{55}$$

For all $u \in \mathfrak{D}$ and, we consider the quantity

$$D_u(x,t) = T_t u(x) - (2it)^{-3/2}e^{i(x^2/4t - c|x|^{-1}\log(|x|^2/t))}\hat{u}(x/2t)$$

Then the definition of $\hat{u}_1$ implies that

$$((H - H_0 - c(2H_0^{1/2}t)^{-1})T_t u)(x) = c|x|^{-1}D_u(x,t) - c(2t)^{-1}D_{u_1}(x,t) \tag{56}$$

We claim that the following estimate holds for $t > \max\{t_u, 2\}$

$$\left|D_u(x,t)\right| \le a_{m,u}\log(|t|)^{s_m}t^{-5/2}(1 + (|x|/t)^2)^{-m} \tag{57}$$

where m is an arbitrary integer and $s_m \in \mathbb{N}$, $a_{m,u} \in \mathbb{R}$. Then

$$\begin{aligned} ||c|x|^{-1} D_u(\cdot,t)|| &\leq 4\pi a_{m,u}^2 \log(|t|)^{2s_m} t^{-5} \int_{\mathbb{R}_+} (1+(|x|/t)^2)^{-2m} d|x| \\ &\leq C \log(|t|)^{2s_m} t^{-4} \end{aligned}$$

and similarly $||c(2t)^{-1} D_{u_1}(\cdot,t)|| \leq C_1 \log(|t|)^{2s_m} t^{-4}$. Hence, the L_2-norm of the right side of (56) lies in $L_1\left([\max\{t_u,2\},\infty)\right)$ and, by (54), $\lim_{t\to\infty} U(-t)T_t u$ exists for all $u \in \mathfrak{D}$. By density of $\mathfrak{D}$ and unitarity of $U(-t)T_t$, the strong limit exists on $\mathcal{H}$ and this proves the first half of (3) in proposition 4.9. The case $t \to -\infty$ is similar.

Then we shall show (57). For $u \in \mathfrak{D}$, we note $u(x,t) = T_t U_0(-t) u(x)$. That is, for $|t| > t_u$, $\hat{u}(\xi,t) = e^{-ic(2|\xi|)^{-1}\log(4|t||\xi|^2)}\hat{u}(\xi)$. Thus we have

$$T_t u(x) = U_0(t) T_t U_0(-t) u(x) = (4i\pi t)^{-3/2} \int_{\mathbb{R}^3} e^{i|x-y|^2/4t} u(y,t) dy$$

$$D_u(x,t) = (4i\pi t)^{-3/2} \int_{\mathbb{R}^3} e^{i|x-y|^2/4t} u(y,t) dy - (2it)^{-3/2} e^{ix^2/4t} \hat{u}(x/2t,t)$$

that is

$$D_u(x,t) = (4i\pi t)^{-3/2} e^{ix^2/4t} \int_{\mathbb{R}^3} e^{-i(x,y)/2t} \left(e^{iy^2/4t} - 1\right) u(y,t) dy \qquad (58)$$

We consider $\mathcal{S}(\mathbb{R}^3)$ equiped with the usual seminorms

$$p_{\alpha,\beta}(v) = \sup_{x\in\mathbb{R}^3} \left| x^\alpha \mathrm{D}^\beta v(x) \right|$$

and we use a property of the Fourier transform on $\mathcal{S}(\mathbb{R}^3)$. The Fourier transform and its inverse are continuous linear mappings from $\mathcal{S}(\mathbb{R}^3)$ to $\mathcal{S}(\mathbb{R}^3)$ and, for any multiindices α, β, there exist multiindices α_j, β_j such that $p_{\alpha,\beta}(v) = \sum_{j=1}^{M} p_{\alpha_j,\beta_j}(\hat{v})$. Then, as $\hat{u} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $\hat{u}(\cdot,t) \in \mathcal{S}(\mathbb{R}^3)$ and therefore $u(\cdot,t) \in \mathcal{S}(\mathbb{R}^3)$, for $t > t_u$ and arbitrary multiindices, we estimate

$$p_{\alpha,\beta}(\hat{u}(\cdot,t)) = \sup_{\xi\in\Delta} \left| \xi^\alpha \mathrm{D}^\beta e^{-ic(2|\xi|)^{-1}\log(4|t||\xi|^2)} \hat{u}(\xi) \right|$$

with Δ compact in $\mathbb{R}^3 \setminus \{0\}$. As we have for $t > t_u$,

$$\frac{\mathrm{d}}{\mathrm{d}\xi_j}(\hat{u}(\xi,t)) = ic(2|\xi|^3)^{-1}\xi_j(2+\log(4|t||\xi|^2))\hat{u}(\xi,t) + \tfrac{\mathrm{d}\hat{u}}{\mathrm{d}\xi_j}(\xi,t)$$

then, for $t > \max\{t_u,2\}$ and sufficiently large $C, \tilde{C} \in \mathbb{R}$:

$$p_{\alpha,\beta}(\hat{u}(\cdot,t)) \leq C \log(|t|)^{|\beta|} \sum_{i=\{0,0,0\}}^{\beta} p_{\alpha,i}(\hat{u}) \leq \tilde{C} \log(|t|)^{|\beta|}$$

Hence, it also holds for some $s_\beta \in \mathbb{N}$:

$$p_{\alpha,\beta}(u(\cdot,t)) \leq C_{\alpha,\beta,u} \log(|t|)^{s_\beta} \qquad (59)$$

We now return to (58), namely we estimate

$$\begin{aligned}\left|\left(\frac{x}{t}\right)^{2m} D_u(x,t)\right| &= \left|(4\pi t)^{-3/2}\int_{\mathbb{R}^3}\left(\frac{x}{t}\right)^{2m} e^{-i(x,y)/2t}\left(e^{iy^2/4t}-1\right)u(y,t)dy\right| \\ &\leq \left|t^{-3/2}\int_{\mathbb{R}^3}(-\Delta_y^m e^{-i(x,y)/2t})\left(e^{iy^2/4t}-1\right)u(y,t)dy\right|\end{aligned}$$

As $u(\cdot,t)\in\mathcal{S}(\mathbb{R}^3)$, this provides by partial integration:

$$\left|\left(\frac{x}{t}\right)^{2m} D_u(x,t)\right| \leq |t|^{-3/2}\int_{\mathbb{R}^3}\left|\Delta_y^m\left(e^{iy^2/4t}-1\right)u(y,t)\right|dy \tag{60}$$

Using Leibnitz's rule, we obtain that

$$\begin{aligned}\left|\Delta_y^m\left(e^{iy^2/4t}-1\right)u(y,t)\right| &= \Bigg|\left(e^{iy^2/4t}-1\right)\Delta_y^m u(y,t)+ \\ &+\sum_{0<|\alpha|\leq 2m}\sum_{0\leq|\beta|<2m} c_{\alpha,\beta}e^{iy^2/4t}y^\alpha \mathrm{D}^\beta u(y,t)\Bigg|\end{aligned}$$

where $\left|c_{\alpha,\beta}\right| = \tilde{c}_{\alpha,\beta}|t|^{-n}$ for some $n\in\mathbb{N}^*$. For $t>t_u$, $\left|c_{\alpha,\beta}\right|\leq k_{\alpha,\beta}|t|^{-1}$. Moreover, using $\left|e^{iy^2/4t}-1\right|\leq\left|y^2/4t\right|$:

$$\left|\Delta_y^m\left(e^{iy^2/4t}-1\right)u(y,t)\right|\leq|t|^{-1}\sum_{\alpha,\beta}k_{\alpha,\beta}\left|y^\alpha D^\beta u(y,t)\right|$$

We rewrite it so that (60) converges:

$$\left|\Delta_y^m\left(e^{iy^2/4t}-1\right)u(y,t)\right|\leq|t|^{-1}(1+|y|)^{-4}\sum_{\tilde{\alpha},\beta}k_{\tilde{\alpha},\beta}p_{\tilde{\alpha},\beta}(u(\cdot,t)) \tag{61}$$

Using (59) and (60), we obtain for arbitrary m

$$\left|\left(\frac{x}{t}\right)^{2m} D_u(x,t)\right|\leq C|t|^{-5/2}\log(|t|)^{s_m} \tag{62}$$

which implies (57). We now note $W_\pm=\lim_{t\to\pm\infty}U(-t)e^{-iX_t}$. Then $W_\pm^*W_\pm=I$ follows.

It remains to prove that $W_\pm W_\pm^*=P^{ac}$. The special form of $W_\pm$ implies the intertwining property $W_\pm H_0=HW_\pm$. Indeed

$$U(-s)U(-t)e^{-iX_t}U_0(s)=U(-(s+t))e^{-iX_{t+s}}e^{+iX_{t+s}-iX_t}U_0(s)$$

For $u\in\mathfrak{D}, |t|>t_u$ and $|t+s|>t_u$,

$$e^{+iX_{t+s}-iX_t}U_0(s)u=e^{ic(2H_0^{1/2})^{-1}\log\left(|1+s/t|\right)}u$$

As $e^{iaH_0^{-1/2}}$ is strongly continuous in a, we have $\lim_{t\to\pm\infty}e^{+iX_{t+s}-iX_t}U_0(s)u=u$. By density of $\mathfrak{D}$ and unitarity, this holds on $\mathcal{H}$. Hence, there holds for any s $U(-s)W_\pm U_0(s)=W_\pm$ which implies the intertwining property. This implies that $W_\pm W_\pm^*$ commutes with H. Moreover, $W_\pm H_0 W_\pm^* = H W_\pm W_\pm^*$ and $H_0 = W_\pm^* H W_\pm W_\pm^* W_\pm$ imply that $\sigma(HW_\pm W_\pm^*) = \sigma(H_0) = \sigma(HP^{ac}) = [0,\infty)$, which are absolutely continuous spectra. Then HP^{ac} and $HW_\pm W_\pm^*$ enjoy the same spectral core. If we find a decomposition of $\mathcal{H}$ as a direct sum of subspaces reducing H and H_0 so that H and H_0 are simple on them and have $\sigma(H_0) = \sigma^{ac}(H) = [0,\infty)$, then we conclude by lemma 2.8 that $P^{ac} = W_\pm W_\pm^*$ on each subspace and then on the whole space. The spherical potential suggests the decomposition $\mathcal{H} = \bigoplus_{\{l,m\}} \mathcal{H}_{l,m}$ into subspaces generated by spherical harmonics $\mathcal{H}_{l,m} = \{f \in L_2(\mathbb{R}^3, dx) :\ \exists u \in L_2(\mathbb{R}_+, dr),\ f(x) = |x|^{-1} u(|x|) Y_{l,m}(\theta,\varphi)\}$. As H_0 and H are rotational invariant, the $\mathcal{H}_{l,m}$ reduce H and H_0. Moreover, H_0 and H have the required properties on $\mathcal{H}_{l,m}$ (see [17]). Then we have shown that $W_\pm$ are generalized wave operators. □

Further, $e^{-iK_\pm t} = U_0(t) W_\pm^* W_\pm + (I - P_0^{ac}) = U_0(t)$ and then, $K_\pm = H_0$, as $P_0^{ac} = I$. Hence there exists a bounded linear operator J on $\mathcal{H}$ and wave operators $\Omega_\pm(H, H_0, J) = W_\pm$.

It was not clear in §1 whether $\Omega_\pm(H,H_0)$ do no exist for Coulomb potentials or our estimates were not accurate enough. This can now be answered.

$$\mathrm{w}-\lim_{|t|\to\infty} U(-t)U_0(t)P_0^{ac}u = \mathrm{w}-\lim_{|t|\to\infty} U(-t)e^{-iX_t}e^{iX_t}U_0(t)P_0^{ac}u = 0 \quad (63)$$

Indeed, we note that for $u \in \mathfrak{D}$, the Riemann-Lebesgue lemma implies that $\mathrm{w}-\lim_{|t|\to\infty} e^{iX_t}U_0(t)u = \mathrm{w}-\lim_{|t|\to\infty} e^{ic(2H_0^{1/2})^{-1}(\log(|t|)+\log(4H_0))}u = 0$, since $H_0^{-1/2}$ is absolutely continuous on $\mathcal{H}$. As $\mathrm{s}\text{-}\lim_{|t|\to\infty} U(-t)e^{-iX_t} = W_\pm$ on $\mathcal{H}$, (63) follows. Hence the strong limits $\Omega_\pm(H,H_0)$ do not exist.

Remark The fact that $K_\pm = H_0$ suggests that the renormalization of H_0 is a little special. We consider for a better understanding the classical scattering of a particle, described by $(\mathbf{r}, \dot{\mathbf{r}})$, by an attractive Coulomb force. The interacting orbits of interest are branches of hyperbola lying in the plane orthogonal to $\dot{\mathbf{r}} \times \mathbf{r}$. Moreover, as the conservation of energy E gives $|\dot{\mathbf{r}}| = \sqrt{2E - 2r^{-1}} \to \sqrt{2E} = |\dot{\mathbf{r}}_{\text{free}}|$, interacting particles have naturally free asymptotics in the phase space and we shall not need to adapt the structure of free dynamics, which is related to $K_\pm = H_0$. Nevertheless, considering the time parametrization, free orbits (straight lines) are given by $\mathbf{r}_{\text{free}}(t) = ct + b$. As interacting orbits have free asymptotics in phase space, there holds for large t, $\dot{\mathbf{r}}(t) = c + o(1)$, that is $\mathbf{r}(t) = ct + o(t)$. Further,

$$\dot{\mathbf{r}}(t) = \sqrt{2E - 2r^{-1}} = \sqrt{2E}(1 + (2Ect)^{-1} + o(t^{-1}))$$

Then $\mathbf{r}(t) = ct + d\log(t) + O(1)$. Because of this $\log(t)$ slippage, free asymptotics cannot catch up the interacting motion and the renormalization consists in adapting the time scale of free dynamics. This motivates the choice of (53). Further, the choice of the quantum mechanical analogue of d, $(2cH_0^{1/2})^{-1}$ is related to the cancellation of the integrand in the right-hand side of (54) as $|t| \to \infty$. Indeed, on the basis of classical results, $H - H_0 = |x|^{-1}$ shall look like $(2|p_{\text{free}}t|)^{-1}$ as $|t| \to \infty$ (Note that $H_0 = p^2$ implies $m = 1/2$).

5 Two space scattering

5.1 Definition and first properties

5.1.1 Basic properties of wave operators

Like in §1, we assume the existence of $\Omega_\pm(H, H_0, J)$ and collect useful properties.

Proposition 5.1 *$\Omega_\pm(H, H_0, J)$ is intertwining for H and H_0. That is $H\Omega_\pm(H, H_0, J) = \Omega_\pm(H, H_0, J)H_0$. In particular, if $\Delta \subseteq \mathbb{R}$, Borel set, then*

$$E(\Delta)\Omega_\pm(H, H_0, J) = \Omega_\pm(H, H_0, J)E_0(\Delta) \tag{64}$$

Then for any bounded Borel function φ,

$$\varphi(H)\Omega_\pm(H, H_0, J) = \Omega_\pm(H, H_0, J)\varphi(H_0) \tag{65}$$

Proof Similarly to the intertwining property in simple scattering, we obtain $U(t)\Omega_\pm = \Omega_\pm U_0(t)$ from which we derive $H\Omega_\pm = \Omega_\pm H_0$. Using the form expression of unitary groups, we have for any $f \in \mathcal{H}_0^{ac}$ and $g \in \mathcal{H}$

$$\begin{aligned}(U(t)\Omega_\pm f, g) = \int_{-\infty}^{\infty} e^{-i\lambda t} d(E(\lambda)\Omega_\pm f, g) &= \int_{-\infty}^{\infty} e^{-i\lambda t} d(E_0(\lambda)f, \Omega_\pm^* g) \\ &= (U_0(t)f, \Omega_\pm^* g)\end{aligned}$$

We note that the integrals are Fourier-Stieltjes transforms respectively of the complex-valued functions of bounded variation on $\mathbb{R}$ $(E(\cdot)\Omega_\pm f, g)$ and $(E_0(\cdot)f, \Omega_\pm^* g)$. Because of the continuity in λ of these functions and of the inversion formula for Fourier-Stieltjes transforms, there is a constant C so that $(E(\cdot)\Omega_\pm f, g) = (E_0(\cdot)f, \Omega_\pm^* g) + C$. As these are generating functions of the spectral measures, we have for any Borel set Δ that $(E(\Delta)\Omega_\pm f, g) = (\Omega_\pm E_0(\Delta)f, g)$. This implies (64). (65) follows from the functional calculus formula $(\varphi(H)f, g) = \int_{-\infty}^{\infty} \varphi(\lambda)d(E(\lambda)f, g)$. □

Proposition 5.2 (Chain rule) *Let H, H_0, H_1 be selfadjoint operators respectively on $\mathcal{H}$, $\mathcal{H}_0$ and $\mathcal{H}_1$, and be J_0: $\mathcal{H}_0 \to \mathcal{H}_1$, J_1: $\mathcal{H}_1 \to \mathcal{H}$. If $\Omega_\pm(H, H_1, J_1)$ and $\Omega_\pm(H_1, H_0, J_0)$ exist, then, for $J = J_1 J_0$, $\Omega_\pm(H, H_0, J)$ exists and $\Omega_\pm(H, H_0, J) = \Omega_\pm(H, H_1, J_1)\Omega_\pm(H_1, H_0, J_0)$.*

Proof Similar to the proof of the chain rule in simple scattering. □

5.1.2 Asymptotical equivalence of identifications

We have seen before that the inclusion of J extends the existence of wave operators to new pairs (H, H_0). For instance, in the extreme case of a compact J, we obtain that $JU_0(t)P_0^{ac}$ strongly converges to zero and then $\Omega_\pm(H, H_0, J)$ exist for any pair of selfadjoint operators (H, H_0). Conversely, for practical purposes, the choice of J given a system (H, H_0) is far from being unique.

Definition Two identifications operators J_1, J_2 are called asymptotically equivalent (with respect to the operator H_0 and the sign $\pm$) or H_0-equivalent if

$$s - \lim_{t\to\pm\infty}(J_1 - J_2)U_0(t)P_0^{ac} = 0 \tag{66}$$

and we note $J_1 \approx_{H_0} J_2$.

This defines equivalence classes of identification operators. We see that the wave operators $\Omega_\pm(H, H_0, J)$ depend only on the equivalence class defined by J.

Proposition 5.3 *If $J_1 \approx_{H_0} J_2$ and $\Omega_\pm(H, H_0, J_1)$ exists, then $\Omega_\pm(H, H_0, J_2)$ exists and $\Omega_\pm(H, H_0, J_1) = \Omega_\pm(H, H_0, J_2)$. Conversely if both wave operators exist and are equal, then $J_1 \approx_{H_0} J_2$.*

Proof The assertions follow from

$$\|\Omega(H, H_0, J_1)(t)f - \Omega(H, H_0, J_2)(t)f\| = \|(J_1 - J_2)U_0(t)P_0^{ac}f\|$$

□

In particular, the intertwining property implies $\|U(-t)\Omega_\pm U_0(t)P_0^{ac}f\| = \|\Omega_\pm f\|$ and $\Omega_\pm(H, H_0, J) = \Omega_\pm(H, H_0, \Omega_\pm)$. Then $\Omega_\pm \approx_{H_0} J$. Then a choice of J reduces to finding the wave operator up to H_0-equivalence. We also note that, since $Ran(\Omega_\pm) \subseteq \mathcal{H}^{ac}$, we have also $P^{ac}J \approx_{H_0} J$, and trivially $J \approx_{H_0} JP_0^{ac}$. Further, we see that the equivalence class defined by J contains the operators of the form $J + K$, with K compact. This brings the following test for the H_0-equivalence of two identifications.

Proposition 5.4 *Two identification operators J_1 and J_2 are H_0-equivalent if, for any bounded interval $\Delta \subseteq \mathbb{R}$, $(J_1 - J_2)E_0(\Delta)$ is compact. If $J_1 - J_2$ is even compact, this holds for any selfadjoint operator H_0.*

Proof If $(J_1-J_2)E_0(\Delta)$ is compact, then s-$\lim_{t\to\pm\infty}(J_1-J_2)E_0(\Delta)U_0(t)P_0^{ac} = 0$. As $E_0(\Delta)$ commutes with H_0 and with P_0^{ac}, there holds for any $f \in \mathcal{H}_0$ that

$$\lim_{t\to\pm\infty}(J_1 - J_2)U_0(t)P_0^{ac}E_0(\Delta)f = 0 \tag{67}$$

It remains to note that the set $\{E_0(\Delta)f,\ f \in \mathcal{H}_0,\ \Delta \text{ bounded interval in } \mathbb{R}\}$ lies dense in $\mathcal{H}_0$. This implies by density that $J_1 \approx_{H_0} J_2$. □

5.1.3 Partial isometricity of wave operators

In view of the unitary equivalence of parts of H and H_0, we turn to the partial isometricity of $\Omega_\pm(H, H_0, J)$ which is not automatic like in §1 because of the presence of J. One does not even know if the wave operators are not trivial on $\mathcal{H}_0^{ac}$. There only holds $\mathcal{H}_0^s \subseteq \ker(\Omega_\pm)$. That's why we now need some assumptions on J. There still holds

$$\|\Omega_\pm f\| = \lim_{t\to\pm\infty}\|\Omega(t)P_0^{ac}f\| = \lim_{t\to\pm\infty}\|\ JU_0(t)P_0^{ac}f\|$$

As J is bounded and $U_0(t)$ unitary, we obtain by the way the boundedness of $\Omega_\pm$ with the estimate $\|\Omega_\pm f\| \leq \||J|\|\|P_0^{ac}f\|$. This shows also that

Lemma 5.5 *$\Omega_\pm$ is isometric on $\mathcal{H}_0^{ac}$ iff for any $f \in \mathcal{H}_0^{ac}$*

$$\lim_{t\to\pm\infty} \| JU_0(t)f\| = \|f\| \tag{68}$$

that is also, iff $J \approx_{H_0} V$ with V partial isometry on $\mathcal{H}_0^{ac}$.

Proof It remains to show the second necessary and sufficient condition for partial isometricity. If $J \approx_{H_0} V$, then $\lim_{t\to\pm\infty} \| JU_0(t)f\| = \|VU_0(t)f\| = \|f\|$. Conversely, if $\Omega_\pm$ is partial isometric, $J \approx_{H_0} \Omega_\pm$ concludes the proof. □

Because of $\lim_{t\to\pm\infty} \| JU_0(t)f\| = \|f\|$, if $\Omega_\pm$ is isometric on $\mathcal{H}_0^{ac}$, then necessarily $\||J|\| \geq 1$. As it is more convenient to have assumptions depending only on J, there is also the following sufficient criterion which we develop with the same ideas as in (5.4).

Proposition 5.6 *$\Omega_\pm$ is isometric on $\mathcal{H}_0^{ac}$ if*

$$s\text{-}\lim_{t\to\pm\infty} (J^*J - I)U_0(t)P_0^{ac} = 0 \tag{69}$$

which is fulfilled if $(J^*J-I)E_0(\Delta)$ is compact for any bounded interval $\Delta \in \mathbb{R}$. If $J^*J - I$ is compact, this even holds for any selfadjoint H_0.

Proof We have

$$\|JU_0(t)f\|^2 = (J^*JU_0(t)f, U_0(t)f) = ((J^*J - I)U_0(t)f, U_0(t)f) + \|f\|^2 \tag{70}$$

and (67) implies (66) for $J_1 = J^*J$ and $J_2 = I$. Then $\Omega_\pm$ is isometric on $\mathcal{H}_0^{ac}$ by 5.5. The sufficient conditions involving compacity are like in 5.4. □

Note that for $\||J|\| = 1$, (67) is a necessary and sufficient condition for the isometricity of $\Omega_\pm$. Indeed $((I-J^*J)u, u) = \|u\|^2 - \|Ju\|^2 \geq (1 - \||J|\|^2)\|u\|^2 \geq 0$ and $I - J^*J$ is selfadjoint. Note that

$$((J^*J - I)U_0(t)f, U_0(t)f) = -\|(I - J^*J)^{1/2}U_0(t)f\|^2 \tag{71}$$

Then, because of (68) and (70), both sides of (71) converge to 0 as $t \to \pm\infty$ if $\Omega_\pm$ is isometric on $\mathcal{H}_0^{ac}$. By boundedness of J, this implies (69).

5.1.4 Unitary equivalence of Hamiltonians

For arbitrary $\Omega_\pm(H, H_0, J)$, $Ran(\Omega_\pm)$ is not necessarily closed. We define the sets $\mathcal{H}_0^\pm = \mathcal{H}_0 \ominus \ker(\Omega_\pm)$ and $\mathcal{H}^\pm = \overline{Ran(\Omega_\pm)}$.

Lemma 5.7 *$\mathcal{H}_0^\pm$ and $\mathcal{H}^\pm$ are respectively reducing subspaces for H_0 and H.*

Proof By the intertwining property, we immediately obtain that $ker(\Omega_\pm)$ is an invariant subspace for H_0 and that $Ran(\Omega_\pm)$ is an invariant subspace for H. By taking the adjoint in the intertwining relation, we obtain $H_0\Omega_\pm^* = \Omega_\pm^* H$. This immediately provides that $ker(\Omega_\pm^*)$ is invariant for H and $Ran(\Omega_\pm^*)$ invariant for H_0. We now use that $\mathcal{H} = \overline{Ran(\Omega_\pm)} \oplus \ker(\Omega_\pm^*)$ and similarly $\mathcal{H}_0 = \overline{Ran(\Omega_\pm^*)} \oplus \ker(\Omega_\pm)$. Then $\mathcal{H}_0^\pm$ and $\mathcal{H}^\pm$ are respectively reducing subspaces for H_0 and H. □

It is to note that $\mathcal{H}_0^s \subseteq \ker(\Omega_\pm)$ implies that $\mathcal{H}_0^\pm \subseteq \mathcal{H}_0^{ac}$.

If $\Omega_\pm$ is a partial isometry on $\mathcal{H}_0^{ac}$, then we obtain immediately from the intertwining relation (65) that H_0^{ac} and $H^\pm$ are unitary equivalent. Though its convenience, the partial isometricity has a rather weak role in the investigation of unitary equivalence. The unitary equivalence of $H^\pm$ and $H_0^\pm$ is in fact only provoked by the intertwining relation.

Proposition 5.8 *Let be $H_0^\pm$ and $H^\pm$ respectively the restrictions of H_0 on $\mathcal{H}_0^\pm$ and H on $\mathcal{H}^\pm$. $H_0^\pm$ and $H^\pm$ are unitarily equivalent.*

Proof Consider the unique polar decomposition $\Omega_\pm = U_\pm P_\pm$ with $P_\pm = (\Omega_\pm^*\Omega_\pm)^{1/2}$ and $U_\pm$ partial isometry with initial subspace $\overline{Ran(P_\pm)} = \overline{Ran(\Omega_\pm^*)} = \mathcal{H}_0^\pm$ and final subspace $\overline{Ran(\Omega_\pm)}$. According to (65) and its adjoint, we obtain for Borel functions φ

$$\Omega_\pm^*\Omega_\pm\varphi(H_0) = \Omega_\pm^*\varphi(H)\Omega_\pm = \varphi(H_0)\Omega_\pm^*\Omega_\pm$$

$\Omega_\pm^*\Omega_\pm$ is a non negative selfadjoint operator, and then, similarly to the proof of 5.1 by considering the measures $(E_{\Omega_\pm^*\Omega_\pm}(\cdot)\varphi(H_0)f, g)$ and $(E_{\Omega_\pm^*\Omega_\pm}f, \overline{\varphi}(H_0)g)$, we obtain that for any Borel function ψ

$$\psi(\Omega_\pm^*\Omega_\pm)\varphi(H_0) = \varphi(H_0)\psi(\Omega_\pm^*\Omega_\pm)$$

and in particular $P_\pm\varphi(H_0) = \varphi(H_0)P_\pm$. Then $\varphi(H)U_\pm P_\pm = U_\pm\varphi(H_0)P_\pm$. Moreover, the partial isometricity of $U_\pm$ implies that, for any $f \in \ker(\Omega_\pm)$, $U_\pm f = 0$ and $U_\pm\varphi(H_0)f = 0$ as $\ker(\Omega_\pm)$ reduces H_0. Hence, we have shown that, on $\overline{Ran(P_\pm)} \oplus \ker(\Omega_\pm) = \mathcal{H}_0^\pm \oplus \ker(\Omega_\pm) = \mathcal{H}_0$, $\varphi(H)U_\pm = U_\pm\varphi(H_0)$. As $U_\pm$ maps unitarily $\mathcal{H}_0^\pm$ onto $\mathcal{H}^\pm$, it concludes the proof. □

As $\mathcal{H}_0^\pm \subseteq \mathcal{H}_0^{ac}$, $H_0^\pm$ and therefore $H^\pm$ are absolutely continuous. Then $\mathcal{H}^\pm \subseteq \mathcal{H}^{ac}$, that is also $P^{ac}\Omega_\pm = \Omega_\pm$. We also note that the unitary equivalence of H_0^{ac} and $H^\pm$ is equivalent to

$$\ker(\Omega_\pm) = \mathcal{H}_0^s \tag{72}$$

Further, if $\Omega_\pm$ has closed range, then $\Omega_\pm$: $\mathcal{H}_0^{ac} \to \mathcal{H}^\pm$ is continuously invertible. Of course, the assumption of partial isometricity on $\mathcal{H}_0^{ac}$ is more convenient and explicites the inverse mapping by $\Omega_\pm^*\Omega_\pm = P_0^{ac}$.

5.1.5 Completeness of wave operators

As seen before the completeness is related to the unitary equivalence of $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$, which motivates this slightly adapted definition for wave operators with identifications.

Definition $\Omega_\pm(H, H_0, J)$ is said to be complete if (72) and

$$Ran(\Omega_\pm) = \mathcal{H}^{ac} \tag{73}$$

are fulfilled.

If $\Omega_\pm$ is continuously invertible on $\mathcal{H}_0^{ac}$, then it has trivial kernel on $\mathcal{H}_0^{ac}$ and the definition of completeness reduces to the condition (73). This is in particular the case if $\Omega_\pm$ is partially isometric on $\mathcal{H}_0^{ac}$.
According to 5.8, the unitary equivalence is not necessarily realized by $\Omega_\pm$ itself. Nevertheless it happens if $\Omega_\pm$ is partially isometric on $\mathcal{H}_0^{ac}$. Since the final projection $\Omega_\pm\Omega_\pm^*$ is the projection on $Ran(\Omega_\pm)$, completeness in this case is equivalent to

$$\Omega_\pm\Omega_\pm^* = P^{ac} \tag{74}$$

Like in simple scattering it is convenient to reduce the completeness of $\Omega_\pm(H, H_0, J)$ to the investigation of the existence of a related wave operator. We first treat our favorite special case.

Proposition 5.9 *Assume that $\Omega_\pm(H, H_0, J)$ is a partial isometry on $\mathcal{H}_0^{ac}$ and $\Omega_\pm(H_0, H, J^*)$ exists. Then the completeness of $\Omega_\pm(H, H_0, J)$ is equivalent to the partial isometricity of $\Omega_\pm(H_0, H, J^*)$ on $\mathcal{H}^{ac}$.*

Proof For arbitrary $\Omega_\pm(H, H_0, J)$, if $\Omega_\pm(H_0, H, J^*)$ exists, then we have $\Omega_\pm(H_0, H, J^*) = \Omega_\pm^*(H, H_0, J)$. Indeed, as the strong limit exists and as $P^{ac}\Omega_\pm(H, H_0, J) = \Omega_\pm(H, H_0, J)$, we can compute

$$(\Omega_\pm(H, H_0, J)f, g) = \lim_{t\to\pm\infty}(P^{ac}U(-t)JU_0(t)P_0^{ac}f, g) = (f, \Omega_\pm(H_0, H, J^*)g)$$

for any $g \in \mathcal{H}$. The completeness of $\Omega_\pm(H, H_0, J)$ is equivalent to (74), that is, to $\Omega_\pm(H_0, H, J^*)^*\Omega_\pm(H_0, H, J^*) = P^{ac}$, which defines the partial isometry. $\Box$

In this special case, $\Omega_\pm(H, H_0, J)$ and $\Omega_\pm(H_0, H, J^*)$ are unitary and mutually inverse as mappings between $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$.
We now turn to the general case. Similarly to H_0-equivalence, we note $A \approx_H B$ for s-$\lim\limits_{t\to\pm\infty}(A-B)U(t)P^{ac} = 0$, which also defines equivalence classes. We first note a lemma

Lemma 5.10 *If there is some bounded J_1:$\mathcal{H} \to \mathcal{H}_0$ so that $J_1J \approx_{H_0} I$, then $\Omega_\pm$ is countinuously invertible on $\mathcal{H}_0^{ac}$.*

Proof The estimate $\|J_1JU_0(t)P_0^{ac}f\| \le |||J_1|||\|JU_0(t)P_0^{ac}f\|$ holds. The right-hand side tends to $|||J_1|||\|\Omega_\pm f\|$ as $t \to \pm\infty$ and the left-hand side to $\|P_0^{ac}f\|$ because of $J_1J \approx_{H_0} I$ and the unitarity of $U_0(t)$. Then $\|P_0^{ac}f\| \le |||J_1|||\|\Omega_\pm\|$, that is, for any $f \in \mathcal{H}_0^{ac}$, $\|\Omega_\pm f\| \ge |||J_1|||^{-1}\|f\|$. So $\Omega_\pm$ is continuously invertible on $\mathcal{H}_0^{ac}$ with left inverse T and norm estimate $|||T||| \le |||J_1|||$. $\Box$

Proposition 5.11 *Suppose that $\Omega_\pm(H, H_0, J)$ exists and for some J_1:$\mathcal{H} \to \mathcal{H}_0$, $J_1J \approx_{H_0} I$. Then $\Omega_\pm(H, H_0, J)$ is complete iff $\Omega_\pm(H_0, H, J_1)$ exists and $JJ_1 \approx_H I$.*
Moreover, $\Omega_\pm(H_0, H, J_1)$ is complete, and $\Omega_\pm(H, H_0, J)$ and $\Omega_\pm(H_0, H, J_1)$ are mutually inverse mappings between $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$.

Proof Note $\Omega_\pm = \Omega_\pm(H, H_0, J)$ and $\tilde{\Omega}_\pm = \Omega_\pm(H_0, H, J_1)$. Because of lemma 5.10, the completeness of $\Omega_\pm$ reduces to the condition (73) under the assumptions of the proposition. The chain rule implies

$$\begin{aligned}\Omega_\pm\tilde{\Omega}_\pm &= \Omega_\pm(H, H, JJ_1) = P^{ac}\\ \tilde{\Omega}_\pm\Omega_\pm &= \Omega_\pm(H_0, H_0, J_1J) = P_0^{ac}\end{aligned} \tag{75}$$

where we have used $J_1 J \approx_{H_0} I$ and $JJ_1 \approx_H I$. Since $Ran(\Omega_\pm) \subseteq \mathcal{H}^{ac}$, it remains to show $\mathcal{H}^{ac} \subseteq Ran(\Omega_\pm)$, that is also $\ker(\Omega_\pm^*) \subseteq \mathcal{H}^s$. This is clear by taking the adjoint in the first relation of (75): $\tilde{\Omega}_\pm^* \Omega_\pm^* = P^{ac}$.

Conversely, if $Ran(\Omega_\pm) = \mathcal{H}^{ac}$, then for any $f \in \mathcal{H}^{ac}$, there exists $f_0 \in \mathcal{H}_0$ such that

$$\lim_{t \to \pm\infty} \|f - U(-t) J U_0(t) P_0^{ac} f_0\| = 0 \tag{76}$$

As $\|f - U(-t)JU_0(t)P_0^{ac} f_0\| = \|U_0(-t)J_1 U(t) f - U_0(-t) J_1 J U_0(t) P_0^{ac} f_0\|$ and $J_1 J \approx_{H_0} I$, this implies that $\lim_{t \to \pm\infty} \|U_0(-t) J_1 U(t) f - P_0^{ac} f_0\| = 0$ and then $\tilde{\Omega}_\pm$ exists.

Multiplying the equation defining $J_1 J \approx_{H_0} I$ on the left by J, we obtain

$$\text{s-}\lim_{t \to \pm\infty} (JJ_1 - I) J U_0(t) P_0^{ac} = 0 \tag{77}$$

(76) implies that for any $f \in \mathcal{H}^{ac}$,

$$\lim_{t \to \pm\infty} \|(JJ_1 - I) U(t) f - (JJ_1 - I) U_0(t) P_0^{ac} f_0\| = 0$$

for some f_0. Using (77), this shows $JJ_1 \approx_H I$. □

Corollary 5.12 *Assume that J is continuously invertible, then the completeness of $\Omega_\pm(H, H_0, J)$ is equivalent to the existence of $\Omega_\pm(H_0, H, J^{-1})$*

Proof If J is continuously invertible, then the conditions involving asymptotical equivalence in 5.11 are trivially fulfilled. □

(75) is a decisive condition in order to prove sufficient conditions. This motivates the following variant

Proposition 5.13 *If $\Omega_\pm$ and $\check{\Omega}_\pm = \Omega_\pm(H_0, H, J^*)$ exist and are both left invertible on $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$, then they are complete.*

Proof $\check{\Omega}_\pm$ exists, then $\check{\Omega}_\pm = \Omega_\pm^*$. $\Omega_\pm$ is left invertible on $\mathcal{H}_0^{ac}$, then completeness reduces to (73). There exists T with $T\check{\Omega}_\pm = T\Omega_\pm^* = P^{ac}$. Then $ker(\Omega_\pm^*) \subseteq \mathcal{H}^s$ which proves (73). □

Corollary 5.14 *Assume that J is continuously invertible. If $\check{\Omega}_\pm$ exists, then $\Omega_\pm$ and $\check{\Omega}_\pm$ are complete.*

Proof By lemma 5.10, if J is continuously invertible (as J^*), then $\Omega_\pm$ is continuously invertible on $\mathcal{H}_0^{ac}$. Similarly, by switching the roles of $\mathcal{H}$ and $\mathcal{H}_0$ in lemma 5.10, we have that $\check{\Omega}_\pm$ is continuous invertible on $\mathcal{H}^{ac}$. □

5.2 Existence and completeness of wave operators I: Cook's criterion

Time-dependent techniques are concerned with the estimation of $\|\Omega(t)f - \Omega(s)f\|$ as $t \to \pm\infty$, where $\Omega(t) = U(-t)JU_0(t)$ and $f \in \mathcal{H}_0^{ac}$. We first turn to a generalization of the Cook's criterion.

5.2.1 Cook's criterion

If we remain in the frame of single-particle scattering by a potential, this part is closely related to the algebraic scattering and in some sense is a concurrent technique. There we modified the free dynamics to force a simple Cook's estimate. Here we rather adapt our estimates to a larger bunch of problems involving identifications.

Like in simple scattering the idea is to obtain the convergence in $L_1([t_0, \pm\infty))$ of $\|(HJ - JH_0)U_0(t)f\|$ for some $t_0 \in \mathbb{R}$ and for f in a dense set of $\mathcal{H}_0^{ac}$. Of course the presence of J implies a little annoyance.

Proposition 5.15 *Let H_0 and H be self-adjoint operators respectively on $\mathcal{H}_0$ and $\mathcal{H}$. Suppose that there exists $\mathfrak{D} \subseteq \mathcal{D}(H_0)$, dense set in $\mathcal{H}_0^{ac}$, and $t_0 \in \mathbb{R}$ such that*

1. $\forall |t| > t_0$, $U_0(t)(\mathfrak{D}) \subseteq \mathfrak{D}$
2. $J(\mathcal{D}(H_0)) \subseteq \mathcal{D}(H)$
3. $\forall f \in \mathfrak{D}$, $\displaystyle\int_{t_0}^{\pm\infty} \|(HJ - JH_0)U_0(t)f\| dt < \infty$

then $\Omega_\pm(H, H_0, J)$ exist.

Proof For any $f \in \mathfrak{D}$, we define $\omega_f(t) = U(-t)JU_0(t)f$. Because of $\mathfrak{D} \subseteq \mathcal{D}(H_0)$ and conditions 1 and 2, ω_f is differentiable for any $f \in \mathfrak{D}$. The rest of the proof is similar to the proof in the simple space case with the derivative $\omega_f'(t) = iU(-t)(HJ - JH_0)U_0(t)f$ and the boundedness estimate $\|\Omega(t)(f - g)\| \leq \||J|\|\|f - g\|$. □

If J does not map $\mathcal{D}(H_0)$ into $\mathcal{D}(H)$, then $HJ - JH_0$ is not well-defined on $\mathfrak{D}$. In this case or if $HJ - JH_0$ is unbounded, a similar condition can be given in terms of the resolvent operators $R_0(\lambda) = (H_0 - \lambda I)^{-1}$ and $R(\lambda) = (H - \lambda I)^{-1}$: $\mathcal{H} \to \mathcal{D}(H)$.

Proposition 5.16 *Suppose that there exists $\mathfrak{D}$ dense set in $\mathcal{H}_0^{ac}$ and $z \in \rho(H_0) \cap \rho(H)$ such that for any $f \in \mathfrak{D}$*

$$\int_0^{\pm\infty} \|(R(z)J - JR_0(z))U_0(t)f\| dt < \infty \tag{78}$$

then $\Omega_\pm(H, H_0, J)$ exist.

Proof We have on the way to the second resolvent equality

$$\begin{aligned}R(z)J - JR_0(z) &= R(z)J(H_0 - zI)R_0(z) - (H - zI)R(z)JR_0(z) \\ &= R(z)JH_0R_0(z) - HR(z)JR_0(z) = (\tilde{J}H_0 - H\tilde{J})R_0(z)\end{aligned}$$

with $\tilde{J} = R(z)J$. Note that $\tilde{J}(\mathcal{D}(H_0)) \subseteq \mathcal{D}(\mathcal{H})$ and let be the set $\tilde{\mathfrak{D}}$ spanned in $\mathcal{H}_0$ by the set of vectors $\{R_0(z)U_0(t)u,\ u \in \mathfrak{D},\ t \in [0, \pm\infty)\}$. For any $u \in \mathfrak{D} \subseteq \mathcal{H}_0^{ac}$, $R_0(z)U_0(t)u$ lies in $\mathcal{H}_0^{ac} \cap \mathcal{D}(H_0)$. Then $\tilde{\mathfrak{D}} \subseteq \mathcal{H}_0^{ac} \cap \mathcal{D}(H_0)$. Moreover $R_0(z)(\mathfrak{D}) \subseteq \tilde{\mathfrak{D}}$ and $R_0(z)(\mathfrak{D})$ is dense in $\mathcal{H}_0^{ac}$, then $\tilde{\mathfrak{D}}$ is dense in $\mathcal{H}_0^{ac}$. By definition, we also have $U_0(t)(\tilde{\mathfrak{D}}) \subseteq \tilde{\mathfrak{D}}$ for any $t \in [0, \pm\infty)$ Then (78) implies for any $u \in \tilde{\mathfrak{D}}$

$$\int_0^{\pm\infty} \|(H\tilde{J} - \tilde{J}H_0)U_0(t)u\| dt < \infty$$

and we can apply the standard Cook's criterion for the identification $\tilde{J}$ and the dense set $\tilde{\mathfrak{D}}$. That is s-$\lim\limits_{t\to\pm\infty} U(-t)R(z)JU_0(t)P_0^{ac}$ exists on $\mathcal{H}_0$.
We have also for any $u \in \mathfrak{D}$

$$\|U(-t)R(z)JU_0(t)u - U(-t)JU_0(t)R_0(z)u\| = \|(R(z)J - JR_0(z))U_0(t)u\|$$

The right-hand side has to tend to 0 as $t \to \pm\infty$ because of (43). Then we obtain that the strong limit s-$\lim\limits_{t\to\pm\infty} U(-t)JU_0(t)u$ exists for $u \in \{R_0(z)v,\ v \in \mathfrak{D}\}$ and coincides with $\Omega_\pm(H, H_0, \tilde{J})v$. As $R_0(z)(\mathfrak{D})$ is dense in $\mathcal{H}_0^{ac}$, the wave operators $\Omega_\pm(H, H_0, J)$ exist in $\mathcal{H}_0$. $\square$

We now see that this proposition can be simplified if we use some property of the perturbation.

5.2.2 Factorization of perturbation

Let be $\mathcal{K}$ an auxiliary Hilbert space. The objective is to express existence conditions when the perturbation $HJ - JH_0$ can be formally written as

$$HJ - JH_0 = G^*G_0 \tag{79}$$

where G: $\mathcal{H} \to \mathcal{K}$ and G_0: $\mathcal{H}_0 \to \mathcal{K}$ are respectively H- and H_0-bounded. As $HJ - JH_0$ may not be well-defined, (79) is always considered in the form sense on dense sets, that is, for any $u \in \mathcal{D}(H_0)$ and $v \in \mathcal{D}(H)$

$$(Ju, Hv) - (JH_0u, v) = (G_0u, Gv) \tag{80}$$

If (80) holds, then for any $f \in \mathcal{H}_0$, $g \in \mathcal{H}$ and $z \in \rho(H) \cap \rho(H_0)$, $Ran(R_{(0)}(z)) = \mathcal{D}(H_{(0)})$ implies

$$\begin{aligned}&(G_0R_0(z)f, GR(\bar{z})g) = (JR_0(z)f, HR(\bar{z})g) - (JH_0R_0(z)f, R(\bar{z})g) \\ &= (JR_0(z)f, g) - (Jf, R(\bar{z})g) + (JR_0(z)f, \bar{z}R(\bar{z})g) - (zJR_0(z)f, R(\bar{z})g)\end{aligned} \tag{81}$$

As $GR(\bar{z})$: $\mathcal{H} \to \mathcal{K}$ is bounded, $(GR(\bar{z}))^*$ is a bounded mapping from $\mathcal{K}$ to $\mathcal{H}$. Then all the operators in (81) are bounded and well-defined and we obtain $((JR_0(z) - R(z)J)f, g) = ((GR(\bar{z}))^*G_0R_0(z)f, g)$ for any f, g. That is

$$JR_0(z) - R(z)J = (GR(\bar{z}))^*G_0R_0(z) \tag{82}$$

The equation (82) shows in particular that the factorization (80) always exists. Indeed, take for any $z \in \rho(H) \cap \rho(H_0)$, $G_0 = (JR_0(z) - R(z)J)(H_0 - zI)$ and $G = (H - \bar{z}I)$, hence $\mathcal{K} = \mathcal{H}$. As for any $\zeta \in \rho(H_0)$, $G_0R_0(\zeta) = (JR_0(z) - R(z)J)\big((\zeta - z)R_0(\zeta) + I\big)$, G_0 is H_0-bounded. Similarly, $GR(\zeta) = (\zeta - \bar{z})R(\zeta) + I$ for any $\zeta \in \rho(H)$ and G is H-bounded.
For such choice of G and G_0, (82) is automatically fulfilled and the factorization formula (80) follows.
The factorization is not unique. If $HJ - JH_0$ is bounded and well-defined on $D(H_0)$, hence on $\mathcal{H}_0$, we have the trivial choice $G_0 = HJ - JH_0$ and $G = I$.

We now have this variant of the last proposition.

Proposition 5.17 *Let (H, H_0, J) be a scattering system. If there is a factorization (G, G_0) of the perturbation and some set $\mathfrak{D} \subseteq \mathcal{D}(H_0)$ dense in $\mathcal{H}_0^{ac}$ such that*

$$\forall u \in \mathfrak{D},\ \exists t_u > 0 : \int_{\pm t_u}^{\pm\infty} \|G_0U_0(t)f\| dt < \infty \tag{83}$$

then $\Omega_\pm(H, H_0, J)$ exist.

Proof Let be $z \in \rho(H_0) \cap \rho(H)$, $(H_0 - zI)$: $\mathcal{D}(H_0) \to \mathcal{H}_0$ is a bijective mapping reduced by $\mathcal{H}_0^{ac}$. As $\mathfrak{D}$ is dense in $\mathcal{H}_0^{ac}$, then $\tilde{\mathfrak{D}} = (H_0 - zI)(\mathfrak{D})$ is dense in $\mathcal{H}_0^{ac}$. Further, for any $u \in \mathcal{D}(H_0)$, (82) provides

$$\|(R(z)J - JR_0(z))U_0(t)(H_0 - z)u\| \leq \||(GR(\bar{z})^*|\|\|G_0U_0(t)u\|$$

Then for any $v \in \tilde{\mathcal{D}}$, note $v = (H_0 - z)u$. Then, by boundedness of $R(z)J - JR_0(z)$ and (83), we obtain

$$\begin{aligned}\int_0^{\pm\infty} \|(R(z)J - JR_0(z))U_0(t)v\| dt &\leq t_u\||R(z)J - JR_0(z)|\|\|v\| dt \\ &+ \||(GR(\bar{z}))^*|\| \int_{\pm t_u}^{\pm\infty} \|G_0U_0(t)u\| dt < \infty\end{aligned}$$

We can now apply the proposition 5.16 for the dense set $\tilde{\mathcal{D}}$ and $\Omega_\pm(H, H_0, J)$ exist. □

Remark The proposition 5.15 constitutes in some sense a special case of 5.17 when $G_0 = HJ - JH_0$ and $G = I$ represent an admissible factorization of the perturbation.

5.2.3 Direct investigation of completeness

In the frame of algebraic scattering, if there were a J such that $\Omega_\pm(H, H_0, J)$ coincided with a generalized wave operator, then the completeness immediately followed. For practical purposes, propositions 5.15-5.17 present like before the inconvenient of calling for rather explicit expressions of $U_0(t)$, which for instance prevent investigation of completeness by mean of existence of $\Omega(H_0, H, J_1)$. Nevertheless, it is sometimes possible to investigate directly the completeness of wave operators.

If the wave operators $\Omega_\pm(H, H_0, J)$ exist and are partial isometries on $\mathcal{H}_0^{ac}$, then completeness reduces to $\Omega_\pm\Omega_\pm^* = P^{ac}$. This motivates the following proposition largely inspired from the method employed to prove $W_\pm W_\pm^* = P_0^{ac}$ in the section 4.4.

Proposition 5.18 (Completeness by spectral conditions) *Given a scattering system (H, H_0, J), suppose that $\Omega_\pm(H, H_0, J)$ exist and are partial isometries on $\mathcal{H}_0^{ac}$. Suppose there exist $n \in \mathbb{N} \cup \{\infty\}$ and decompositions in direct sums of $\mathcal{H}_0 = \bigoplus_{j=1}^{n} \mathcal{H}_{0j}$ and $\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j$. Then note H_{0j}^{ac} and H_j^{ac} the restrictions of H_0^{ac} and H^{ac} onto the subspaces $\mathcal{H}_{0j}$ and $\mathcal{H}_j$. If*

1. *$\mathcal{H}_{0j}$ and $\mathcal{H}_j$ reduce respectively H_0 and H*
2. *$J(\mathcal{H}_{0j}) \subseteq \mathcal{H}_j$*
3. *$\hat{\sigma}_j^{ac} \subseteq \hat{\sigma}_{0j}^{ac}$*
4. *H_j^{ac} has simple spectrum.*

then $\Omega_\pm(H, H_0, J)$ are complete.

Proof It suffices to prove $\Omega_\pm\Omega_\pm^* = P_0^{ac}$. First note that the conditions 1 and 2 imply that $\Omega_\pm$ map $\mathcal{H}_{0j}$ to $\mathcal{H}_j$. Define the projections onto $\mathcal{H}_{(0)j}$ by $P_{(0)j}$ and $\Omega_{j\pm} = \Omega_\pm P_{0j}$. $\Omega_{j\pm}$ is a partial isometry defined by

$$\Omega_{j\pm}^*\Omega_{j\pm} = P_{0j}\Omega_\pm^*\Omega_\pm P_{0j} = P_0^{ac}P_{0j}$$
$$\Omega_{j\pm}\Omega_{j\pm}^* = \Omega_\pm P_{0j}\Omega_\pm^* = P^{ac}P_j\Omega_{j\pm}\Omega_{j\pm}^*$$

This implies that $\Omega_{j\pm}\Omega_{j\pm}^* \leq P^{ac}P_j$ and $H\Omega_{j\pm}\Omega_{j\pm}^*$ is a part of H_j^{ac}. By the mapping $\Omega_{j\pm}$, H_{0j}^{ac} and $H\Omega_{j\pm}\Omega_{j\pm}^*$ are unitary equivalent. Then

$$\hat{\sigma}_{0j}^{ac} = \hat{\sigma}(H\Omega_{j\pm}\Omega_{j\pm}^*) \subseteq \hat{\sigma}_j^{ac}$$

The condition 3 implies the equality $\hat{\sigma}_j^{ac} = \hat{\sigma}(H\Omega_{j\pm}\Omega_{j\pm}^*)$. As H_j^{ac} has simple spectrum, we obtain from Lemma 2.8 that $\Omega_{j\pm}\Omega_{j\pm}^* = P^{ac}P_j$. This proves $\Omega_\pm\Omega_\pm^* = P^{ac}$. □

This proposition is meant in the view of scattering by spherical potential $v(|x|)$, with $\mathcal{H} = \mathcal{H}_0 = L_2(\mathbb{R}^3)$ and the decomposition $\mathcal{H} = \bigoplus_{l,m} \mathcal{H}_{l,m}$ into spherical harmonics.

In the application 3.9, we showed the existence of simple (hence partial isometric) wave operators $\Omega_\pm(H, H_0)$ for potentials decreasing asymptotically faster than the Coulomb potential. If such a potential is moreover spherically symmetric, then the assumptions of 5.18 are fulfilled with $\hat{\sigma}_{0lm}^{ac} = \hat{\sigma}_{lm}^{ac} = [0, \infty)$ and $\Omega_\pm(H, H_0)$ are complete.

In the case of the Coulomb potential, conditions 1, 3 and 4 also hold. Because of proposition 5.3, we can make the choice $J = W_\pm = f(H, H_0)$ in section 4.4, hence condition 2 holds and $\Omega_\pm(H, H_0, J)$ is partial isometric. $\Omega_\pm(H, H_0, J)$ are complete.

5.3 Existence and completeness of wave operators II: Trace class methods

In this section, the existence of the wave operators is derived under the condition that $HJ - JH_0$ is a trace operator. If $HJ - JH_0$ is not well-defined, the trace class property is meant in the form sense: there exists $C \in \mathfrak{S}_1(\mathcal{H}_0, \mathcal{H})$ such that

$$(Ju, Hv) - (JH_0 u, v) = (Cu, v) \tag{84}$$

5.3.1 Pearson's theorem

We first need to find a dense set in $\mathcal{H}_0^{ac}$ where we can estimate "$(HJ - JH_0)U_0(t)f$".

Lemma 5.19 *Let $\mathcal{H}$ be an Hilbert space and H a self-adjoint operator on it. Consider the set*

$$\mathfrak{R}_H = \{f \in \mathcal{H}^{ac},\ r_H(f)^2 = \operatorname{ess\,sup} \frac{d(E(\lambda)f, f)}{d\lambda} < \infty\}$$

$\mathfrak{R}_H$ is a dense subspace of $\mathcal{H}^{ac}$. Moreover for any $f \in \mathfrak{R}$ and $g \in \mathcal{H}$, the inequality

$$\int_{\mathbb{R}} \left|(U(t)f, g)\right|^2 dt \leq 2\pi r_H(f)^2 \|P^{ac} g\| \tag{85}$$

holds.

Proof For any $f \in \mathcal{H}^{ac}$, define the set $\Delta_n = \{\lambda, \frac{d(E(\lambda)f, f)}{d\lambda} \leq n\}$ and $X_n = \mathbb{R} \setminus \Delta_n$. Then $X_\infty = \bigcap_n X_n$ denotes the set where the derivative $\frac{d(E(\lambda)f, f)}{d\lambda}$ is infinite or not defined. As the Radon-Nikodym derivative lies in $L_1(\mathbb{R})$, X_∞ has to be a Lebesgue-nullset. Hence the sequence $|X_n|$ has to tend to zero as $n \to \infty$. It remains to produce a sequence lying in $\mathfrak{R}_H$ tending to f. Consider for any $n \in \mathbb{N}$, $f_n = E(\Delta_n)f$. Then, by

$$\frac{d(E(\lambda)f_n, f_n)}{d\lambda} = \frac{d(E(\Delta_n)E(\lambda)f, f)}{d\lambda} = \gamma_{\Delta_n}(\lambda)\frac{d(E(\lambda)f, f)}{d\lambda} \leq n$$

f_n lies in $\mathfrak{R}_H$. Further $\|f - f_n\| = \|(I - E(\Delta_n))f\| = \|E(X_n)f\|$. As $f \in \mathcal{H}^{ac}$, $(E(X_n)f, g) \to 0$ as $n \to \infty$ for any g. Then $\|f - f_n\| \to 0$ as $n \to \infty$. This proves the first part.

For any $f \in \mathfrak{R}_H$ and $g \in \mathcal{H}$, we have

$$(U(t)f, g) = \int_{\mathbb{R}} e^{-i\lambda t} \frac{d(E(\lambda)f, g)}{d\lambda} d\lambda$$

Then $\frac{d(E(\lambda)f, g)}{d\lambda}$ is the Fourier transform of $(U(t)f, g)$ and by Parseval's equation

$$\int_{\mathbb{R}} \left|(U(t)f, g)\right|^2 dt = 2\pi \int_{\mathbb{R}} \left|\frac{d(E(\lambda)f, g)}{d\lambda}\right|^2 d\lambda$$

Further we note that $(E(\lambda)f, g) = (E(\lambda)f, P^{ac}g)$ and

$$\left|\frac{d(E(\lambda)f, P^{ac}g)}{d\lambda}\right|^2 \leq \left|\frac{d(E(\lambda)f, f)}{d\lambda}\right| \left|\frac{d(E(\lambda)P^{ac}g, P^{ac}g)}{d\lambda}\right|$$

Hence we have

$$\int_{\mathbb{R}} \left| \frac{d(E(\lambda)f,g)}{d\lambda} \right|^2 d\lambda \leq r_H(f)^2 \int_{\mathbb{R}} \frac{d(E(\lambda)P^{ac}g, P^{ac}g)}{d\lambda} d\lambda \leq r_H(f)^2 \|P^{ac}g\|$$

This proves (85). □

This opens the way to the next lemma

Lemma 5.20 (Rosenblum's lemma) *Let $\mathcal{K}$ be an auxiliary Hilbert space. Let H be a selfadjoint operator in $\mathcal{H}$ and $G : \mathcal{H} \to \mathcal{K}$ a Hilbert-Schmidt operator. Then for any $f \in \mathfrak{R}_H$, there holds*

$$\int_{\mathbb{R}} \|GU(t)f\|^2 dt \leq 2\pi r_H(f)^2 \|G\|_2^2 \tag{86}$$

Proof As $G \in \mathfrak{G}_2$, we have the Schmidt representation $GU(t)f = \sum_{j=1}^{\infty} s_j (U(t)f, u_j) v_j$ where $\{u_j\}$ and $\{v_j\}$ are orthonormal systems with $\|G\|_2^2 = \sum_{j=1}^{\infty} s_j^2$. Then $\|GU(t)f\| = \sum_{j=1}^{\infty} s_j^2 \big|(U(t)f, u_j)\big|^2$ and by lemma 5.19

$$\int_{\mathbb{R}} \|GU(t)f\|^2 dt = \int_{\mathbb{R}} \sum_{j=1}^{\infty} s_j^2 \big|(U(t)f, u_j)\big|^2 dt = 2\pi r_H(f)^2 \|P^{ac}u_j\| \sum_{j=1}^{\infty} s_j^2$$

where the interchange of time integration and summation on j is justified by Fubini as the integrand on the right-hand side converges absolutely. With $\|P^{ac}u_j\| \leq 1$, we have proved (86). □

We can now investigate the existence of wave operators under trace class conditions. The keys of the next result are an appropriate factorization of the perturbation and the preceeding lemma.

Theorem 5.21 (Pearson) *Let H_0 and H be selfadjoint operators respectively in $\mathcal{H}_0$ and $\mathcal{H}$. If $HJ - JH_0 \in \mathfrak{G}_1$ in the form sense, then $\Omega_\pm(H, H_0, J)$ exist.*

Proof As $HJ - JH_0$ is trace class in the form sense, that is $C \in \mathfrak{G}_1$, there are G,$G_0 \in \mathfrak{G}_2$ so that $C = G^* G_0$. In other words, there is a factorization (G, G_0) of the perturbation with Hilbert-Schmidt factors. This also implies that

$$(Ju, Hv) - (JH_0 u, v) = (Cu, v) \leq |||C||| \|u\| \|v\|$$

so that the sesquilinear form defined in (84) extends by continuity from $\mathcal{D}(H_0) \times \mathcal{D}(H)$ to $\mathcal{H}_0 \times \mathcal{H}$. This gives sense to the computation (88). It suffices to show that $\|\Omega(t)f - \Omega(s)f\| \to 0$ when $t, s \to \pm\infty$ on the dense set $\mathfrak{R}_{H_0}$. We have

$$\|\Omega(t)f - \Omega(s)f\|^2 \leq (f, \Omega(t)^*(\Omega(t) - \Omega(s))f) - (f, \Omega(s)^*(\Omega(t) - \Omega(s))f) \tag{87}$$

For $u \in \mathcal{D}(H_0)$ and $v \in \mathcal{D}(H)$, there holds

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(\Omega(t)u, v) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(JU_0(t)u, U(t)v) = -i(JH_0U_0(t)u, U(t)v) \\ &\quad + (JU_0(t)u, Hu(t)v) = i(G_0U_0(t)u, GU(t)v) = i(U(-t)CU_0(t)u, v) \end{aligned}$$

Hence $((\Omega(t)-\Omega(s))u,v) = i\int\limits_s^t (U(-\tau)CU_0(\tau)u,v)d\tau$. By boundedness of all the operators, this equation extends from $\mathcal{D}(H_0)\times\mathcal{D}(H)$ to $\mathcal{H}_0\times\mathcal{H}$. We take arbitrary orthonormal basis $\{\phi_j\}$ and $\{\psi_j\}$ respectively in $\mathcal{H}_0$ and $\mathcal{H}$. As $\|U(-\tau)CU_0(\tau)\|_1 = \|C\|_1$, there holds

$$\begin{aligned}\sum_j \left|((\Omega(t)-\Omega(s))\phi_j,\psi_j)\right| &\le \int\limits_t^s \sum_j \left|(U(-\tau)CU_0(\tau)\phi_j,\psi_j)\right| d\tau \\ &\le (t-s)\|C\|_1 < \infty\end{aligned}$$

Hence, $\Omega(t)-\Omega(s)\in\mathfrak{G}_1$ is compact and so is $\Omega(t)^*(\Omega(t)-\Omega(s))$. We focus on the first term in the right-hand side of (87). The treatment is similar for the second term. The compactness above implies

$$\begin{aligned}&(f,\Omega(t)^*(\Omega(t)-\Omega(s))f) \\ &\quad= \lim_{\tau\to\pm\infty}(f,\Omega(t)^*(\Omega(t)-\Omega(s)) - U_0(-\tau)\Omega(t)^*(\Omega(t)-\Omega(s))U_0(\tau)f)\end{aligned}$$

Put $X_{s,t}(f) = \lim_{\tau\to\pm\infty}(f,\Omega(t)^*\Omega(s)-U_0(-\tau)\Omega(t)^*\Omega(s)U_0(\tau)f)$ for abbreviation. Then $(f,\Omega(t)^*(\Omega(t)-\Omega(s))f) = X_{t,t}(f)-X_{s,t}(f)$. It suffices to show $X_{s,t}(f)\to 0$ as $s,t\to\infty$. The form of $X_{s,t}(f)$ suggests an integral form. For any $f\in\mathcal{D}(H_0)$, we have using the definition of $\Omega(t)$

$$\begin{aligned}\tfrac{\partial}{\partial\tau}(U_0(\tau)f,\Omega(t)^*\Omega(s)U_0(\tau)f) &= -i(H_0U_0(\tau)f,\Omega(t)^*\Omega(s)U_0(\tau)f) \\ &\quad +i(U_0(\tau)f,\Omega(t)^*\Omega(s)H_0U_0(\tau)f) \\ &= -i(JH_0U_0(t+\tau)f,U(t-s)JU_0(s+\tau)f) \\ &\quad +i(U(s-t)JU_0(t+\tau)f,JH_0U_0(s+\tau)f) \\ &= i(CU_0(t+\tau)f,U(t-s)JU_0(s+\tau)f) \\ &\quad -i(U(s-t)JU_0(t+\tau)f,CU_0(s+\tau)f)\end{aligned} \tag{88}$$

where we have used the relation (84) in the last equality. Then

$$X_{s,t}(f) = -i\lim_{\tau\to\pm\infty}\int_0^\tau (U_0(t+\rho)f,\left[C^*U(t-s)J-J^*U(t-s)C\right]U_0(s+\rho)f)d\rho \tag{89}$$

and by boundedness of all operators, this representation extends to $f\in\mathcal{H}_0$. Assume now that $f\in\mathfrak{R}_{H_0}$. Again the integrand is a sum of two terms and we focus on the first one. Using $C=G^*G_0$ and Cauchy-Schwarz inequality, we have the estimate

$$\begin{aligned}\left|\int_0^{\pm\infty}(U_0(t+\rho)f,C^*U(t-s)JU_0(s+\rho)f)d\rho\right| &\le \pm\int_0^{\pm\infty}\|G_0U_0(t+\rho)f\| \\ &\quad\times\|GU(t-s)JU_0(s+\rho)f\|d\rho\end{aligned} \tag{90}$$

Lemma 5.20 grants that both terms of the integrand are square-integrable so that

$$(55)\le\left(\pm\int_t^{\pm\infty}\|G_0U_0(\rho)f\|^2d\rho\right)^{1/2}\underbrace{\left(\pm\int_s^{\pm\infty}\|GU(t-s)JU_0(\rho)f\|^2d\rho\right)^{1/2}}_{\le(2\pi)^{1/2}r_{H_0}(f)\|G\|_2\||J\||}$$

The right-hand side tends to zero as $s \to \pm\infty$ uniformly with respect to t. By conjuging the integrand in (90) and interchanging the role of s and t, we obtain the same estimate for the second term of the integrand in (89). It shows that $X_{s,t}(f) \to 0$ as $s, t \to \pm\infty$ and the proof is complete. □

Remark The proof provides the estimate $\|(\Omega(t) - \Omega(s))f\|^2 \le |X_{t,t}(f)| + |X_{s,t}(f)| + |X_{t,s}(f)| + |X_{s,s}(f)|$ for $f \in \mathfrak{R}_{H_0}$. That is, using (89) and (90),

$$\|(\Omega(t) - \Omega(s))f\|^2 \le 4(2\pi)^{1/2} r(f) \|G\|_2 \||J\|| \times \left(\left| \int_t^{\pm\infty} \|G_0 U_0(\rho) f\|^2 d\rho \right|^{1/2} + \left| \int_s^{\pm\infty} \|G_0 U_0(\rho) f\|^2 d\rho \right|^{1/2} \right) \quad (91)$$

If $HJ - JH_0 \in \mathfrak{G}_1$ in the sense of (84), we also have

$$(J^* v, H_0 u) - (J^* H v, u) = (-C^* v, u)$$

and $H_0 J^* - J^* H \mathfrak{G}_1$ in the form sense. Then under the assumptions of the last theorem, $\Omega_\pm(H, H_0, J)$ and $\Omega_\pm(H_0, H, J^*)$ both exist. Then, under the practical assumptions $J^* J \approx_{H_0} I$ and $JJ^* \approx_H I$, $\Omega_\pm(H, H_0, J)$ and $\Omega_\pm(H_0, H, J^*)$ are complete. This is the main advantage of the trace class method. If $J = I$, this proves

Corollary 5.22 (Kato-Rosenblum) *Suppose H and H_0 are selfadjoint operators on $\mathcal{H}$ and $H - H_0 \in \mathfrak{G}_1$. Then $\Omega_\pm(H, H_0)$ exist and are complete.*

Nevertheless, the assumption that the perturbation is trace class is too much strong for practical purposes. Potential scattering seems to be in particulary bad shape. The spectrum of multiplication operators by an essentially bounded function ϕ is the essential range of ϕ and if ϕ is continuous on some interval, the perturbation cannot even be compact. The next propositions present relaxed conditions susceptible to be fulfilled in applications.

5.3.2 Resolvent-based formula

Based on resolvents of H and H_0, we have immediate corollaries of the Pearson's theorem. The key ideas in this part are that $R_0(z)(\mathcal{H}_0)$ lies dense in $\mathcal{H}_0$ and the equivalence classes in 5.1.2.

Corollary 5.23 (Kuroda-Birman) *Let be H and H_0 self-adjoint operators so that $R(z)J - JR_0(z) \in \mathfrak{G}_1$ for some $z \in \rho(H) \cap \rho(H_0)$. Then $\Omega_\pm(H, H_0, J)$ (and $\Omega_\pm(H_0, H, J^*)$) exist.*

Proof Note $\tilde{J} = R(z) J R_0(z)$: $\mathcal{H}_0 \to \mathcal{D}(H)$. Then

$$\begin{aligned} H\tilde{J} - \tilde{J} H_0 &= (H - z + z) R(z) J R_0(z) - R(z) J R_0(z) (H_0 - z + z) \\ &= J R_0(z) - R(z) J \in \mathfrak{G}_1 \quad (92) \end{aligned}$$

and we can apply Pearson's theorem, that is the limits

$$s - \lim_{t \to \pm\infty} U(-t) R(z) J R_0(z) U_0(t) P_0^{ac}$$

exist. Since elements in the dense set $\mathcal{D}(H_0)$ are of the form $R_0(z)f$, we obtain the existence of $\Omega_\pm(H, H_0, R(z)J)$. Since $R(z)J - JR_0(z)$ is trace class, hence compact, $R(z)J$ and $JR_0(z)$ are in the same equivalence class and $\Omega_\pm(H, H_0, JR_0(z))$ exist. Again $R_0(z)(\mathcal{H}_0)$ lies dense in $\mathcal{H}_0$ and $\Omega_\pm(H, H_0, J)$ exist. □

The idea that stacking up $R_0(z)$ on the right and $R(z)$ on the left of J does not change a thing if we have appropriate trace class assumptions generalizes the preceeding corollary to higher powers of resolvents.

Proposition 5.24 *Let be H and H_0 self-adjoint operators. Suppose further that for some $z \in \rho(H) \cap \rho(H_0)$ and some $p \in \mathbb{N}$, $T_p(z) = R(\zeta)^p J - JR_0(\zeta)^p \in \mathfrak{G}_1$ for any ζ in a neighbourhood $U(z) \subset \rho(H) \cap \rho(H_0)$ of z. Then $\Omega_\pm(H, H_0, J)$ (and $\Omega_\pm(H_0, H, J^*)$) exist.*

Proof First recall for an holomorphic function on G the special form of the Laurent series as Taylor expansion

$$\frac{f^n(z)}{n!} = (2\pi i)^{-1} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

where $\Gamma \subset G$ is a counterclockwise oriented closed contour enclosing z one time and apply this to $R_0(z)^p$ and $R(z)^p$ with Γ lying in $U(z)$. Note first that $\frac{\mathrm{d}}{\mathrm{d}z} R(z)^p = pR(z)^{p+1}$. Then we have for $m > p$

$$(m-p)! \int_\Gamma \frac{R(\zeta)^p}{(\zeta - z)^{m-p+1}} d\zeta = \frac{\mathrm{d}^{m-p}}{\mathrm{d}z^{m-p}} R(z)^p = \frac{(m-1)!}{(p-1)!} R(z)^m$$

and similarily for $R_0(z)^p$, this provides for any integer $m > p$

$$T_m(z) = \frac{(m-p)!(p-1)!}{2\pi i (m-1)!} \int_\Gamma \frac{T_p(\zeta)}{(\zeta - z)^{m-p+1}} d\zeta$$

Hence $T_m(z)$ is trace class. We define $\tilde{J} = R(z)^{p-1} \sum_{j=0}^{p-1} R(z)^{j+1} JR_0(z)^{p-j}$ and evaluate

$$\begin{aligned} H\tilde{J} - \tilde{J}H_0 &= R(z)^{p-1} \sum_{j=0}^{p-1} R(z)^j JR_0(z)^{p-j} - R(z)^{p-1} \sum_{j=1}^{p} R(z)^j JR_0(z)^{p-j} \\ &= R(z)^{p-1} \left(JR_0(z)^p - R(z)^p J \right) = R(z)^{p-1} T_p(z) \end{aligned}$$

which is trace class. Then $\Omega_\pm(H, H_0, \tilde{J})$ exist. Note $J' = pJR_0(z)^{2p}$. Then

$$\tilde{J} - J' = \sum_{j=0}^{p-1} \left(R(z)^{p+j} JR_0(z)^{p-j} - JR_0^{2p} \right) = \sum_{j=0}^{p-1} T_{p+j}(z) R_0^{p-j}$$

is also trace class since the $T_{p+j}(z)$ are trace class. Hence it is compact, $\tilde{J}$ and J' are in the same equivalence class so that $\Omega_\pm(H, H_0, JR_0(z)^{2p})$ exist. It remains to note $2p$ times that $R_0(z)(\mathcal{H}_0)$ lies dense in $\mathcal{H}_0$.
As our criteria also hold if H and H_0 change roles and J is replaced by J^*, the existence of $\Omega_\pm(H_0, H, J^*)$ also follows. □

Assume now that $\mathcal{H}_0 = \mathcal{H}$ and $J = I$. For practical purposes, it is easier to use $R_0(z)$ than $R(z)$ since the first one is often the only to be known explicitely. This shall not be a problem since $R(z) = R(z)(H_0 - z)R_0(z)$ with $R(z)(H_0 - z)$ bounded. We resume the problem of potential scattering treated at the beginning in order to illustrate this. We have already shown that under certain conditions on V, $\Omega_\pm$ exist. Their completeness was nevertheless still uncertain except for spherical potentials.

Application 5.25 *Let be $H_0 = -\Delta$ and $H = -\Delta + V$ on $L_2(\mathbb{R}^3)$ with V multiplication operator by $v(x)$. We show that if $v \in L_2(\mathbb{R}^3) \cap L_1(\mathbb{R}^3)$, then $\Omega_\pm(H, H_0)$ exist and are complete.*

Proof We want to show that $R_0(z) - R(z) \in \mathfrak{S}_1$ for some $z \in \rho(H) \cap \rho(H_0)$. The particular form of H and H_0 allows the choice $z = c < 0$ with $|c|$ large enough so that $R(c)$ and $R_0(c)$ are defined. Equation (92) with $J = I$ provides

$$R_0(c) - R(c) = R(c)VR_0(c) = R(c)(H_0 - z)R_0(c)VR_0(c) \tag{93}$$

It remains to show that $R_0(c)VR_0(c)$ is trace class. Consider the polar decomposition of $V = UP$ with $P = (V^*V)^{1/2}$ so that P turns to be the multiplication operator by $\left|v(x)\right|$ and U the multiplication by $sign(v(x))$. Then U is bounded and we further take the square root $P^{1/2}$, multiplication by $\left|v(x)\right|^{1/2}$, since P is selfadjoint and non-negative, so that we have

$$R_0(c)VR_0(c) = U\left(R_0(c)P^{1/2}\right)\left(P^{1/2}R_0(c)\right)$$

Note that $\Phi R_0(c)\Phi^*$ is multiplication by $\frac{1}{\xi^2 - c}$ in $L_2(\mathbb{R}^3, d\xi)$ and we have

$$\Phi R_0(c)P^{1/2}u(\xi) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\frac{e^{-ix\cdot\xi}}{\xi^2 - c}\left|v(x)\right|^{1/2}u(x)dx$$

as an integral operator with kernel $k(x,\xi) = \dfrac{e^{-ix\cdot\xi}}{\xi^2 - c}\left|v(x)\right|^{1/2}$ of Hilbert-Schmidt type since $v(x) \in L_1(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|k(x,\xi)\right|^2 dxd\xi = \int_{\mathbb{R}^3}\left|v(x)\right| dx \int_{\mathbb{R}^3}\frac{1}{(\xi^2 - c)^2}d\xi \leq \infty$$

then $R_0(c)P^{1/2}$ and its adjoint $P^{1/2}R_0(c)$ are Hilbert-Schmidt, then $R_0(c)VR_0(c)$ is trace class and by Kuroda-Birman theorem $\Omega_\pm(H, H_0)$ and $\Omega_\pm(H_0, H)$ exist and are then complete. □

5.3.3 Spectral cutoffs

A second natural and convenient strenghtening of the existence theorem is to consider spectral cutoffs of the trace class assumption.

Proposition 5.26 *If for any bounded interval Δ, $(HJ - JH_0)E_0(\Delta) \in \mathfrak{S}_1$ in the form sense, then $\Omega_\pm(H, H_0, J)$ exist.*

Proof Note $J' = JE_0(\Delta)$. Then by theorem 5.21, $\Omega(H, H_0, J')$ exist. It is then clear that $\lim_{t\to\pm\infty}\|\Omega_\pm(H, H_0, J')f - U(-t)JU_0(t)P_0^{ac}E_0(\Delta)f\| = 0$. It remains to note that the set $\{E_0(\Delta)f,\ f \in \mathcal{H}_0,\ \Delta \text{ bounded interval in } \mathbb{R}\}$ lies dense in $\mathcal{H}_0$. Then $\Omega_\pm(H, H_0, J)$ exist. □

The inconvenient there is that the assumption is not symmetrical in H and H_0 and prevent the investigation of completeness. In this view, we consider the symmetric condition $E(\Delta)(HJ - JH_0)E_0(\Delta) \in \mathfrak{S}_1$. If we note $J'' = E(\Delta)JE_0(\Delta)$, then the wave operators $\Omega_\pm(H, H_0, J'')$ exist. In other words, the limits

$$\mathrm{s} - \lim_{t\to\pm\infty} E(\Delta)U(-t)JU_0(t)E_0(\Delta)P_0^{ac} \tag{94}$$

exist. We now need an additional condition in order to remove the projection $E(\Delta)$ on the left.

Proposition 5.27 *If for any bounded interval Δ, $E(\Delta)(HJ - JH_0)E_0(\Delta) \in \mathfrak{S}_1$ in the form sense and*

$$s - \lim_{t\to\pm\infty} E(\mathbb{R} \setminus \Delta)JU_0(t)E_0(\Delta)P_0^{ac} = 0 \tag{95}$$

then $\Omega_\pm(H, H_0, J)$ exists.

Proof The limits (94) exist. Then by (95), we have $\lim\limits_{t\to\pm\infty} \|\Omega_\pm(H, H_0, J''f - U(-t)JU_0(t)P_0^{ac}E_0(\Delta)f\| = 0$. Then by density of the elements $E_0(\Delta)f$, $\Omega_\pm(H, H_0, J)$ exist. $\square$.

Remark The mechanism of spectral cutoffs can be extended to this additional condition. Namely the equation (95) for an arbitrary interval Δ is equivalent to

$$\mathrm{s} - \lim_{t\to\pm\infty} E(\mathbb{R} \setminus \Delta)JU_0(t)E_0(\Delta_0)P_0^{ac} = 0 \tag{96}$$

for any strict subinterval $\Delta_0 \subset \Delta$, since the set $\{E_0(\Delta_0)f,\ f \in \mathcal{H}_0^{ac},\ \Delta_0$ strict subinterval of $\Delta\}$ lies dense in $E_0(\Delta)(\mathcal{H}_0^{ac})$.
By the Riemann-Lebesgue lemma, this condition is clearly satisfied if $E(\mathbb{R} \setminus \Delta)JE_0(\Delta_0)$ is compact. The next lemma points out concrete criteria for this.

Lemma 5.28 *$E(\mathbb{R} \setminus \Delta)JE_0(\Delta_0)$ is compact for any bounded interval Δ and strict subintervals $\Delta_0 \subset \Delta$ if one of the following conditions holds:*

1. *for any $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(H)J - J\varphi(H_0)$ is compact*
2. *there exists $z \in \rho(H) \cap \rho(H_0)$ so that $R(z)J - JR_0(z)$ is compact*

Proof If the first condition holds, let choose for fixed Δ and Δ_0 a function $\varphi \in C_0^\infty(\mathbb{R})$ so that $\varphi(\lambda) = 1$ for $\lambda \in \Delta_0$ and $\varphi(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus \Delta$. This is possible since $\mathbb{R} \setminus \Delta$ and Δ_0 are disjoint. Then recalling the definition $(\varphi(H)E(\Lambda)f, g) = \displaystyle\int_{\mathbb{R}} \varphi(\lambda)\gamma_\Lambda(\lambda)d(E(\lambda)f, g)$ for any self-adjoint operator H and Borel set Λ, we obtain $E(\mathbb{R} \setminus \Delta)\varphi(H) = 0$ and $\varphi(H_0)E_0(\Delta_0) = I$, hence $E(\mathbb{R} \setminus \Delta)JE_0(\Delta_0) = E(\mathbb{R} \setminus \Delta)(J\varphi(H_0) - \varphi(H)J)E_0(\Delta_0)$, which is then compact.
If the second condition holds, then for any $\zeta \in \rho(H) \cap \rho(H_0)$, there holds

$$\begin{aligned} R(\zeta)J - JR(\zeta) &= (H - z)R(\zeta)R(z)JR_0(\zeta)(H_0 - z + z - \zeta) - (H - z + z - \zeta) \\ &\quad \times R(z)JR_(\zeta)R_0(\zeta)R_0(z)(H_0 - z) \\ &= (H - z)R(\zeta)\left[R(z)J - JR(z)\right]R_0(\zeta)(H_0 - z) \end{aligned} \tag{97}$$

and $R(\zeta)J - JR(\zeta)$ is compact. In view of the Riesz projection, we consider a closed contour Γ enclosing Δ_0 one time counterclockwise and leaving the segments of $\mathbb{R} \setminus \Delta$ outside. Take the self-adjoint operator $H_0 E_0(\Delta_0)$ on the space $E_0(\Delta_0)(\mathcal{H}_0)$ with the resolvent $R_{H_0 E_0(\Delta_0)}(z) = R_0(z)E_0(\Delta)$ (in a precise sense, this should be defined on $\mathbb{R} \setminus \Delta$ as a limit). Then $\sigma(H_0 E_0(\Delta_0)) \cap \Gamma = \emptyset$ and we have

$$E_0(\Delta_0) = -(2\pi i)^{-1} \oint_\Gamma R_0(z)E_0(\Delta)dz$$

Similarly, for the operator $HE(\mathbb{R} \setminus \Delta)$ on $E(\mathbb{R} \setminus \Delta)(\mathcal{H})$, we obtain

$$\oint_\Gamma R(z)E(\Delta)dz = 0$$

since the spectrum of $HE(\mathbb{R} \setminus \Delta)$ lies outside the contour Γ. Hence we obtain

$$E(\mathbb{R} \setminus \Delta)JE_0(\Delta_0) = (2\pi i)^{-1} \oint_\Gamma E(\mathbb{R} \setminus \Delta)[R(z)J - JR_0(z)]E_0(\Delta_0)dz$$

By assumption, the integrand is continuous and takes compact values. Hence, $E(\mathbb{R} \setminus \Delta)JE_0(\Delta_0)$ is compact. $\square$

A condition similar to (95) can be obtained by other means. We first need a technical definition.

Definition Let H and H_0 be self-adjoint operators. The operator H is called subordinate (with respect to J) to H_0 if there exist a pair of locally bounded functions f, f_0 on $\mathbb{R}$ such that $|f(\lambda)| \geq 1$, $|f_0(\lambda)| \geq 1$ and $\lim_{|\lambda|\to\infty} |f(\lambda)| = \lim_{|\lambda|\to\infty} |f(\lambda)| = \infty$ and $J(\mathcal{D}(f_0(H_0)) \subset \mathcal{D}(f(H))$.
If H_0 is simultaneously subordinate (with respect to J^*) to H, then H and H_0 are called mutually subordinate.

Note that the conditions on f and f_0 imply that $f_0(H_0)^{-1}$ and $f(H)^{-1}$ are bounded mappings from $\mathcal{H}_0$ (resp. $\mathcal{H}$) to $\mathcal{D}(f_0(H_0))$ (resp. $\mathcal{D}(f(H))$). Then $J(\mathcal{D}(f_0(H_0)) \subset \mathcal{D}(f(H))$ implies that $f(H)Jf_0(H_0)^{-1}$ is well-defined and bounded.
Note further that mutual subordinacy is a quite weak condition. If $\mathcal{H}_0 = \mathcal{H}$ and $J = I$, then $\mathcal{D}(H) = \mathcal{D}(H_0)$ guarantee mutual subordinacy of H and H_0.

Lemma 5.29 *Define the interval* $\Delta_r = (-r, r)$. *If* H *is subordinate to* H_0, *then for any bounded interval* Δ

$$\lim_{r\to\infty} |||E(\mathbb{R} \setminus \Delta_r)JE_0(\Delta)||| = 0 \tag{98}$$

Proof We estimate the quantity on the right-hand side by

$$|||E(\mathbb{R} \setminus \Delta_r)JE_0(\Delta)||| \leq |||E(\mathbb{R} \setminus \Delta_r)f(H)^{-1}||| \, |||f(H)Jf_0(H_0)^{-1}||| \\ \times |||f_0(H_0)E_0(\Delta)|||$$

where f, f_0 are the functions for which H is subordinate to H_0. Note that $E(\mathbb{R}\setminus\Delta_r)f(H)^{-1} = \int_\mathbb{R} \underbrace{\gamma_{\mathbb{R}\setminus(-r,r)}(\lambda)f(\lambda)^{-1}}_{\xrightarrow[r\to\infty]{} 0} dE(\lambda)$ since $f(\lambda)^{-1} \to 0$ as $|\lambda| \to \infty$.

Then the first factor on the right-hand side tends to zero as $r \to \infty$. The second factor is bounded since H is subordinate to H_0. Further the third factor is bounded since Δ is a bounded interval and $f_0(H_0)E_0(\Delta)$ is a bounded operator. Then we obtain (98). $\square$

It is time to put everything together.

Theorem 5.30 *Assume that for any bounded interval Δ, $E(\Delta)(HJ - JH_0)E_0(\Delta) \in \mathfrak{S}_1$ in the form sense and that one of the following condition is fulfilled:*

1. *for any $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(H)J - J\varphi(H_0)$ is compact*
2. *there is $z \in \rho(H) \cap \rho(H_0)$ such that $R(z)J - JR_0(z)$ is compact*
3. *H and H_0 are mutually subordinate*

then $\Omega_\pm(H, H_0, J)$ and $\Omega_\pm(H_0, H, J^*)$ exist.

Proof It remains to prove it in the case of the association of the trace class assumption and mutual subordinacy. Taking the identification operator $J = E(\Delta_r)JE_0(\Delta)$, the limit

$$\mathrm{s} - \lim_{t \to \pm\infty} E(\Delta_r)U(-t)JU_0(t)E_0(\Delta)P_0^{ac} \tag{99}$$

exists for any bounded interval Δ and $r \in \mathbb{R}_+$ because of Pearson's theorem. Note that for any $f \in \mathcal{H}_0$

$$\lim_{r \to \infty} \|(I - E(\Delta_r))U(-t)JU_0(t)P_0^{ac}E_0(\Delta)f\| \leq \lim_{r \to \infty} \||E(\mathbb{R} \setminus \Delta_r)JE_0(\Delta)|\|\|f\|$$

where the right-hand side is equal to zero. Then $\Omega_\pm(H, H_0, J)$ exists on the dense set $\{E_0(\Delta)f,\ f \in \mathcal{H}_0,\ \Delta \text{ bounded interval in } \mathbb{R}\}$, then on $\mathcal{H}_0$. Lastly we note that if one of the assumptions of the theorem holds, then it holds also if the operators H and H_0 change roles and J is replaced by J^*. Then $\Omega_\pm(H_0, H, J^*)$ exist simultaneously. $\square$

Remark Combination of the trace class condition and mutual subordinacy appears in the litterature as the Birman's theorem.

5.4 Stationary wave operators

In the last part, we have constructed the wave operators in terms of the time-dependent unitary evolution groups $U(t)$ and $U_0(t)$. The first aim of a stationary theory is to express the wave operators in terms of the time-independent resolvents $R(z)$, $R_0(z)$.

5.4.1 Weak and weak abelian wave operators

Until here, we only considered the limit (1) in the strong sense because it was the natural frame for developing a time-dependent scattering theory. Resolvent-based definitions for the wave operators accomodate a much weaker concept of wave operators. First, weak wave operators are defined by

$$\tilde{\Omega}_\pm = \tilde{\Omega}_\pm(H, H_0, J) = \text{w-}\lim_{t\to\pm\infty} P^{ac}U(-t)JU_0(t)P_0^{ac} \tag{100}$$

Clearly, if $\Omega_\pm(H, H_0, J)$ exist, then $\tilde{\Omega}_\pm$ exist. Moreover, $\tilde{\Omega}_\pm$ are bounded with the same estimate $\|\tilde{\Omega}_\pm f\| \leq \||J|\|\|f\|$. Contrary to the strong wave operators, it is clear that $\tilde{\Omega}_\pm(H, H_0, J)$ and $\tilde{\Omega}_\pm(H_0, H, J^*)$ exist simultaneously and $\tilde{\Omega}_\pm(H, H_0, J)^* = \tilde{\Omega}_\pm(H_0, H, J^*)$ holds. Nevertheless, conditions for partial isometricity like lemma 5.5 do not hold since the norm is not continuous with respect to weak convergence. More important, we cannot give a chain rule since the product of weak-convergent sequences is not necessarily weak-convergent. The fact that a chain rule is an attribute of strong wave operators is illustrated by the next proposition

Proposition 5.31 *The strong wave operator* $\Omega_\pm(H, H_0, J)$ *exists iff* $\tilde{\Omega}_\pm(H, H_0, J)$ *and* $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$ *exist and verify the equality*

$$\tilde{\Omega}_\pm(H, H_0, J)^*\tilde{\Omega}_\pm(H, H_0, J) = \tilde{\Omega}_\pm(H_0, H_0, J^*J) \tag{101}$$

Proof Assume that the weak operators exist and (101) holds. Then for any $f \in \mathcal{H}_0$, in the equality

$$\|U(-t)JU_0(t)P_0^{ac}f - \tilde{\Omega}_\pm f\|^2 = (P_0^{ac}U_0(-t)J^*JU_0(t)P_0^{ac}f, f) + \|\tilde{\Omega}_\pm f\|^2 - 2\Re(U(-t)JU_0(t)P_0^{ac}f, \tilde{\Omega}_\pm f) \tag{102}$$

The first term on the right-side tends as $t \to \pm\infty$ to $(\tilde{\Omega}_\pm(H_0, H_0, J^*J)f, f)$ and the second term tends to $-2\|\tilde{\Omega}_\pm f\|$ since $P^{ac}\tilde{\Omega}_\pm = \tilde{\Omega}_\pm$. Since by (101)

$$(\tilde{\Omega}_\pm(H_0, H_0, J^*J)f, f) = \|\tilde{\Omega}_\pm f\|^2 \tag{103}$$

the strong wave operators exist and coincide with $\tilde{\Omega}_\pm$. Conversely, if the strong wave operators exist, $\tilde{\Omega}_\pm$ exist and (102) holds with the left-hand side converging to zero as $t \to \pm\infty$. Then the wave operators $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$ exist and (103) holds and implies (101). □

The definition of wave operators can still be weakened.

Definition ε is a positive parameter. The non-negative function ω_ε is called an averaging kernel if $\omega_\varepsilon = \varepsilon\omega(\varepsilon t)$ with $\int_0^\infty \omega(t)dt = 1$ and if the Fourier transform $\hat{k}(\lambda)$ of $k(t) = e^{-t}\omega(e^{-t})$ has no real zeros.

The weak limit $X_\pm$ of the family of operators $X(t)$: $\mathcal{H}_0 \to \mathcal{H}$ with respect to the averaging process is defined by

$$(X_\pm f, g) = \lim_{\varepsilon \to 0} \int_0^\infty \omega_\varepsilon(t)(X(\pm t)f, g)dt \tag{104}$$

for any $f \in \mathcal{H}_0$, $g \in \mathcal{H}$. If the limits $X_\pm$ exist for some averaging kernel ω_ε, then they exist for any averaging kernel and their value is independent from the choice of ω_ε. This is a consequence of Wiener's Tauberian theorem (see [1], p. 97 and [8], p. 156). In particular, we are interested in the Abel kernel $\omega_\varepsilon = \varepsilon e^{-\varepsilon t}$. The weak abelian wave operator $\mathscr{A}_\pm(H, H_0, J)$ are then defined by

$$(\mathscr{A}_\pm f, g) = \lim_{\varepsilon \to 0} \varepsilon \int_0^\infty e^{-\varepsilon t}(P^{ac}U(\mp t)JU_0(\pm t)P_0^{ac}f, g)dt \tag{105}$$

This sesquilinear form is indeed bounded with the estimate $\left|(\mathscr{A}_\pm f, g)\right| \le \|\|J\|\|\|f\|\|g\|$ and defines well the operators $\mathscr{A}_\pm$. Clearly, if the weak wave operators $\tilde{\Omega}_\pm$ exist, we have

$$((\mathscr{A}_\pm - \tilde{\Omega}_\pm)f, g) = \lim_{\varepsilon \to 0} \int_0^\infty e^{-s}(P^{ac}U(\mp s/\varepsilon)JU_0(\pm s/\varepsilon)P_0^{ac}f - \tilde{\Omega}_\pm f, g)ds = 0$$

and $\tilde{\Omega}_\pm$ coincide with $\mathscr{A}_\pm$.

5.4.2 Stationary representations

If the weak abelian wave operators exist, for any $u \in \mathcal{H}_0$ and $v \in \mathcal{H}$, there holds

$$(\mathscr{A}_\pm u, v) = \lim_{\varepsilon \to 0} 2\varepsilon \int_0^\infty (e^{-\varepsilon t}JU_0(\pm t)P_0^{ac}u, e^{-\varepsilon t}U(\pm t)P^{ac}v)dt$$

We define the functions $f_u(t) = \gamma_{[0,\infty)}(t)e^{-\varepsilon t}JU_0(\pm t)P_0^{ac}u$ and $g_v(t) = \gamma_{[0,\infty)}(t)e^{-\varepsilon t}U(\pm t)P^{ac}v$, which clearly belongs to $L_1(\mathbb{R}, \mathcal{H}) \cap L_2(\mathbb{R}, \mathcal{H})$. By (22), the functions $f_u(t)$ and $g_v(t)$ have Fourier transforms $\hat{f}_u(\lambda) = (2\pi)^{-1}$ $JR_0(\lambda \pm i\varepsilon)P_0^{ac}u$ and $\hat{g}_v(\lambda) = (2\pi)^{-1}R(\lambda \pm i\varepsilon)P^{ac}v$.
Applying Parseval's equality $\int_\mathbb{R}(f(t), g(t))dt = \int_\mathbb{R}(\hat{f}(\lambda), \hat{g}(\lambda))d\lambda$, we obtain the representation

$$(\mathscr{A}_\pm u, v) = \lim_{\varepsilon \to 0} \pi^{-1}\varepsilon \int_\mathbb{R} (JR_0(\lambda \pm i\varepsilon)P_0^{ac}u, R(\lambda \pm i\varepsilon)P^{ac}v)d\lambda \tag{106}$$

for any $u \in \mathcal{H}_0$ and $v \in \mathcal{H}$. If the strong wave operators exist, they coincide with the weak abelian ones. Then the equation (106) is the stationary representation of the sesquilinear form associated to the wave operators. We know from the preliminaries that radial limit values $\lim_{\varepsilon \to 0}(R(\lambda \pm i\varepsilon)u, v)$ exist almost everywhere on the real axis. The first question arising is whether the limit on ε and the integration over λ in the expression (106) can be interchanged or not, assuming the limit of the integrands exist.
When they exist for a pair $(u, v) \in \mathcal{H}_0 \times \mathcal{H}$ and for some $\lambda \in \mathbb{R}$, we denote the following limits

$$a_\pm(u, v; \lambda) = \lim_{\varepsilon \to 0} \pi^{-1}\varepsilon(JR_0(\lambda \pm i\varepsilon)u, R(\lambda \pm i\varepsilon)v) \tag{107}$$

We now investigate some properties of these limits. We notice that there holds by Cauchy-Schwarz

$$\left|\pi^{-1}\varepsilon(JR_0(\lambda \pm i\varepsilon)u, R(\lambda \pm i\varepsilon)v)\right|^2 \leq |||J|||(\pi^{-1}\varepsilon\|R_0(\lambda \pm i\varepsilon)u\|^2)(\pi^{-1}\varepsilon\|R(\lambda \pm i\varepsilon)v\|^2)$$

By relation (25), we obtain that the right-hand side of (108) tends to

$$|||J|||^2 \frac{d(E_0(\lambda)u,u)}{d\lambda}\frac{d(E(\lambda)v,v)}{d\lambda}$$

as $\varepsilon \to 0$. Then, if $a_\pm(u,v;\lambda)$ exist for almost every $\lambda \in \mathbb{R}$, (29) implies that there holds for a.e. $\lambda \in \mathbb{R}$

$$\left|a_\pm(u,v;\lambda)\right|^2 \leq |||J|||^2 \frac{d(E_0(\lambda)u,u)}{d\lambda}\frac{d(E(\lambda)v,v)}{d\lambda} \tag{108}$$

In particular, the measures $(E_0(\cdot)u,u)$ and $(E(\cdot)v,v)$ are absolutely continuous with respect to their respective spectral measures, hence they are respectively concentrated on the spectral cores $\hat{\sigma}_0$ and $\hat{\sigma}$. That is, if λ belongs to $\mathbb{R}\setminus(\hat{\sigma}_0\cap\hat{\sigma})$, it is necessarily a point of constancy for both generating functions $(E_0(\lambda)u,u)$ and $(E(\lambda)v,v)$, hence the derivatives of the generating functions exist and are equal to zero. By taking the limit on ε of (108), this implies that for any $u \in \mathcal{H}_0$, $v \in \mathcal{H}$ and a.e. $\lambda \in \mathbb{R}\setminus(\hat{\sigma}_0 \cap \hat{\sigma})$, $a_\pm(u,v;\lambda)$ exist and is equal to zero. This idea extends to the next lemma, in which the domain of λ is this time restrained by arbitrary Borel sets.

Lemma 5.32 *For any $u \in \mathcal{H}_0$, $v \in \mathcal{H}$, Borel sets Δ_0, Δ, there holds for a.e. $\lambda \in \mathbb{R}\setminus(\Delta_0 \cap \Delta)$*

$$a_\pm(E_0(\Delta_0)u, E(\Delta)v;\lambda) = 0 \tag{109}$$

Moreover, if $a_\pm(u,v;\lambda)$ exists for a pair (u,v) and for a.e. $\lambda \in \mathbb{R}$, then there also exists for a.e. $\lambda \in \mathbb{R}$

$$a_\pm(E_0(\Delta_0)u, E(\Delta)v;\lambda) = \gamma_{\Delta_0\cap\Delta}(\lambda)a_\pm(u,v;\lambda) \tag{110}$$

Proof The first part is similar to (108). We write $\tilde{u} = E_0(\Delta_0)u$ and $\tilde{v} = E(\Delta)v$. Then the right-hand side of (108) where we replaced (u,v) by $(\tilde{u},\tilde{v})$ tends for a.e. $\lambda \in \mathbb{R}$ as $\varepsilon \to 0$ to

$$\begin{aligned}|||J|||^2 &\frac{d(E_0(\lambda)E_0(\Delta_0)u,u)}{d\lambda}\frac{d(E(\lambda)E(\Delta)v,v)}{d\lambda}\\ &= |||J|||^2\gamma_{\Delta_0\cap\Delta}(\lambda)\frac{d(E_0(\lambda)u,u)}{d\lambda}\frac{d(E(\lambda)v,v)}{d\lambda}\end{aligned}$$

by the equation (13). For a.e. $\lambda \in \mathbb{R}\setminus(\Delta_0\cap\Delta)$, this expression is equal to zero, hence $a_\pm(E_0(\Delta_0)u, E(\Delta)v);\lambda)$ exists and is equal to zero.
Hence, it remains to show that (110) holds for a.e. $\lambda \in \Delta_0\cap\Delta$. By the sesquilinearity of $a_\pm$, we can formally write

$$\begin{aligned}a_\pm(E_0(\Delta_0)u, E(\Delta)v;\lambda) &= -a_\pm(E_0(\mathbb{R}\setminus\Delta_0)u, E(\Delta)v;\lambda) + a_\pm(u, E(\Delta)v;\lambda)\\ &= -a_\pm(E_0(\mathbb{R}\setminus\Delta_0)u, E(\Delta)v;\lambda) - a_\pm(u, E(\mathbb{R}\setminus\Delta_0)v;\lambda) + a_\pm(u,v;\lambda)\end{aligned}$$

where by (109) and assumptions, the three terms on the right-hand side exist for a.e. $\lambda \in \Delta_0\cap\Delta$, the two first ones being equal to zero. Then we get (110). □

Note the decomposition $u = P_0^{ac}u + P_0^s u$ in its absolutely continuous and singular parts. We note like in the preliminaries for the spectral projections $P_0^{ac} = E_0(X_{ac})$, $P_0^s = E_0(X_s)$ where X_{ac} is a full Lebesgue measure set and X_s a Lebesgue nullset.
Assume that $a_\pm(u, v; \lambda)$ exist for a.e. $\lambda \in \mathbb{R}$. Then, by (110), $a_\pm(P_0^{ac}u, v; \lambda)$ exist and coincide for a.e. $\lambda \in \mathbb{R}$ with $a_\pm(u, v; \lambda)$.
Conversely, assume $a_\pm(P_0^{ac}u, v; \lambda)$ exist almost everywhere. Note that by (109), $a_\pm(P_0^s u, v; \lambda)$ exists and is equal to zero almost everywhere. Then by sesquilinearity of $a_\pm$, $a_\pm(u, v; \lambda)$ exist and coincide with $a_\pm(P^{ac}u, v; \lambda)$ almost everywhere. The same facts holds for v.
Hence, the existence of $a_\pm(u, v; \lambda)$ almost everywhere is equivalent to the existence of $a_\pm$ for the pairs $(P_0^{ac}u, v)$, $(u, P^{ac}v)$ or $(P_0^{ac}u, P^{ac}v)$. The values of $a_\pm$ for the four pairs coincide almost everywhere. The set of full Lebesgue measure on which it happens depends on u, v and the spectral measure $E(\cdot)$.

Moreover, if $a_\pm(u, v; \lambda)$ exists almost everywhere for a pair (u, v), there holds by (108)

$$\left|a_\pm(u, v; \lambda)\right| \leq |||J||| \left(\frac{d(E_0(\lambda)P_0^{ac}u, P_0^{ac}u)}{d\lambda}\right)^{1/2} \left(\frac{d(E(\lambda)P^{ac}v, P^{ac}v)}{d\lambda}\right)^{1/2}$$

Hence, by Cauchy-Schwarz inequality and equation (12), $a_\pm(u, v; \lambda)$ is absolutely integrable with the estimate

$$\int_{\mathbb{R}} \left|a_\pm(u, v; \lambda)\right| d\lambda \leq |||J||| \|P_0^{ac}u\| \|P^{ac}v\| \tag{111}$$

We can now conclude the question of the interchange of limit and integral in (17)

Lemma 5.33 *Suppose that $a_\pm(u, v; \lambda)$ exists almost everywhere for the pair (u, v). Then*

$$\lim_{\varepsilon \to 0} \pi^{-1}\varepsilon \int_{\mathbb{R}} (JR_0(\lambda \pm i\varepsilon)\tilde{u}, R(\lambda \pm i\varepsilon)\tilde{v}) d\lambda = \int_{\mathbb{R}} a_\pm(u, v; \lambda) d\lambda \tag{112}$$

where the pair $(\tilde{u}, \tilde{v})$ on the left-hand side is one of the three pairs $(P_0^{ac}u, v)$, $(u, P^{ac}v)$ or $(P_0^{ac}u, P^{ac}v)$.

Proof We note for simplicity

$$f_\varepsilon(\lambda) = \pi^{-1}\varepsilon(JR_0(\lambda \pm i\varepsilon)\tilde{u}, R(\lambda \pm i\varepsilon)\tilde{v})$$

and we choose without loss of generality $\tilde{v} = P^{ac}v$. Then, by (108), Cauchy-Schwarz inequality and (27), we obtain for any Borel set $X \subseteq R$

$$\int_X \left|f_\varepsilon(\lambda)\right| d\lambda \leq |||J||| \int_X \left((\delta_0(\lambda, \varepsilon)\tilde{u}, \tilde{u})\right)^{1/2} \left((\delta(\lambda, \varepsilon)\tilde{v}, \tilde{v})\right)^{1/2} d\lambda$$
$$\leq |||J||| \|\tilde{u}\| \left(\int_X (\delta(\lambda, \varepsilon)\tilde{v}, \tilde{v}) d\lambda\right)^{1/2}$$

This relation implies that the f_ε are integrable with the estimate $\int_X \left|f_\varepsilon(\lambda)\right| d\lambda \leq |||J||| \|\tilde{u}\| \|\tilde{v}\|$. Moreover, since $\tilde{v} \in \mathcal{H}^{ac}$, $(\delta(\lambda, \varepsilon)\tilde{v}, \tilde{v})$ is the Poisson integral of the function $\dfrac{d(E(\lambda)\tilde{v},\tilde{v})}{d\lambda}$. Since $\dfrac{d(E(\lambda)\tilde{v},\tilde{v})}{d\lambda}$ belongs to $L_1(\mathbb{R})$, $(\delta(\lambda,\varepsilon)\tilde{v},\tilde{v})$ converges to $\dfrac{d(E(\lambda)\tilde{v},\tilde{v})}{d\lambda}$ as $\varepsilon \to 0$ in L_1-norm (see [10], p. 123). This implies that the limit on the left-hand side of (112) exists with

$$\lim_{\varepsilon\to 0}\int_X \left|f_\varepsilon(\lambda)\right| d\lambda \leq |||J||| \|\tilde{u}\| \left(\int_X \frac{d(E(\lambda)\tilde{v},\tilde{v})}{d\lambda} d\lambda\right)^{1/2}$$

Together with the fact that $(\delta(\lambda,\varepsilon)\tilde{v},\tilde{v}) \leq C(\lambda,\tilde{v})$ for any $\varepsilon > 0$, this implies that $\int_X \left|f_\varepsilon(\lambda)\right| d\lambda$ tends to zero uniformely with respect to ε, when $|X| \to 0$ or when $X = \mathbb{R} \setminus (-n,n)$ with $n \to \infty$. We can now apply the Vitali's convergence theorem and interchange the limit on ε and the integral over λ in (112). The relation (112) is immediately proved since for any admissible pair $(\tilde{u},\tilde{v})$, $a_\pm(\tilde{u},\tilde{v};\lambda) = a_\pm(u,v;\lambda)$. $\square$

We are now in position to define stationary wave operators.

Definition Assume that for any $u \in \mathfrak{D}_0$ and $v \in \mathfrak{D}$, $\mathfrak{D}_0$ and $\mathfrak{D}$ dense sets respectively in $\mathcal{H}_0$ and $\mathcal{H}$, the limits $a_\pm(u,v;\lambda)$ exist almost everywhere. Then we define on $\mathfrak{D}_0 \times \mathfrak{D}$ the following sesquilinear forms

$$\mathfrak{u}_\pm(u,v) = \int_{\mathbb{R}} a_\pm(u,v;\lambda) d\lambda \tag{113}$$

It is of course equivalent to take the integral only onto $\hat{\sigma}_0 \cap \hat{\sigma}$.
Note that by the estimate (111), $\left|\mathfrak{u}_\pm(u,v)\right| \leq |||J||| \|P_0^{ac}u\| \|P^{ac}v\|$. Therefore, to the bounded forms $\mathfrak{u}_\pm$ correspond bounded operators $\mathscr{U}_\pm = \mathscr{U}_\pm(H,H_0,J)$ so that $(\mathscr{U}_\pm u, v) = \mathfrak{u}_\pm(u,v)$. There holds the norm estimate $|||\mathscr{U}_\pm||| \leq |||J|||$. We call $\mathscr{U}_\pm(H,H_0,J)$ the stationary wave operators for the operators H and H_0 and the identification J.

Using the notations of lemma 5.33, we can define a bounded operator $\mathscr{U}_\pm(\varepsilon)$: $\mathcal{H}_0 \to \mathcal{H}$ by the sesquilinear form

$$(\mathscr{U}_\pm(\varepsilon)\tilde{u},\tilde{v}) = \int_{\mathbb{R}} f_\varepsilon(\lambda) d\lambda \tag{114}$$

defined on $\mathcal{H}_0 \times \mathcal{H}$. Indeed, we showed during the proof the boundedness estimate $\left|(\mathscr{U}_\pm(\varepsilon)\tilde{u},\tilde{v})\right| \leq |||J||| \|\tilde{u}\| \|\tilde{v}\|$. We hence have a uniform in ε norm estimate $|||\mathscr{U}_\pm(\varepsilon)||| \leq |||J|||$. Then, by applying lemma 5.33 in the special case $\tilde{u} = P_0^{ac}u$ and $\tilde{v} = v$, we obtain

Proposition 5.34 *Assume that the stationary wave operators $\mathscr{U}_\pm$ exist, then*

$$\mathscr{U}_\pm = w - \lim_{\varepsilon\to 0} \mathscr{U}_\pm(\varepsilon) P_0^{ac} \tag{115}$$

and $\mathscr{U}_\pm$ is then independent of the choice of the dense sets $\mathfrak{D}_0$ and $\mathfrak{D}$. Because of the relation (110), we assume from here that $\mathfrak{D}_0$ and $\mathfrak{D}$ are invariant respectively with respect to the spectral measures $E_0(\cdot)$ and $E(\dot{})$.

5.4.3 Properties of stationary wave operators

Proposition 5.35 *The stationary wave operators* $\mathscr{U}_\pm(H, H_0, J)$ *and* $\mathscr{U}_\pm(H_0, H, J^*)$ *exist simultaneously. Moreover, we have*

$$\mathscr{U}_\pm(H_0, H, J^*) = \mathscr{U}_\pm(H, H_0, J)^* \tag{116}$$

Proof It suffices to note that the limits $a_\pm(u, v; \lambda) = \lim_{\varepsilon\to 0} \pi^{-1}\varepsilon(JR_0(\lambda \pm i\varepsilon)u, R(\lambda \pm i\varepsilon)v)$ and $\bar{a}_\pm(u, v; \lambda) = \lim_{\varepsilon\to 0} \pi^{-1}\varepsilon(J^*R(\lambda \pm i\varepsilon)v, R_0(\lambda \pm i\varepsilon)u)$ simultaneously exist and are obviously conjugate. □

Proposition 5.36 *If the wave operators* $\mathscr{U}_\pm = \mathscr{U}_\pm(H, H_0, J)$ *exist, then there holds* $\mathcal{H}_0^s \subseteq \ker(\mathscr{U}_\pm)$ *and* $Ran(\mathscr{U}_\pm) \subseteq \mathcal{H}^{ac}$.

Proof Recall that $\mathscr{U}_\pm$ is a bounded operator. From

$$\left|(\mathscr{U}_\pm u, v)\right| \leq \| |J| \| \|P_0^{ac} u\| \|P^{ac} v\|$$

we get that $\mathcal{H}_0^s \subseteq \ker(\mathscr{U}_\pm)$ and $\mathcal{H}^s \subseteq \ker(\mathscr{U}_\pm^*)$. □

Proposition 5.37 *The operators* $\mathscr{U}_\pm$ *are intertwining for* H *and* H_0*, that is for any Borel set* Δ*, we have* $H\mathscr{U}_\pm = \mathscr{U}_\pm H_0$ *or equivalently*

$$E(\Delta)\mathscr{U}_\pm = \mathscr{U}_\pm E_0(\Delta) \tag{117}$$

Proof By definition of $\mathscr{U}_\pm$ and by (110), we have for any Borel sets Λ_0, Λ the representation on the dense set $\mathfrak{D}_0 \times \mathfrak{D}$

$$(\mathscr{U}_\pm E_0(\Delta_0)u, E(\Delta)v) = \int_{\Delta_0\cap\Delta} a_\pm(u, v; \lambda)d\lambda \tag{118}$$

This implies that $(\mathscr{U}_\pm E_0(\Delta)u, v) = (\mathscr{U}_\pm u, E(\Delta)v)$ on the dense set $\mathfrak{D}_0 \times \mathfrak{D}$. This extends by continuity to $\mathcal{H}_0 \times \mathcal{H}$ and we obtain $\mathscr{U}_\pm E_0(\Delta) = E(\Delta)\mathscr{U}_\pm$. □

By considering the stationary representation for weak abelian wave operators and the special case $(\tilde{u}, \tilde{v}) = (P_0^{ac}u, P^{ac}v)$ in the lemma 5.33, we immediately obtain

Proposition 5.38 *Suppose that the stationary wave operators* $\mathscr{U}_\pm$ *exist, then the weak abelian wave operators* $\mathscr{A}_\pm$ *exist and coincide with* $\mathscr{U}_\pm$.

5.5 Existence and completeness of wave operators III: stationary methods

We can now develop a strategy for an investigation of existence of strong wave operators based on stationary methods. The first step is to obtain conditions for the existence of $\mathscr{U}_\pm = \mathscr{U}_\pm(H, H_0, J)$ (that is for the existence of the limits $a_\pm$). Then we shall see that under a little stronger assumptions, we can get $\mathscr{U}_\pm(H_0, H_0, J^*J)$ to exist with the chain rule relation $\mathscr{U}_\pm^* \mathscr{U}_\pm = \mathscr{U}_\pm(H_0, H_0, J^*J)$. The third step consists in expressing conditions for the existence of the weak wave operators $\tilde{\Omega}_\pm(H, H_0, J)$ and $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$. If all these conditions are fulfilled, we obtain the existence of the strong wave operators by theorem 5.31 since stationary, weak abelian and weak wave operators coincide.

5.5.1 Existence of the stationary wave operators

The existence of $\mathscr{U}_\pm$ is equivalent to the existence of the limits $a_\pm$ on a dense set $\mathfrak{D}_0 \times \mathfrak{D}$. Like in the previous parts, the existence of $a_\pm$ shall depend on the perturbation "$HJ - JH_0$". We use the technique of factorization of the perturbation already developed in 3.2.2. Namely, we consider some factorization of the perturbation (G_0, G), with G_0 H_0-bounded and G H-bounded, defined by the relation (45). Then by relation (47), we obtain

$$\pi^{-1}\varepsilon(JR_0(\lambda \pm i\varepsilon)u, R(\lambda \pm i\varepsilon)v) = \pi^{-1}\varepsilon\left(G_0R_0(\lambda \pm i\varepsilon)u, GR(\lambda \mp i\varepsilon)R(\lambda \pm i\varepsilon)v\right)$$
$$+\pi^{-1}\varepsilon(R(\lambda \pm i\varepsilon)Ju, R(\lambda \pm i\varepsilon)v) = (\delta(\lambda,\varepsilon)Ju, v) + (G_0R_0(\lambda \pm i\varepsilon)u, G\delta(\lambda,\varepsilon)v) \tag{119}$$

The first term on the right-hand side has a limit as $\varepsilon \to 0$ for almost every $\lambda \in \mathbb{R}$. Then the existence of $a_\pm$ for a pair (u, v) reduces to the existence of limits on ε for the functions $G_0R_0(\lambda \pm i\varepsilon)u$ and $G\delta(\lambda,\varepsilon)v$ (or equivalently for $GR(\lambda \pm i\varepsilon)v$ and $G_0\delta_0(\lambda,\varepsilon)u$ by the simultaneous existence of the conjugate limit $\bar{a}_\pm$). Precisely one of the limits can be weak while the other remains strong. Because of the definition of $\mathscr{U}_\pm$, we have

Lemma 5.39 *Suppose that for $u \in \mathfrak{D}_0$, dense set in $\mathcal{H}_0$ and for $v \in \mathfrak{D}$, dense set in $\mathcal{H}$, there exist*

$$s - \lim_{\varepsilon\to 0} G_0R_0(\lambda \pm i\varepsilon)u, \quad w - \lim_{\varepsilon\to 0} G\delta(\lambda,\varepsilon)v, \quad \textit{for a.e.}\lambda \in \mathbb{R} \tag{120}$$

or

$$s - \lim_{\varepsilon\to 0} GR(\lambda \pm i\varepsilon)v, \quad w - \lim_{\varepsilon\to 0} G\delta_0(\lambda,\varepsilon)u, \quad \textit{for a.e.}\lambda \in \mathbb{R} \tag{121}$$

then $\mathscr{U}_\pm$ exist.

On the way to the chain rule relation, we shall need the following intermediate result which will also be useful in the investigation of the scattering operator.

Lemma 5.40 *Suppose that the conditions (120) of 5.39 hold with $\mathfrak{D} = \mathcal{H}$. Assume further that G: $\mathcal{H} \to \mathcal{K}$ is bounded. Then there holds for $u \in \mathfrak{D}_0$*

$$w - \lim_{\varepsilon\to 0} G\delta(\lambda,\varepsilon)\mathscr{U}_\pm u = w - \lim_{\varepsilon\to 0} \pi^{-1}\varepsilon GR(\lambda \mp i\varepsilon)JR_0(\lambda \pm i\varepsilon)u \tag{122}$$

where both limits exist for almost every $\lambda \in \mathbb{R}$. Moreover, for any $u_1, u_2 \in \mathfrak{D}_0$, we have the representation

$$(\mathscr{U}_\pm u_1, \mathscr{U}_\pm u_2) = \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (J R_0(\lambda \pm i\varepsilon) u_1, J R_0(\lambda \pm i\varepsilon) u_2) d\lambda \tag{123}$$

Proof By lemma 5.39 the stationary wave operators $\mathscr{U}_\pm$ exist, and since $\mathfrak{D} = \mathcal{H}$, $\text{w} - \lim_{\varepsilon \to 0} G\delta(\lambda, \varepsilon) u$ exists for any $v \in \mathcal{H}$. Then the limit on the left-hand side of (122) exists for any $u \in \mathcal{H}_0$.
For the right-hand side, we have to study the convergence of the scalar product

$$\pi^{-1} \varepsilon (J R_0(\lambda \pm i\varepsilon) u, R(\lambda \pm i\varepsilon) G^* w) \tag{124}$$

for $w \in \mathcal{K}$. To obtain the desired expression, the full measure set of λ where the scalar product converges has to be independent of the choice of the element w. We consider a basis $\{w_i\}$ of the Hilbert space $\mathcal{K}$. Because the conditions (120) hold, we have that the limit when $\varepsilon \to 0$ of the expression (124) exists for $w = w_i$ and for any $u \in \mathfrak{D}_0$ on a set of full measure Λ_i. Notice that $\Lambda_\infty = \bigcap_i \Lambda_i$ remains a set of full measure. Then we have that the limit of (124) exists for any $u \in \mathfrak{D}_0$ and any w belonging to the linear span S of the basis $\{w_i\}$, always on the full measure set Λ_∞. It remains to show that for a given $u \in \mathfrak{D}_0$, $f_\varepsilon = \pi^{-1} \varepsilon G R(\lambda \mp i\varepsilon) J R_0(\lambda \pm i\varepsilon) u$ is a bounded sequence. Indeed we have for a.e. λ

$$\|f_\varepsilon\| \leq \|\|J\|\|\|\|G\|\| \left(\pi^{-1}\varepsilon \|\|R(\lambda \pm i\varepsilon)\|\|^2\right)^{1/2} \left(\pi^{-1}\varepsilon \|R_0(\lambda \pm i\varepsilon) u\|^2\right)^{1/2} \leq C(\lambda)$$

where we used the relation (26) and (29). We remove the nullset on which the bounded estimate does not hold from Λ_∞ and then obtain the weak-convergence still on a set of full measure. This concludes the proof of existence of the limits on both sides of (122).
Under the conditions (120), we have for any Borel set Δ, $u \in \mathfrak{D}_0$ and $v \in \mathcal{H}$

$$\int_\Delta \lim_{\varepsilon \to 0} (\delta(\lambda, \varepsilon) \mathscr{U}_\pm u, v) d\lambda = (E(\Delta) \mathscr{U}_\pm u, v)$$

because of the relation (29) and $\mathscr{U}_\pm u \in \mathcal{H}^{ac}$. By the intertwining relation and the representation (118), we obtain for any Borel set Δ and $v \in \mathcal{H}$

$$\int_\Delta \lim_{\varepsilon \to 0} (\delta(\lambda, \varepsilon) \mathscr{U}_\pm u, v) d\lambda = \int_\Delta \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (R(\lambda \mp i\varepsilon) J R_0(\lambda \pm i\varepsilon) u, v) d\lambda$$

hence the integrands are equal. As this holds in particular for any $v = G^* w$, this proves the equality (122).

Writing the equality of the integrands with $v = J u_1$ and $u = u_2$, we get

$$\lim_{\varepsilon \to 0} (J u_1, \delta(\lambda, \varepsilon) \mathscr{U}_\pm u_2) = \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (J u_1, R(\lambda \mp i\varepsilon) J R_0(\lambda \pm i\varepsilon) u_2) \tag{125}$$

where the limits on both sides exist. Since $G_0 R_0(\lambda \pm i\varepsilon) u$ converges strongly and (122) holds, we obtain

$$\lim_{\varepsilon \to 0} (G_0 R_0(\lambda \pm i\varepsilon) u_1, G\delta(\lambda, \varepsilon) \mathscr{U}_\pm u_2)$$
$$= \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon (G_0 R_0(\lambda \pm i\varepsilon) u_1, G R(\lambda \mp i\varepsilon) J R_0(\lambda \pm i\varepsilon) u_2) \tag{126}$$

Taking the limit as $\varepsilon \to 0$ in the expression (119) with $u = u_1$ and $v = \mathscr{U}_\pm u_2$ and substituting the terms on the right-hand side thanks to the relations (125) and (126), we obtain in view of the resolvent identity for any $u_1, u_2 \in \mathfrak{D}_0$

$$a_\pm(u, \mathscr{U}_\pm v; \lambda) = \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon \Big[\big((GR(\lambda \mp i\varepsilon))^* G_0 R_0(\lambda \pm i\varepsilon) u_1, JR_0(\lambda \pm i\varepsilon) u_2 \big)$$

$$+ \big(R(\lambda \pm i\varepsilon) J u_1, JR_0(\lambda \pm i\varepsilon) u_2 \big) \Big] = \lim_{\varepsilon \to 0} \pi^{-1} \varepsilon \left(JR_0(\lambda \pm i\varepsilon) u_1, JR_0(\lambda \pm i\varepsilon) u_2 \right)$$

with proves relation (123). □

Under the conditions of lemma 5.40, the integrand on the right-hand side of (123) exists for a dense set in $\mathcal{H}_0 \times \mathcal{H}_0$, this is exactly the condition for the existence of the wave operator $\mathscr{U}_\pm(H_0, H_0, J^*J)$. Then we get

Proposition 5.41 *Suppose that the conditions of lemma 5.40 holds. Then the stationary operators $\mathscr{U}_\pm = \mathscr{U}_\pm(H, H_0, J)$ and $\mathscr{U}_\pm(H_0, H_0, J^*J)$ exist and there holds*

$$\mathscr{U}_\pm^* \mathscr{U}_\pm = \mathscr{U}_\pm(H_0, H_0, J^*J) \tag{127}$$

We can now turn to the third step of our investigation.

5.5.2 Existence of weak wave operators by stationary means

The factors G_0, G are still assumed to be respectively H_0-bounded and H-bounded. The first lemma establishes the connection between stationary conditions and time-falloff ones.

Lemma 5.42 *Suppose H selfadjoint on $\mathcal{H}$ and assume that G is bounded. Suppose that further that there is a dense set $\mathfrak{D}$ in $\mathcal{H}$ such that for any $u \in \mathfrak{D}$, $w - \lim_{\varepsilon \to 0} G\delta(\lambda, \varepsilon) u$ exists for almost every $\lambda \in \mathbb{R}$.*
Then, there exists a dense subset $\mathfrak{E}$ of $\mathcal{H}^{ac}$ such that for any $v \in \mathfrak{E}$

$$\int_{\mathbb{R}} \|GU(t)v\|^2 dt < \infty \tag{128}$$

Proof Fix $u \in \mathfrak{D}$ and denote $l(u, \lambda) \in \mathcal{K}$ for the weak limit of $G\delta(\lambda, \varepsilon)u$. We define the set $\Delta_{N,n} = \{\lambda, \ |\lambda| \le n \text{ and } \|l(u,\lambda)\| \le N\}$. Define $X_{N,n} = (n, n) \setminus \Delta_{N,n}$. Then $X_{\infty,n} = \bigcap_N X_{N,n}$ denotes the set where $l(u, \lambda)$ is infinite or not defined. By definition, $\left| X_{\infty,n} \right| = 0$. Then for any $n \in \mathbb{N}$, $\left| X_{N,n} \right| \to 0$ as $N \to \infty$. Consider for any $N, n \in \mathbb{N}$, $u_{N,n} = E(\Delta_{N,n}) P^{ac} u$. Then $\|P^{ac}u - u_{N,n}\| = \|E(\mathbb{R} \backslash \Delta_{N,n}) P^{ac} u\| = \lim_{n \to \infty} (E(X_{N,n}) P^{ac} u, u) \to 0$ as $N \to \infty$, since $(E(\cdot) P^{ac} u, u)$ is absolutely continuous.
We denote $\mathfrak{E}$ the linear span of all elements of the form $E(\Delta_{N,n}) P^{ac} u$ for all possible N, n and $u \in \mathfrak{D}$. Then $\mathfrak{E}$ is dense in $P^{ac}\mathfrak{D}$, which is itself dense in $\mathcal{H}^{ac}$.
For arbitrary N, n take an element $u_{N,n} = E(\Delta_{N,n}) P^{ac} u$ in $\mathfrak{E}$ and a basis $\{w_i\}$ in the Hilbert space $\mathcal{K}$. Then we get

$$\begin{aligned}
(GU(t) u_{N,n}, w_i) &= \int_{\mathbb{R}} e^{-i\lambda t} d(E(\lambda) E(\Delta_{N,n}) P^{ac} u, G^* w_i) \\
&= \int_{\mathbb{R}} e^{-i\lambda t} \gamma_{\Delta_{N,n}}(\lambda) \frac{d(E(\lambda)u, G^* w_i)}{d\lambda} d\lambda \\
&= \int_{\mathbb{R}} e^{-i\lambda t} \gamma_{\Delta_{N,n}}(\lambda) (l(u, \lambda), w_i) d\lambda
\end{aligned}$$

$(GU(t)u_{N,n}, w_i)$ is the Fourier transform of $\gamma_{\Delta_{N,n}}(\lambda)(l(u,\lambda), w_i)$. By Parseval equality, we obtain $\int_{\mathbb{R}} \left|(GU(t)u_{N,n}, w_i)\right|^2 dt = 2\pi \int_{\mathbb{R}} \left|\gamma_{\Delta_{N,n}}(\lambda)(l(u,\lambda), w_i)\right|^2 d\lambda$. Since $\|v\| = \sum_i |v, w_i|^2$, we obtain

$$\int_{\mathbb{R}} \|GU(t)u_{N,n}\|^2 dt = 2\pi \int_{\Delta_{N,n}} \|l(u,\lambda)\|^2 d\lambda \leq 4\pi n N^2 < \infty$$

by construction of the set $\Delta_{N,n}$. Then for any $v \in \mathfrak{E}$, (128) holds. □

Proposition 5.43 *Assume that the conditions of lemma 5.42 hold for both bounded G_0 and G, respectively on dense sets $\mathfrak{D}_0$ and $\mathfrak{D}$. Then the weak wave operators $\tilde{\Omega}_\pm(H, H_0, J)$ exist.*

Proof In the time-dependent proof of Pearson's theorem, we showed that

$$\big((\Omega(t) - \Omega(s))v_0, v\big) \leq \int_s^t (G_0 U_0(\tau) v_0, GU(\tau)v) d\tau \tag{129}$$

By the definition $\tilde{\Omega}_\pm = \text{w-}\lim_{t\to\pm\infty} P^{ac}\Omega(t)P_0^{ac}$ and the uniform bound in t provided by $|||P^{ac}\Omega(t)P_0^{ac}||| \leq |||J|||$, it suffices to verify that the left hand side of (129) tends to zero as $t, s \to \pm\infty$ on dense sets of elements v_0, v respectively in $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$.
Under the conditions of lemma 5.42, there are dense subsets $\mathfrak{E}_0$ and $\mathfrak{E}$ respectively in $\mathcal{H}_0^{ac}$ and $\mathcal{H}^{ac}$ such that for any $v_0 \in \mathfrak{E}_0$ and $v \in \mathfrak{E}$, $\|G_0 U_0(t) v_0\|$ and $\|GU(t)v\|$ are square integrable. For such elements v_0, v, we obtain by Cauchy-Schwarz

$$\big|(129)\big| \leq \left(\int_s^t \|G_0 U_0(\tau)v_0\|^2 d\tau \int_s^t \|GU(\tau)v\|^2 d\tau\right)^{1/2} \xrightarrow[s,t\to\pm\infty]{} 0$$

which closes the proof. □

It now remains to express conditions for the existence of the weak wave operator $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$.

Proposition 5.44 *Suppose that G_0 and G are bounded and that for $u \in \mathfrak{D}_0$, dense set in $\mathcal{H}_0$ and $v \in \mathcal{H}$, there exist*

$$w - \lim_{\varepsilon\to 0} G_0 R_0(\lambda \pm i\varepsilon)u, \quad w - \lim_{\varepsilon\to 0} GR(\lambda \pm i\varepsilon)G^* v, \quad \textit{for a.e. } \lambda \in \mathbb{R} \tag{130}$$

then the weak wave operators $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$ exist.

Proof The idea is to adapt the preceeding proof by finding an appropriate factorization of the perturbation "$H_0 J^* J - J^* J H_0$". By the boundedness of G, we can write for any $z \in \rho(H) \cap \rho(H_0)$

$$JR_0(z) - R(z)J = (GR(\bar{z}))^* G_0 R_0(z) = R(z) G^* G R_0(z) \tag{131}$$

Hence $JR_0(z) = R(z)\big(J + G^* G_0 R_0(z)\big)$, which implies that J maps $\mathcal{D}(H_0)$ into $\mathcal{D}(H)$. This gives sense to the operator $HJ - JH_0 = G^* G_0$ as a bounded mapping on $\mathcal{H}_0$. We can compute the perturbation

$$H_0 J^* J - J^* J H_0 = J^*(HJ - JH_0) - (HJ - JH_0)^* J = (GJ)^* G_0 - G_0^*(GJ)$$

We note $\Omega(t) = U_0(-t)J^*JU_0(t)$. Similarly to (129), we obtain by derivating the quantity $(\Omega(t)u, v)$

$$\big((\Omega(t) - \Omega(s))v_0, v\big) \leq \int_s^t (G_0U_0(\tau)v_0, GJU_0(\tau)v) - (GJU_0(\tau)v_0, G_0U_0(\tau)v)d\tau$$

By our assumptions, the conditions of lemma 5.42 hold for G_0. It suffices to show that they also hold for GJ. Then we can conclude like in the last proposition that $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$ exist.
GJ is bounded. Since G is H-bounded, we can multiply (131) on the left by G which gives

$$GJR_0(z) = GR(z)J + GR(z)G^*G_0R_0(z)$$

Multiplying on the right by $R_0(\bar{z})$ and setting $z = \lambda \pm i\varepsilon$, we get for any $u \in \mathcal{H}_0$

$$GJ\delta_0(\lambda, \varepsilon)u = \pi^{-1}\varepsilon GR(\lambda \pm i\varepsilon)JR_0(\lambda \mp i\varepsilon)u + GR(\lambda \pm i\varepsilon)G^*G_0\delta_0(\lambda, \varepsilon)u \quad (132)$$

For any $w \in \mathcal{K}$, $(GJ\delta(\lambda, \varepsilon)u, w)$ has a limit for almost every λ. Like in lemma 5.40, the set of full measure on which the limit exists is taken independent from the choice of w in the dense set S. It then remains to find an uniform bound in ε holding almost everywhere for $GJ\delta(\lambda, \varepsilon)u$. By our assumptions, there exists also $\operatorname{w-lim}_{\varepsilon\to 0} G\delta(\lambda, \varepsilon)G^*$ for a.e. λ. The existence of the limit implies that for any $v \in \mathcal{H}$ $\|G\delta(\lambda, \varepsilon)G^*v\|$ is uniformly bounded in ε by the Banach-Steinhaus theorem. Hence $|||G\delta(\lambda, \varepsilon)G^*||| \leq C_1(\lambda)$ for a.e. λ. Similarly, $|||GR(\lambda \pm i\varepsilon)G^*|||$ and $G_0\delta_0(\lambda, \varepsilon)u$ for $u \in \mathfrak{D}_0$ are uniformly bounded in ε for a.e. λ. Further for any $v \in \mathcal{H}_0$

$$\pi^{-1}\varepsilon\|R(\lambda \mp i\varepsilon)G^*v\|^2 = (G\delta(\lambda, \varepsilon)G^*v, v) \leq C_1(\lambda)\|v\|^2 \quad (133)$$

which shows $\varepsilon^{1/2}|||GR(\lambda \pm i\varepsilon)||| \leq C_2(\lambda)$ for a.e. λ. Further $\|\varepsilon^{-1}R_0(\lambda \mp i\varepsilon)u\| \leq C_3(\lambda)$ for a.e. λ by relations (26) and (30).
Hence, there exists $\operatorname{w-lim}_{\varepsilon\to 0} GJ\delta(\lambda, \varepsilon)u$ for any $u \in \mathfrak{D}_0$ and a.e. λ. By lemma 5.42 we conclude that $\tilde{\Omega}_\pm(H_0, H_0, J^*J)$ exist. □

This concludes the last step. We can now sum up the conditions. By assembling propositions 5.41, 5.43 and 5.44, we obtain the existence of the strong wave operators $\Omega_\pm(H, H_0, J)$ under the following conditions

1. G and G_0 are bounded
2. $\exists \operatorname{s-lim}_{\varepsilon\to 0} G_0R_0(\lambda \pm i\varepsilon)u$ for a.e. λ for any $u \in \mathfrak{D}_0$, dense in $\mathcal{H}_0$
3. $\exists \operatorname{w-lim}_{\varepsilon\to 0} G\delta(\lambda, \varepsilon)v$ for a.e. λ for any $v \in \mathcal{H}$
4. $\exists \operatorname{w-lim}_{\varepsilon\to 0} GR(\lambda \pm i\varepsilon)G^*v$ for a.e. λ for any $v \in \mathcal{H}$

This can be simplified.

Lemma 5.45 *Condition 4 implies condition 3.*

Proof Fix $u \in \mathcal{H}_0$. Since $(G\delta(\lambda, \varepsilon)u, w)$ converges on a full measure set independent of the choice of w in S, it suffices to find an uniform bound in ε holding almost everywhere for $G\delta(\lambda, \varepsilon)u$. Condition 4 implies the existence of the weak limit of $G\delta(\lambda, \varepsilon)G^*$. In the equality

$$\|G\delta(\lambda, \varepsilon)u\| \leq \pi^{-1}(\varepsilon^{1/2}\||GR(\lambda - i\varepsilon)|\|)(\varepsilon^{1/2}\|R(\lambda + i\varepsilon)u\|) \tag{134}$$

the first factor on the right is uniformly bounded by the existence of the weak limit of $G\delta(\lambda, \varepsilon)G^*$ while the second is bounded uniformly by relations (26) and (30). □

Thus we have proved

Theorem 5.46 *Assume that the factors of the perturbation G and G_0 are bounded and that for elements $u \in \mathfrak{D}_0$, dense set in $\mathcal{H}_0$, and $v \in \mathcal{H}$ there exist*

$$s - \lim_{\varepsilon \to 0} G_0 R_0(\lambda \pm i\varepsilon)u, \quad w - \lim_{\varepsilon \to 0} GR(\lambda \pm i\varepsilon)G^* v, \quad \textit{for a.e. } \lambda \tag{135}$$

then the strong wave operators $\Omega_\pm(H, H_0, J)$ exist.

5.6 Scattering operators

5.6.1 Basic properties of the scattering operator

Definition Given a scattering system (H, H_0, J), the scattering operator S: $\mathcal{H}_0 \to \mathcal{H}_0$ is defined by $S = \Omega_+^*(H, H_0, J)\Omega_-(H, H_0, J)$.

Since $\mathcal{H}_0^s \subseteq \ker(\Omega_\pm)$, we see that $\mathcal{H}_0^s \subseteq \ker(S)$ and $Ran(S) \subseteq \mathcal{H}_0^{ac}$. Some basic properties follow from this definition.

Proposition 5.47 *S commutes with H_0.*

Proof The intertwining relation implies $SH_0 = \Omega_+^* H\Omega_- = H_0 S$. $\square$

Proposition 5.48 *S is a bounded operator of $\mathcal{H}_0$.*

Proof Since $\|\Omega_\pm f\| \leq |||J||| \|P_0^{ac} f\|$, $|||\Omega_\pm||| = |||\Omega_\pm^*||| \leq |||J|||$. Then $|||S||| = |||\Omega_+^*\Omega_-||| \leq |||J|||^2$. $\square$

Interesting properties of S are isometricity on $\mathcal{H}_0^{ac}$ (outcoming states are normalized) and unitarity (convenient for the inversion of scattering).

Proposition 5.49 *Assume that $\Omega_\pm$ are partial isometries on $\mathcal{H}_0^{ac}$. S is isometric iff $Ran(\Omega_-) \subseteq Ran(\Omega_+)$. S: $\mathcal{H}_0^{ac} \to \mathcal{H}_0^{ac}$ is an unitary mapping iff $Ran(\Omega_-) = Ran(\Omega_+)$.*

Proof $\|Sf\| = \|\Omega_+^*\Omega_- f\| = \|\Omega_- f\| = \|f\|$ for any $f \in \mathcal{H}_0^{ac}$. Then the isometricity of S on $\mathcal{H}_0^{ac}$ is equivalent to $\|\Omega_+^* g\| = \|g\|$ for any $g \in Ran(\Omega_-)$. By assumption and adjointness, Ω_+^* is a partial isometry on $Ran(\Omega_+)$. Then the isometricity of S on $\mathcal{H}_0^{ac}$ is equivalent to $Ran(\Omega_-) \subseteq Ran(\Omega_+)$.

If S: $\mathcal{H}_0^{ac} \to \mathcal{H}_0^{ac}$ is a unitary mapping, then $Ran(\Omega_-) \subseteq Ran(\Omega_+)$ by isometricity and $Ran(S) = \mathcal{H}_0^{ac}$ by surjectivity. Then, for any $g \in \mathcal{H}_0^{ac}$, $\exists h \in \mathcal{H}_0^{ac}$: $\Omega_+^*\Omega_- h = g$, that is $\Omega_+\Omega_+^*\Omega_- h = \Omega_+ g$. Note that $\Omega_+\Omega_+^*$ is the final projection on $Ran(\Omega_+) \supset Ran(\Omega_-)$, then for any $g \in \mathcal{H}_0^{ac}$, $\Omega_+ g \in Ran(\Omega_-)$. Then $Ran(\Omega_-) = Ran(\Omega_+)$.
Conversely, if $Ran(\Omega_+) = Ran(\Omega_-)$, S is isometric on $\mathcal{H}_0^{ac}$ and for any $f \in \mathcal{H}_0^{ac}$, $\exists g \in \mathcal{H}_0^{ac}$: $\Omega_+ f = \Omega_- g$, that is $\Omega_+^*\Omega_- g = P_0^{ac} f = f$, and S is surjective, hence unitary on $\mathcal{H}_0^{ac}$. $\square$

Corollary 5.50 *If $\Omega_\pm$ are partial isometric on $\mathcal{H}_0^{ac}$ and complete, then S is unitary, with $SS^* = S^*S = P_0^{ac}$.*

Proof In this case, $Ran(\Omega_-) = Ran(\Omega_+) = \mathcal{H}_0^{ac}$. And we have

$$SS^* = \Omega_+^*\Omega_-\Omega_-^*\Omega_+ = \Omega_+^* P^{ac}\Omega_+ = \Omega_+^*\Omega_+ = P_0^{ac}$$
$$S^*S = \Omega_-^*\Omega_+\Omega_+^*\Omega_- = \Omega_-^* P^{ac}\Omega_- = \Omega_-^*\Omega_- = P_0^{ac}$$

$\square$.

We now consider the decomposition of $\mathcal{H}_0^{ac}$ into a direct integral of Hilbert spaces $\int_{\hat{\sigma}_0} \bigoplus \mathfrak{h}_\lambda d\lambda$. Since S commutes with H_0^{ac}, it goes over into multiplication by an operator-valued function $S(\lambda)$. $S(\lambda)$ is called the scattering matrix. Before turning to stationary representations of S and $S(\lambda)$ we discuss a necessary technical issue.

5.6.2 Integral operators on direct integral representation

In this part, we still consider G: $\mathcal{H} \to \mathcal{K}$ bounded and H-bounded and we assume that there exist $\text{w-}\lim_{\varepsilon\to 0} GR(\lambda \pm i\varepsilon)G^*$ for almost every $\lambda \in \mathbb{R}$.
In the view of expressing the scattering matrix by stationary means we are interested in writing operators on an Hilbert space $\mathcal{H}_0^{ac}$ as integral operators with respect to the direct integral of Hilbert spaces $\mathfrak{h}^{ac} = \int_{\hat\sigma} \bigoplus \mathfrak{h}_\lambda d\lambda$.
We denote by $\mathscr{F}$ the unitary transformation from $\mathcal{H}^{ac}$ to $\mathfrak{h}^{ac}$ extended on the whole space $\mathcal{H}$ by setting $\mathscr{F}(\mathcal{H}^s) = \{0\}$. Given $u \in \mathcal{H}$, we note when it is possible $\mathscr{F}u = \mathscr{F}P^{ac}u = \tilde u$. We require a few preliminary facts.

Lemma 5.51 *For any $u, v \in \mathcal{H}$, there holds for almost every $\lambda \in \hat\sigma$*

$$\lim_{\varepsilon\to 0}(\delta(\lambda,\varepsilon)u, v) = \left\langle \tilde u(\lambda), \tilde v(\lambda)\right\rangle_\lambda \tag{136}$$

Proof We know that

$$\lim_{\varepsilon\to 0}(\delta(\lambda,\varepsilon)u, v) = \frac{d(E(\lambda)u,v)}{d\lambda} = \frac{d(E(\lambda)P^{ac}u,v)}{d\lambda}$$

Further, there holds for any Borel set Δ

$$(E(\Delta)P^{ac}u, v) = \int_{\Delta\cap\hat\sigma} \frac{d(E(\lambda)u,v)}{d\lambda} d\lambda$$

since $\hat\sigma$ is also a spectral core for the absolutely continuous part of $E(\cdot)$. We also have

$$(E(\Delta)P^{ac}u, v) = \int_{\Delta\cap\hat\sigma} \left\langle \tilde u(\lambda), \tilde v(\lambda)\right\rangle_\lambda d\lambda$$

Since this holds for any Borel set Δ, comparison of the two equations concludes the proof. □

Lemma 5.52 *Under the assumptions of this part, there exist for any $u \in \mathcal{H}$ and for almost every $\lambda \in \mathbb{R}$*

$$w\text{-}\lim_{\varepsilon\to 0} G\delta(\lambda,\varepsilon)u = w\text{-}\lim_{\varepsilon\to 0}(2\varepsilon)^{-1}G(E(\lambda+\varepsilon) - E(\lambda-\varepsilon))u \tag{137}$$

Proof We have already proved before that the existence of $\text{w-}\lim_{\varepsilon\to 0} GR(\lambda \pm i\varepsilon)G^*$ implies the existence of the limit on the left-hand side. On the right-hand side, we apply the standard scheme. Given a basis $\{w_i\}$ of $\mathcal{K}$, the measure $(E(\cdot)u, G^*w_i)$ has the symmetric derivative of its generating function almost everywhere defined on a full measure set Λ_i depending on w_i. By taking $\Lambda_\infty = \bigcap_i \Lambda_i$, we obtain the existence of $\lim_{\varepsilon\to 0}(2\varepsilon)^{-1}(G(E(\lambda+\varepsilon) - E(\lambda-\varepsilon))u, w)$ on a full measure set independent of the choice of w in S. It then remains to find an uniform bound in ε holding almost everywhere. It is based on the inequality

$$\gamma_{(\lambda-\varepsilon,\lambda+\varepsilon)}(\mu) \le \frac{2\varepsilon^2}{(\mu-\lambda)^2 + \varepsilon^2}$$

for any $\mu \in \mathbb{R}$. This implies by the spectral theorem that for any $u \in \mathcal{H}$

$$(E(\lambda-\varepsilon, \lambda+\varepsilon)u, u) \le 2\pi\varepsilon(\delta(\lambda,\varepsilon)u, u)$$

that is also $\|E(\lambda-\varepsilon,\lambda+\varepsilon)u\| \leq \sqrt{2}\varepsilon\|R(\lambda\pm i\varepsilon)u\|$. In particular, for $u = G^*w$, we get the estimate $\|E(\lambda-\varepsilon,\lambda+\varepsilon)G^*w\| \leq \sqrt{2}\varepsilon|||R(\lambda\pm i\varepsilon)G^*|||\|w\|$.
We have proved before that the existence of w-$\lim_{\varepsilon\to 0} GR(\lambda\pm i\varepsilon)G^*$ implies the norm estimate $\varepsilon^{1/2}|||GR(\lambda\pm i\varepsilon)||| \leq C(\lambda)$. Hence we obtain

$$|||GE(\lambda-\varepsilon,\lambda+\varepsilon)||| \leq C_1(\lambda)\varepsilon^{1/2}$$

Further on the full measure set on which the symmetric derivative is defined the sequence $(2\varepsilon)^{-1}(E(\lambda-\varepsilon,\lambda+\varepsilon)u,u)$ has to be bounded, that is $\|E(\lambda-\varepsilon,\lambda+\varepsilon)u\| \leq C_2(\lambda)\varepsilon^{1/2}$.
We can now compute for any $u \in \mathcal{H}$

$$(2\varepsilon)^{-1}\|G(E(\lambda+\varepsilon)-E(\lambda-\varepsilon))u\| \leq (2\varepsilon)^{-1}|||GE(\lambda-\varepsilon,\lambda+\varepsilon)|||\|E(\lambda-\varepsilon,\lambda+\varepsilon)u\|$$

By the estimate we computed, we obtain the uniform bound.
The equality of the two limits now follows directly from (29). □

We now turn to the construction of integral operators. In particular we are interested in operators of the form $A = G^*BG$ with B bounded operator on $\mathcal{K}$.
Given a basis $\{w_i\}$ on $\mathcal{K}$ we denote by Λ_i the set of full measure in $\hat{\sigma}$ for which $\mathscr{F}G^*w_i$ is defined. Note that since G is bounded, $\|\mathscr{F}G^*u\|_{\mathfrak{h}^{ac}} \leq |||G|||\|u\|$ and $\mathscr{F}G^*w_i$ belongs to the direct integral.
On the dense set of elements $w \in S$ for $\lambda \in \Lambda_\infty = \bigcap_i \Lambda_i$, we then define the operator $\Phi_G(\lambda)$ by

$$\Phi_G(\lambda)w = (\mathscr{F}G^*w)(\lambda)$$

By lemma 5.51, we obtain for any $w_1, w_2 \in S$

$$\langle\Phi_G(\lambda)w_1, \Phi_G(\lambda)w_2\rangle_\lambda = \lim_{\varepsilon\to 0}(G\delta(\lambda,\varepsilon)G^*w_1, w_2) \tag{138}$$

on a set Λ of full measure in $\hat{\sigma}$. Since the $G\delta(\lambda,\varepsilon)G^*$ are a bounded family of operators, $\Phi_G(\lambda)$ is bounded for $\lambda \in \Lambda$ and its domain of definition extends by continuity to $\mathcal{H}$. The relation (138) then turns to $\Phi_G(\lambda)^*\Phi_G(\lambda) =$w-$\lim_{\varepsilon\to 0} G\delta(\lambda,\varepsilon)G^*$ for $\lambda \in \Lambda$.
Note that for any $u \in \mathcal{H}$ and $w \in S$, there holds by lemma 5.51 and definition of $\Phi_G(\lambda)$

$$\frac{d(E(\lambda)u, G^*w)}{d\lambda} = \langle\tilde{u}(\lambda), \mathscr{F}G^*w\rangle_\lambda = (\Phi_G(\lambda)^*\tilde{u}(\lambda), w)$$

for a.e. $\lambda \in \hat{\sigma}$. By the boundedness of G and $\Phi_G(\lambda)$, this extends to any $w \in \mathcal{K}$.
We now define the operators $a(\mu,\nu) = \Phi_G(\mu)B\Phi_G(\nu)^*$ for $(\mu,\nu) \in \Lambda\times\Lambda$. By this definition, these are bounded mappings on $\mathfrak{h}_\nu$ into $\mathfrak{h}_\mu$. Then for any $u_1, u_2 \in \mathcal{H}^{ac}$

$$(Au_1,u_2) = (P^{ac}Au_1,u_2) = \int_{\hat{\sigma}} \frac{d(E(\mu)Au_1,u_2)}{d\mu}d\mu = \int_{\hat{\sigma}} \frac{d(G^*BGu_1, E(\mu)u_2)}{d\mu}d\mu$$

By relation (139), this gives

$$(Au_1,u_2) = \int_{\hat{\sigma}} (BGu_1, \Phi_G(\mu)^*\tilde{u}_2(\mu))d\mu$$

Similarly, there holds

$$\begin{aligned}(BGu_1, \Phi_G(\mu)\tilde{u}_2) &= (P^{ac}u_1, G^*B^*\Phi_G(\mu)^*\tilde{u}_2(\mu)) \\ &= \int_{\hat{\sigma}} \frac{d(E(\nu)u_1, G^*B^*\Phi_G(\mu)^*\tilde{u}_2(\mu))}{d\nu} d\nu \\ &= \int_{\hat{\sigma}} (\Phi_G(\nu)^*\tilde{u}_1(\nu), B^*\Phi_G(\mu)^*\tilde{u}_2(\mu)) d\nu \\ &= \int_{\hat{\sigma}} \langle a(\mu,\nu)\tilde{u}_1(\nu), \tilde{u}_2(\mu)\rangle_\mu d\nu\end{aligned}$$

Substituting this in the first relation we obtain

$$(Au_1, u_2) = \int_{\hat{\sigma}}\int_{\hat{\sigma}} \langle a(\mu,\nu)\tilde{u}_1(\nu), \tilde{u}_2(\mu)\rangle_\mu d\nu d\mu \tag{139}$$

We can clearly extend this scheme to the case $A = G_1^* B G_2$ where G_1 and G_2 satisfy the same assumptions as G. Then for any $\lambda \in \Lambda$ all the properties of $\Phi_G(\lambda)$ copy to $\Phi_{G_1}(\lambda)$ and $\Phi_{G_2}(\lambda)$. By defining the kernel $a_{12}(\mu,\nu) = \Phi_{G_1}(\mu)B\Phi_{G_2}(\nu)^*$, we obtain (139) just in the same way.
Similarly, we can extend this to finite sums of operators $A = G_1^{1*}BG_2^1 + \ldots + G_1^{r*}BG_2^r$.

We return to the case $A = G_1^* B G_2$. We intend to obtain a more useful representation of the sesquilinear form corresponding to the kernel $a(\mu,\nu)$.

$$\langle a_{12}(\mu,\nu)\tilde{u}_1(\nu), \tilde{u}_2(\mu)\rangle_\mu = (\Phi_{G_2}(\nu)^*\tilde{u}_1(\nu), B^*\Phi_{G_1}(\mu)^*\tilde{u}_2(\mu))$$

By relation (139) this is equal to

$$\frac{d}{d\nu}(E(\nu)u_1, G_2^*B^*\Phi_{G_1}(\mu)\tilde{u}_2(\mu)) = \frac{d}{d\nu}\frac{d}{d\mu}(G_1^*BG_2E(\nu)u_1, E(\mu)u_2)$$

for $(\mu,\nu) \in \Lambda_1 \times \Lambda_2$ where Λ_1 and Λ_2 are sets of full measure in $\hat{\sigma}$. By lemma 5.52, this is equal to

$$\frac{d}{d\nu}\lim_{\varepsilon\to 0}(BG_2E(\nu)u_1, \delta(\mu,\varepsilon)u_2) = \lim_{\eta\to 0}\lim_{\varepsilon\to 0}(B\delta(\nu,\eta)u_1, \delta(\mu,\varepsilon)) \tag{140}$$

for $(\mu,\nu) \in \tilde{\Lambda}_1 \times \tilde{\Lambda}_2$ where $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are sets of full measure in $\hat{\sigma}$.
We conclude this part

Theorem 5.53 *Suppose that G_1 and G_2 are bounded, H-bounded operators for which there exists for any $u \in \mathcal{H}$, any $j \in \{1,2\}$ and almost every $\lambda \in \hat{\sigma}$*

$$w - \lim_{\varepsilon\to 0} G_j\delta(\lambda,\varepsilon)G_j^* u \tag{141}$$

Suppose further that there exist both of the strong limits

$$s\text{-}\lim_{\varepsilon\to 0} G_1\delta(\lambda,\varepsilon)u_2, \quad s\text{-}\lim_{\varepsilon\to 0} G_2\delta(\lambda,\varepsilon)u_1 \tag{142}$$

Suppose there is a family $B(\zeta)$ of bounded operators on $\mathcal{K}$ such that $B(\zeta)$ converges weakly to B. Define further the operator $A(\zeta) = G_1B(\zeta)G_2^*$. Then the kernel a_{12} of the operator $A = G_1^*BG_2$ exists in the sense of relation (139) and there exists the limit

$$\lim_{\varepsilon\to 0}\big(A(\varepsilon)\delta(\nu,\varepsilon)u_1, \delta(\mu,\varepsilon)u_2\big) = \langle a_{12}(\mu,\nu)\tilde{u}_1(\nu), \tilde{u}_2(\mu)\rangle_\lambda \tag{143}$$

for a.e. $\mu \in \hat{\sigma}$ and a.e. $\nu \in \hat{\sigma}$.

Proof We study the convergence of the term

$$\big(A(\varepsilon)\delta(\nu,\varepsilon)u_1, \delta(\mu,\varepsilon)u_2\big) = \big(B(\varepsilon)G_2\delta(\nu,\varepsilon)u_1, G_1\delta(\mu,\varepsilon)u_2\big) \tag{144}$$

The right factor on the right-hand side converges strongly for a.e. $\lambda \in \hat{\sigma}$ by assumptions. The left factor $B(\varepsilon)$ converges weakly and $G_2\delta(\nu,\varepsilon)$ converges strongly so that $B(\varepsilon)G_2\delta(\nu,\varepsilon)$ converges weakly for a.e. $\lambda \in \hat{\sigma}$. Then the limit of the scalar product exists and coincides with

$$\lim_{\varepsilon\to 0} \big(BG_2\delta(\nu,\varepsilon)u_1, G_1\delta(\mu,\varepsilon)u_2\big) \tag{145}$$

Under the assumptions the iterated limit (140) exists and coincides with the limit (145). □

When $A = G_1BG_2^*$ does not depend on an additional parameter, the conditions can be relaxed.

Theorem 5.54 *Suppose that G_1 and G_2 are bounded, H-bounded operators for which there exists for any $u \in \mathcal{H}$, any $j \in \{1,2\}$ and almost every $\lambda \in \hat{\sigma}$*

$$w-\lim_{\varepsilon\to 0} G_j\delta(\lambda,\varepsilon)G_j^*u \tag{146}$$

Suppose further that there exists one of the strong limits

$$s\text{-}\lim_{\varepsilon\to 0} G_1\delta(\lambda,\varepsilon)u_2, \quad s\text{-}\lim_{\varepsilon\to 0} G_2\delta(\lambda,\varepsilon)u_1 \tag{147}$$

Suppose that B is a bounded operator on $\mathcal{K}$. Then the kernel a_{12} of the operator $A = G_1^*BG_2$ exists in the sense of relation (139) and there exists the limit

$$\lim_{\varepsilon\to 0} \big(A\delta(\nu,\varepsilon)u_1, \delta(\mu,\varepsilon)u_2\big) = \langle a_{12}(\mu,\nu)\tilde{u}_1(\nu), \tilde{u}_2(\mu)\rangle_\lambda \tag{148}$$

for a.e. $\mu \in \hat{\sigma}$ and a.e. $\nu \in \hat{\sigma}$.

Proof Similar to the proof of 5.53. By our assumptions, one strong limit in (147) is replaced with a weak one. Since A does not depend on an additional parameter, it is sufficient to obtain the convergence of the scalar product. □

In anticipation of the computation carried in section 5.6.4, we define the following operators for any $z \in \rho(H)$

$$T_+(z) = (J^* - G_0^*GR(z))G^*G_0, \quad T_-(z) = G_0^*G(J - R(z)G^*G_0)$$

We assume that the conditions of theorem 5.46 hold. Assume further that there also exists w-$\lim_{\varepsilon\to 0} G_0G_0^*$. $T_\pm(\lambda + i\varepsilon)$ splits in two parts. For instance, $T_+(\lambda+i\varepsilon) = (GJ)^*G_0 - G_0^*(GR(\lambda+i\varepsilon)G^*)G_0$. Under our assumptions, there exists w-$\lim_{\varepsilon\to 0} GR(\lambda+i\varepsilon)G^*$ and the conditions of theorem 5.53 are fulfilled with $G_1 = G_2 = G_0$ for any u_1, u_2 belonging to the dense set $\mathfrak{D}_0$. It remains to prove that there exists under our conditions w-$\lim_{\varepsilon\to 0}(GJ)\delta(\lambda,\varepsilon)(GJ)^*$. Then the conditions of theorem 5.54 are fulfilled with $G_1 = GJ$ and $G_2 = G_0$ for any u_1, u_2 belonging to the dense set $\mathfrak{D}_0$.

Lemma 5.55 *Suppose that there exist* $w\text{-}\lim_{\varepsilon\to 0} G_0R_0(\lambda\pm i\varepsilon)G_0^*$ *and* $w\text{-}\lim_{\varepsilon\to 0} GR(\lambda\pm i\varepsilon)G^*$. *Then there exist* $w\text{-}\lim_{\varepsilon\to 0}(GJ)R_0(\lambda\pm i\varepsilon)(GJ)^*$.

Proof It is sufficient to find a bound of $(GJ)\delta(\lambda,\varepsilon)(GJ)^*$ uniform in ε and holding almost everywhere. By the resolvent identity, we have

$$|||GJR_0(\lambda\pm i\varepsilon)||| \leq \underbrace{|||GR(\lambda\pm i\varepsilon)|||}_{\leq\varepsilon^{-1/2}C_1(\lambda)}\,|||J|||+\underbrace{|||GR(\lambda\pm i\varepsilon)G^*|||}_{\leq C_2(\lambda)}\,\underbrace{|||G_0R_0(\lambda\pm i\varepsilon)|||}_{\leq\varepsilon^{-1/2}C_3(\lambda)}$$

where the bounds were already computed before under these assumptions and hold together almost everywhere. Hence

$$|||GJ\delta(\lambda,\varepsilon)(GJ)^*||| = \pi^{-1}\varepsilon|||GJR_0(\lambda+i\varepsilon)||| \leq C_4(\lambda)$$

holding for a.e. $\lambda\in\mathbb{R}$. □

Then by theorems 5.53 and 5.54 w-$\lim_{\varepsilon\to 0} T_\pm(\lambda+i\varepsilon)$ are integral operators on $\mathfrak{h}_0^{ac}$ with kernels $\mathfrak{t}_\pm(\mu,\nu;\lambda)$ for a.e. $\lambda\in\mathbb{R}$. For any $u_1,u_2\in\mathfrak{D}_0$, there holds

$$\lim_{\varepsilon\to 0}(T_\pm(\lambda+i\varepsilon)\delta(\nu,\varepsilon)u_1,\delta(\mu,\varepsilon)u_2) = \langle\mathfrak{t}_\pm(\mu,\nu;\lambda)\tilde{u}_1(\nu),\tilde{u}_2(\mu)\rangle_\mu \tag{149}$$

for a.e. $\mu\in\hat{\sigma}_0$ and a.e. $\nu\in\hat{\sigma}_0$.

5.6.3 Stationary representation of the scattering operator

We assume that the conditions of theorem (5.46) hold so that the strong wave operators $\Omega_\pm(H,H_0,J)$ exist. We recall that the existence and the equality of the limits we compute are defined for almost every λ. Under these assumptions, we can compute by the resolvent identity for any u_1,u_2 in $\mathfrak{D}_0$, dense set of $\mathcal{H}_0$

$$\begin{aligned}\pi^{-1}\varepsilon(JR_0(\lambda-i\varepsilon)u_1,R(\lambda-i\varepsilon)\mathscr{U}_+u_2) &= (Ju_1,\delta(\lambda,\varepsilon)\mathscr{U}_+u_2)\\ &\quad+(G_0R_0(\lambda-i\varepsilon)u_1,G\delta(\lambda,\varepsilon)\mathscr{U}_+u_2)\end{aligned}$$

We have already shown that the first term on the right-hand side has a limit, namely

$$\lim_{\varepsilon\to 0}(Ju_1,\delta(\lambda,\varepsilon)\mathscr{U}_+u_2) = \lim_{\varepsilon\to 0}\pi^{-1}\varepsilon(Ju_1,R(\lambda-i\varepsilon)JR_0(\lambda+i\varepsilon)u_2)$$

For the second term on the right-hand side, we know that

$$\mathrm{w}-\lim_{\varepsilon\to 0}G\delta(\lambda,\varepsilon)\mathscr{U}_\pm u = \mathrm{w}-\lim_{\varepsilon\to 0}\pi^{-1}\varepsilon GR(\lambda\mp i\varepsilon)JR_0(\lambda\pm i\varepsilon)u$$

and under our assumptions, $G_0R_0(\lambda-i\varepsilon)u_1$ converges strongly. Hence we obtain

$$\begin{aligned}&a_-(u_1,\mathscr{U}_+u_2;\lambda)\\ &\quad=\lim_{\varepsilon\to 0}\pi^{-1}\varepsilon\Big(\big(J+G^*G_0R_0(\lambda-i\varepsilon)\big)u_1,R(\lambda-i\varepsilon)JR_0(\lambda+i\varepsilon)u_2\Big)\end{aligned} \tag{150}$$

Substituting this in the integral representation of $\mathscr{U}_-$, we obtain for any Borel set Δ

$$
\begin{aligned}
&(\mathscr{U}_- E_0(\Delta)u_1, \mathscr{U}_+ u_2) \\
&\quad = \int_\Delta \lim_{\varepsilon\to 0} \pi^{-1}\varepsilon \left(\left(J + G^* G_0 R_0(\lambda - i\varepsilon)\right) u_1, R(\lambda - i\varepsilon) J R_0(\lambda + i\varepsilon) u_2 \right) d\lambda
\end{aligned}
\tag{151}
$$

Since $S = \Omega_+^* \Omega_- = \mathscr{U}_+^* \mathscr{U}_-$, we have in particular for $\Delta = \mathbb{R}$

$$
\begin{aligned}
&(Su_1, u_2) \\
&\quad = \int_{\mathbb{R}} \lim_{\varepsilon\to 0} \pi^{-1}\varepsilon \left(\left(J + G^* G_0 R_0(\lambda - i\varepsilon)\right) u_1, R(\lambda - i\varepsilon) J R_0(\lambda + i\varepsilon) u_2 \right) d\lambda
\end{aligned}
\tag{152}
$$

We now note for simplicity $R = R(\lambda + i\varepsilon)$ and $R^* = R(\lambda - i\varepsilon)$ and similarly for R_0. Note that the integrand can be transformed by means of the relation

$$
\begin{aligned}
\pi^{-1}\varepsilon R_0^* J^* R(J + G^* G_0 R_0^*) &= \pi^{-1}\varepsilon (J^* R^* + R_0^* G_0^* G R^*) R (J + G^* G_0 R_0^*) \\
= (J + R_0^* G_0^* G)\delta(\lambda,\varepsilon)(J + G^* G_0 R_0^*) &= \pi^{-1}\varepsilon (J^* + R_0^* G_0^* G) R (R^* J + R^* G^* G_0 R_0^*) \\
&= \pi^{-1}\varepsilon (J^* + R_0^* G_0^* G) R J R_0^*
\end{aligned}
$$

Substituting it in the integral representation, we obtain another stationary representation for the scattering operator.

5.6.4 Stationary representation of the scattering matrix

We resume our computation of the integrand of the integral representation by using $JR_0 - RJ = RG^* G_0 R_0$ a couple of times

$$
\begin{aligned}
&\pi^{-1}\varepsilon (J^* + R_0^* G_0^* G) R J R_0^* \\
&\qquad = \pi^{-1}\varepsilon (J^* + R_0 G_0^* G) R J R_0^* + \pi^{-1}\varepsilon (R_0^* - R_0) G_0^* G R J R_0^*
\end{aligned}
$$

By the relation $J^* R + R_0 G_0^* G R = R_0 J^*$ and by the definition of $\delta(\lambda,\varepsilon)$, we obtain

$$
\pi^{-1}\varepsilon (J^* + R_0^* G_0^* G) R J R_0^* = \pi^{-1}\varepsilon R_0 J^* J R_0^* - 2i\varepsilon \delta_0(\lambda,\varepsilon) G_0^* G R J R_0^*
$$

By the relation $RJ = JR_0 - RG^* G_0 R_0$ and by the definition of $T_-(\lambda + i\varepsilon)$, we obtain

$$
\pi^{-1}\varepsilon (J^* + R_0^* G_0 G) R J R_0^* = \pi^{-1}\varepsilon R_0 J^* J R_0^* - 2\pi i \delta_0(\lambda,\varepsilon) T_-(\lambda + i\varepsilon) \delta_0(\lambda,\varepsilon)
$$

Similarly we have

$$
\begin{aligned}
&\pi^{-1}\varepsilon R_0^* J^* R(J + G^* G_0 R_0^*) \\
&\qquad = \pi^{-1}\varepsilon R_0^* J^* R (J + G^* G_0 R_0) + \pi^{-1}\varepsilon R_0^* J^* R G^* G_0 (R_0^* - R_0)
\end{aligned}
\tag{153}
$$

$$
\pi^{-1}\varepsilon R_0^* J^* R(J + G^* G_0 R_0^*) = \pi^{-1}\varepsilon R_0^* J^* J R_0 - 2\pi i \delta_0(\lambda,\varepsilon) T_+(\lambda + i\varepsilon) \delta_0(\lambda\varepsilon)
$$

All in all, we have shown

$$\begin{aligned}&\pi^{-1}\varepsilon\left((J+G^*G_0R_0^*)u_1, R^*JR_0u_2\right)\\&=\pi^{-1}\varepsilon(JR_0(\lambda\pm i\varepsilon)u_1, JR_0(\lambda\pm i\varepsilon)u_2)-2\pi i(T_\pm(\lambda+i\varepsilon)\delta_0(\lambda,\varepsilon)u_1,\delta_0(\lambda,\varepsilon)u_2)\end{aligned} \tag{154}$$

Under our assumptions the second term on the right-hand side has a limit as $\varepsilon \to 0$ since the two other terms of the equality have one. We take the limit in (154) and integrate over λ on a Borel set Δ. Using the integral representations we already gathered for the scattering operator and for $\mathscr{U}_\pm^0 = \mathscr{U}_\pm(H_0, H_0, J^*J)$, we obtain for any Borel set Δ

$$\begin{aligned}(\mathscr{U}_- E_0(\Delta)u_1, \mathscr{U}_+ u_2) &= (\mathscr{U}_\pm^0 E_0(\Delta)u_1, u_2)\\&\quad - 2\pi i\int_\Delta \lim_{\varepsilon\to 0}(T_\pm(\lambda+i\varepsilon)\delta_0(\lambda,\varepsilon)u_1, \delta_0(\lambda,\varepsilon)u_2)d\lambda\end{aligned}$$

We recall that in fact the integral is taken over $\Delta\cap\hat\sigma_0$. Indeed the integrand vanishes outside of $\hat\sigma_0$ as the difference of $a_-(E_0(\Delta)u_1, \mathscr{U}_+u_2;\lambda)$ and $a_\pm^0(E_0(\Delta)u_1, u_2) = \lim_{\varepsilon\to 0}(J^*JR_0(\lambda\pm i\varepsilon)E_0(\Delta)u_1, R_0(\lambda\pm i\varepsilon)u_2)$.

$\mathscr{U}_\pm^0$ commutes with H_0^{ac} by its intertwining property and by $\mathcal{H}_0^s \subseteq \ker(\mathscr{U}_\pm^0)$, $Ran(\mathscr{U}_\pm^0)\subseteq \mathcal{H}_0^{ac}$. Then to $\mathscr{U}_\pm^0$ corresponds an operator-valued function $\mathfrak{u}_\pm^0(\lambda)$ in the direct integral of Hilbert spaces $\int_{\hat\sigma_0}\bigoplus \mathfrak{h}_\lambda d\lambda$. The operators $\mathfrak{u}_\pm^0(\lambda)$ are bounded for almost every $\lambda\in\hat\sigma_0$. We note again $\tilde u_1(\lambda)$ and $\tilde u_2(\lambda)$ the vector-valued functions images respectively of $P_0^{ac}u_1$ and $P_0^{ac}u_2$ by the unitary transformation $\mathcal{H}_0^{ac}\to\int_{\hat\sigma_0}\bigoplus\mathfrak{h}_\lambda d\lambda$. This unitarity implies that $(P_0^{ac}u_1, u_2) = \int_{\hat\sigma_0}\langle\tilde u_1(\lambda), \tilde u_2(\lambda)\rangle_\lambda d\lambda$. In particular, there holds in our case for any Borel set Δ

$$\begin{aligned}\Big(E_0(\Delta)(S-\mathscr{U}_\pm^0)u_1, u_2\Big) &= \int_{\hat\sigma_0}\Big\langle\gamma_\Delta(\lambda)(S(\lambda)-\mathfrak{u}_\pm^0(\lambda))\tilde u_1(\lambda), \tilde u_2(\lambda)\Big\rangle_\lambda d\lambda\\&= \int_{\hat\sigma_0\cap\Delta}\Big\langle(S(\lambda)-\mathfrak{u}_\pm^0(\lambda))\tilde u_1(\lambda), \tilde u_2(\lambda)\Big\rangle_\lambda d\lambda\end{aligned}$$

As this holds for any Borel set Δ, we obtain by comparison with the precedent relation for almost every $\lambda\in\hat\sigma_0$

$$\begin{aligned}&\Big\langle(S(\lambda)-\mathfrak{u}_\pm^0(\lambda))\tilde u_1(\lambda), \tilde u_2(\lambda)\Big\rangle_\lambda\\&\qquad = -2\pi i\lim_{\varepsilon\to 0}(T_\pm(\lambda+i\varepsilon)\delta_0(\lambda,\varepsilon)u_1, \delta_0(\lambda,\varepsilon)u_2)\end{aligned} \tag{155}$$

Since $\lambda\in\hat\sigma_0$, there is some set of full measure in $\hat\sigma_0$ on which we can write the relation (149) in the special case $\mu=\nu=\lambda$. By substituting this relation in (155), we obtain for any $u_1, u_2\in\mathfrak{D}_0$ and a.e. $\lambda\in\hat\sigma_0$

$$\Big\langle(S(\lambda)-\mathfrak{u}_\pm^0(\lambda))\tilde u_1(\lambda), \tilde u_2(\lambda)\Big\rangle_\lambda = -2\pi i\left\langle\mathfrak{t}_\pm(\lambda,\lambda;\lambda)\tilde u_1(\lambda), \tilde u_2(\lambda)\right\rangle_\lambda$$

$\mathfrak{D}_0$ is dense in $\mathcal{H}_0$ and $S(\lambda)$, $\mathfrak{u}_\pm^0(\lambda)$ and $\mathfrak{t}_\pm(\lambda,\lambda;\lambda)$ are bounded for a.e. $\lambda\in\hat\sigma_0$, then we obtain by lemma (2.7)

$$S(\lambda) = \mathfrak{u}_\pm^0(\lambda) - 2\pi i\mathfrak{t}_\pm(\lambda,\lambda;\lambda) \tag{156}$$

In quantum physics the kernel $\mathfrak{t}_\pm(\lambda,\lambda;\lambda)$ is called the transfer matrix.

References

[1] H. Baumgärtel and M. Wollenberg. *Mathematical scattering theory.* Birkhäuser Verlag, Basel, 1983.

[2] M.S. Birman and M.Z. Solomyak. *Spectral theory of selfadjoint operators.* D. Reidel Publishing Company, Dordrecht, 1987.

[3] M. Blümlinger. Funktional Analysis 2, Skriptum, WS 2007.

[4] N. Dunford and J. Schwartz. *Linear operators Parts 1,2.* Interscience, New York, 1958,1963.

[5] G. B. Folland. *Real analysis: modern techniques and their applications.* Wiley-Interscience, New York, 1999.

[6] P.R. Halmos. *Measure theory.* Springer Verlag, New York, 1974.

[7] P.R. Halmos. *Introduction to Hilbert space and the theory of spectral multiplicity.* American Mathematical Society, 2000.

[8] E. Hille and R.S. Phillips. *Functional analysis and semi-groups.* American Mathematical Society, 1957.

[9] P.D. Hislop and I.M. Sigal. *Introduction to spectral theory with applications to Schrödinger operators.* Springer-Verlag, New York, 1955.

[10] K. Hoffman. *Banach spaces of analytic functions.* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.

[11] T. Kato. *Perturbation theory for linear operators.* Springer-Verlag, Berlin Heidelberg, 1980.

[12] D. Pearson. *Quantum scattering and spectral theory.* Academic Press, San Diego, CA, 1988.

[13] I.I. Priwalow. *Randeigenschaften analytischer Funktionen.* Deutscher Verlag der Wissenschaft, Berlin, 1956.

[14] M. Reed and B. Simon. *Methods of modern mathematical physics, Vols. 1, 2, 3, 4.* Academic Press, San Diego, CA, 1972,1975,1979,1978.

[15] G. Teschl. *Mathematical methods in quantum mechanics with applications to Schrödinger operators.* American Mathematical Society, 2009.

[16] J. Weidmann. *Linear operators in Hilbert spaces.* Springer Verlag, New York, 1980.

[17] J. Weidmann. *Spectral theory of ordinary differential operators.* Springer Verlag, Berlin Heidelberg, 1987.

[18] D.R. Yafaev. *Mathematical scattering theory.* American Mathematical Society, 1992.

[19] D.R. Yafaev. Lecture notes on scattering theory, 2001.